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Chapter 1: Pure Computable Model Theory

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1 Introduction

Exploiting the fundamental concepts of computability theory, computable model theory introduces effective analogues of model theoretic notions. By combining methods from both fields, it has enabled the development of machinery for investigating the *effective* content of model theoretic constructions. While some model theoretic constructions can be replaced by effective ones, for others such replacement is impossible. Thus, another important objective for computable model theory is the discovery of effective counterexamples to model theoretic results. For instance, Vaught's theorem (no complete theory has exactly two non-isomorphic countable models) cannot be effectivized.

The article begins with the foundations of computable model theory: the definitions and examples of decidable theories, and computable and decidable models. It then presents the effective completeness theorem and the effective omitting types theorem; and characterizations of decidable theories with decidable prime models, and then with decidable saturated models. The next sections characterize decidable homogeneous models, and give examples of decidable theories with exactly two non-isomorphic decidable models. The following sections present the results on decidable theories with only finitely many, and on decidable theories with only countably many, non-isomorphic countable models, and investigate the model theoretic nature and the computability theoretic complexity of models of such theories. Later sections study indiscernibles from the computability theoretic point of view, and the degrees of models. Finally, we consider the isomorphisms of effective models and related subtopics, such as intrinsically c.e. relations, computably stable models, and computably categorical models.

Computable model theory was developed simultaneously and for the most part independently in the West, mainly in the United States and Australia, and in Russia. Because of poor communication between the two groups, many results were independently discovered by both groups. This article looks at computable model theory from the Western perspective. (There are articles in this volume on the Russian approach.) However, the article also presents some results of the Russian group, and often emphasizes the connections with and gives references to their results.

Almost every section contains a detailed proof with a survey of the computability theoretic and model theoretic background needed. The bibliography contains both Western and Russian papers in pure computable model theory, but not papers in computable algebra nor in computable combinatorics. Another survey article on this subject has been recently and independently written by Millar [149].

2 History

The goal of computable mathematics is to find the extent to which certain classical results of mathematics are effectively true. Although many consider the modern study of computability of algebraic constructions to have started with Fröhlich and Shepherdson in 1955–56 and Rabin in 1958–60, even van der Waerden in his book [202] from 1930, see also [203], discussed the problem of carrying out certain field-theoretic procedures effectively. He also defined an *explicitly* given field as one whose elements are uniquely represented by distinguishable symbols with which one can perform the field operations effectively. In a pioneering paper from 1930, van der Waerden [201] proved that there does not exist a splitting algorithm applicable to every explicit field. Fröhlich and Shepherdson [62, 63] used the precise notion of a *computable* function to obtain a collection of results and examples about explicit fields.

Rabin [177, 178] did a systematic study of *computable* groups and *computable* fields. In Russia, a systematic study of *constructive* algebraic systems and their *enumerations* was initiated by Mal'cev [125] in the 1960's, and continued by Ershov and his collaborators, see [56, 57].

In the 1970's, Nerode and his collaborators revived the study of computability of algebraic constructions. At the 1974 *Recursive Model Theory Symposium* at Monash University (Melbourne, Australia), Metakides and Nerode announced that, in addition to other computability theoretic tools, they started using the priority method as an important tool in the algorithmic part of computable mathematics, see [129]. Thus, they founded in the West the field of the post-Friedberg-Muchnik computable mathematics. Metakides and Nerode used the priority method in their systematic study of the effective content of specific structures, such as vector spaces [130], fields [131], and structures with a dependence relation [132]. For more information on the development of computable mathematics see [38, 133, 184]. In Russia, the post-Friedberg-Muchnik constructive mathematics was founded by Nurtazin and Goncharov in the 1970's [79, 160].

In the West, the computability of ordered sets has also been studied by Ash, Case, Chen, Crossley, Downey, Feiner, Feldman, Fellner, Hay, Hingston, Hird, Jockusch, Kierstead, Knight, Lerman, Manaster, Metakides, McNulty, Moses, Remmel, Richter, Rosenstein, Roy, Schmerl, Schwarz, Soare, Tennenbaum, Trotter and Watnick; the computability of vector spaces by Ash, Guhl, Guichard, Dekker, Downey, Hamilton, Kalantari, Remmel, Retzlaff, Shore, Smith and Welch; the computability of rings and fields by Ash, Hodges, Jockusch, MacIntyre, Madison, Marker, Mines, Rosenthal, Seidenberg, Shlapentokh, Smith, Staples, Tucker and van den Dries; the computability of the structures with a dependence relation by Baldwin, Downey and Remmel. The computability in other mathematical structures is also extensively studied: in groups by Ash, Barker, Ge, Kent, Knight, Lin, Oates, Richards, Richman and Smith;

in graphs by Aharoni, Bean, Beigel, Burr, Carstens, Gasarch, Golze, Kierstead, Lockwood, Manaster, Magidor, Päppinghaus, Remmel, Rosenstein, Schmerl and Shore; in Boolean algebras by Carroll, Feiner, LaRoche, Remmel and Thurber; in topological spaces by Kalantari, Legett, Remmel, Retzlaff and Weitkamp. Computable Ramsey's theory has been studied by Clote, Hummel, Jockusch, Seetapun, Simpson, Solovay and Specker. Computability in analysis and physics has also been studied, see [176].

The generalization of the definition of a particular computable algebraic structure to an arbitrary model yields one of the basic concepts of pure computable model theory, an area of logic developed in the last twenty-five years. That is, the notion of a *computable* model, and a stronger notion of a *decidable* model. Lerman and Schmerl have given examples of theories with computable models. The first general results in computable model theory have been obtained by following the fundamental notions and results of classical model theory. For example, Millar has obtained the effective version of the omitting types theorem, and Harrington, Goncharov and Nurtazin have found when a complete decidable theory with a prime model has a decidable prime model. Millar and Morley have characterized decidable theories with decidable saturated models, and Goncharov and Peretyat'kin have characterized decidable homogeneous models. Barwise, Schlipf and Ressayre have introduced the notion of a computably saturated model. Although developed in the context of admissible sets and admissible fragments of infinitary logic, computably saturated models have also provided a useful tool for research and exposition in classical model theory.

In the West, Millar has further produced an extensive body of work on topics including effective Vaught's theorem, the structure of types in decidable models, decidability and prime, saturated and homogeneous models, decidable theories with finitely many and decidable theories with countably many non-isomorphic countable models. Reed has also studied decidable theories with finitely many non-isomorphic countable models. Kierstead and Remmel have investigated the degrees of sets of indiscernibles in decidable models. Ash, Knight, Macintyre, Marker, Nadel, Nies, Richter, Jockusch, Lachlan, Scott, Shoenfield, Shore, Soare and Tennenbaum have studied the degrees of models of various theories, including the theory of linear orders, Peano arithmetic, true arithmetic, and the theory of Boolean algebras.

The whole spectrum of questions involving the isomorphisms of abstract computable models has been investigated by Ash, Barker, Chisholm, Cholak, Crossley, Downey, Eisenberg, Guichard, Harizanov, Hird, Khoussainov, Knight, Manasse, Manaster, Millar, Moses, Nerode, Remmel, Shore, Slaman and Wehner. The lattices of computably enumerable submodels have been studied by Ash, Guichard, Carroll, Downey, Metakides, Nerode, Remmel and Smith. More recently, Nerode, Remmel and Cenzer [31, 158] have been developing feasible model theory (as a part of feasible mathematics), the theory of models with bounded space and time resources. They have investigated how feasible models differ from computable models. The feasible models studied include Boolean algebras, abelian groups, linear orders, models of arithmetic, and graphs.

3 Notation and Basic Definitions

The set $\{0, 1, 2, \ldots\}$ of all natural numbers is denoted by ω. Unless explicitly stated otherwise, it is assumed that all languages considered are first-order and computable (hence at most countable), and that the domains of the considered models are subsets of ω. For a set of sentences T, by $L(T)$ we denote its language. A set of sentences T is deductively closed if T contains every sentence σ of $L(T)$ such that $T \vdash \sigma$. A consistent deductively closed set of sentences is called a *theory*.

Models are denoted by script letters, and their domains by the corresponding capital Latin letters. By $\mathcal{A} \subseteq \mathcal{B}$ we denote that $\mathcal{A}$ is a submodel of $\mathcal{B}$, and by $\mathcal{A} \preceq \mathcal{B}$ that $\mathcal{A}$ is an elementary submodel of $\mathcal{B}$. By $\mathcal{A} \equiv \mathcal{B}$ we denote that $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent, and by $\mathcal{A} \cong \mathcal{B}$ that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. A model is *prime* if it can be elementarily embedded in every model of its theory. Hence a prime model for a countable language must be countable. Two prime models of the same complete theory are isomorphic.

Let $\mathcal{A}$ be a model (with domain A) for L. By $Th(\mathcal{A})$ we denote the theory of $\mathcal{A}$. For $X \subseteq A$, let L_X be the language $L \cup \{\mathbf{a} : a \in X\}$, L expanded by adding a constant $\mathbf{a}$ for every $a \in X$. Let $\mathcal{A}_X = (\mathcal{A}, a)_{a \in X}$ be the expansion of $\mathcal{A}$ to the language L_X such that for every $a \in X$, $\mathbf{a}$ is interpreted by a. The *atomic diagram* of $\mathcal{A}$ is the set of all atomic and negated atomic sentences of L_A which are true in $\mathcal{A}_A$. It is denoted by $\Delta_{\mathcal{A}}$. The *complete diagram* of $\mathcal{A}$ is the set of all sentences of L_A which are true in $\mathcal{A}_A$. The complete diagram of $\mathcal{A}$ is often called an elementary diagram of $\mathcal{A}$.

A sequence of variables displayed after a formula or after a set of formulae includes all the free variables occurring in any of the formulae. For two sequences $\overline{x}$ and $\overline{y}$ of the same length k, by writing $\overline{x}/\overline{y}$ after a formula or a set of formulae, we denote the result of replacing every occurrence of $\overline{y}(i)$ by $\overline{x}(i)$ for $i < k$. To simplify the notation, instead of $\theta(y)(x/y)$ we often write only $\theta(x)$. For a set of formulae Θ, $\wedge\Theta$ is the conjunction of all formulae in Θ. For a formula θ, let $\theta^1 =_{def} \theta$ and $\theta^0 =_{def} \neg\theta$.

A formula is in a $\Sigma^0_0 = \Pi^0_0$ form if it contains no quantifiers. For $n > 0$, a formula is in a Σ^0_n (Π^0_n, respectively) form if it is logically equivalent to a formula in a prenex normal form which begins with an existential (universal) quantifier and has $n-1$ alternations of quantifiers. Σ^0_1 (Π^0_1, respectively) sentences are also called existential (universal, respectively). $T_\exists$ ($T_\forall$, respectively) denotes the set of all existential (universal, respectively) sentences in T. For infinite cardinals κ and λ, $L_{\kappa\lambda}$ denotes the infinitary logic which has κ individual variables, allows conjunction and disjunction of a set of $< \kappa$ formulae, and allows universal and existential quantification over a set of $< \lambda$ individual variables. In particular, $L_{\omega\omega}$ is classical first-order logic, and $L_{\omega_1\omega}$ allows countable conjunctions and disjunctions but only finite quantification. For more information on infinitary logic see [99].

A *type* of a theory T in variables $x_0, \ldots, x_{n-1}$ is a maximal consistent set of formulae containing T, with free variables among $x_0, \ldots, x_{n-1}$. To emphasize its maximality, it is often called a *complete* type in the literature. An

n–type is a type in n variables, and a (finite) type is an n–type for some $n \in \omega$. If $\Gamma(x_0, \ldots, x_{n-1})$ is a type and $x_{i_0}, \ldots, x_{i_{k-1}} \in \{x_0, \ldots, x_{n-1}\}$, then $\Gamma \restriction \{x_{i_0}, \ldots, x_{i_{k-1}}\}$ is the subtype of Γ in variables $x_{i_0}, \ldots, x_{i_{k-1}}$. A formula $\theta(x_0, \ldots, x_{n-1})$ is *complete* in T if for every formula $\psi(x_0, \ldots, x_{n-1})$, exactly one of

$$T \vdash (\forall x_0) \ldots (\forall x_{n-1})[\theta(x_0, \ldots, x_{n-1}) \Rightarrow \psi(x_0, \ldots, x_{n-1})],$$

$$T \vdash (\forall x_0) \ldots (\forall x_{n-1})[\theta(x_0, \ldots, x_{n-1}) \Rightarrow \neg\psi(x_0, \ldots, x_{n-1})]$$

holds. That is, there is exactly one (complete) type of T in $x_0, \ldots, x_{n-1}$ which contains θ. A type which contains a complete formula is called *principal*. The set of all (complete finite) types realized in a model $\mathcal{A}$ is called the *type spectrum* of $\mathcal{A}$. A type spectrum of a theory is the type spectrum of one of its models.

Let X be a set. $|X|$ denotes the cardinality of X. X is *countable* if $|X| = \omega$. X is *at most countable* if $|X| \leq \omega$. Let κ be an infinite cardinal, and let T be a complete theory in a countable language. T is called *stable in power* κ, or κ*–stable*, if for an arbitrary model $\mathcal{U}$ of T, for every subset X of U with $|X| = \kappa$, the model $\mathcal{U}_X$ realizes exactly κ many 1-types. T is called *stable* if it is stable in some power. If T is $\aleph_0$–stable, then T is stable in every infinite power (see Chapter VII of [32]). T is called *superstable* if it is κ–stable for every $\kappa \geq 2^{\aleph_0}$. For more information on stability theory see [22, 121, 175].

The quantifier $\exists! x$ abbreviates "there exists a unique x". The empty set is denoted by $\emptyset$. For a set X, $\mathcal{P}(X)$ is its power set. If f is a partial function, then $dom(f)$ is the domain of f, $rng(f)$ is the range of f, and $f(a)\downarrow$ denotes that $a \in dom(f)$. The length of a sequence $\overline{x}$ is denoted by $lh(\overline{x})$. If $\overline{x} = (x_0, \ldots, x_{n-1})$ and f is a unary function, then $f(\overline{x}) =_{def} (f(x_0), \ldots, f(x_{n-1}))$. The concatenation of sequences is denoted by $\frown$. A set $\mathcal{T}$ of sequences of numbers is a *tree* if it is closed under subsequences. The empty sequence is the *root* of $\mathcal{T}$. Elements of $\mathcal{T}$ are also called *nodes*. A *branch* of $\mathcal{T}$ is a maximal linearly ordered subset of $\mathcal{T}$. The terminal node of a finite branch of $\mathcal{T}$ is a *leaf*.

Let $\phi_0^{(n)}, \phi_1^{(n)}, \phi_2^{(n)}, \ldots$ be a fixed effective enumeration of all n–ary partial computable functions. If $X \subseteq \omega$, let $\phi_0^{(n),X}, \phi_1^{(n),X}, \phi_2^{(n),X}, \ldots$ be a fixed effective enumeration of all n–ary X–computable functions. The superscripts are usually omitted for $n = 1$ or when it is clear from the context. ϕ_e (ϕ_e^X) is also denoted by $\{e\}$ ($\{e\}^X$), and e is called the Gödel number or index of ϕ_e. We write $\phi_{e,s}(n) = m$ if $e, n, m < s$ and m is the output of $\phi_e(n)$ after $< s$ steps in the corresponding computation. Let $W_e =_{def} dom(\phi_e)$ and $W_{e,s} =_{def} dom(\phi_{e,s})$. Thus, $W_0, W_1, W_2, \ldots$ is a computable enumeration of all c.e. sets. We fix $\langle \cdot\, , \cdot \rangle$ to be a computable bijection from ω^2 onto ω, which is strictly increasing with respect to both arguments. For $X \subseteq \omega$ and $i \in \omega$, we define $X^{[i]} = \{k : \langle k, i \rangle \in X\}$.

Let $X \subseteq \omega$ and $Y \subseteq \omega$. The join $X \oplus Y$ is

$$\{2n : n \in X\} \cup \{2n+1 : n \in Y\}.$$

By $X \leq_T Y$ we denote that X is Turing reducible to Y. $X <_T Y$ denotes $X \leq_T Y$ but $Y \not\leq_T X$. $X \equiv_T Y$ if $X \leq_T Y$ and $Y \leq_T X$. $\deg(X)$ denotes the Turing degree of X. Let $\mathbf{0} =_{def} \deg(\emptyset)$. If $\mathbf{x} = \deg(X)$ and $n \geq 1$, then $\mathbf{x}^{(n)} =_{def} \deg(X^{(n)})$, where $X^{(n)}$ is the n-th jump of X. Define $X^{(\omega)} = \{\langle k, n\rangle : k \in X^{(n)} \wedge k, n \in \omega\}$ and $\mathbf{x}^{(\omega)} = \deg(X^{(\omega)})$. A degree $\mathbf{x}$ is *low* if $\mathbf{x}' = \mathbf{0}'$. Turing degrees $\mathbf{x}$ and $\mathbf{y}$ form a *minimal pair* if they are nonzero and for every Turing degree $\mathbf{z}$,

$$(\mathbf{z} \leq \mathbf{x} \wedge \mathbf{z} \leq \mathbf{y}) \Longrightarrow \mathbf{z} = \mathbf{0}.$$

The set of all Turing degrees is denoted by $\mathcal{D}$. For more information on classical computability theory see [109, 161, 98, 196]. An ordinal is *computable* if it is finite or is the order type of a computable well-ordering on ω. The computable ordinals form a countable initial segment of the ordinals. Kleene's $\mathcal{O}$ is the set of notations for computable ordinals, with the corresponding partial ordering $<_{\mathcal{O}}$, see [98, 189]. The least non-computable ordinal is denoted by ω_1^{CK}, where CK stands for Church-Kleene. To obtain hyperarithmetic sets, we define the representative sets in the hyperarithmetic hierarchy, H_a for $a \in \mathcal{O}$. The definition is recursive, and is based on iterating the Turing jump:

$$H_1 = \emptyset,$$

$$H_{2^a} = (H_a)',$$

$$H_{3\cdot 5^e} = \{2^x \cdot 3^n : x \in H_{\{e\}(n)}\}.$$

A set of natural numbers X is *hyperarithmetic* if $(\exists a \in \mathcal{O})[X \leq_T H_a]$. The hyperarithmetic sets coincide with the Δ^1_1 sets.

4 Decidable Theories, and Computable and Decidable Models

Computable model theory explores the effectiveness of constructions and theorems in model theory, see [32, 44, 92, 188], and in universal algebra, see [37, 80, 126]. It begins by defining effective analogues of classical concepts of algebra and model theory. Three of its fundamental concepts are: decidable theories, computable models and decidable models. One of the basic problems is determining whether computable or decidable models satisfying certain conditions exist.

Definition 4.1. (i) A theory T is *decidable* if T is a computable set of sentences.

(ii) A model A is *computable* if its domain A is computable and its relations and functions are uniformly computable. That is, A is computable if A is computable and there is a computable enumeration $(a_i)_{i\in\omega}$ of A such that the atomic diagram of $\mathcal{A}$ is decidable.

(iii) A model $\mathcal{A}$ is *decidable* if A is computable and there is a computable enumeration $(a_i)_{i\in\omega}$ of A such that the complete diagram of $\mathcal{A}$ (that is, $Th((\mathcal{A}, a_i)_{i\in\omega})$) is decidable.

We assume that a formula is identified with its Gödel number, so a set of formulae is thought of as a subset of ω. Thus, a theory is decidable (resp. belongs to $\mathcal{P}$, where $\mathcal{P}$ is a complexity class) if the set of Gödel numbers of its sentences is computable (resp. belongs to $\mathcal{P}$). Hence, if Ax is a set of axioms of a theory T, then T is decidable if there is an algorithm which determines for every sentence σ of L, whether $Ax \vdash \sigma$. Clearly, a computably axiomatizable theory is computably enumerable. Hence a complete computably axiomatizable theory is decidable. In particular, a complete finitely axiomatizable theory is decidable. An example of such a theory is the theory of dense linear order. Peretyat'kin [168, 169, 170, 171, 172, 173, 174] has developed intricate methods for constructing finitely axiomatizable theories satisfying various additional properties. In [168], he constructed a complete, finitely axiomatizable, $\aleph_1$–categorical theory which is not $\aleph_0$–categorical. Well-known and important examples of decidable theories in mathematics include the theory of equality, the theory of unary predicates, the additive number theory, the theory of the field of real numbers, the theory of the field of complex numbers, the theory of algebraically closed fields, the theory of real-closed fields, the theory of p-adic fields, the theory of Boolean algebras, the theory of linear order, the theory of abelian groups, and the theory of free commutative algebras. Well-known and important examples of undecidable theories in mathematics include number theory, the theory of simple groups, the theory of semigroups, the theory of rings, the theory of fields, the theory of distributive lattices, and the theory of partial order. For more information on decidable and undecidable theories see [58] and Part III in [150]. For computability theoretic complexity of various sets of sentences satisfied in certain classes of models see [204].

A model $\mathcal{A}$ is *computable* if A is computable, and if there is a computable enumeration $(a_i)_{i\in\omega}$ of A and an algorithm which determines, for every quantifier-free formula $\theta(x_0,\ldots,x_{n-1})$ and every sequence $(a_{i_0},\ldots,a_{i_{n-1}}) \in A^n$, whether $\mathcal{A}_A \models \theta(\mathbf{a}_{i_0},\ldots,\mathbf{a}_{i_{n-1}})$. A model $\mathcal{A}$ is *decidable* if A is computable and there is a computable enumeration $(a_i)_{i\in\omega}$ of A and an algorithm which determines for every formula $\theta(x_0,\ldots,x_{n-1})$ and every sequence $(a_{i_0},\ldots,a_{i_{n-1}}) \in A^n$, whether $\mathcal{A}_A \models \theta(\mathbf{a}_{i_0},\ldots,\mathbf{a}_{i_{n-1}})$. Clearly, every decidable model is computable. The converse is not true. For example, $(\omega,+,\times)$ is a computable model which is not decidable (by Gödel's incompleteness theorem [64]). Peretyat'kin [163] has constructed a decidable linear order without a computable proper elementary extension. In [160], Nurtazin characterized decidable models which are isomorphic to computable non-decidable models. Peretyat'kin [166] has shown that there is a complete decidable theory T which is neither $\aleph_0$-categorical nor $\aleph_1$-categorical, and which has, up to isomorphism, a unique decidable model. Moreover, all computable models of T are decidable.

A model is *computably presentable* if it is isomorphic to a computable model. Goncharov [66] has constructed an $\aleph_1$–categorical theory which is not $\aleph_0$–categorical and whose only computably presentable model is the prime model. On the other hand, Khoussainov, Nies and Shore [101] have shown that there is an $\aleph_1$–categorical theory which is not $\aleph_0$–categorical and whose only countable non-computably presentable model is the prime model. It is sometimes convenient to call a model computable (decidable, etc.) even if it is only computably (decidably, etc.) presentable.

Morozov [152, 153, 154] has extensively studied the automorphisms of computable models. He constructed a decidable model [152] whose theory is $\aleph_0$–categorical and which does not have non-trivial computable automorphisms. He also constructed a computable model [153] with $2^{\aleph_0}$ many automorphisms and without a non-trivial hyperarithmetic automorphism.

The notion of a computable (resp. decidable) model corresponds to the notion of a *constructive* (resp. *strongly constructive*) model used by the group in Novosibirsk. A constructive (resp. strongly constructive) model is a pair $(\mathcal{A}, \nu)$, where $\mathcal{A}$ is a countable model, and ν is a function from ω onto the domain of $\mathcal{A}$, such that the model "induced on ω by $\mathcal{A}$ via ν^{-1}" is computable (resp. decidable). ν is called a *constructivization* (resp. *strong constructivization*) of $\mathcal{A}$. For example, the field of rational numbers has a constructivization, while the group of all computable permutations of ω does not.

In general, the *Turing degree* of a model $\mathcal{A}$ with finite language is the least upper bound of the Turing degrees of its universe, and its relations and functions. Hence a model is computable if its Turing degree is zero. Isomorphic models may have different Turing degrees. Tennenbaum [199] has proved that there is no computable nonstandard model of Peano arithmetic. Scott and Tennenbaum [193] have established that every degree $\mathbf{d}$ such that $\mathbf{d} > \mathbf{0}'$ is a degree of a complete extension of Peano arithmetic, and that no computably enumerable degree $\mathbf{d}$ such that $\mathbf{d} < \mathbf{0}'$ can be a degree of a complete extension of Peano arithmetic. Jockusch and Soare [94] have shown that there is a nonstandard model of Peano arithmetic of low degree. Jockusch and Soare [96] have proved that for every non-zero c.e. degree $\mathbf{d}$, there is a linear order of degree $\mathbf{d}$ which is not isomorphic to any computable linear order. Lerman and Schmerl [122] have given a number of examples of important theories with computable models.

By a *theory of linear order* we mean a theory whose language consists of a binary relation symbol, and which contains the axioms of linear order. Lerman and Schmerl [122] have extended Peretyat'kin's [164] result that every c.e. (Σ^0_1) theory of linear order has a computable model, by showing that every Σ^0_2 theory of linear order has a computable model. They have also constructed a Δ^0_3 theory of linear order without a computable model. Lerman and Schmerl have further shown that if $\mathbf{x}$ is a Turing degree such that $\mathbf{x} \not\leq \mathbf{0}''$, then there is a theory of linear order of degree $\mathbf{x}$ without a computable model.

Definition 4.2. (Millar [140]) Let $\mathcal{P}$ be a class of theories. A theory T is *persistently* $\mathcal{P}$ if for every $n \in \omega$, for every complete n–type $\Gamma(x_0, \ldots, x_{n-1})$ of T and a sequence $c_0, \ldots, c_{n-1}$ of new constants, the theory $\Gamma(c_0, \ldots, c_{n-1})$ belongs to $\mathcal{P}$.

In [55], Ershov has studied persistently $\forall$–*finitely axiomatizable* theories. A theory T is $\forall$–finitely axiomatizable if for every theory S extending T, $S_\forall$ is finitely axiomatizable. For examples of persistently $\forall$–finitely axiomatizable theories see [55, 103]. Ershov [55] has established that every c.e. theory extending a persistently $\forall$–finitely axiomatizable theory has a computable model. This result implies the previously mentioned result that every c.e. theory of linear order has a computable model. It also implies that every c.e. (Σ^0_1) theory of trees has a computable model. By a *theory of trees* we mean a theory whose language consists of a binary relation symbol, and which contains the axioms of a partially ordered set such that the set of all predecessors of any element is linearly ordered. Lerman and Schmerl [122] have constructed a Δ^0_2 theory of trees without a computable model. Lerman and Schmerl have further shown that for every Turing degree $\mathbf{x}$ such that $\mathbf{x} \not< \mathbf{0}'$, there is a complete theory of trees of degree $\mathbf{x}$ without a computable model.

Lerman and Schmerl [122] have proved that if T is an arithmetic $\aleph_0$–categorical theory such that for every $n \in \omega$, the set of all Σ^0_{n+2} sentences in T is a Σ^0_{n+1} set, then T has a computable model. They have also shown that for every $n \in \omega$, and a Turing degree $\mathbf{x}$ such that $\mathbf{x} \not\leq \mathbf{0}^{(n)}$, there is an $\aleph_0$–categorical theory T of degree $\mathbf{x}$ such that the set of all Σ^0_{n+1} sentences in T is computable and T does not have a computable model. In particular, for every Turing degree $\mathbf{x}$, there is an $\aleph_0$–categorical theory of degree $\mathbf{x}$ such that the set of all existential sentences in T is computable and every model of T has the degree $\geq \mathbf{x}$.

Feldman [60, 61] has constructed a complete decidable $\aleph_0$–categorical theory T of a partial order with the greatest lower bound operator. T has a decidable model in which every countable lower semilattice can be embedded. Knight [111] has constructed a complete, decidable, superstable theory T with $2^{\aleph_0}$ many types, such that no independent sequence of formulae (with respect to T) is computable in a type of T. A sequence $(\sigma_n(\overline{x}))_{n\in\omega}$ of formulae in $L(T)$ is independent with respect to T if for every $\alpha \in 2^{<\omega}$,

$$T \vdash (\exists \overline{x})[\bigwedge_{\alpha(n)=1} \sigma_n(\overline{x}) \wedge \bigwedge_{\alpha(n)=0} \neg\sigma_n(\overline{x})].$$

Hurlburt [93] has given some general conditions which are sufficient to construct computable models for highly non-decidable theories.

According to the Ryll-Nardzewski theorem, a complete theory T is $\aleph_0$–categorical if and only if for every $n \in \omega$, the set of all n–types of T is finite. For such a theory T, the function which assigns to every n the number of all n–types of T is called *Ryll-Nardzewski function.* Schmerl [191], Herrmann [90] and Venning [205] have proved independently that a complete decidable $\aleph_0$–categorical theory does not necessarily have a computable Ryll-Nardzewski function. More generally, the following relativized result holds.

Theorem 4.3. (*Schmerl [191]*) *For every Turing degree* $\mathbf{x}$*, there is a function* $f : \omega \to \omega$ *of degree* $\mathbf{x}$ *such that for every Turing degree* $\mathbf{y}$ *with the property that* $\mathbf{x}$ *is c.e. in* $\mathbf{y}$*, there is a complete* $\aleph_0$*–categorical theory of degree* $\mathbf{y}$ *(in a language consisting of one binary relation symbol) whose Ryll-Nardzewski function is* f*.*

We can assume that the characteristic function of a consistent set $\Gamma(\overline{x})$ of formulae in L is a function $\chi : \omega \to \{0, 1\}$, defined by:

$$\chi_{\Gamma(\overline{x})}(k) = \begin{cases} 1 & \text{if } \theta_k(\overline{x}) \in \Gamma(\overline{x}), \\ 0 & \text{if } \theta_k(\overline{x}) \notin \Gamma(\overline{x}), \end{cases}$$

where $\theta_0(\overline{x}), \theta_1(\overline{x}), \theta_2(\overline{x}), \ldots$ is an effective enumeration of all formulae in L whose free variables are among those in $\overline{x}$. The set $\Gamma(\overline{x})$ is computable if its characteristic function is computable. Equivalently, $\Gamma(\overline{x})$ is computable if the set $\{n : \theta_n(\overline{x}) \in \Gamma(\overline{x})\}$ is computable.

Proposition 4.4. *Every type realized in a decidable model is computable.*

Proof. Let $\mathcal{A}$ be a decidable model such that a type $\Gamma(x_0, \ldots, x_{n-1})$ of $Th(\mathcal{A})$ is realized in $\mathcal{A}$ by some $a_0, \ldots, a_{n-1} \in A$. Since $\mathcal{A}$ is decidable and

$$\gamma(x_0, \ldots, x_{n-1}) \in \Gamma \Longleftrightarrow \mathcal{A}_A \models \gamma(\mathbf{a}_0, \ldots, \mathbf{a}_{n-1}),$$

Γ must be computable. □

A set of codes of a set of computable (complete) types of a theory T is a set of Gödel numbers of characteristic functions (which are computable) of these types, containing at least one index for each type. We say that a set of computable types belongs to $\mathcal{P}$, where $\mathcal{P}$ is a complexity class, if it has a set of codes which belongs to $\mathcal{P}$. The following proposition follows from a more general proposition in the theory of enumerations (see Chapter VI of [57]).

Proposition 4.5. *Every Σ^0_{n+1} set of codes of a set of computable types of a theory T is a Π^0_n set of codes.*

Hence, every c.e. set of codes of a set of computable types is a computable set of codes. To determine the complexity of the set of types realized in a decidable model, we need from *computability theory* the *s-m-n* theorem.

Theorem 4.6. (*i*) (*s-m-n theorem*) *For every $m, n \geq 1$, there is an $(m+1)$–ary computable function, denoted by s^m_n, such that*

$$\phi^{(m+n)}_e(l_1, \ldots, l_m, k_1, \ldots, k_n) = \phi^{(n)}_{s^m_n(e,l_1,\ldots,l_m)}(k_1, \ldots, k_n),$$

where $e, l_1, \ldots, l_m, k_1, \ldots, k_n \in \omega$.

(*ii*) (*Relativized s-m-n theorem*) *For every $m, n \geq 1$ and every oracle $X \subseteq \omega$, there is an $(m+1)$–ary computable function, denoted by s^m_n, such that*

$$\phi^{(m+n),X}_e(l_1, \ldots, l_m, k_1, \ldots, k_n) = \phi^{(n),X}_{s^m_n(e,l_1,\ldots,l_m)}(k_1, \ldots, k_n),$$

where $e, l_1, \ldots, l_m, k_1, \ldots, k_n \in \omega$.

Proposition 4.7. *The set of all types of T realized in a decidable model of T is computable.*

Proof. Let $\mathcal{A}$ be a decidable model of T and let $a_0, a_1, a_2, \ldots$ be an effective enumeration of A. Choose $g : A^{<\omega} \to \omega$ to be a computable bijection. Define a computable function $h : \omega^2 \to \{0, 1\}$ by:

$$h(n,k) = \begin{cases} 1 & \text{if } \mathcal{A} \models \theta_k[\overline{a}], \\ 0 & \text{if } \mathcal{A} \not\models \theta_k[\overline{a}], \end{cases}$$

where $g(\overline{a}) = n$, and $\theta_0, \theta_1, \theta_2, \ldots$ is an effective enumeration of all formulae of $L(T)$ whose free variables are among $\overline{x} = (x_{i_0}, \ldots, x_{i_{l-1}})$, corresponding to $\overline{a} = (a_{i_0}, \ldots, a_{i_{l-1}})$. By the *s-m-n* theorem, $h(n,k) = \phi_{f(n)}(k)$ for some computable function f. Clearly, $\{f(n) : n \in \omega\}$ is a c.e. set which is a set of codes of all (computable) types of T realized in $\mathcal{A}$. □

Proposition 4.8. *Let T be a decidable theory.*

(*i*) *The set of all Gödel numbers of all computable types of T is a Π^0_2 set.*

(*ii*) *Every principal type of T is a computable type, and the set of all principal types of T is a Π^0_1 set.*

Proof. For a given sequence $\overline{x}$ of variables, let $\theta_0(\overline{x}), \theta_1(\overline{x}), \theta_2(\overline{x}), \ldots$ be a computable enumeration of all formulae of $L(T)$ with all free variables contained in $ran(\overline{x})$.

(i) For $e \in \omega$, ϕ_e is the characteristic function of a computable type of T in variables $\overline{x}$ if and only if

$$\forall n \exists s \forall j \leq n \exists k_j \in \{0,1\}[\phi_{e,s}(j) \downarrow = k_j \wedge T \vdash \exists \overline{x}(\bigwedge \{\theta_j^{k_j}(\overline{x}) : j \leq n\})].$$

(ii) Every principal type of T is computable because it is generated by a complete formula. For every $i \in \omega$, use $\emptyset'$ to determine whether $\theta_i(\overline{x})$ is a complete formula. That is, $\theta_i(\overline{x})$ is a complete formula if and only if

$$\forall j \exists k \in \{0,1\}[T \vdash \forall \overline{x}(\theta_i(\overline{x}) \Rightarrow \theta_j^k(\overline{x}))].$$

Hence by the relativized *s-m-n* theorem, we can enumerate with oracle $\emptyset'$:

The principal type that $\theta_i(\overline{x})$ generates, if $\theta_i(\overline{x})$ is a complete formula;

Any fixed principal type of T, if $\theta_i(\overline{x})$ is not a complete formula.

Thus, since the sets which are computably enumerable in $\emptyset'$ are Σ^0_2, it follows that the set of all principal types is Σ^0_2. □

Proposition 4.9. (*Millar [145]*) *Every Σ^0_2 set of computable types of a decidable theory T is contained in a computable set of computable types of T.*

Nerode and his collaborators have also initiated the study of the lattice of all computably enumerable submodels of a computable model. Models whose computably enumerable submodels have been investigated include vector spaces, fields, Boolean algebras, and linear orders. For more information see [7, 30, 45, 47, 48, 82, 81, 157, 159].

Moses [155] has generalized the concepts of computable and decidable models to "Γ–computably enumerable models," where Γ is a computably enumerable set of formulae. For such a set Γ, a model $\mathcal{A}$ for $L(\Gamma)$ is Γ–computably enumerable if the universe of $\mathcal{A}$ is computable, and its satisfaction predicate restricted to Γ is computably enumerable. For other notions of an "effective model" and of an "effective isomorphism," see [181] and [50].

5 Effective Completeness Theorem

One of the major tasks of computable model theory is to obtain effective versions of or effective counterexamples to various classical model theoretic results. To obtain an effective version of the completeness theorem, we use from *model theory*, Henkin's method of constructing models; and from *computability theory*, the notion of a computable set and Church's thesis.

Theorem 5.1. *(Effective Completeness Theorem) A decidable theory has a decidable model.*

Proof. Let T be a decidable theory. A corresponding model of T will be obtained in an effective way by Henkin's method. Let $c_0, c_1, c_2, \dots$ be an effective one-to-one enumeration of an infinite set C of new constants. Let $\sigma_0, \sigma_1, \sigma_2, \dots$ be an effective enumeration of all sentences in $L(T) \cup C$. We will construct effectively, by induction, a complete theory Ψ in $L(T) \cup C$ such that $\Psi \supseteq T$. Ψ will be the complete diagram of a model $\mathcal{A}_A$, where $\mathcal{A}$ is a desired model for T. As usual, the domain A consists of the equivalence classes of the constants in C, where two constants $c, d \in C$ are equivalent if and only if $(c = d) \in \Psi$. We will arrange that $\Psi = \{\delta_0, \delta_1, \delta_2, \dots\}$, where δ_s is defined at stage s. For $s > 0$, let $\psi^s =_{def} \delta_0 \wedge \delta_1 \wedge \dots \wedge \delta_{s-1}$. *Construction*

Stage 0:
Let $\delta_0 =_{def} (c_0 = c_0)$.

Stage $s = 2e + 1$ for $e \in \omega$ (Henkin's witnesses requirement):
If δ_e is of the form $\delta_e = \exists x \theta(x)$, we effectively find the least i such that c_i does not occur in ψ^s and let $\delta_s =_{def} \theta(c_i)$. Otherwise, let $\delta_s =_{def} (c_0 = c_0)$.

Stage $s = 2e + 2$ for $e \in \omega$ (Completeness of the diagram requirement):
Let $\overline{c}$ be a sequence of all constants in C which occur in $(\psi^s \Rightarrow \sigma_e)$. Let $\overline{x}$ be the first sequence of variables of the same length as $\overline{c}$ (in some fixed effective enumeration of the finite sequences of all variables) which do not occur in $(\psi^s \Rightarrow \sigma_e)$. We effectively check whether

$$T \vdash \forall \overline{x}[(\psi^s \Rightarrow \sigma_e)(\overline{x}/\overline{c})]. \qquad (*)$$

If this is true, let $\delta_s =_{def} \sigma_e$. Otherwise, let $\delta_s =_{def} \neg\sigma_e$. End of the construction. Condition $(*)$ can be verified effectively because T is a decidable theory. We describe the action at stage $2e + 1$ as effectively providing a Henkin's witness for δ_e, and the action at stage $2e + 2$ as effectively satisfying the e–th completeness requirement. □

Proposition 5.2. (*Millar [145]*) *Every computable type of a theory T is realized in some decidable model of T.*

Proof. Assume that $\Gamma = \Gamma(x_0, \dots, x_{n-1})$ is a computable type of a theory T. Let $c_0, \dots, c_{n-1}$ be constants which do not occur in Γ. $T \cup \Gamma(c_0, \dots, c_{n-1})$ is a complete decidable theory in $L(T) \cup \{c_0, \dots, c_{n-1}\}$, so it has a decidable model $\mathcal{A}$. The reduct of $\mathcal{A}$ to $L(T)$ is a decidable model of T realizing Γ. □

6 Model Completeness and Decidability

Many examples of decidable theories constructed to illustrate certain model theoretic or computability theoretic properties are obtained as model completions of universal theories, which allow the elimination of quantifiers.

Definition 6.1. A theory T is *model complete* if for any two models $\mathcal{A}$ and $\mathcal{B}$ of T,

$$\mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A} \preceq \mathcal{B}.$$

Neither one of completeness and model completeness implies the other.

Theorem 6.2. *A theory T in a language L is model complete*
$\iff$ *For every $\mathcal{A} \models T$, the theory $T \cup \Delta_{\mathcal{A}}$ is complete in L_A.*
$\iff$ *If $\mathcal{A}$ and $\mathcal{B}$ are models of T and $\mathcal{A} \subseteq \mathcal{B}$, then every existential sentence of L_A true in $\mathcal{B}_A$ is also true in $\mathcal{A}_A$.*
$\iff$ *For every formula $\theta(\overline{x})$, there is a universal formula $\psi(\overline{x})$ such that*

$$T \vdash \forall \overline{x}[\theta(\overline{x}) \Leftrightarrow \psi(\overline{x})].$$

Definition 6.3. T is a *model completion* of a theory T' if
$(\forall \mathcal{A} \models T)(\mathcal{A} \models T')$,
$(\forall \mathcal{A} \models T')(\exists \mathcal{B} \models T)[\mathcal{A} \subseteq \mathcal{B}]$, and
$(\forall \mathcal{D} \models T')(\forall \mathcal{A}, \mathcal{B} \models T)[(\mathcal{D} \subseteq \mathcal{A} \wedge \mathcal{D} \subseteq \mathcal{B}) \Rightarrow \mathcal{A}_D \equiv \mathcal{B}_D]$.

A model completion of a theory is a model complete theory.

Theorem 6.4. (*Robinson*) *If T_1 and T_2 are* model completions *of T', then $T_1 = T_2$.*

A theory T^* is a *model companion* of a theory T if T^* is model complete and $T_\forall = T^*_\forall$. For example, the theory of atomless Boolean algebras is a model companion of the theory of Boolean algebras, and the theory of algebraically closed fields is a model companion of the theory of fields. Both the theory of atomless Boolean algebras and the theory of algebraically closed fields are decidable. Burris [29] has established some general criteria for a model companion to be decidable.

Definition 6.5. T is *submodel complete* if for every model $\mathcal{B}$ of T and every $\mathcal{A} \subseteq \mathcal{B}$, the theory $T \cup \Delta_{\mathcal{A}}$ is complete in L_A.

Hence a submodel complete theory is both complete and model complete.

Definition 6.6. (Robinson) A model completion of a universal theory is submodel complete.

We say that T *admits the elimination of quantifiers* if for every formula $\theta(x_0, \dots, x_{n-1})$, there is a quantifier-free formula $\psi(x_0, \dots, x_{n-1})$ such that

$$T \vdash \forall x_0, \dots, x_{n-1}[\theta(x_0, \dots, x_{n-1}) \Leftrightarrow \psi(x_0, \dots, x_{n-1})].$$

If there is an algorithm which for every formula $\theta(\overline{x})$ finds the corresponding quantifier-free formula $\psi(\overline{x})$, then we say that T *effectively admits the elimination of quantifiers*.

Proposition 6.7. (i) *Let T be a theory which effectively admits the elimination of quantifiers. Then every computable model of T is a decidable model of T.*

(ii) *Let T be a computably enumerable theory which admits the elimination of quantifiers. Then every computable model of T is a decidable model of T.*

Proof. (i) The statement follows immediately from the definitions of a computable and of a decidable model.

(ii) The statement follows from (i) because if T is a computably enumerable theory which admits the elimination of quantifiers, then T effectively admits the elimination of quantifiers. ☐

Theorem 6.8. *A theory T is submodel complete*

$\Longleftrightarrow (\forall \mathcal{A}, \mathcal{B} \models T)(\forall \mathcal{D} \subseteq \mathcal{A}, \mathcal{B})$ [*$\mathcal{A}$ and $\mathcal{B}$ satisfy the same existential sentences in $L(T)$ with parameters from D*]

$\Longleftrightarrow$ *T admits the elimination of quantifiers.*

Millar has characterized universal theories which have decidable model completions, thus providing a uniform approach for producing specific examples of decidable theories.

To state this characterization, we fix a language L and let $\theta_0, \theta_1, \theta_2, \ldots$ be an effective enumeration of all quantifier-free formulae of L in variables $x_0, x_1, x_2, \ldots; y_0, y_1, y_2, \ldots$. The convention will be that if the free variables of a formula are displayed, then the free x–variables (if any) are displayed before the free y–variables (if any).

Theorem 6.9. (*Millar [135]*) *Assume that T' is a universal theory in L. T' has a* (*complete*) *decidable model completion if and only if there is a unary computable function f such that for every $i \in \omega$, $\theta_{f(i)}$ does not contain any y–variable and for all $i, j \in \omega$:*

(i) *(θ_i is inconsistent with T') $\Leftrightarrow \theta_{f(i)} = \neg(x_0 = x_0)$,*

(ii) *$T' \vdash \forall\overline{x}[\exists\overline{y}\theta_i(\overline{x}, \overline{y}) \Rightarrow \theta_{f(i)}(\overline{x})]$,*

(iii) *If θ_i does not contain any x–variable and is consistent with T', then $\theta_{f(i)} = (x_0 = x_0)$,*

(iv) *($T' \cup \{\theta_{f(i)}(\overline{x}), \theta_j(\overline{x}, \overline{y}^*)\}$ is consistent)*

$\Longrightarrow$ ($T' \cup \{\theta_i(\overline{x}, \overline{y}), \theta_j(\overline{x}, \overline{y}^*)\}$ is consistent),

where $rng(\overline{y}) \cap rng(\overline{y}^*) = \emptyset$.

Notice that, by (ii), the implication in (iv) can be replaced by the equivalence. Property (iv) is often called the *amalgamation property*.

Proof. Assume that T is a decidable model completion of T'. By Theorem 6.6, T admits the elimination of quantifiers. Thus, there is a unary computable function f which has the following properties:

(a) (θ_i is inconsistent with T') $\Leftrightarrow \theta_{f(i)} = \neg(x_0 = x_0)$;

(b) If θ_i is consistent with T and does not contain an x–variable, then $\theta_{f(i)} = (x_0 = x_0)$;

(c) If θ_i is consistent with T, then $T \vdash \forall\overline{x}[\exists\overline{y}\theta_i(\overline{x},\overline{y}) \Leftrightarrow \theta_{f(i)}(\overline{x})]$.

Clearly, (i) and (iii) are satisfied. Let us prove (ii). Assume otherwise. If follows that

$$T' \cup \{\exists\overline{x}\exists\overline{y}[\theta_i(\overline{x},\overline{y}) \wedge \neg\theta_{f(i)}(\overline{x})]\}$$

is consistent. Hence it has a model $\mathcal{A}$. Since T is a model completion of T', there is a model $\mathcal{B}$ of T such that $\mathcal{A} \subseteq \mathcal{B}$. Hence $\mathcal{B}$ is a model of

$$T \cup \{\exists\overline{x}\exists\overline{y}[\theta_i(\overline{x},\overline{y}) \wedge \neg\theta_{f(i)}(\overline{x})]\}.$$

Since T is complete, we have

$$T \vdash \exists\overline{x}\exists\overline{y}[\theta_i(\overline{x},\overline{y}) \wedge \neg\theta_{f(i)}(\overline{x})].$$

That is,

$$T \vdash \neg\forall\overline{x}[\forall\overline{y}\neg\theta_i(\overline{x},\overline{y}) \vee \theta_{f(i)}(\overline{x})]$$

or, equivalently,

$$T \vdash \neg\forall\overline{x}[\exists\overline{y}\theta_i(\overline{x},\overline{y}) \Rightarrow \theta_{f(i)}(\overline{x})],$$

thus contradicting (c). Hence (ii) holds. Finally, let us prove (iv). Assume that $rng(\overline{y}) \cap rng(\overline{y}^\star) = \emptyset$. Let

$$T' \cup \{\exists\overline{x}\exists\overline{y}^\star[\theta_{f(i)}(\overline{x}) \wedge \theta_j(\overline{x},\overline{y}^\star)]\}$$

be consistent. By the same argument as in the proof of (ii), we conclude that $T \vdash \exists\overline{x}\exists\overline{y}^\star[\theta_{f(i)}(\overline{x}) \wedge \theta_j(\overline{x},\overline{y}^\star)]$. Hence, by (c),

$$T \vdash \exists\overline{x}\exists\overline{y}^\star[\exists\overline{y}\theta_i(\overline{x},\overline{y}) \wedge \theta_j(\overline{x},\overline{y}^\star)].$$

That is, $T \cup \{\theta_i(\overline{x},\overline{y}), \theta_j(\overline{x},\overline{y}^\star)\}$ is consistent. Thus, since T is a model completion of T', $T' \cup \{\theta_i(\overline{x},\overline{y}), \theta_j(\overline{x},\overline{y}^\star)\}$ is consistent. To prove the converse, we assume that a universal theory T' and a unary computable function f satisfy (i)-(iv). Let T be obtained by adding to T' the following two sets of axioms:

Ax I $\forall\overline{x}\forall\overline{y}\neg\theta_i(\overline{x},\overline{y})$ for all $i \in \omega$ such that $\theta_{f(i)} = \neg(x_0 = x_0)$;

Ax II $\forall\overline{x}[\theta_{f(i)}(\overline{x}) \Rightarrow \exists\overline{y}\theta_i(\overline{x},\overline{y})]$ for all $i \in \omega$ such that $\theta_{f(i)} \neq \neg(x_0 = x_0)$.

Clearly $T \supseteq T'$. We will show that T is a decidable model completion of T'.

Lemma 6.10. *T is consistent.*

Proof. We will prove that the union of T' and the two sets of axioms is consistent. If σ is an axiom in Ax I, then $T' \vdash \sigma$ by (i). Therefore, by the compactness argument, it is enough to prove that for every finite set S of axioms in Ax II, $T' \cup S$ is consistent. Let

$$S = \{\forall\overline{x}\exists\overline{y}[\theta_{f(i_k)}(\overline{x}) \Rightarrow \theta_{i_k}(\overline{x},\overline{y})] : 0 \leq k \leq n-1\}$$

for some $n \geq 1$. We will construct a model $\mathcal{A}$ for $T' \cup S$ by Henkin's method. We choose an infinite set C of new constants. For each $k \in \{0, \ldots, n-1\}$, let C_k be an enumeration of all sequences of elements in C of the same length as the length of $\overline{x}$ in $\theta_{i_k}(\overline{x}, \overline{y})$, such that every such sequence appears in C_k infinitely often. Let $\sigma_0, \sigma_1, \sigma_2, \ldots$ be an enumeration of all sentences in $L(T') \cup C$. We will construct the complete diagram Ψ of $\mathcal{A}_A$, where A will consist of the equivalence classes of the constants in C. We will arrange that $\Psi = \{\delta_0, \delta_1, \delta_2, \ldots\}$, where δ_s is defined at stage s. For $s > 0$, let $\Psi^s = \{\delta_0, \delta_1, \ldots, \delta_{s-1}\}$ and let ψ^s be $\wedge \Psi^s$.

Construction

Stage 0:

Let $\delta_0 =_{def} (c_0 = c_0)$.

Stage $s = (n+2)e$ for $e \geq 1$:

Satisfy the $(e-1)$–st completeness of the diagram requirement.

Stage $s = (n+2)e + 1$ for $e \in \omega$:

Provide a Henkin's witness for δ_e.

Stage $s = (n+2)e + k$ for $e \in \omega$ and $k \in \{2, ..., n+1\}$:

Let C_{k-2} be $\overline{c}_0, \overline{c}_1, \overline{c}_2, \ldots$. If $\theta_{f(i_{k-2})}(\overline{c}_e) \notin \Psi^s$, then $\delta_s =_{def} (c_0 = c_0)$. If $\theta_{f(i_{k-2})}(\overline{c}_e) \in \Psi^s$, then $\delta_s =_{def} \theta_{i_{k-2}}(\overline{c}_e, \overline{c})$, where $\overline{c}$ is a sequence of constants in C which do not occur in ψ^s such that $\overline{c}$ is of the same length as $\overline{y}$ in $\theta_{i_{k-2}}(\overline{x}, \overline{y})$. End of the construction. We can prove inductively that for every $s \in \omega$, $T' \cup \Psi^s$ is consistent. In the proof, we use property (iv) at stages of the form $(n+2)\,e + k$ for $k \in \{2, \ldots, n+1\}$. Hence Ψ is consistent. The corresponding model $\mathcal{A}$ satisfies $T' \cup S$. □

Lemma 6.11. *Every model of T' can be isomorphically embedded in a model of T.*

Proof. Assume that $\mathcal{B}$ is a model of T'. Let Ω be the atomic diagram of $\mathcal{B}$. To prove that there is a model for $\Omega \cup T$, we use the same argument as in the proof of Lemma 6.10 to construct a model for $\Omega \cup S$, where S is a finite set of axioms in Ax II. □

Lemma 6.12. $(\forall \mathcal{A}, \mathcal{B} \models T)(\forall \mathcal{D} \subseteq \mathcal{A}, \mathcal{B})$ [*$\mathcal{A}$ and $\mathcal{B}$ satisfy the same existential sentences in $L(T)$ with parameters from D*]

Proof. If follows from (ii) and Ax II that T admits the elimination of quantifiers, which is equivalent to this statement. □

Although the following lemma follows from Theorem 6.6, we also give an easy direct proof.

Lemma 6.13. *T is complete.*

Proof. Let σ be a sentence in L such that $T \cup \{\sigma\}$ is consistent. Since T admits the elimination of quantifiers, the formula $\sigma \wedge (y_0 = y_0)$ is T–equivalent to $\theta_i = \theta_i(y_0)$ for some $i \in \omega$. By (iii), $\theta_{f(i)} = (x_0 = x_0)$. By the definition of axioms in Ax II, $T \vdash \forall x_0[x_0 = x_0 \Rightarrow \exists y_0(\sigma \wedge (y_0 = y_0))]$. Hence $T \vdash \sigma$. □

T is decidable because it is complete and the given sets of axioms are computable. T is a model completion of T' by Lemma 6.10, Lemma 6.11 and Lemma 6.12.

7 Omitting Types and Decidability

Let Γ be a nonprincipal type of a complete theory T. Then there is a countable model $\mathcal{A}$ of T which omits Γ. However, $\mathcal{A}$ does not have to be computable even if Γ is. The following theorem shows that if T is decidable and Γ is computable, then Γ is omitted in some decidable model of T.

Theorem 7.1. *Let Γ be a computable nonprincipal type of a complete decidable theory T. There is a decidable model of T which omits Γ.*

Proof. Without loss of generality, we assume that Γ is a 1–type, $\Gamma(x)$. Let C, $(\sigma_i)_{i\in\omega}$, $\Psi = \{\delta_0, \delta_1, \delta_2, \ldots\}$, ψ^s and $\mathcal{A}$ be as in the proof of Theorem 5.1.

Construction

Stage 0:

Let $\delta_0 =_{def} (c_0 = c_0)$.

Stage $s = 3e + 1$ for $e \in \omega$:

We effectively provide a Henkin's witness for δ_e.

Stage $s = 3e + 2$ for $e \in \omega$ (Omitting the types requirement):

Let ψ^s be of the form $\psi^s(c_e, \overline{c})$, where c_e does not occur in $\overline{c}$. We effectively find the first formula $\gamma(x) \in \Gamma$ such that

$$(\circ) \quad T \nvdash \forall z[\exists \overline{y} \psi^s(z, \overline{y}) \Rightarrow \gamma(z)],$$

where $(z, \overline{y})$ is an appropriate effectively chosen sequence of new variables. Let $\delta_s =_{def} \neg\gamma(c_e)$.

Stage $s = 3e + 3$ for $e \in \omega$:

We effectively satisfy the e–th completeness of the diagram requirement. End of the construction. At stage $3e + 2$, the corresponding formula γ exists because Γ is a nonprincipal type and, by the construction, $T \cup \{\exists z \exists \overline{y} \psi^s(z, \overline{y})\}$ is a consistent set. Condition $(\circ)$ can be verified effectively because T is a decidable theory. Stage $3e + 2$ guarantees that the interpretation of c_e in $\mathcal{A}$ does not realize Γ. Since every element in the domain of $\mathcal{A}$ is the interpretation of some constant in C, $\mathcal{A}$ omits Γ. Clearly, for an arbitrary n–type Γ, stage $3e + 2$ should be modified so that instead of $(c_i)_{i\in\omega}$ some effective enumeration of all n–tuples of elements of C is considered. □

A *partial type* of T is a subset of a (complete) type of T. Millar has established the following general result.

Theorem 7.2. (*Effective Omitting Types Theorem, Millar [137]*) *Let T be a complete decidable theory. If Φ_1 is a Σ^0_2 set of computable nonprincipal partial types of T, and Φ_2 is a Σ^0_2 set of computable types of T, then there is a decidable model of T which omits all types in Φ_1 and all nonprincipal types in Φ_2.*

The completeness of types in Φ_2 plays an important role in Theorem 7.2, as demonstrated by the next theorem.

Theorem 7.3. (*Millar [137]*) *There is a complete decidable theory T and a computable set Φ of computable partial types of T such that no decidable model of T omits all nonprincipal types in Φ.*

The following two theorems can be obtained using the Effective Omitting Types Theorem.

Theorem 7.4. (*Millar [137]*) *Let T be a complete decidable theory without a decidable prime model. There are infinitely many distinct decidable models of T such that the set of all types realized in any two of these models simultaneously is exactly the set of all principal types of T.*

Theorem 7.5. (*Millar [137]*) *Let T be a complete decidable theory and let Φ be a Σ^0_2 set of computable nonprincipal types of T. Assume that for every decidable model $\mathcal{A}$ of T which omits Φ, for every finite $X \subseteq A$, $\mathcal{A}_X$ is not a prime model. Then there are $2^{\aleph_0}$ distinct type spectra of decidable models of T which omit Φ.*

8 Decidable Prime Models

Definition 8.1. Let $\mathcal{U}$ be an arbitrary (possibly uncountable) model. $\mathcal{U}$ is *atomic* if every n–tuple of elements of U satisfies a complete formula in the theory of $\mathcal{U}$.

Proposition 8.2. *Let T be a complete theory in at most countable language.*
(*i*) *A countable model $\mathcal{A}$ of T is prime if and only if $\mathcal{A}$ is atomic.*
(*ii*) *T has a prime model if and only if every formula consistent with T is a member of a principal type of T.*

Definition 8.3. An arbitrary model $\mathcal{U}$ is $\aleph_0$–*homogeneous* if for every two sequences of elements of U of the same length,

$$(a_0, \dots, a_{n-1}) \text{ and } (b_0, \dots, b_{n-1}),$$

with the property

$$(\mathcal{U}, a_0, \dots, a_{n-1}) \equiv (\mathcal{U}, b_0, \dots, b_{n-1}),$$

for every $a \in U$, there is $b \in U$ such that

$$(\mathcal{U}, a_0, \dots, a_{n-1}, a) \equiv (\mathcal{U}, b_0, \dots, b_{n-1}, b).$$

A countable model $\mathcal{A}$ which is $\aleph_0$–homogeneous is also called a *homogeneous* model.

Proposition 8.4. (i) *Every atomic model is* $\aleph_0$*–homogeneous.*

(ii) *Two countable homogeneous models which realize exactly the same types are isomorphic.*

Proposition 8.5. *Let* T *be a complete theory in at most countable language.*

(i) *If* T *has* $> \aleph_0$ *types, then* T *has* $2^{\aleph_0}$ *non-isomorphic countable homogeneous models.*

(ii) *If all countable models of* T *are homogeneous, then the number of non-isomorphic countable models of* T *is either* 1*, or* $\aleph_0$*, or* $2^{\aleph_0}$*.*

The following theorem, obtained independently by Harrington, and Goncharov and Nurtazin, is an effective version of Proposition 8.2 (ii). It establishes that a complete decidable theory T has a decidable prime model if there is an algorithm which for a given formula $\theta(\overline{x})$ consistent with T, outputs Gődel number of the characteristic function of a computable principal type $\Gamma(\overline{x})$ containing $\theta(\overline{x})$. In the proof of this result we use from *model theory*, Henkin's method of constructing models; and from *computability theory*, the finite injury priority method.

Theorem 8.6. (*Goncharov-Nurtazin [79], Harrington [88]*) *Let* T *be a complete decidable theory. The following are equivalent.*

(i) T *has a decidable prime model.*

(ii) T *has a prime model and the set of all principal types of* T *is computable.*

Proof. ($\Longrightarrow$): The conclusion follows from Proposition 4.7, since the set of all types realized in a decidable prime model of T is the set of all principal types of T.

($\Longleftarrow$): Let f be a computable function such that $\{f(n) : n \in \omega\}$ is a set of codes of the set of all principal types $\{\Gamma_n : n \in \omega\}$ of T, where $\phi_{f(n)} = \chi_{\Gamma_n}$. We will use Henkin's method to construct a decidable prime model of T. Let $C = \{c_0, c_1, c_2, \ldots\}$ be a set of new constants, and let $\sigma_0, \sigma_1, \sigma_2, \ldots$ be an effective enumeration of all sentences in $L(T) \cup C$. As usual, the domain of the resulting model $\mathcal{A}$ will be $\{[c_0], [c_1], \ldots\}$, where $[c]$ is the corresponding equivalence class of c. We will ensure that in $\mathcal{A}$, for every $e \geq 0, ([c_0], \ldots, [c_e])$ realizes a principal type of T, that is, a type from $\{\Gamma_n : n \in \omega\}$. This is sufficient since, for example, if a (complete) formula $\xi(x_0, x_1)$ generates a principal 2–type, then $\exists x_0 \xi(x_0, x_1)$ generates a principal 1–type. That is because $T \vdash \xi(x_0, x_1) \Rightarrow \zeta(x_1)$ implies $T \vdash \exists x_0 \xi(x_0, x_1) \Rightarrow \zeta(x_1)$. Hence every finite sequence of elements in the domain of $\mathcal{A}$ will satisfy a principal type. We will construct the complete diagram Ψ of $\mathcal{A}$. At every stage s ($s \geq 0$) of the construction, we will have a finite set Ψ^s of sentences such that

$$\Psi^0 \subseteq \Psi^1 \subseteq \Psi^2 \subseteq \cdots \qquad \text{and} \qquad \Psi = {}_{s\geq 0}\Psi^s.$$

Let $\psi^s = \wedge\Psi^s$. If $\psi^s = \psi^s(c_0, \ldots, c_{n_s})$, then for every $e \in \{0, \ldots, n_s\}$, we set

$$\psi^s_e =_{def} \exists y_{e+1} \ldots \exists y_{n_s} \psi^s(c_0, \ldots, c_e, y_{e+1}, \ldots, y_{n_s}).$$

For every $e \geq 0$, at almost every stage s of the construction, we have a type $\Omega^s_e \in \{\Gamma_n : n \in \omega\}$ which is a candidate for a principal type realized by $([c_0], \ldots, [c_e])$. We will allow Ω^s_e to be undefined for finitely many s. Because of the consistency property, if Ω^s_e is defined then $\psi^s_e(\overline{x}/\overline{c}) \in \Omega^s_e$. The construction will satisfy the following requirements for every $e \geq 0$.

P^1_e: $\sigma_e \in \Psi$ or $\neg\sigma_e \in \Psi$;

P^2_e: If $\sigma_e \in \Psi$ and $\sigma_e = \exists x\theta(x)$, then $\theta(c) \in \Psi$ for some $c \in C$;

Q_e: $([c_0], \ldots, [c_e])$ realizes a principal type of T.

The priority ranking of the requirements in the decreasing order is:

$$P^1_0\,,\, P^2_0\,,\, Q_0\,,\, \ldots\,,\, P^1_e\,,\, P^2_e\,,\, Q_e\,,\, \ldots$$

We attempt to satisfy the requirements in the order of their priority. We say that at stage $s > 0$:

P^1_e *requires attention* if $\sigma_e \notin \Psi^{s-1}$ and $\neg\sigma_e \notin \Psi^{s-1}$;

P^2_e *requires attention* if $\sigma_e \in \Psi^{s-1}$ and $\sigma_e = \exists x\theta(x)$ for some θ such that $\theta(c) \notin \Psi^{s-1}$ for every $c \in C$;

Q_e *requires attention* if Ω^{s-1}_e is undefined.

Once satisfied at some stage, requirements P^1_e and P^2_e are never injured again. However, we say that

Q_e *is injured* at stage $s > 0$ if Ω^{s-1}_e is defined, but $\psi^s_e(\overline{x}/\overline{c}) \notin \Omega^{s-1}_e$.

Construction

Stage 0:

Let $\Psi^0 = \emptyset$ and let Ω^0_e be undefined for every $e \in \omega$.

Stage s>0:

Let *Req* be the highest priority requirement which requires attention at stage s. We now attack *Req* as follows. Let $Req = P^1_e$

(a) If $T \vdash \forall\overline{x}[(\psi^{s-1} \Rightarrow \sigma_e)(\overline{x}/\overline{c})]$, then $\Psi^s = \Psi^{s-1} \cup \{\sigma_e\}$.

(b) If $T \vdash \forall\overline{x}[(\psi^{s-1} \Rightarrow \neg\sigma_e)(\overline{x}/\overline{c})]$, then $\Psi^s = \Psi^{s-1} \cup \{\neg\sigma_e\}$.

The properties on the left-hand side of (a) and (b) can be checked effectively because T is decidable.

(c) If neither (a) nor (b) is satisfied, we add either σ_e or $\neg\sigma_e$ to Ψ^{s-1} such that if some Q–requirement must be injured, then the first such injured requirement is of the highest priority. (Since the types in $\{\Gamma_n : n \in \omega\}$ are computable, we can effectively check whether a given Q–requirement is injured.)

We effectively check whether some Q_n is injured at stage s. Let n_0 be the least such n, if it exists. For every $n \geq n_0$, Ω^s_n will be undefined. Let $Req = P^2_e$ Thus, $\sigma_e \in \Psi^{s-1}$ and $\sigma_e = \exists x\theta(x)$ for some θ. Let c be the first constant in C which has not been used in the construction before stage s. We define $\Psi^s = \Psi^{s-1} \cup \{\theta(c)\}$. Let $Req = Q_e$ Thus, Ω^{s-1}_e is undefined. We find the first $(e+1)$–type $\Gamma(x_0, \ldots, x_e) \in \{\Gamma_n : n \in \omega\}$ such that $\psi_e^{s-1}(x_0/c_0, \ldots, x_e/c_e) \in \Gamma(x_0, \ldots, x_e)$, and set $\Omega_e^s =_{def} \Gamma(x_0, \ldots, x_e)$. This can be done effectively because of the following two facts.

(1) Such Γ exists because T has a prime model, so it is an atomic theory, hence every formula consistent with T belongs to some principal type.

(2) For every computable complete type, we can effectively decide whether a given formula or its negation belongs to that type. End of the construction.

Lemma 8.7. *For every e, $\Omega_e =_{def} \lim_s \Omega_e^s$ exists. Hence every Q_e is satisfied.*

Proof. Assume that $e = 0$. Let t_0 be the least stage such that $\Omega_0^{t_0}$ is defined. Let $\Omega_0^{t_0} = \Gamma_{n_0}$. Then $\psi_0^{t_0-1}(x_0/c_0) \in \Gamma_{n_0}$. Hence, by construction, Q_0 will never be injured, so $\Omega_0 = \Gamma_{n_0}$. Assume that $e = 1$. Let ξ_0 be a complete formula such that $\xi_0 \in \Omega_0 (= \Gamma_{n_0})$. Clearly, $\neg\xi_0(x_0)$ is inconsistent with Ω_0, and Q_0 is never injured. Choose the least stage s_0 such that $\xi_0(c_0) \in \Psi^{s_0}$. Let t_1 be the least stage $> s_0$ such that $\Omega_1^{t_1}$ is defined. If $\Omega_1^{t_1} = \Gamma_{n_1}$, then $\psi_1^{t_1-1}(x_0/c_0, x_1/c_1) \in \Gamma_{n_1}$. Since every formula consistent with $\Psi^{s_0}(\overline{x}/\overline{c})$ is also consistent with Ω_0, it follows that Q_1 is not injured after t_1. The general proof is by induction on e. If $e > 0$, choose the least s such that

$\forall t > s(\Omega_{e-1}^t = \Omega_{e-1}^s = \Omega_{e-1})$,

$\psi_{e-1}^s(x_0/c_0, \ldots, x_{e-1}/c_{e-1})$ is a complete formula for Ω_{e-1},

Ω_e^s is defined.

Let $t > s$. It follows that $\Omega_e^t = \Omega_e^s$ since $\psi_e^s(x_0/c_0, \ldots, x_{e-1}/c_{e-1}, x_e/c_e) \in \Omega_e^s$, so $\psi_{e-1}^s(x_0/c_0, \ldots, x_{e-1}/c_{e-1}) \in \Omega_e^s$, and hence $\Omega_{e-1}^s \subseteq \Omega_e^t$. □

Theorem 8.8. (*Millar [145]*) *There is a complete decidable theory T with a prime model which does not have a computable prime model. In addition, all types of T are computable.*

Proof. The language of T is $L = \{P_n(\cdot) : n \in \omega\}$, where every P_n is a unary relation symbol. Let $\theta_0(x), \theta_1(x), \theta_2(x), \ldots$ be a computable enumeration of all quantifier-free formulae in L whose only free variable is x. For a quantifier-free formula $\theta(x)$ in L, let

$$\ulcorner\theta(x)\urcorner =_{def} \mu k(\theta(x) = \theta_k(x)).$$

For a finite sequence $\alpha \in 2^m$, let

$$\theta_\alpha(x) =_{def} \bigwedge\{P_k(x)^{\alpha(k)} : 0 \le k \le m-1\}.$$

The set of sentences T is defined using a computable tree $\mathcal{T} \subseteq 2^{<\omega}$ which will be constructed later. The idea is to use the nodes in $\mathcal{T}$ to define certain formulae which are consistent with T and to use the nodes in $(2^{<\omega} - \mathcal{T})$ to define certain formulae which are inconsistent with T. Namely, the axioms of T fall into the following two groups:

Ax I $\forall x \neg\theta_\beta(x)$ for every $\beta \in 2^{<\omega} - \mathcal{T}$,

Ax II $\exists x_0 \ldots \exists x_{n-1} [\bigwedge_{0\le i<j\le n-1} x_i \neq x_j \quad \wedge \bigwedge_{0\le i\le n-1} \theta_\alpha(x_i)]$
for every $\alpha \in \mathcal{T}$ and every $n \geq 1$.

In addition to being a computable tree, $\mathcal{T}$ will satisfy the following condition:

$$\forall\beta[\beta \in \mathcal{T} \Rightarrow \beta\hat{\ }1 \in \mathcal{T}]. \qquad (*)$$

This allows us to conclude that T has the properties stated in the following four lemmas. □

Lemma 8.9. *T is consistent. Hence T is a theory.*

Proof. Consistency of T will follow easily from the construction of $\mathcal{T}$. We can also use the compactness theorem to prove that the set of all axioms of T has a model. Assume that S is a finite set of axioms. Let $\sigma_0, \ldots, \sigma_{k-1}$ be a list of all axioms in S from Ax II. For every $i \in \{0, \ldots, k-1\}$, let α_i be the node in $\mathcal{T}$ and n_i the natural number corresponding to σ_i. Define a finite model $\mathcal{A}$ of L as follows. The domain A is $A_0 \cup \ldots \cup A_{k-1}$, where $A_0, \ldots, A_{k-1}$ are pairwise disjoint sets, and for every $i \in \{0, \ldots, k-1\}$, A_i has n_i elements. Fix $i \in \{0, \ldots, k-1\}$. Let $m_i = lh(\alpha_i)$. We define the unary relations on A_i in such a way that σ_i is true, and for every $j \geq m_i$, we have $P_j^{\mathcal{A}} \supseteq A_i$. The sentences in S from Ax I are then automatically satisfied, because $\mathcal{T}$ is a tree with property $(*)$.

Lemma 8.10. *T admits the elimination of quantifiers.*

Proof. We will use Theorem 6.8. Let $\mathcal{A}, \mathcal{B} \models T$ and $\mathcal{D} \subseteq \mathcal{A}, \mathcal{B}$. We will prove that $\mathcal{A}_D$ and $\mathcal{B}_D$ satisfy the same existential sentences. Let $\exists\overline{y}\psi(\overline{x}, \overline{y})$ be a formula of L and let $\overline{d} \in D^{<\omega}$ be such that $\psi(\overline{x}, \overline{y})$ is a quantifier-free formula, and $\mathcal{A} \models \exists\overline{y}\psi(\overline{x}, \overline{y})[\overline{d}]$. Assume that $\psi(\overline{x}, \overline{y})$ is in a disjunctive normal form. Choose a disjunct $\delta(\overline{x}, \overline{y})$ of $\psi(\overline{x}, \overline{y})$ such that $\mathcal{A} \models \delta(\overline{x}, \overline{y})[\overline{d}, \overline{a}]$ for some $\overline{a} \in A^{<\omega}$. Let $m \geq 1$ be the largest number such that P_{m-1} occurs in δ. Let $a \in rng(\overline{a}) - B$, and let y be the variable in $rng(\overline{y})$ assigned to a. Assume that $\theta(y)$ is the largest subformula of $\delta(\overline{x}, \overline{y})$, containing only variable y. Let $\alpha \in 2^m$ be such that $\mathcal{A}_A \models \theta_\alpha(\mathbf{a})$. Clearly, $\theta(y)$ is a subformula of $\theta_\alpha(y)$. Since $\theta_\alpha(y)$ is consistent with T, we conclude that $\alpha \in \mathcal{T}$, so $\mathcal{B} \models \exists y\theta_\alpha(y)$. Now it is easy to see that $\mathcal{B} \models \exists\overline{y}\psi(\overline{x}, \overline{y})[\overline{d}]$. □

Lemma 8.11. *T is complete.*

Proof. Let σ be a sentence in L. If σ is inconsistent with T, then $T \vdash \neg\sigma$. Therefore, assume that σ is consistent with T. We will prove that $T \vdash \sigma$. By Lemma 8.10, there is quantifier-free formula $\psi(x)$ such that

$$T \vdash \forall x(\sigma \Leftrightarrow \psi(x)).$$

Hence $T \vdash (\sigma \Leftrightarrow \exists x\psi(x))$. $\psi(x)$ can be written as a disjunction of conjunctions of atomic formulae or negations of atomic formulae. Let $\mathcal{A}$ be a model of T such that $\mathcal{A} \models \sigma$. Let a disjunct $\theta(x)$ of $\psi(x)$ and $a \in A$ be such that $\mathcal{A}_A \models \theta(\mathbf{a})$. Assume that $m \geq 1$ is the largest number such that P_{m-1} occurs in $\theta(x)$. As in the proof of the previous lemma, there is $\alpha \in 2^m$ such that $\mathcal{A}_A \models \theta_\alpha(\mathbf{a})$ and $\theta(x)$ is a subformula of $\theta_\alpha(x)$. Thus, $\alpha \in \mathcal{T}$, so $\exists x \theta_\alpha(x)$ is an axiom of T. Therefore $T \vdash \exists x \theta(x)$. Hence $T \vdash \exists x \psi(x)$. □

Lemma 8.12. *(i) T is decidable.*
(ii) Every computable model of T is decidable.

Proof. (i) The sets of axioms in Ax I and Ax II are computable because $\mathcal{T}$ is computable. Hence the decidability of T follows from Lemma 8.11.

(ii) The statement follows from Proposition 6.7 (ii) since T is decidable and admits the elimination of quantifiers. □

Since T admits the elimination of quantifiers, every 1–type of T is uniquely determined by a function $f \in 2^\omega$ such that for every $k \in \omega$:

$$P_k(x)^{f(k)} \in \Gamma(x).$$

We now construct a computable binary tree in such a way that every type of T is computable, and every decidable model of T realizes a nonprincipal 1–type. Since every type of T is computable, T has countably many types, so it has a countable saturated model, and hence a prime model. However, a prime model of T cannot be decidable because it does not realize a nonprincipal type. $\mathcal{T}$ is computably enumerated, where for every $s \in \omega$, $\mathcal{T}_s$ is the part of $\mathcal{T}$ enumerated by stage s. The construction satisfies the following requirements for $e \in \omega$:

$$\begin{aligned} R_e \quad : \quad & [\phi_e^{(2)} \text{ is the satisfaction predicate of a decidable model } \mathcal{A} \models T \\ \Longrightarrow \quad & \mathcal{A} \text{ realizes a nonprincipal type of } T]. \end{aligned}$$

To achieve this, for every $\phi_e^{(2)}$ which is a satisfaction predicate of a decidable model $\mathcal{A}$ of T, we define an infinite set of *e–marked nodes* and a unique *e–marked element*. The e–marked nodes belong to a single infinite branch which determines a nonprincipal type of T, satisfied by the element of A which corresponds to the e–marked element. For $e \in \omega$, exactly one new e–marked node is defined at every stage s of the construction such that $s \in \{0\} \cup \{e+1, e+2, \ldots\}$. For any a, let a^0 denote the empty sequence, and for $m \geq 1$, let a^m abbreviate the sequence of m consecutive a's.

Construction

Stage 0:

$\mathcal{T}_0$ is the tree consisting of the nodes $1^e\hat{\ }0$ for all $e \in \omega$, and of their initial segments. For every $e \in \omega$, the node $1^e\hat{\ }0$ is e–marked. *Stage* $s+1$:

Step 1: For every $\beta \in \mathcal{T}_s$:

Enumerate $\beta\hat{\ }1$ into $\mathcal{T}_{s+1}$; and

Declare that $\beta\hat{\ }0$ (and hence every $\gamma \in \mathcal{T}_s$ such that $\beta\hat{\ }0$ is an initial segment of γ) is in the complement of $\mathcal{T}$.

Step 2: Consider each $e \leq s$. Let α be the e–marked node at s.

Case (a): No e–marked element has been defined at any previous stage.
Action: Search for the least $j \leq s$ (if it exists) such that

$$\phi^{(2)}_{e,s}(\lceil \theta_\alpha(x) \rceil, j) \downarrow = 1.$$

If such j does not exist, then α is the e–marked node at $s+1$.
If such j is found, define the e–marked element to be j. Let β be the node, enumerated in $\mathcal{T}_{s+1}$ by Step 1, of the maximal length such that α is the initial segment of β. The construction guarantees the uniqueness of β. Define β to be the e–marked node at $s+1$, and enumerate both $\beta\hat{\ }0$ and $\beta\hat{\ }1$ into $\mathcal{T}_{s+1}$.

Case (b): Assume that j is the e–marked element.
Action: Let $lh(\alpha) = n$. Find the least $b \in \{0,1\}$ (if it exists) such that

$$\phi^{(2)}_{e,s}(\lceil \theta_\alpha(x) \wedge P_n(x)^b \rceil, j) \downarrow = 1.$$

If such b does not exist, then α is the e–marked node at $s+1$.
Now assume that b exists. Let β be the node, enumerated in $\mathcal{T}_{s+1}$ by Step 1, of the maximal length such that $\alpha\hat{\ }b$ is an initial segment of β. The construction guarantees the uniqueness of β. Define β to be the e–marked node at $s+1$, and enumerate both $\beta\hat{\ }0$ and $\beta\hat{\ }1$ into $\mathcal{T}_{s+1}$. End of the construction.

Let $\mathcal{T} =_{def} \bigcup_{s\in\omega} \mathcal{T}_s$. $\mathcal{T}$ is a computable tree by construction.

Lemma 8.13. (*i*) *Every* 1*–type of* T *is computable.*
(*ii*) *Every type of* T *is computable.*

Proof. (i) It is easy to see that the principal types of T are computable. Assume that $\Gamma(x)$ is a nonprincipal type of T. Since T admits the elimination of quantifiers, $\Gamma(x)$ is uniquely determined by a function $f_\Gamma \in 2^\omega$. That is, for every $i \in \omega$,

$$P_i(x) \in \Gamma \Longleftrightarrow f_\Gamma(i) = 1.$$

To prove that $\Gamma(x)$ is computable, it is sufficient to prove that f is computable. Let $e \geq 0$ be such that $1^e\hat{\ }0$ is an initial segment of f. If there were only finitely many e–marked nodes on the infinite branch of $\mathcal{T}$ determined by f, then, since every e-marked node "branches", f would determine a principal type. Therefore, there is an infinite set E of such e–marked nodes. Since E is computable by the construction of $\mathcal{T}$, f is computable.

(ii) Let $\Omega(x_0, \ldots, x_{n-1})$ be an arbitrary type of T. Since T admits the elimination of quantifiers, Ω is uniquely determined by its 1–subtypes and by the set Ψ of all inequalities among $x_0, \ldots, x_{n-1}$, which are in Ω. Since, by (i), all 1–types are computable, and Ψ is finite, it follows that Ω is computable. □

Lemma 8.14. *Every decidable model of* T *realizes a nonprincipal type of* T.

Proof. Assume that $\mathcal{A}$ is a decidable model of T. Then there is an effective enumeration $(a_i)_{i\in\omega}$ of A, and $e \in \omega$ such that for every formula $\theta(x)$ of L and every $i \in \omega$:

$$\mathcal{A} \models \theta(\mathbf{a}_i) \Longleftrightarrow \phi_e^{(2)}(\ulcorner\theta(x)\urcorner, i) \downarrow= 1.$$

Let α be the e–marked node of the least length. Since $T \vdash \exists x\theta_\alpha(x)$, there is $i \in \omega$ such that $\mathcal{A}_A \models \theta_\alpha(\mathbf{a}_i)$. Hence there is $j \in \omega$ such that j is the e–marked element and for some stage $s \in \omega$, $\phi_{e,s}^{(2)}(\ulcorner\theta_\alpha(x)\urcorner, j) \downarrow= 1$. It follows from the construction of $\mathcal{T}$ that there are infinitely many e–marked nodes. For every such node β, both $\beta\hat{\ }0$ and $\beta\hat{\ }1$ belong to $\mathcal{T}$. Hence $\mathcal{T}$ has an infinite branch determining a nonprincipal type which is realized in $\mathcal{A}$ by a_j. □

Now, let us show that the Σ_2^0 complexity assumption in the Effective Omitting Types Theorem (Theorem 7.2) cannot be replaced by a Π_2^0 one. Assume that a theory T is as in the previous theorem. Then the set of all nonprincipal types of T is not omitted in any decidable model of T. However, by Proposition 4.8, the set of all nonprincipal types of T is a Π_2^0 set.

Goncharov and Nurtazin [79] have also given an example of a decidable theory without a computable prime model. The language of the theory is infinite, and the theory is $\aleph_0$–stable. In [73], Goncharov has established a criterion for the computability of a prime model of a complete decidable theory. Let us first state a model theoretic result about $\aleph_0$–stable theories.

Theorem 8.15. *Let T be an $\aleph_0$–stable theory in at most countable language, and let $\mathcal{U}$ be an arbitrary model of T. For every set $X \subseteq U$, the complete theory of $\mathcal{U}_X$ has an atomic prime model.*

Theorem 8.16. (*Goncharov [73]*) *There is a complete decidable $\aleph_0$–stable theory in a finite language* (*consisting of four unary relation symbols and one binary relation symbol*)*, which does not have a computable prime model.*

Theories obtained in Theorem 8.8 and in Theorem 8.16 have infinite sets of axioms. However, Peretyat'kin [169] has found a finitely axiomatizable complete (hence decidable) theory T with a prime model, which does not have a decidable prime model. In Peretyat'kin's example, T is associated with a computably enumerable binary tree $\mathcal{T}$ which has the following properties. For every node α of $\mathcal{T}$, either both or none of $\alpha\frown 0$ and $\alpha\frown 1$ belong to $\mathcal{T}$. Every node of $\mathcal{T}$ is an initial segment of a leaf of $\mathcal{T}$, and the set of all finite branches of $\mathcal{T}$ is non-computable. A tree with these properties was first used by Goncharov and Nurtazin [79]. To prove that the described tree suffices for the result, Peretyat'kin has invented a general method for constructing finitely axiomatizable theories whose properties are determined by Turing machine computations.

Definition 8.17. Let $X \subseteq \omega$. A model $\mathcal{A}$ is *decidable in* X if $A \leq_T X$ and there is an enumeration $(a_i)_{i\in\omega}$ of A such that the theory of $(\mathcal{A}, a_i)_{i\in\omega}$ is $\leq_T X$.

Theorem 8.18. (*Denisov [43], Millar [145], Drobotun [49]*) *Let T be a complete decidable theory with a prime model. Then T has a prime model which is decidable in $\emptyset'$.*

Millar [142] has introduced a different concept of the effectiveness of a model, which is weaker than the concept of decidability.

Definition 8.19. A countable model $\mathcal{A}$ for L is *almost decidable* if there is a computable function F which assigns to every finite binary sequence α a finite set $F(\alpha)$ of formulae in $L \cup \{c_0, c_1, c_2, \ldots\}$, where $c_0, c_1, c_2, \ldots$ are new constants, such that the following conditions are satisfied.

(1) For $\alpha \in 2^\omega$, if β is an initial segment of α , then $F(\beta) \subseteq F(\alpha)$.

(2) We can assign to every $f \in 2^\omega$ a model $\mathcal{A}_f$ such that

$$\{F(\alpha) : \alpha \text{ is an initial segment of } f\}$$

determines the complete diagram of $\mathcal{A}_f$, and for all but countably many $f \in 2^\omega$, $\mathcal{A}_f$ is isomorphic to $\mathcal{A}$.

Every decidable model is almost decidable, and there is an almost decidable model which is not decidable. In fact, the concept of almost decidability is introduced to capture a class of models which fail to be decidable because, although there are computable strategies for their construction, the strategies are not uniformly computable.

Theorem 8.20. (*Millar [142]*)

(*i*) *If a complete decidable theory T has fewer than continuum many complete types, then T has an almost decidable prime model.*

(*ii*) *There is a complete decidable theory which has a prime model but does not have an almost decidable prime model.*

9 Computable Saturated Models and Computably Saturated Models

In 1961, Vaught introduced the notion of a countably saturated model. In 1970's, Barwise, Schlipf and Ressayre [28, 27, 185, 190] introduced the notion of a computably saturated model. Barwise and Schlipf have extensively used computably saturated models to study computability over admissible sets.

Definition 9.1. (i) Let $\mathcal{U}$ be an arbitrary model. $\mathcal{U}$ is $\aleph_0$*–saturated* if for every finite subset X of its domain, $\mathcal{U}_X$ realizes every type $\Phi(x)$ of the theory $Th(\mathcal{U}_X)$.

(ii) Let $\mathcal{U}$ be a model for a computable language L. $\mathcal{U}$ is *computably saturated* if for every finite subset X of its domain, every computable set of formulae $\Phi(x)$ in L_X consistent with $Th(\mathcal{U}_X)$ is realized in $\mathcal{U}_X$.

Hence, every $\aleph_0$–saturated model for a computable language is computably saturated. A countable $\aleph_0$–saturated model is simply called *saturated*.

Theorem 9.2. (*i*) *A complete theory in a computable language whose models are infinite has a countable computably saturated model*

(*ii*) *Every computably saturated model is $\aleph_0$–homogeneous.*

(*iii*) *Elementarily equivalent countable saturated models are isomorphic.*

(*iv*) *A complete theory with a countable saturated model has a prime model.*

Hence it follows from (ii) of the previous theorem that every countable saturated model is homogeneous. It follows from (i) of the previous theorem that every countable model for a computable language has a countable computably saturated elementary extension.

Theorem 9.3. (*Engeler, Ryll-Nardzewski, Svenonius*) *The following statements are equivalent for a complete theory T.*

(*i*) *T is $\aleph_0$–categorical.*

(*ii*) *There is a countable model of T which is both prime and saturated.*

(*iii*) *All types of T are principal.*

(*iv*) *For every finite sequence $\overline{x}$ of variables, there are only finitely many types of T in $\overline{x}$.*

(*v*) *For every finite sequence $\overline{x}$ of variables, there are only finitely many formulae with free variables among the elements of the sequence $\overline{x}$, which are not pairwise equivalent with respect to T.*

(*vi*) *All models of T are atomic.*

Theorem 9.4. *A complete theory T has a countable saturated model if and only if for every $n \in \omega$, T has only countably many n–types.*

Hence, every complete theory with only countably many non-isomorphic countable models has a countable saturated model. While countable saturated models do not exist for complete theories with uncountably many types, countable computably saturated models always exist. Thus, the proofs of many early results in model theory are simplified if countable computably saturated models are used to replace certain large models which exist only under specific assumptions of set theory.

Theorem 9.5. (*Millar [135]*) *Let T be a theory in a computable language L. Suppose that T has a complete extension T' in the language $L \cup \{c_0, \ldots, c_{n-1}\}$, where $c_0, \ldots, c_{n-1}$ are new constants, such that T' does not have an atomic model. Then T has a model which is not computably saturated.*

Proof. Since T does not have an atomic model, there is a formula

$$\psi(c_0, \ldots, c_{n-1}; x_0, \ldots, x_{m-1}) \text{ in } L(T')$$

which is consistent with T' and not contained in any principal type of T'. Let $\theta_0, \theta_1, \theta_2, \ldots$ be a computable enumeration of all formulae in L in free variables $(\overline{y}, \overline{x}) = (y_0, \ldots, y_{n-1}, x_0, \ldots, x_{m-1})$. We define a computable set of formulae $\Phi(\overline{y}, \overline{x}) = \{\psi_0, \psi_1, \psi_2, \ldots\}$ by:

$$\psi_0 = \psi(\overline{y}, \overline{x}),$$
$$\psi_{k+1} = [\theta_k(\overline{y}, \overline{x}) \Leftrightarrow \exists \overline{z}(\theta_k(\overline{y}, \overline{z}) \wedge \psi_0(\overline{y}, \overline{z}) \wedge \ldots \wedge \psi_k(\overline{y}, \overline{z}))] \text{ for } k \geq 0.$$

$\Phi(c_0, \ldots, c_{n-1}, \overline{x})$ generates an m–type of T'. It must be a nonprincipal type since no principal type of T' contains $\psi(c_0, \ldots, c_{n-1}, \overline{x})$. So there is a model

$(\mathcal{A}, a_0, \ldots, a_{n-1})$ of T' omitting $\Phi(c_0, \ldots, c_{n-1}, \overline{x})$, such that $\mathcal{A}$ is a model of T. Since $\Phi(\overline{y}, \overline{x})$ is a computable set of formulae, $\mathcal{A}$ is not computably saturated. □

Proposition 9.6. (*Millar [145]*) *Let T be a complete decidable theory with a countable saturated model. Every consistent computably enumerable set $\Phi(x_0, \ldots, x_{n-1})$ of formulae, $n \in \omega$, is contained in a computable type of T.*

Proof. Assume that $\Phi(x_0, \ldots, x_{n-1})$ is a consistent computably enumerable set of formulae which is not contained in any computable type of T. Then there is no formula $\phi = \phi(x_0, \ldots, x_{n-1})$ of $L(T)$ such that $\Phi \cup \{\phi\}$ is contained in exactly one n–type of T in variables $x_0, \ldots, x_{n-1}$. We can use the splitting along the nodes of a binary tree to show that T has $2^{\aleph_0}$ many n-types. Hence T does not have a countable saturated model, contradicting the assumption. □

Theorem 9.7. (*Morley [151], Millar [144, 145]*) *Let T be a complete decidable theory such that all types of T are computable. If the set of all types of T is computably enumerable, then T has a decidable saturated model.*

Proof. Let $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ be an effective enumeration of all types of T such that every type appears infinitely often. Also, consider an effective enumeration of all finite sequences of constants from an infinite set C of new constants. Modify the construction in the proof of Theorem 8.6 so that the constructed decidable model is saturated. □

Theorem 9.8. (*Millar [145]*) *There is a complete decidable theory T with a countable saturated model which does not have a computable saturated model. In addition, all types of T are computable.*

Proof. The example in the proof of Theorem 8.8 can be modified to guarantee that every decidable model of T omits a (nonprincipal) 1–type of T. □

Theorem 9.9. (*Millar [145]*) *Let T be a complete decidable theory.*

(*i*) *If all types of T are computable, then T has a countable saturated model which is decidable in $\emptyset'$.*

(*ii*) *If T has a countable saturated model, then T has a saturated model which is decidable in some hyperarithmetic set.*

Morley introduced a very important classification of formulae according to their complexity.

Definition 9.10. Let $\mathcal{U}$ be an $\aleph_1$–saturated model for a countable language L.

(i) Let $\theta(x)$ be a formula in L_U. We say that an ordinal α is the *Morley rank* or the *transcendence rank* of $\theta(x)$ if the set of formulae

$$\{\theta(x)\} \cup \{\psi(x) : \neg\psi(x) \text{ has the Morley rank } < \alpha\}$$

in L_U is consistent and has finitely many maximal consistent extensions in the theory of $\mathcal{U}_U$.

We assign ∞ to $\theta(x)$ as its Morley rank if $\theta(x)$ is consistent with the theory of $\mathcal{U}_U$, but no ordinal is assigned to it as its Morley rank.

(ii) The *Morley rank* of $\mathcal{U}$ is the Morley rank of the formula $x = x$.

It is convenient to work with $\aleph_1$–saturated models because a formula has the same Morley rank in two elementary equivalent $\aleph_1$-saturated models. The valid formula $x = x$ is chosen to "represent the model" because it has the largest Morley rank. Clearly, a formula $\theta(x)$ in L_U has the Morley rank 0 if it is satisfied in $\mathcal{U}$ by at least one but at most finitely many distinct elements. Such a formula is also called *algebraic*.

It follows that all $\aleph_1$–saturated models of a complete theory T have the same Morley rank, called the *Morley rank of T*. It can be shown that if Morley rank of T is not ∞, then it is a countable ordinal. A theory T whose Morley rank is not ∞ is called *totally transcendental*. A complete theory is totally transcendental if and only if it is $\aleph_0$–stable.

Theorem 9.11. (*Peretyat'kin [171]*) *There is a complete finitely axiomatizable $\aleph_0$–stable theory of finite Morley rank, which has neither a computable prime model nor a computable saturated model.*

Schlipf [190] has established that if $\mathcal{A}$ is a countable, computably saturated model and S is a computably axiomatizable theory consistent with $Th(\mathcal{A})$, then $\mathcal{A}$ can be expanded to a computably saturated model of S. For example, a countable nonstandard model of additive number theory can be expanded to a model of Peano arithmetic if and only if it is computably saturated (see [123]). For applications of computably saturated models see [114, 118, 124].

10 Decidable Homogeneous Models

While countable homogeneous models are relatively simple objects in model theory, they can be very complex from a computability theoretic point of view. Classical model theory has established that every countable model has a countable homogeneous elementary extension. Two countable homogeneous models are isomorphic if and only if they realize the same finite types. Thus, a countable homogeneous model is uniquely determined, up to isomorphism, by a set of types it realizes. Therefore, the following question, first posed by Morley, is a very natural one.

Let T be a complete decidable theory. Assume that the type spectrum of a countable homogeneous model $\mathcal{A}$ of T consists only of computable types and is computable. Is $\mathcal{A}$ necessarily decidable?

(The converse is obviously true.)

Goncharov, Peretyat'kin and Millar have independently answered Morley's question negatively by providing examples of a non-computable countable homogeneous model of a complete decidable theory such that the type spectrum of the model consists only of computable types and is computable. Millar [134] has used the infinite injury priority method to construct his counterexample.

In addition, Goncharov [68] and Peretyat'kin [167] have characterized a decidable countable homogeneous model of a complete decidable theory. While Peretyat'kin's counterexample has not used this characterization, Goncharov has used the characterization to find his counterexample.

Goncharov [72] has later given an example of a complete $\aleph_0$–stable decidable theory which does not have a computable homogeneous model. Notice that such theory has neither a computable prime nor a computable saturated model. Another consequence of Goncharov's example is the existence of a decidable model without a homogeneous computable elementary extension.

To present Peretyat'kin's counterexample to Morley's question, we use from *model theory*, the Łoś-Vaught Test and a characterization of submodel complete theories from Theorem 6.8; and from *computability theory*, the notion of an *approximable* set and the existence of a non-approximable c.e. set, both of which are due to Peretyat'kin.

Theorem 10.1. (*Łoś-Vaught Test*) *If a theory S of an arbitrary language has only infinite models and for some infinite cardinal $\kappa \geq |L(S)|$, S is κ–categorical, then S is complete.*

Definition 10.2. Let $X \subseteq \omega$. X is *approximable* if there is a computable function f such that for every $m \in \omega$,

$$|\{0, \dots, m-1\} \cap X| \geq f(m)$$

and for infinitely many m,

$$|\{0, \dots, m-1\} \cap X| = f(m).$$

(If $m = 0$, then $\{0, \dots, m-1\} =_{def} \emptyset$.)

Hence, a set X is not approximable if and only if for every computable function f,

$$[\forall m\, |\{0, \dots, m-1\} \cap X| \geq f(m)] \Longrightarrow [\exists m_0 \forall m \geq m_0\, |\{0, \dots, m-1\} \cap X| > f(m)].$$

Theorem 10.3. (*Peretyat'kin [167]*) *There is a computably enumerable set X which is not approximable.*

Proof. We will algorithmically enumerate X at stages. *Construction*

Stage 0: Let $X_0 =_{def} \emptyset$.

Stage $s+1$: Let X_s be the part of X enumerated by stage s. For every $e \leq s$, consider all k such that

$$\text{for every } n \in \{\langle e, k\rangle + 1, \dots, \langle e, k+1\rangle\}, \quad \phi_{e,s}(n)\downarrow \wedge |\{0, \dots, n-1\} \cap X_s| \geq \phi_{e,s}(n).$$

Enumerate all such $\langle e, k\rangle$ in X_{s+1}. End of the construction.

Let $X = \bigcup_{s\in\omega} X_s$.

Lemma 10.4. *X is not approximable.*

Proof. Let $e \in \omega$ be such that ϕ_e is total. Assume that

$$\forall n\, |\{0,\ldots,n-1\} \cap X| \geq \phi_e(n).$$

Let $m_0 =_{def} \langle e, 0\rangle + 1$. We will prove that

$$\forall m \geq m_0[|\{0,\ldots,m-1\} \cap X| > \phi_e(m)].$$

Let $m \geq m_0$. There is a unique k such that $m \in I$, where

$$I = \{\langle e,k\rangle + 1,\ldots,\langle e,k+1\rangle\}.$$

Consider the least $s \geq e$ such that

$$(\forall n \in I)[\phi_{e,s}(n) \downarrow \wedge |\{0,\ldots,n-1\} \cap X_s| \geq \phi_{e,s}(n)].$$

By construction, $\langle e,k\rangle \in X_{s+1} - X_s$. Hence

$$|\{0,\ldots,m-1\} \cap X| > |\{0,\ldots,m-1\} \cap X_s| \geq \phi_{e,s}(m) = \phi_e(m)].$$

□

Theorem 10.5. (*Goncharov [68], Millar [134], Peretyat'kin [167]*) *There is a complete decidable theory T and a countable homogeneous model $\mathcal{M}$ of T such that $\mathcal{M}$ is not computable, and the type spectrum of $\mathcal{M}$ consists only of computable types and is computable.*

Proof. We will present the example from [167]. First, we will define a complete and decidable theory T in L which admits the elimination of quantifiers. Then for an arbitrary c.e. set X, we will define a c.e. set $\mathcal{S}_X$ of types of T such that there is a homogeneous model $\mathcal{M}$ with the following properties. $\mathcal{M}$ realizes precisely the types in $\mathcal{S}_X$; and if $\mathcal{M}$ is computable, then X must be approximable. However, by Theorem 10.3, X can be chosen to be non-approximable, thus forcing $\mathcal{M}$ to be noncomputable.

Theory T

Definition of T. The language of T is $L = \{=, R, P_0, P_1, P_2, \ldots\}$, where R is a binary relation symbol and for $i \in \omega$, P_i is a unary predicate symbol. We will also consider the finite sublanguages $L_0 = \{=, R\}$, and $L_s = \{=, R, P_0, \ldots, P_{s-1}\}$ for $s > 0$. Let $T =_{def}\, _{s\geq 0}T_s$, where T_s is a set of sentences in L_s defined as follows. T_0 has the following two axioms.

Ax 1 $\forall x \neg R(x,x)$;

Ax 2 $\forall x \forall y[R(x,y) \Rightarrow R(y,x)]$.

For $s > 0$, T_s has, in addition to the above two axioms, the following axiom schema:
Ax$_s$ $\forall x_{0 \ldots} \forall x_{n-1}[\delta'(x_0, x_1, \ldots, x_{n-1}) \Longrightarrow \exists x_n \delta(x_0, x_1, \ldots, x_{n-1}, x_n)]$,
where δ is a conjunction of atomic formulae and negations of atomic formulae in L_s which is consistent with $\{Ax\ 1, Ax\ 2\}$, and δ' is a subformula of δ. We will call δ a finite diagram in L_s and δ' a subdiagram of δ.

Lemma 10.6. *For $s \geq 0$, T_s is consistent.*

Proof. Clearly, T_0 is consistent. Assume that $s > 0$. Let $\mathcal{A}_0$ be a model in L_s of axioms Ax 1 and Ax 2. We will construct a model $\mathcal{A}$ of T_s. Let $\theta_1, \theta_2, \ldots$ be an enumeration of all axioms Ax$_s$ in which each axiom appears infinitely often. Let θ_1 be of the form

$$\forall x_0 \ldots \forall x_{n-1}[\delta'(x_0, x_1, \ldots, x_{n-1}) \Longrightarrow \exists x_n \delta(x_0, x_1, \ldots, x_{n-1}, x_n)].$$

We will extend $\mathcal{A}_0$ to $\mathcal{A}_1$ in such a way that $\mathcal{A}_1$ satisfies the matrix of θ_1 on all n–tuples from A_0. Let $A_1 = A_0 \cup \{a\}$, where $a \notin A_0$. Let

$$\mathcal{A}_0 \models \delta'(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{n-1})$$

for $a_0, a_1, \ldots, a_{n-1} \in A_0$. Extend the definitions of the predicates in L_s to the set $\{a_0, a_1, \ldots, a_{n-1}, a\}$ so that

$$A_1 \models \delta(\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{n-1}, \mathbf{a}).$$

Continuing in a similar fashion, we construct a chain of models

$$\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots .$$

Let $\mathcal{A} =_{def} \bigcup_{s \geq 0} \mathcal{A}_s$. □

Lemma 10.7. *For $s \geq 0$, T_s is $\aleph_0$–categorical.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be countable models of T_s. We will prove that they are isomorphic. Assume that f is a finite (partial) isomorphism from $\mathcal{A}$ to $\mathcal{B}$ and $dom(f) = \{a_0, a_1, \ldots, a_{n-1}\}$. Let $\delta'(x_0, x_1, \ldots, x_{n-1})$ be the finite diagram of $\mathcal{A}$ determined by $dom(f)$, let $a \in A - dom(f)$, and $\delta(x_0, x_1, \ldots, x_{n-1}, x_n)$ be the finite diagram of $\mathcal{A}$ determined by $dom(f)\ \cup \{a\}$. We have that $\mathcal{B} \models \delta'[f(a_0), f(a_1), \ldots, f(a_{n-1})]$. Thus, there is $b \in B$ such that

$$\mathcal{B} \models \delta[f(a_0), f(a_1), \ldots, f(a_{n-1}), b].$$

Then $f_1 = f \cup \{(a, b)\}$ is a finite isomorphism from $\mathcal{A}$ to $\mathcal{B}$. Similarly, if $b_1 \in B - ran(f)$, there is $a_1 \in A$ such that $f_2 = f_1 \cup \{(a_1, b_1)\}$ is a finite isomorphism from $\mathcal{A}$ to $\mathcal{B}$. □

Lemma 10.8. *For $s \geq 0$, T_s is complete.*

Proof. T_s has no finite models since $\{x_i \neq x_j : i \neq j \wedge i, j \leq n\}$ belongs to a finite diagram of T_s. Since T_s is $\aleph_0$–categorical, by the Łoś-Vaught Test, it is complete. $\square$

Lemma 10.9. *For $s \geq 0$, T_s is decidable.*

Proof. T_s is complete and computably axiomatizable. Hence, it is decidable. $\square$

Lemma 10.10. *For $s \geq 0$, T_s admits the elimination of quantifiers.*

Proof. We will prove that T_s is submodel complete by showing that

$$(\forall \mathcal{A}, \mathcal{B} \models T_s)(\forall \mathcal{D} \subseteq \mathcal{A}, \mathcal{B})\ [\mathcal{A} \text{ and } \mathcal{B} \text{ satisfy the same existential sentences in } L(T_s) \text{ with parameters from } D].$$

Let $\mathcal{A} \models \theta(\mathbf{d}_0, \ldots, \mathbf{d}_{n-1}, \mathbf{a}_0, \ldots, \mathbf{a}_{m-1})$, where

$$d_0, \ldots, d_{n-1} \in D \qquad \text{and} \qquad a_0, \ldots, a_{m-1} \in A - D.$$

Extend the identity function on $\{d_0, \ldots, d_{n-1}\}$ to a finite isomorphism f from $\mathcal{A}$ to $\mathcal{B}$ such that $a_0, \ldots, a_{m-1} \in dom(f)$. Then

$$\mathcal{B} \models \exists x_0 \ldots \exists x_{m-1} \theta(\mathbf{d}_0, \ldots, \mathbf{d}_{n-1}, x_0, \ldots, x_{m-1}).$$

Types of T

Description of the types of T. The fact that T admits the elimination of quantifiers allows us to easily describe all finite types of T. A 1–type $\Gamma(x)$ of T is uniquely determined by the sequence $f \in \{0,1\}^\omega$ such that

$$\{P_0^{f(0)}(x), P_1^{f(1)}(x), P_2^{f(2)}(x), \ldots\} \subseteq \Gamma.$$

For $n \geq 2$, an n–type $\Gamma(x_0, \ldots, x_{n-1})$ of T is uniquely determined by the 1–types $\Gamma \restriction \{x_0\}, \ldots, \Gamma \restriction \{x_{n-1}\}$ and some finite L_0–diagram $\delta(x_0, \ldots, x_{n-1})$.

Description of a set $\mathcal{S}_X$ of types of T. Let X be a c.e. set of natural numbers.

Definition 10.11. (i) A sequence $f \in \{0,1\}^\omega$ is compatible with X if there is $l \in \omega$ such that for $i \in \{0, \ldots, l-1\}$,
$f(l+i) = 1$ if $i \in X$, and $f(l+i) = 0$ if $i \notin X$,
$f(2l) = 0$ and for $i > 2l$, $f(i) = 1$.
(ii) A 1–type is compatible with X if the infinite binary sequence which determines it is compatible with X.

That is, f is compatible with X if f is defined arbitrarily on some initial segment of length l, then "follows" X on length l, after that has value 0, and then its value becomes and remains 1 forever. A 1–type belongs to $\mathcal{S}_X$ if and only if it is determined by an almost constant 1–sequence (that is, $\exists n_0 \forall n \geq n_0 f(n) = 1$). A 2–type $\Gamma = \Gamma(x, y)$ of T with $\Gamma_1(x) = \Gamma \restriction \{x\}$ and $\Gamma_2(y) = \Gamma \restriction \{y\}$ belongs to

$\mathcal{S}_X$ if and only if $\Gamma_1, \Gamma_2 \in \mathcal{S}_X$ and the following condition is satisfied: $\neg R(x,y) \in \Gamma(x,y)$, or neither $\Gamma_1(x)$ nor $\Gamma_2(y)$ is determined by the constant 1–sequence, or if one of $\Gamma_1(x)$ and $\Gamma_2(y)$ is determined by the constant 1–sequence, then the other one is compatible with X. An n–type belongs to $\mathcal{S}_X$ if and only if each of its 2-subtypes belongs to $\mathcal{S}_X$.

Lemma 10.12. *There is a homogeneous model which realizes precisely the types in* $\mathcal{S}_X$*.*

Proof. The existence of such a model follows from the next two properties. Property (1) will guarantee Henkin's witnesses, and property (2) will guarantee the homogeneity of the model which can be constructed by Henkin's method.

(1) If $\Gamma(x_0, \ldots, x_{n-1}) \in \mathcal{S}_X$ and $\theta(x_0, \ldots, x_{n-1}, x)$ is a formula consistent with Γ (that is, $\exists x \theta(x_0, \ldots, x_{n-1}, x) \in \Gamma)$), then there is an $(n+1)$–type $\Omega(x_0, \ldots, x_{n-1}, x) \in \mathcal{S}_X$ containing Γ and θ. Let us prove (1). Let the language of $\theta(x_0, \ldots, x_{n-1}, x)$ be L_s. Since T_s eliminates the quantifiers, there is a finite diagram $\delta(x_0, \ldots, x_{n-1}, x)$ of T_s such that

$$\vdash \delta(x_0, \ldots, x_{n-1}, x) \Rightarrow \theta(x_0, \ldots, x_{n-1}, x).$$

We can extend $\Gamma(x_0, \ldots, x_{n-1}) \cup \{\delta(x_0, \ldots, x_{n-1}, x)\}$ to a type

$$\Omega(x_0, \ldots, x_{n-1}, x)$$

in $\mathcal{S}_X$. If $x = x_i$ for some $i < n$, then the required extension Ω is unique. Otherwise, choose Ω in such a way that $\Omega(x_0, \ldots, x_{n-1}, x) \restriction \{x\}$ is compatible with X.

(2) If $\Gamma_1(x_0, \ldots, x_{n-1}, x_n) \in \mathcal{S}_X$ and $\Gamma_2(x_0, \ldots, x_{n-1}, x_n) \in \mathcal{S}_X$ and if

$$\Gamma_1 \restriction \{x_0, \ldots, x_{n-1}\} = \Gamma_2 \restriction \{x_0, \ldots, x_{n-1}\},$$

then there is an $(n+2)$–type $\Omega(x_0, \ldots, x_{n-1}, x_n, x) \in \mathcal{S}_X$ such that Ω contains $\Gamma_1(x_0, \ldots, x_{n-1}, x_n)$ and $\Gamma_2(x_0, \ldots, x_{n-1}, x)$. Let us prove (2). Let

$$\Gamma(x_0, \ldots, x_{n-1}, x_n, x) =_{def} \Gamma_1(x_0, \ldots, x_{n-1}, x_n) \cup \Gamma_2(x_0, \ldots, x_{n-1}, x).$$

If for some $i < n$,

$$(x_i = x_n) \in \Gamma_1(x_0, \ldots, x_{n-1}, x_n) \cup \Gamma_2(x_0, \ldots, x_{n-1}, x_n),$$

then the required extension Ω of Γ is uniquely determined. Otherwise, Ω will be determined by Γ, $x_n \neq x$, and $\neg R(x_n, x)$. □

Lemma 10.13. *The set* $\mathcal{S}_X$ *of types is computably enumerable.*

Proof. It is sufficient to prove that the set of all 2–types in $\mathcal{S}_X$ is computably enumerable. To prove this fact, it is enough to prove that a family $\mathcal{T}$ of 2–types in $\mathcal{S}_X$ is computably enumerable, where

$\mathrm{T} \supseteq \{\Gamma(x,y) : R(x,y) \in \Gamma$

$\wedge(\Gamma \restriction \{x\}$ is determined by the constant 1–sequence)
$\wedge(\Gamma \restriction \{y\}$ is determined by a sequence compatible with $X)\}$.

Let $\{X_t\}_{t\in\omega}$ be a computable enumeration of X. For every pair $(\overline{p}, t)$ of a finite sequence $\overline{p} = (p_0, \ldots, p_{l-1})$ and a number t, we define $f, g \in 2^\omega$ as follows:

$$f(0) = p_0, \ldots, f(l-1) = p_{l-1},$$
$$f(l) = 1 \text{ if } 0 \in X_t, \text{ and } \quad f(l) = 0 \text{ if } 0 \notin X_t,$$
$$\ldots$$
$$f(2l-1) = 1 \text{ if } l-1 \in X_t, \text{ and } \quad f(2l-1) = 0 \text{ if } l-1 \notin X_t,$$
$$f(2l) = 0, \text{ and } f(i) = 1 \quad \text{if } i > 2l.$$

Clearly, f is compatible with X if no new elements among $\{0, \ldots, l-1\}$ are enumerated in X after stage t. The sequence g keeps track of that part of the enumeration. Namely,

$$\begin{aligned} g(s) &= 0 \text{ if } (X_s - X_{s-1}) \cap \{0, \ldots, l-1\} \neq \emptyset \text{ for } s > t, \text{ and} \\ g(s) &= 1 \text{ otherwise.} \end{aligned}$$

Notice that g is determined by an almost constant 1–sequence. Also, by the above remark, if g is determined by the constant 1–sequence, then f is compatible with X. Let $\Gamma_{(\overline{p},t)}(x, y)$ be the 2–type such that

$$\Gamma_{(\overline{p},t)}(x, y) \supseteq \{R(x, y), P_k^{f(k)}, P_k^{g(k)} : k \geq 0\}.$$

Then $\mathcal{T} = \{\Gamma_{(\overline{p},t)}(x, y) : \overline{p} \in 2^{<2} \wedge t \in \omega\}$. □

Lemma 10.14. *Assume that a homogeneous model of T realizing precisely the types in $\mathcal{S}_X$ is computable. Then the set of types*

$$\{\Gamma : \Gamma \textit{ is a } 1\textit{-type of } T \textit{ compatible with } X\}$$

is computably enumerable.

Proof. Let $\mathcal{M}$ be a homogeneous model of T realizing precisely the types in $\mathcal{S}_X$. Let a' be an element of M which realizes in $\mathcal{M}$ the 1–type Θ of T determined by the constant 1–sequence. For every $a \in M$, let Γ_a be the 1–type realized in $\mathcal{M}$ by a. Since $\mathcal{M}$ is computable by assumption, it is enough to prove that

$$\{\Gamma : \Gamma \text{ is a 1-type of } T \text{ compatible with } X\} = \{\Gamma_a : a \in M \wedge \mathcal{M} \models R(\mathbf{a}', \mathbf{a})\}.$$

We first assume that $\mathcal{M} \models R(\mathbf{a}', \mathbf{a})$ for some $a \in M$. Since the 2–type realized in $\mathcal{M}$ by (a', a) belongs to $\mathcal{S}_X$, Γ_a is compatible with X (by the choice of a'). We now assume that Γ is a 1–type of T compatible with X. Let a 2–type $\Omega(x, y)$ be such that $R(x, y) \in \Omega$, $\Omega \restriction \{x\} = \Theta(x)$ and $\Omega \restriction \{y\} = \Gamma(y)$. Since Ω belongs to $\mathcal{S}_X$, it is realized in $\mathcal{M}$ by some (b', b). Thus, a' and b' realize the same 1–type in the homogeneous model $\mathcal{M}$. Let f be an automorphism of $\mathcal{M}$ such that $f(b') = a'$. Let $f(b) = a$ for some $a \in M$. Since (a', a) and (b', b) realize in M the same 2–types, it follows that $\Gamma = \Gamma_a$. Also, since $R^{\mathcal{M}}(\mathbf{b}', \mathbf{b})$, we have that $R^{\mathcal{M}}(\mathbf{a}', \mathbf{a})$. □

Lemma 10.15. *Let $\mathcal{M}$ be a homogeneous model realizing precisely the types in $\mathcal{S}_X$. If $\mathcal{M}$ is computable, then X is approximable.*

Proof. By Lemma 10.14, the set

$$\{\Gamma : \Gamma \text{ is a 1–type of } T \text{ compatible with } X\}$$

is computably enumerable. Therefore, we can algorithmically enumerate the infinite binary sequences which determine the 1–types of T compatible with X. We choose such a computable enumeration $\alpha_0, \alpha_1, \alpha_2, \ldots$ in such a way that for every $e \geq 0$, the length of agreement of α_e with X is at least e. Hence $(\exists i \geq 2e)[\alpha_e(i) = 0]$. We will define, by recursion, a unary computable function g as follows.

$g(0) = 0$

For $n > 0$, $g(n)$ is the least number such that

$$g(n) > g(n-1), \qquad g(n) > 2n + 2,$$

and for every $e < n$, there is $l = l_{e,n}$ which satisfies the following conditions:
$g(n) > 2l$,
$\alpha_e(2l) = 0, \alpha_e(2l + 1) = \alpha_e(2l + 2) = \ldots = \alpha_e(g(n)) = 1,$ $(*)$
$\{i : 0 \leq i \leq l - 1 \wedge \alpha_e(l + i) = 1\} = \{0, 1, \ldots, l - 1\} \cap X_{g(n)}$.

Thus, at stage n, we look at the initial segment of length $g(n)$ of each of the sequences $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ and, within this segment, obtain the compatibility of the sequences with $X_{g(n)}$. However, past this initial segment, it is still possible to have value 0 in certain α_e's for $e < n$.

We define a function f by $f(n) = |\{0, 1, \ldots, n-1\} \cap X_{g(n)}|$. Clearly, f is computable and for every $n \in \omega$, $f(n) \leq |\{0, 1, \ldots, n-1\} \cap X|$. Let $e \in \omega$. We define n_e to be the least number such that $n_e > n$ and α_e does not have a value 0 past the initial segment of length $g(n_e)$. We will prove that $f(n_e) = |\{0, 1, \ldots, n_e - 1\} \cap X|$.

Since α_e is compatible with X, we have the following equality for $l = l_{e,n_e}$:

$$\{i : 0 \leq i \leq l - 1 \wedge \alpha_e(l + i) = 1\} = \{0, 1, \ldots, l - 1\} \cap X.$$

On the other hand, by the definition of g, we have

$$\{i : 0 \leq i \leq l - 1 \wedge \alpha_e(l + i) = 1\} = \{0, 1, \ldots, l - 1\} \cap X_{g(n_e)}.$$

Therefore, to prove that $f(n_e) =| \{0, 1, \ldots, n_e - 1\} \cap X |$, it is enough to prove that $n_e \leq l_{e,n_e}$.

Assume that $e = n_e - 1$. The required inequality follows from the length of the compatibility of α_e.

Now assume that $e < n_e - 1$. By the definition of g, $g(n_e - 1) > 2n_e$. By the definition of n_e, α_e must have value 0 past the initial segment of length $g(n_e - 1)$. Hence, the desired inequality follows from the condition $(*)$ in the definition of g. □

In [139], Millar has given an example of a complete decidable theory with only computable complete types and with only countably many non-isomorphic countable models, which has an undecidable countable homogeneous model.

To state Goncharov's and Peretyat'kin's characterization of a decidable countable homogeneous model of a complete decidable theory, we introduce the following definition.

Definition 10.16. A computable set T of computable types of a theory T in L has the *effective extension property* if the following condition is satisfied for an effective enumeration $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ of all types in T, and an effective enumeration $\theta_0, \theta_1, \theta_2, \ldots$ of all formulae in L. There is a partial computable binary function f such that for every $n, i \in \omega$, if $\Gamma_n = \Gamma_n(x_0, \ldots, x_{k-1})$ for some $k \in \omega$, and Γ_n is consistent with $\theta_i = \theta_i(x_0, \ldots, x_{k-1}, x_k)$, then $f(n, i)$ is defined, $\Gamma_{f(n,i)}$ is a $(k+1)$–type and

$$(\Gamma_n \cup \{\theta_i\}) \subseteq \Gamma_{f(n,i)}.$$

Theorem 10.17. (*Goncharov [68], Peretyat'kin [167]*) *Let $\mathcal{A}$ be a countable homogeneous model with the type spectrum $\mathcal{T}$. Then $\mathcal{A}$ is decidable if and only if $\mathcal{T}$ is a computable set of computable types and $\mathcal{T}$ has the effective extension property.*

As a consequence of this characterization, Goncharov-Nurtazin's, and Harrington's characterization of a decidable prime model, as well as Morley's and Millar's characterization of a decidable countably saturated model can be obtained. Another consequence of this characterization is the next theorem, also obtained by Millar [147] as a consequence of a more general result.

Theorem 10.18. (*Goncharov [68], Millar [147]*) *Let the set of all computable types of a complete theory T be computable. If the set of all complete types realized in a countable homogeneous model $\mathcal{A}$ of T is a Σ^0_2 set of computable types, then $\mathcal{A}$ is decidable.*

Theorem 10.19. (*Millar [143]*) *Assume that T is a complete decidable theory all of whose types are computable and which has only countably many type spectra. Let $\mathcal{A}$ be a countable homogeneous model of T. If the type spectrum of $\mathcal{A}$ is Σ^0_2, then $\mathcal{A}$ is almost decidable.*

Algorithmic complexity of countable homogeneous models has also been studied by Denisov [41, 42, 43]. The following result is a computable analogue of the classical model theoretic result that every theory in a countable language has a countable homogeneous model.

Theorem 10.20. (*Denisov [43]*) *Every complete decidable theory has a countable homogeneous model which is decidable in $\emptyset'$.*

Theorem 10.21. (*Tusupov [200]*) *Let $\mathcal{A}$ be a countable homogeneous model of a decidable theory, such that the type spectrum of $\mathcal{A}$ is a computable family of computable types. Then $\mathcal{A}$ is decidable in $\emptyset'$.*

While every countable model has a countable homogeneous elementary extension, Goncharov and Drobotun [76] have constructed a computable linear order which does not have a computable homogeneous elementary extension. They have also constructed a decidable model which does not have a computable homogeneous elementary extension. (Also see [163].)

11 Vaught's Theorem Computably Visited

Theorem 11.1. (*Vaught*) *There is no complete theory which has exactly two non-isomorphic countable models.*

Proof. By contradiction. Assume that T has exactly two non-isomorphic countable models. Then T must have a countable saturated model $\mathcal{A}$ and a prime model $\mathcal{C}$. Clearly, $\mathcal{A}$ and $\mathcal{C}$ are not isomorphic. Since $\mathcal{A}$ is not prime, there is an n–tuple of elements of A which realizes a nonprincipal type of T. Without loss of generality, assume that $n = 1$. Thus, there is $a \in A$ which realizes a nonprincipal type $\Gamma(x)$. Let c be a new constant. Since $(\mathcal{A}, a)$ is a countable saturated model of $\Gamma(c)$, $\Gamma(c)$ also has a prime model $(\mathcal{B}, b)$. However, $\mathcal{B}$ is not prime because it realizes a nonprincipal type $\Gamma(x)$. Hence $\mathcal{B}$ and $\mathcal{C}$ are not isomorphic. Finally, $(\mathcal{B}, b)$ is not saturated because T is not $\aleph_0$–categorical, so T and, hence, $\Gamma(c)$ satisfy (v) of Theorem 9.3, so $\Gamma(c)$ is not $\aleph_0$-categorical. Hence $\mathcal{B}$ and $\mathcal{A}$ are not isomorphic. The existence of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ contradicts the assumption at the beginning of the proof. □

Theorem 11.2. (*Ehrenfeucht*) *For every $n \geq 3$, there is a complete theory with exactly n non-isomorphic models.*

On the other hand, Millar and Kudaibergenov have constructed a complete decidable theory with exactly two non-isomorphic decidable models. However, the effective version of Ehrenfeucht's result remains true [119]. To present Millar's and Kudaibergenov's result, we use from *model theory,* a characterization of submodel complete theories from Theorem 6.8; and from *computability theory,* the existence of two computably inseparable c.e. sets. That is, there are c.e. sets X and Y such that

$$X \cap Y = \emptyset \quad \text{and} \quad \neg(\exists R)[R \text{ is computable } \wedge X \subseteq R \wedge R \cap Y = \emptyset].$$

Theorem 11.3. (*Millar [146], Kudaibergenov[119]*) *There is a complete decidable theory T with exactly two non-isomorphic decidable models.*

Proof. We present the example from [146]. We will define a theory T such that the following conditions are satisfied.

(1) T has only one nonprincipal 1–type, $\Gamma(x)$. $\Gamma(x)$ is a computable type. (Notice that $T \subseteq \Gamma(x)$.)

(2) There is no computable 2–type $\Omega(x, y)$ of T such that

$$\Gamma(x) \cup \Gamma(y) \cup \{x \neq y\} \subseteq \Omega(x, y).$$

(3) $\Gamma(c)$ has a decidable prime model, where c is a new constant.

(4) If a model of T realizes a computable nonprincipal type of T, then it realizes all computable nonprincipal types of T.

Lemma 11.4. *Conditions (1)–(4) imply the theorem.*

Proof. Let $(\mathcal{B}, b)$ be a decidable prime model of $\Gamma(c)$, $\Gamma(c) = T \cup \Gamma(c)$, which exists by (3). Then $\mathcal{B}$ is a decidable prime model of T which realizes $\Gamma(x)$. $\Gamma(x)$ is a nonprincipal type, hence there is a decidable model $\mathcal{A}$ which omits $\Gamma(x)$. Since $\mathcal{A}$ is decidable, all types realized in $\mathcal{A}$ are computable. Since every type realized in $\mathcal{A}$ is principal, $\mathcal{A}$ is a prime model of T. We will prove that every decidable model of T is either isomorphic to $\mathcal{A}$ or to $\mathcal{B}$. Let $\mathcal{D}$ be a decidable model of T.

Case (a): $\mathcal{D}$ omits $\Gamma(x)$. Since $\mathcal{D}$ is decidable, all types realized in $\mathcal{D}$ are computable. By (4), $\mathcal{D}$ omits all nonprincipal types of T. Since every type realized in $\mathcal{D}$ is principal, $\mathcal{D}$ is a prime model of T, hence $\mathcal{D} \cong \mathcal{A}$.

Case (b): $\mathcal{D}$ realizes $\Gamma(x)$. Let $d \in D$ be such that $\mathcal{D} \models \Gamma(x)[d]$. We claim that $(\mathcal{D}, d) \cong (\mathcal{B}, b)$ and, hence, that $\mathcal{D} \cong \mathcal{B}$. Assume otherwise, that is, $(\mathcal{D}, d) \ncong (\mathcal{B}, b)$. Then $(\mathcal{D}, d)$ is not a prime model of $\Gamma(c)$. Thus, $(\mathcal{D}, d)$ must realize a nonprincipal type $\Omega(c, \overline{x}) = \Omega(c, x_1, \ldots, x_n)$ of $\Gamma(c)$. Hence $\Gamma(x) \subseteq \Omega(x, \overline{x})$, and $\Omega(x, \overline{x})$ is a computable type. Also, $\Omega(x, \overline{x})$ is a nonprincipal type of T, hence it is realized in $\mathcal{B}$. Let $b', b'_1, \ldots, b'_n \in B$ be such that $\mathcal{B} \models \Omega(x, \overline{x})[b', b'_1, \ldots, b'_n]$. It follows by (2) that $\Gamma(x)$ cannot be realized in a decidable model $\mathcal{B}$ by two different elements, b and b', since the 2-type determined by (b, b') in B would be computable. Thus, $(\mathcal{B}, b) \models \Omega(c, \overline{x})[b'_1, \ldots, b'_n]$. This is a contradiction, since $(\mathcal{B}, b)$ is a decidable prime model of $\Gamma(c)$, and $\Omega(c, \overline{x})$ is a nonprincipal type of $\Gamma(c)$. □

The language of T is $L = \{P_n(\cdot), S_n(\cdot, \cdot) : n \in \omega\}$, where each $P_n(\cdot)$ is a unary relation symbol and each $S_n(\cdot, \cdot)$ is a binary relation symbol. Let $X \subseteq \omega$ and $Y \subseteq \omega$ be computably inseparable c.e. sets. We will encode X and Y into 2–types of T. Let $(X_t)_{t\in\omega}$ and $(Y_t)_{t\in\omega}$ be computable enumerations of X and Y, respectively, such that if $n \in X_t$ or $n \in Y_t$, then $n < t$. (We have $X_0 \subseteq X_1 \subseteq X_2 \subseteq \ldots$ and ${}_{t\in\omega}X_t = X$, and similar relations for Y.) We first define T' such that $T' \subseteq T$. The axioms of T' are the universal closures with respect to x and y of the following formulae. Let $n, t \in \omega$.

Ax 1 $P_t(x) \Rightarrow P_{t+1}(x)$;

Ax 2 $\neg S_n(x, x)$;

Ax 3 $S_n(x,y) \Rightarrow S_n(y,x)$;
Ax 4 $P_t(x) \Rightarrow \neg S_t(x,y)$;
Ax 5 $(\neg P_t(x) \wedge \neg P_t(y) \wedge x \neq y) \Rightarrow S_n(x,y)$ if $n \in X_t$;
Ax 6 $(\neg P_t(x) \wedge \neg P_t(y) \wedge x \neq y) \Rightarrow \neg S_n(x,y)$ if $n \in Y_t$.

Clearly, T' is a universal set of sentences. T' is obviously consistent, since a nonempty set A with $P_t^{\mathcal{A}} = A$ for $t \in \omega$, and $S_n^{\mathcal{A}} = \emptyset$ for $n \in \omega$ is a model of T'. We will now extend T' to T in such a way that T is submodel complete, and, therefore, admits the elimination of quantifiers. We will add a new set of axioms. First we introduce some notation. Let $\mathcal{M}$ be a finite model of T'. Let $\Delta_{\mathcal{M}^n}(\overline{a})$ be the conjunction of all atomic and negated atomic sentences true in $\mathcal{M}_M$, in which only relation symbols $P_0, \dots, P_n$ and $S_0, \dots, S_n$ may occur. There are only finitely many such sentences. T' is extended to T by adding a new group of axioms for every $n \in \omega$:

Ax 7 $(\forall \overline{x})(\exists \overline{y})[\Delta_{\mathcal{M}^n}(\overline{x}) \Rightarrow \Delta_{\mathcal{N}^n}(\overline{x}, \overline{y})]$,

where $\mathcal{M}$ and $\mathcal{N}$ are finite models of T' such that $\mathcal{M} \subseteq \mathcal{N}$, allowing $M = \emptyset$.

Lemma 11.5. *T is consistent.*

Proof. By compactness. □

Lemma 11.6. *T is computably axiomatizable.*

Lemma 11.7. *T is submodel complete.*

Proof. We will prove

$$(\forall \mathcal{A}, \mathcal{B} \models T)(\forall \mathcal{D} \subseteq \mathcal{A}, \mathcal{B})[\mathcal{A} \text{ and } \mathcal{B} \text{ satisfy the same existential sentences with parameters from } D].$$

Let $\mathcal{A}, \mathcal{B} \models T$ and $\mathcal{D} \subseteq \mathcal{A}, \mathcal{B}$. Let $\delta(\overline{x}, \overline{y})$ be a conjunction of atomic and negated atomic formulae, and $\overline{d} \in D^{h(\overline{x})}$ such that $\mathcal{A} \models (\exists \overline{y})\delta(\overline{x}, \overline{y})[\overline{d}]$. Let $\overline{a} \in A^{lh(\overline{y})}$ be such that $\mathcal{A} \models \delta(\overline{x}, \overline{y})[\overline{d}, \overline{a}]$. Let $\mathcal{M}$ be the submodel of $\mathcal{A}$ whose domain consists of the elements in $\overline{d}$. Let $\mathcal{N}$ be the submodel of $\mathcal{A}$ whose domain consists of the elements in $\overline{d}$ and $\overline{a}$. Since T' is a universal theory, we have $\mathcal{M}, \mathcal{N} \models T'$. Let n be the largest subscript of a P–predicate symbol or of an S–predicate symbol occurring in $\delta(\overline{x}, \overline{y})$. (If no P–predicate symbol and no S–predicate symbol occurs in $\delta(\overline{x}, \overline{y})$, let $n = 0$.) Clearly,

$$\models [\Delta_{\mathcal{N}^n}(\overline{x}, \overline{y}) \Rightarrow \delta(\overline{x}, \overline{y})].$$

Since $(\forall \overline{x})(\exists \overline{y})[\Delta_{\mathcal{M}^n}(\overline{x}) \Rightarrow \Delta_{\mathcal{N}^n}(\overline{x}, \overline{y})]$ is an axiom of T, we have that

$$T \models (\forall \overline{x})(\exists \overline{y})[\Delta_{\mathcal{M}^n}(\overline{x}) \Rightarrow \delta(\overline{x}, \overline{y})].$$

Since $\mathcal{B} \models \Delta_{\mathcal{M}^n}(\overline{x})[\overline{d}]$ and $\mathcal{B} \models T$, we have $\mathcal{B} \models (\exists \overline{y})\delta(\overline{x}, \overline{y})[\overline{d}]$. Let us now prove that T satisfies the conditions (1)–(4).

(1) *T has only one nonprincipal 1–type, $\Gamma(x)$. Furthermore, $\Gamma(x)$ is a computable type.* Let $\Gamma^*(x)$ be a 1–type of T. Since T admits the elimination of quantifiers and $\neg S_n(x,x)$ is an axiom, every formula in $L(T)$ with one free variable is equivalent to a quantifier-free formula whose only relation symbols are from $\{P_n : n \in \omega\}$. Therefore, $\Gamma^*(x)$ is uniquely determined by the set $\{t : P_t(x) \in \Gamma^*(x)\}$. Assume that this set is nonempty, and let t_0 be its smallest element. Since

$$P_{t_0}(x) \Rightarrow P_{t_0+1}(x) \Rightarrow P_{t_0+2}(x) \Rightarrow \cdots,$$

we have that $\neg P_0(x) \wedge \ldots \wedge \neg P_{t_0-1}(x) \wedge P_{t_0}(x)$ is a complete formula of $\Gamma^*(x)$. Hence, $\Gamma^*(x)$ is a principal type. Thus, there is exactly one nonprincipal 1–type $\Gamma(x)$, where $\Gamma(x)$ contains $\{\neg P_t(x) : t \in \omega\}$. Since T is decidable and admits the elimination of quantifiers, we can effectively find for each formula a corresponding quantifier-free formula. Thus, Γ is computable.

(2) *There is no computable 2–type $\Omega(x,y)$ of T such that*

$$\Gamma(x) \cup \Gamma(y) \cup \{x \neq y\} \subseteq \Omega(x,y).$$

Assume otherwise for some $\Omega(x,y)$. Define

$$R = \{n \in \omega : S_n(x,y) \in \Omega(x,y)\}.$$

R is a computable set since $\Omega(x,y)$ is a computable type. By Ax 5 and Ax 6, $X \subseteq R$ and $Y \cap R = \emptyset$, contradicting the computable inseparability of X and Y.

(3) *Let c be a new constant. Then $\Gamma(c)$ has a decidable prime model.* To prove (3), it is enough to prove the following lemma.

Lemma 11.8. *Every computable type of $\Gamma(c)$ is principal.*

Let us first prove that Lemma 11.8 implies (3). Since $\Gamma(x)$ is a computable type, $\Gamma(c)$ has a decidable model $(\mathcal{B}, b)$. Since every type realized in a decidable model is computable, by Lemma 11.8, every type realized in $(\mathcal{B}, b)$ is principal. Hence $(\mathcal{B}, b)$ is a prime model of $\Gamma(c)$. Let us next prove Lemma 11.8.

Assume otherwise. Let $\Xi = \Xi(c, x_2, \ldots, x_n)$ be a computable nonprincipal type of $\Gamma(c)$. It is an $(n-1)$–type for $n \geq 2$. By (2), for each $i \in \{2, \ldots, n\}$, there is the least $k(i)$ with $P_{k(i)}(x_i) \in \Xi$. Thus,

$$[\bigwedge_{2 \leq i \leq n} P_{k(i)}(x_i)] \in \Xi.$$

Since Ξ is nonprincipal, $\bigwedge_{2 \leq i \leq n} P_{k(i)}(x_i)$ is not a complete formula. Hence there are infinitely many distinct $(n-1)$–types of $\Gamma(c)$ which contain

$$\{P_{k(i)}(x_i) : 2 \leq i \leq n\}.$$

Thus, there are infinitely many distinct n–types of T which contain

$$\{P_{k(i)}(x_i) : 2 \leq i \leq n\} \cup \Gamma(x_1).$$

Since T eliminates quantifiers, every formula of $L(T)$ in n free variables for $n \geq 2$ is uniquely determined by the $\frac{1}{2}n(n-1)$ many 2–types it determines. Hence infinitely many 2–types of T contain

$$\{P_{k(i)}(x_i), P_{k(j)}(x_j), x_i \neq x_j\}$$

for some $i, j \in \{2, \ldots, n\}$, or infinitely many 2–types of T contain $\{P_{k(i)}(x_i)\} \cup \Gamma(x_1)$ for some $i \in \{2, \ldots, n\}$. Again, by the elimination of quantifiers, each of these implies that for some $k(i)$ and for infinitely many n, both $\{P_{k(i)}(x_i), S_n(x, y)\}$ and $\{P_{k(i)}(x_i), \neg S_n(x, y)\}$ are consistent. This contradicts the axioms of T'.

(4) *If a model of T realizes a computable nonprincipal type of T, then it realizes all computable nonprincipal types of T.* For every nonprincipal computable type $\Theta(x_1, \ldots, x_n)$ of T, there is $i \in \{1, \ldots, n\}$ such that $\Gamma(x_i) \subseteq \Theta(x_1, \ldots, x_n)$. Therefore, any model of T realizing a nonprincipal computable type must realize Γ. As before, we can conclude that all decidable models of T realizing Γ are isomorphic. Hence the statement follows. □

While the theory T constructed in the previous theorem has only two non-isomorphic decidable models, it has $2^{\aleph_0}$ non-isomorphic countable models. Millar [136] has further shown that there is a complete decidable theory with only countably many non-isomorphic countable models, which has exactly two non-isomorphic decidable models.

12 Decidable Ehrenfeucht Theories

Definition 12.1. An *Ehrenfeucht theory* is a complete theory with only finitely many non-isomorphic countable models.

Clearly, if a complete decidable theory T is $\aleph_0$–categorical, then T has, up to isomorphism, only one countable model which can be chosen to be decidable. In 1971, Baldwin and Lachlan [23] established Vaught's conjecture that every complete $\aleph_1$–categorical theory has either exactly one or exactly ω many non-isomorphic countable models. The following result is an effective version of the Baldwin-Lachlan's result.

Theorem 12.2. (*Harrington [88], Khisamiev [100]*) *If a complete decidable theory T is $\aleph_1$–categorical, then every countable model of T is isomorphic to a decidable model.*

Proof. Every countable model of T can be viewed as a prime model of some other $\aleph_1$–categorical decidable theory. □

Nerode posed the following problem:

Let T be a complete decidable theory which has only finitely many non-isomorphic countable models. Can all of these models be chosen to be decidable?

By Vaught's theorem, T cannot have exactly two non-isomorphic countable models. We will prove that T must have a decidable prime model. Assume otherwise. Then T has a decidable model realizing a nonprincipal type, which is omitted in another decidable model of T realizing another nonprincipal type, etc. Here we use the fact that every finite set of nonprincipal types of T is omitted in some decidable model of T. Thus, contrary to the assumption, T has infinitely many non-isomorphic decidable models.

Morley gave an example of a theory T with exactly six non-isomorphic countable models, of which only the prime one can be chosen to be computable (even decidable). Moreover, Lachlan has found a simple example of such a theory, using the fact that there is a computable linear ordering of order type $\omega + \omega^*$ whose ω–part is not computable. Peretyat'kin [165] has generalized this result. He has obtained for every $n \geq 3$, a theory T in a finite language, with exactly n non-isomorphic countable models, of which only the prime one can be chosen to be computable (decidable). To construct such theories, Peretyat'kin has used a least upper bound operator to obtain an underlying $\aleph_0$–categorical theory which admits the elimination of quantifiers, and in which a binary tree can be distinguished by constants.

The countable non-isomorphic models of decidable Ehrenfeucht theories in all mentioned examples can be chosen to be decidable in $\mathbf{0}'$. The question then arises whether all countable models of an arbitrary complete decidable Ehrenfeucht theory can be chosen to be, for example, arithmetic. Millar has answered this question negatively by showing that there is a complete decidable theory with only finitely many non-isomorphic countable models, some of which must be chosen to be of arbitrarily high hyperarithmetic degree. Moreover, the theory in Millar's example is persistently Ehrenfeucht (see Definition 4.2). Persistently Ehrenfeucht theories are also called *persistently finite* theories and have been introduced and first studied by Benda. It can be shown that every persistently decidable Ehrenfeucht theory has a decidable saturated model.

Definition 12.3. Let $X \subseteq \omega$. We say that a model $\mathcal{A}$ is decidable exactly in X if $\mathcal{A}$ is decidable in X and for every $Y \subseteq \omega$, if $\mathcal{A}$ is decidable in Y then $X \leq_T Y$.

Theorem 12.4. (*Millar [138]*) *Let H_n be a hyperarithmetic set, where $n \in \omega$. Then there is a complete decidable persistently Ehrenfeucht theory T with an undecidable countable model, such that for every undecidable countable model $\mathcal{A}$ of T, A is decidable exactly in H_n.*

For every H_n, the corresponding theory in the previous theorem has eighteen countable non-isomorphic models, exactly three of which are decidable. To define such a theory, Millar has used the existence of a computable subtree of $\omega^{<\omega}$ having exactly one infinite branch f, where $f \equiv_T H_n$ (see [98], page 456).

In Morley's, Lachlan's and Peretyat'kin's counterexamples to Nerode's question, the theories have non-computable types. Therefore Morley raised the following question:

Let T be a complete decidable theory with all types computable, which has only finitely many non-isomorphic countable models. Can all of these models be chosen to be decidable?

Assume that one of the finitely many non-isomorphic models of T must be undecidable. Clearly, T is not $\aleph_0$–categorical. The fact that T has a finite number of non-isomorphic decidable models has several implications. As shown before, T has a decidable prime model. The set of all types of T is computably enumerable, because every computable type of T is realized in some decidable model of T. Since the set of all types of T is computably enumerable, T has a decidable saturated model. We can mimic Vaught's construction to obtain a third non-isomorphic decidable model. Therefore, if the answer to Morley's question is negative, then a counterexample must have at least four non-isomorphic countable models. Indeed, Goncharov has recently announced a negative answer to Morley's question.

Theorem 12.5. (*Goncharov [75]*) *There is a decidable Ehrenfeucht theory with all types computable, whose non-isomorphic countable models cannot be chosen to be all decidable.*

Millar asked the following question:

If T is an arithmetic Ehrenfeucht theory whose types are all arithmetic, are all countable models of T arithmetic?

Ash and Millar have proven that if the answer to this question is negative, then a counterexample must have at least five non-isomorphic countable models. Ash and Millar have also proven that the answer to this question is positive when every type of T is realized in only finitely many non-isomorphic countable models.

Theorem 12.6. (*Ash-Millar [20]*) *If T is a complete, arithmetic, persistently Ehrenfeucht theory with a countable non-arithmetic model, then T has at least five non-isomorphic countable models.*

Theorem 12.7. (*Ash-Millar [20]*) *If T is a complete, persistently Ehrenfeucht theory with only arithmetic complete types, then all countable models of T are arithmetic.*

Theorem 12.8. (*Millar [140]*) *If T is a decidable Ehrenfeucht theory with a countable model which is not decidable in $\emptyset''$, then T has at least five non-isomorphic countable models.*

Theorem 12.9. (*Reed [179, 180]*) *Let H_n be a hyperarithmetic set, where $n \in \omega$. Then there is a decidable persistently Ehrenfeucht theory T with exactly five non-isomorphic countable models:*

Theorem 12.10. (*i*) *A decidable prime model;*

(*ii*) *A decidable non-homogeneous model which is the reduct of the prime model of* $\Gamma(c)$*, where* c *is a new constant and* $\Gamma(x)$ *is a computable nonprincipal type of* T*;*

(*iii*) *A decidable homogeneous model which realizes all computable types of* T*;*

(*iv*) *A non-homogeneous model decidable exactly in* H_n*, which is the reduct of the prime model of* $\Omega(d)$*, where* d *is a new constant and* $\Omega(x)$ *is the only non-computable* 1*–type of* T*;*

(*v*) *A saturated model decidable exactly in* H_n*.*

Thus, the theory in the previous theorem has, up to isomorphism, three decidable models and two models which are decidable exactly in H_n. It follows from Theorem 12.8 that this is an example of a decidable Ehrenfeucht theory with the least possible number of non-isomorphic countable models which are not all decidable in $\emptyset''$. It is not known whether a decidable Ehrenfeucht theory whose undecidable countable models are decidable exactly in $\emptyset''$ can have fewer than five countable models.

Closely related to the notion of an $\aleph_0$–homogeneous model is the notion of an almost homogeneous model.

Definition 12.11. A model is *almost homogeneous* if some finite expansion of the model by constants is $\aleph_0$–homogeneous.

It is not known whether there is an Ehrenfeucht theory with a model which is not almost homogeneous. Millar [140] has shown that if T is a persistently Ehrenfeucht, persistently decidable theory whose every model is almost homogeneous, then every countable model of T is isomorphic to a decidable model.

13 Decidable Theories with Countably Many Countable Models

Millar has constructed a complete decidable theory with exactly two non-isomorphic decidable models and only countably many non-isomorphic countable models. To present Millar's construction, we use from: *computable model theory*, a characterization of a universal theory with a complete decidable model completion, as stated in Theorem 6.9; and from *computability theory*, the existence of a certain computable binary tree, as will be established by Theorem 13.3.

Definition 13.1. For a tree T, an infinite branch f of T is called a limit branch if for every initial segment α of f, there is a node $\beta \in T$ such that α is an initial segment of β, and β is not an initial segment of f.

For finite or infinite binary sequences f and g, we write $f <_L g$ if there is a (finite) binary sequence α such that $\alpha\hat{\ }0$ is an initial segment of f and $\alpha\hat{\ }1$ is an initial segment of g.

Theorem 13.2.

Theorem 13.3. (*Millar [136]*) *There is a computable binary tree $\mathcal{T}$ whose leaves form a computable set $\mathcal{L}$, and a unary computable function h such that the following conditions are satisfied.*

(*i*) $\forall \alpha \in \mathcal{T}[\alpha \notin \mathcal{L} \Rightarrow \alpha\widehat{\ }1 \in \mathcal{T}]$

(*ii*) *There is exactly one limit branch of $\mathcal{T}$, which we denote by f. Moreover, f is not computable.*

(*iii*) *If g is an infinite branch of $\mathcal{T}$ different from f, then all but finitely many values of g are 1, and $f <_L g$.*

(*iv*) *If $\beta \in \mathcal{L}$ and $\gamma \in \mathcal{T}$ are such that $\gamma <_L \beta$, then $h(\gamma) < lh(\beta)$.*

(*v*) *There is at most one element of a given length in $\mathcal{L}$. If $\alpha_0, \alpha_1, \alpha_2, \ldots$ is an enumeration of $\mathcal{L}$ in the order of the increasing length of nodes, then for all $i, j \in \omega$, $h(i) = lh(\alpha_i)$ and $(i < j \Rightarrow \alpha_i <_L \alpha_j)$.*

Theorem 13.4. (*Millar [136]*) *There is a complete decidable theory T with exactly two non-isomorphic decidable models, which has only countably many non-isomorphic countable models.*

Proof. We will define a complete decidable theory T such that the following conditions are satisfied.

(1) T has a nonprincipal computable 1–type, $\Gamma(x)$.

(2) Every countable model of T is homogeneous.

(3) There is a sequence $(\Gamma_n)_{n\in\omega}$ of types of T such that:

(3.1) $\Gamma_0 = \Gamma$;

(3.2) Γ_1 is non-computable;

(3.3) If $i < j$, then every model which realizes Γ_j also realizes Γ_i;

(3.4) For every $n \in \omega$, there is a countable model which realizes Γ_n and omits Γ_{n+1};

(3.5) The type spectrum of a countable model $\mathcal{A}$ of T is exactly the set of all types in $\{\Gamma_n : n \in \omega\}$ which $\mathcal{A}$ realizes.

Lemma 13.5. *Conditions (1)–(3) imply the Theorem.*

Proof. Since Γ is a computable type, T has a decidable model $\mathcal{A}$ which realizes Γ. Since the computable type Γ is nonprincipal, T has a decidable model $\mathcal{B}$ which omits Γ. Clearly, $\mathcal{A}$ and $\mathcal{B}$ are non-isomorphic. Let $\mathcal{D}$ be a decidable model of T. $\mathcal{D}$ must omit Γ_1, because Γ_1 is not computable. Hence, by (3.3), $\mathcal{D}$ omits every Γ_k for $k \geq 1$. Thus, since all countable models of T are homogeneous, if $\mathcal{D}$ realizes Γ, $\mathcal{D}$ is isomorphic to $\mathcal{A}$, and if $\mathcal{D}$ omits Γ, $\mathcal{D}$ is isomorphic to $\mathcal{B}$. Hence, T has exactly two decidable non-isomorphic models. For every $n \in \omega$, let $\mathcal{A}_n$ be a countable model of T which realizes Γ_n and omits Γ_{n+1}. Hence, by (3.3), $\mathcal{A}_n$ realizes every Γ_k for $k \leq n$, and omits every Γ_k for $k > n$. Hence, by (3.5), $\{\Gamma_k : k \leq n\}$ is the type spectrum of $\mathcal{A}_n$. Thus, since all countable models are homogeneous, T has exactly countably many countable models. The language of T is $L = \{P_n(\cdot), S_n(\cdot,\cdot) : n \in \omega\}$, where for $n \in \omega$, $P_n(\cdot)$ is a unary relation symbol and $S_n(\cdot,\cdot)$ is a binary relation symbol. Let a computable binary tree $\mathcal{T}$ whose leaves form a computable set $\mathcal{L}$, and a unary function h be as in Theorem

13.3. We first define T' such that $T' \subseteq T$. The axioms of T' are the universal closures of the following formulae:

Ax 1 $P_t(x) \Rightarrow P_{t+1}(x)$ for $t \in \omega$;
Ax 2 $\neg S_n(x,x)$ for $n \in \omega$;
Ax 3 $S_n(x,y) \Rightarrow S_n(y,x)$ for $n \in \omega$;
Ax 4 $P_t(x) \Rightarrow \neg S_n(x,y)$ for $n \geq h(t)$;
Ax 5 $[\neg P_t(x) \wedge \neg P_t(y) \wedge x \neq y] \Longrightarrow \neg \bigwedge_{i<lh(\alpha)} S_i(x,y)^{\alpha(i)}$
for $\alpha \notin \mathcal{T}$ such that $lh(\alpha) = h(t+1)$;
Ax 6 $\bigwedge_{i<lh(\alpha)} S_i(x,y)^{\alpha(i)} \Longrightarrow \neg S_n(x,y)$
for $\alpha \notin \mathcal{T} - \mathcal{L}$ such that $lh(\alpha) \leq n$;
Ax 7 $\bigwedge_{i<lh(\alpha)} S_i(x,y)^{\alpha(i)} \Longrightarrow [P_t(x) \Leftrightarrow P_t(y)]$
for $\alpha \in \mathcal{T} - \mathcal{L}$ such that $lh(\alpha) \leq h(t)$;
Ax 8 $\bigwedge_{i<lh(\alpha)} S_i(x,y)^{\alpha(i)} \Longrightarrow [P_t(x) \vee P_t(y)]$
for $\alpha \in \mathcal{L}$ such that $lh(\alpha) = h(t)$;
Ax 9 $[\bigwedge_{i<lh(\alpha)} S_i(x,y)^{\alpha(i)} \wedge \bigwedge_{i<lh(\beta)} S_i(y,z)^{\beta(i)} \wedge x \neq y]$
$\Longrightarrow \bigwedge_{i<lh(\alpha)} S_i(x,z)^{\alpha(i)}$ for $\alpha, \beta \in 2^{<\omega}$ such that $\alpha <_L \beta$.

Now it can be shown that Theorem 6.9 applies to T'. T will be a complete decidable model completion of T'. □

Let T be a complete decidable theory with all complete types computable. It is known that there is such a theory which has, up to isomorphism, $2^{\aleph_0}$ countable models. Hence it has undecidable models. Millar (see Theorem 7.5) has shown that if T does not have a decidable model whose finite expansion by constants is prime, then T must have, up to isomorphism, $2^{\aleph_0}$ countable models. The question then arises whether there is a T with only countably many non-isomorphic countable models and with an undecidable countable model. First we introduce the following

Definition 13.6. Let Γ and Ω be types of T. The type ordering is defined by

$$\Gamma \leq \Omega \Longleftrightarrow (\forall \mathcal{A} \models T)[\mathcal{A} \text{ realizes } \Gamma \Rightarrow \mathcal{A} \text{ realizes } \Omega].$$

Theorem 13.7. (*Millar [139]*) *There is a complete decidable theory T with all complete types computable, and with only countably many non-isomorphic countable models such that its countable saturated model is undecidable.*

The theory T is the model completion of a universal theory T', obtained using Theorem 6.9. The ordering of all nonprincipal 1-types of T is linear with order type ω^*. The set of all complete types of T is not Σ^0_2. Every decidable model of T omits a type of T, and, therefore, every countable saturated model of T is undecidable.

Theorem 13.8. (*Millar [141]*) *There is a complete decidable theory T with all complete types computable, and with only countably many non-isomorphic countable models, such that T has a decidable saturated model and a countable undecidable homogeneous model.*

The theory T is the model completion of a universal theory T', obtained using Theorem 6.9. The set of all complete types of T is computably enumerable. This guarantees the existence of a decidable saturated model. The set of all types realized by a countable undecidable homogeneous model is not Σ^0_2. However, both the set of all 1-types realized and the set of all 1–types omitted by a countable undecidable homogeneous model are linearly ordered by the type ordering relation, with order type ω^*.

14 Indiscernibles and Decidability

The notion of order indiscernibles, introduced by Ehrenfeucht and Mostowski, plays an important role in generating models with certain properties.

Let T be a fixed complete theory in L and let $\mathcal{U}$ be an $\aleph_1$–saturated model of T. Since all countable models of T are elementarily embeddable in $\mathcal{U}$, we can assume that all countable models considered in this section are elementary submodels of $\mathcal{U}$.

Definition 14.1. Let $D \subseteq U$.

(i) A set of formulae $\Gamma = \Gamma(x_0, \ldots, x_{n-1})$ is a *type over D* if there are $a_0, \ldots, a_{n-1} \in U$ such that for every formula $\theta(x_0, \ldots, x_{n-1})$ in L_D we have $\theta(x_0, \ldots, x_{n-1}) \in \Gamma \Longleftrightarrow \mathcal{U}_D \models \theta(x_0, \ldots, x_{n-1})[a_0, \ldots, a_{n-1}]$.

(ii) Let $B \subseteq U$, where $B = \{b_0, b_1, b_2, \ldots\}$ is a fixed enumeration of B. A set Γ of formulae with free variables among $x_0, x_1, x_2, \ldots$ is the *ω–type of B over D* (with respect to the enumeration of B) if for every $n \in \omega$, for every finite sequence $(k_0, \ldots, k_{n-1})$ of natural numbers and every formula θ in L_D in n free variables, we have

$$\theta(x_{k_0}, \ldots, x_{k_{n-1}}) \in \Gamma \Longleftrightarrow \mathcal{U}_D \models \theta(x_{k_0}, \ldots, x_{k_{n-1}})[b_{k_0}, \ldots, b_{k_{n-1}}].$$

Definition 14.2. Let $D \subseteq U$ and $I \subseteq U$, where $I = \{i_0, i_1, i_2, \ldots\}$ is a fixed enumeration of I.

(i) I is a set of (*order*) *indiscernibles over D* if for every $n \in \omega$ and every increasing n–tuple $k_0 < \ldots < k_{n-1}$ of natural numbers:

$(i_0, \ldots, i_{n-1})$ and $(i_{k_0}, \ldots, i_{k_{n-1}})$ satisfy the same formulae in L_D.

(ii) I is a set of *total indiscernibles over D* if for every $n \in \omega$ and every n–tuple $(k_0, \ldots, k_{n-1})$ of distinct natural numbers:

$(i_0, \ldots, i_{n-1})$ and $(i_{k_0}, \ldots, i_{k_{n-1}})$ satisfy the same formulae in L_D.

(iii) The indiscernibles over $\emptyset$ are simply called the *indiscernibles*.

Proposition 14.3. *Every theory with an infinite model has a model $\mathcal{A}$ with an infinite set I of indiscernibles such that $I \subseteq A$.*

Kierstead and Remmel [108] have studied computable analogues of the previous proposition. They have shown that the problem of determining whether a decidable model of T has an infinite set of indiscernibles is a Σ_1^1 question. They have investigated decidable theories which have decidable models with infinite computable sets of indiscernibles, as well as the possible Turing degrees of the sets of indiscernibles in decidable models.

Let us recall that an ω–branching tree is a tree whose nodes belong to $\omega^{<\omega}$. Kierstead and Remmel [108] have shown that the problem of finding an infinite set of indiscernibles in an infinite decidable model of T is, in some sense, equivalent to the problem of finding an infinite branch in a computable ω–branching tree. More precisely, a decidable model $\mathcal{A}$ of T is equivalent to an ω–branching tree $\mathcal{T}$ if there are oracle algorithms $\phi_{e_1}^{(\,)}$ and $\phi_{e_2}^{(\,)}$, such that the following is true:

(i) For every infinite set I of indiscernibles in $\mathcal{A}$, $\phi_{e_1}^{(I)}$ outputs an infinite branch f_I of $\mathcal{T}$;

(ii) For every infinite branch f of $\mathcal{T}$, $\phi_{e_2}^{(f)}$ outputs an infinite set I_f of indiscernibles in $\mathcal{A}$;

(iii) For every infinite branch f of $\mathcal{T}$, $f_{I_f} = f$.

Kierstead and Remmel have proven that for every decidable model of a complete theory, there exists an equivalent computable ω–branching tree; and for every computable ω–branching tree $\mathcal{T}$, there exists a complete decidable theory whose every decidable model is equivalent to $\mathcal{T}$.

Definition 14.4. Let $\mathcal{A}$ be a countable model of T, and let $D \subseteq U$ be such that $A \subseteq D$. Let $\Gamma = \Gamma(x_0, \ldots, x_{n-1})$ be a type of T over D.

(i) Γ is *definable over* $\mathcal{A}$ if for every $k \in \omega$, for every formula

$$\theta(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{k-1})$$

in L_A there exists a formula $\delta_\theta = \delta_\theta(y_0, \ldots, y_{k-1})$ in L_A such that for every $d_0, \ldots, d_{k-1} \in D$

$$\theta(x_0, \ldots, x_{n-1}, \mathbf{d}_0, \ldots, \mathbf{d}_{k-1}) \in \Gamma \iff \mathcal{U} \models \delta_\theta(y_0, \ldots, y_{k-1})[d_0, \ldots, d_{k-1}].$$

We call the set

$$\{\delta_\theta(y_0, \ldots, y_{k-1}) : \theta(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{k-1}) \text{ is a formula in } L_A\}$$

a definition of Γ over $\mathcal{A}$.

(ii) Γ is *computably definable over* $\mathcal{A}$ if there is an algorithm which assigns to every formula $\theta(x_0, \ldots, x_{n-1}, \overline{y})$ in L_A a formula $\delta_\theta(\overline{y})$ such that $\{\delta_\theta(\overline{y}) : \theta(x_0, \ldots, x_{n-1}, \overline{y})$ is a formula in $L_A\}$ is a definition of Γ over $\mathcal{A}$.

To prove that certain theories have decidable models with infinite computable sets of indiscernibles, we use from *model theory*, a result in stability theory which establishes that the range of a sequence whose every member realizes a certain type, forms an infinite set of indiscernibles.

This result is stated in part (ii) of the following theorem, and it uses the basic fact about the unique definable extensions, stated in part (i) of the same theorem.

Theorem 14.5. *(i) Let $\mathcal{A}$ be a countable model of T, and let B and D be subsets of U such that $A \subseteq B \subseteq D$. Let $\Gamma(\overline{x})$ be a type of T over B which is definable over $\mathcal{A}$. There is a unique type over D, denoted by $\Gamma_D(\overline{x})$, which is definable over $\mathcal{A}$, such that $\Gamma(\overline{x}) \subseteq \Gamma_D(\overline{x})$.*

(ii) Let $\mathcal{A}$ be a countable model of T such that there is a type $\Gamma(x)$ which is definable over $\mathcal{A}$ and not realized in $\mathcal{A}$. Let $(b_0, b_1, b_2, \ldots)$ be a sequence of elements in U such that for every $n \in \omega$, b_n realizes $\Gamma_{A_n}(x)$, where $A_n = A \cup \{b_k : k < n\}$. Then $\{b_0, b_1, b_2, \ldots\}$ is an infinite set of indiscernibles over $\mathcal{A}$.

Proof. Let $\{\delta_\theta(\overline{y}) : \theta(\overline{x}, \overline{y}) \text{ is a formula in } L_A\}$ be a definition of $\Gamma(\overline{x})$ over $\mathcal{A}$. Define $\Gamma_D(\overline{x})$ to be the following set of formulae in L_A.

$$\{\theta(\overline{x}, \mathbf{d}_0, \ldots, \mathbf{d}_{k-1}) : (k \in \omega) \wedge (d_0, \ldots, d_{k-1} \in D) \\ \wedge (\mathcal{U} \models \delta_\theta(y_0, \ldots, y_{k-1})[d_0, \ldots, d_{k-1}])\}.$$

Since $B \subseteq D$, we have that $\Gamma(\overline{x}) \subseteq \Gamma_D(\overline{x})$. $\Gamma_D(\overline{x})$ is a consistent set of formulae by compactness. $\Gamma_D(\overline{x})$ is complete because $\mathcal{U} \models (\neg\delta_\theta \Leftrightarrow \delta_{\neg\theta})$. The uniqueness of $\Gamma_D(\overline{x})$ follows from the definition of a type over a model.

Notice that, by (i), if $i, j \in \omega$ are such that $i < j$, then the restriction of the type $\Gamma_{A_j}(x)$ to A_i is the type $\Gamma_{A_i}(x)$. Let $\{\delta_\theta(\overline{y}) : \theta(x, \overline{y}) \text{ is a formula in } L_A\}$ be a definition of $\Gamma(x)$ over $\mathcal{A}$. To show that $\{b_0, b_1, b_2, \ldots\}$ is a set of indiscernibles over $\mathcal{A}$, it is enough to show that for every two increasing sequences $f, g \in 2^\omega$, for every $n \geq 1$, and every formula θ in L_A in n free variables

$$\mathcal{U} \models \theta[b_{f(0)}, \ldots, b_{f(n-1)}] \iff \mathcal{U} \models \theta[b_{g(0)}, \ldots, b_{g(n-1)}]. \qquad (*)$$

Let such f and g be given, and fix n. Assume that $f(n) < g(n)$. Let $\{\gamma_\theta(\overline{y}) : \theta(x, \overline{y}) \text{ is a formula in } L_A\}$ be a definition of $\Gamma_{A_{g(n)}}$ over $\mathcal{A}$. Then for a formula θ in L_A in $(n+1)$ free variables, we have

$$\begin{aligned} \mathcal{U} \models \theta[b_{f(0)}, \ldots, b_{f(n-1)}, b_{g(n)}] &\iff \mathcal{U} \models \gamma_\theta[b_{f(0)}, \ldots, b_{f(n-1)}] \\ &\iff \theta(\mathbf{b}_{f(0)}, \ldots, \mathbf{b}_{f(n-1)}, x) \in \Gamma_{A_{g(n)}}(x) \\ &\iff \theta(\mathbf{b}_{f(0)}, \ldots, \mathbf{b}_{f(n-1)}, x) \in \Gamma_{A_{f(n)}}(x) \\ &\iff \mathcal{U} \models \theta[b_{f(0)}, \ldots, b_{f(n-1)}, b_{f(n)}]. \end{aligned}$$

Now the equivalence in $(*)$ follows inductively. To prove that $\{b_0, b_1, b_2, \ldots\}$ is an infinite set, consider the formula $\theta(x, y) = \neg(x = y)$. To prove that $\mathcal{A} \models \forall y \delta_\theta(y)$, we assume otherwise. Hence

$$\begin{aligned} \mathcal{A} \models \neg\delta_\theta(y)[a] \text{ for some } a \in A &\implies \mathcal{U} \models \neg\delta_\theta(y)[a] \\ &\implies (x = \mathbf{a}) \in \Gamma(x) \\ &\implies a \text{ realizes } \Gamma(x) \text{ in } \mathcal{U} \\ &\implies a \text{ realizes } \Gamma(x) \text{ in } \mathcal{A}. \end{aligned}$$

However, the last statement contradicts the assumption of the theorem. Hence $\mathcal{A} \models \forall y \delta_\theta(y)$ and, thus, $\mathcal{U} \models \forall y \delta_\theta(y)$. Let $i, j \in \omega$ be such that $i < j$. Then $\mathcal{A} \models \delta_\theta(y)[b_i]$ and $\neg(x = \mathbf{b}_i) \in \Gamma_{A_j}(x)$. Since b_j realizes $\Gamma_{A_j}(x)$, we have that $b_i \neq b_j$. □

Theorem 14.6. (*Kierstead-Remmel [107]*) *Let $\mathcal{A}$ be a decidable model of T such that there is a computably definable type $\Gamma(x)$ over $\mathcal{A}$, which is not realized in $\mathcal{A}$. Then T has a decidable model with an infinite computable set of indiscernibles.*

Proof. Let $\{\delta_\theta(\overline{y}) : \theta(x, \overline{y})$ is a formula in $L_A\}$ be a definition of $\Gamma(x)$ over $\mathcal{A}$ such that there is an algorithm which to every formula $\theta(x, \overline{y})$ in L_A assigns $\delta_\theta(\overline{y})$. Let L' be $L_A \cup \{c_0, c_1, c_2, \ldots\}$, where $c_0, c_1, c_2, \ldots$ is a computable enumeration of new constants. We inductively define the following sets of sentences in L'

$$T_0 = \text{the theory of } \mathcal{A}_A \text{ in } L_A;$$

$$T_{n+1} = T_n \cup \{\theta(c_n, c_{n-1}, \ldots, c_0) : (\theta(x_n, x_{n-1}, \ldots, x_0) \text{ is in } L_A)$$
$$\wedge\, \delta_\theta(c_{n-1}, \ldots, c_0) \in T_n\} \text{ for } n \geq 0.$$

Let $T' =_{def} \bigcup_{n\in\omega} T_n$. T' is a consistent complete theory in L'. T' is decidable because $\mathcal{A}$ is decidable and the considered definition of $\Gamma(x)$ is algorithmic. By the Effective Completeness Theorem, there is a decidable model $\mathcal{B}$ of T'. As mentioned before, we assume that $\mathcal{B} \preceq \mathcal{U}$. Let $I =_{def} \{b_0, b_1, b_2, \ldots\}$, where for every $i \in \omega$, b_i is the interpretation in $\mathcal{B}$ of the constant c_i. Since $\mathcal{B}$ satisfies T_0, we can assume that $\mathcal{A} \preceq \mathcal{B}$. Clearly, I is a computable set. We use Theorem 14.5 (ii) to show that I is an infinite set of indiscernibles. Let $A_0 = A$, and $A_{n+1} = A \cup \{b_0, \ldots, b_n\}$. We show that

$$\Gamma_{A_n}(x) = \{\theta(x, c_{n-1}, \ldots, c_0) : (\theta(x_n, x_{n-1}, \ldots, x_0) \text{ is in } L_A)$$
$$\wedge\, \delta_\theta(c_{n-1}, \ldots, c_0) \in T_n\}.$$

Thus, b_n realizes $\Gamma_{A_n}(x)$. □

Examples of theories to which Theorem 14.6 applies are the theory of dense linear order without endpoints, and the theory of real closed fields.

Theorem 14.7. (*Kierstead-Remmel [107]*) *Let Q be a generalized quantifier whose interpretation is "there are infinitely many". Assume that T is a stable theory which also satisfies the following decidability condition (D).*

There is an effective procedure which decides for every formula in L of the form $\phi(x, y_0, \ldots, y_{k-1})$, whether

$$T \cup \{(\exists y_0) \ldots (\exists y_{k-1})(Qx)\phi(x, y_0, \ldots, y_{k-1})\}$$

has a model.

Then T has a decidable model with an infinite computable set of total indiscernibles.

Proof. Such a theory T has a decidable model $\mathcal{A}$ and a type Γ over A, such that Γ is computably definable over $\mathcal{A}$, and not realized in $\mathcal{A}$. Also, since T is stable, every set of order indiscernibles is a set of total indiscernibles. □

The strong decidability condition (D) in the previous theorem cannot be replaced by the usual decidability condition. Also, the stability condition cannot be omitted from the assumption of the theorem, as shown by the following counterexample.

Proposition 14.8. (*Kierstead-Remmel [107]*) *There is a complete theory T satisfying the decidability condition (D) such that T has infinitely many decidable models, none of which has an infinite computable set of indiscernibles, although each of them has an infinite set of indiscernibles.*

It is well known from the classical model theory that every $\aleph_0$–stable theory is stable in all infinite powers. It is easy to show that there is an $\aleph_0$–stable decidable theory which does not satisfy the decidability condition in Theorem 14.7.

Proposition 14.9. (*Kierstead-Remmel [107]*) *If T is an $\aleph_0$–stable and decidable theory, then T has a decidable model with an infinite computable set of total indiscernibles.*

Kierstead and Remmel have shown that $\aleph_0$–stability in the previous theorem can be replaced neither by stability nor even by superstability.

Proposition 14.10. (*Kierstead-Remmel [107]*) *There is a complete decidable superstable theory which has an infinite decidable model, but it does not have a decidable model with an infinite computable set of indiscernibles.*

The following result illustrates an application of Theorem 14.9.

Proposition 14.11. (*Kierstead-Remmel [107]*) *If T is $\aleph_0$–stable and decidable, then T has models of arbitrarily large cardinality, which realize only computable types.*

Proof. By Theorem 14.9, T has a decidable model $\mathcal{A}$ with an infinite computable set of indiscernibles I. Let κ be an arbitrary infinite cardinal. There are a model $\mathcal{B}$ of T, and a subset J of B of cardinality κ such that J is the set of indiscernibles satisfying the same ω–type of T as I. Since T is $\aleph_0$–stable, by a result from model theory, there is a prime model $\mathcal{C}$ over J. Clearly, $\mathcal{A}$ and $\mathcal{C}$ realize the same types. Since $\mathcal{A}$ is decidable, the types that they realize are computable. □

Theorem 14.12. (*Kierstead-Remmel [108]*) *If $\mathcal{A}$ is a decidable model with an infinite set of indiscernibles, then $\mathcal{A}$ has an infinite set I of indiscernibles such that the hyperdegree of I is strictly less than the hyperdegree of Kleene's $\mathcal{O}$.*

Kierstead and Remmel have also investigated the degrees of sets of indiscernibles in decidable models of $\aleph_0$–categorical theories.

Definition 14.13. A decidable theory T has *decidable atoms* if there is an effective procedure which decides whether a given formula is an atom in the Lindenbaum algebra of formulae with the corresponding free variables.

Kierstead and Remmel [108] have shown that the problem of finding an infinite set of indiscernibles in an infinite decidable model of an $\aleph_0$–categorical theory with decidable atoms is, in some sense, equivalent to the problem of finding an infinite branch in an infinite computable tree. In particular, for every infinite computable binary tree $\mathcal{T}$, there is a decidable model $\mathcal{A}$ of an $\aleph_0$–categorical decidable theory with decidable atoms, such that there is an effective one-to-one correspondence between the infinite branches of $\mathcal{T}$ and the ω–types of infinite sets of indiscernibles in $\mathcal{A}$.

Thus, the set of Turing degrees realized by the sets of ω–types of infinite sets of order indiscernibles in a decidable model of an $\aleph_0$–categorical theory coincides with the set of degrees realized by recursively bounded Π^0_1 classes. Thus, the following result follows from Jockusch-Soare's work [95] on Turing degrees of Π^0_1 classes.

Theorem 14.14. (*Kierstead-Remmel [108]*) *Let $\mathcal{A}$ be a decidable model of an $\aleph_0$–categorical theory with decidable atoms. $\mathcal{A}$ has an infinite set of indiscernibles of low Turing degree, and $\mathcal{A}$ has an infinite set of indiscernibles of a c.e. degree. If $\mathcal{A}$ does not have an infinite computable set of indiscernibles, then there are continuum many ω–types of infinite sets of indiscernibles, which have mutually incomparable Turing degrees.*

There are decidable models of T with infinite sets of indiscernibles which have no hyperarithmetic infinite sets of indiscernibles. However, it is not true if T is $\aleph_0$–categorical.

Theorem 14.15. (*Kierstead-Remmel [108]*) *If $\mathcal{A}$ is a decidable model of an $\aleph_0$–categorical complete theory, then $\mathcal{A}$ has an infinite set I of indiscernibles such that* $\deg(I) \leq \mathbf{0}'$.

15 Degrees of Models

Clearly, a computably axiomatizable complete theory is computably enumerable. Kleene [109] and Hasenjaeger [89] have independently shown that if T is a computably axiomatizable theory, then T has a countable model whose domain is a set of natural numbers, such that every relation and function of the model is Δ^0_2. On the other hand, there is a computably axiomatizable theory which does not have a model in which every relation and function is c.e. or co-c.e.

Unless otherwise stated, we consider only models whose domain is ω. (For such a model $\mathcal{A}$, a set of formulae in L_A can be thought of as a set of natural numbers.) This allows us to define the (*Turing*) *degree* of $\mathcal{A}$, denoted by $\deg(\mathcal{A})$, as the Turing degree of the atomic diagram $\Delta_{\mathcal{A}}$ of $\mathcal{A}$. Thus, $\mathcal{A}$ is computable if and only if $\deg(\mathcal{A}) = \mathbf{0}$.

It is easy to see that the theory of a model $\mathcal{A}$ is computable in the complete diagram of $\mathcal{A}$, and that the complete diagram of $\mathcal{A}$ is computable in $(\Delta_{\mathcal{A}})^{(\omega)}$. Henkin's construction of a model of a complete theory T produces a model $\mathcal{B}$ whose atomic diagram and complete diagram are both computable in T (see Theorem 5.1). Hence T and the complete diagram of $\mathcal{B}$ have the same Turing degree. The atomic diagram of a model of T may be of much lower Turing degree than T. For example, *true arithmetic* is the theory of the standard model of natural numbers, and its Turing degree is $\mathbf{0}^{(\omega)}$.

Shoenfield has used the following lemma from computability theory to improve Hasenjaeger's and Kleene's result.

Lemma 15.1. (*Kreisel's Basis Lemma*) *An infinite computable binary tree has a Δ^0_2 infinite branch.*

Shoenfield has first strengthened Kreisel's Basis Lemma by proving that an infinite computable binary tree has an infinite branch of Turing degree $< \mathbf{0}'$.

Theorem 15.2. (*Shoenfield [194]*) *If T is a computably axiomatizable theory, then T has a countable model whose degree is $< \mathbf{0}'$.*

Proof. Extend T to a complete theory S in the same language such that the Turing degree of S is $< \mathbf{0}'$. This can be done using Shoenfield's strengthening of Kreisel's Basis Lemma. □

Jockusch and Soare [94] have generalized Kreisel-Shoenfield Basis Theorem.

Theorem 15.3. (*Low Basis Theorem*) *An infinite computable binary tree has an infinite branch of low Turing degree.*

The Low Basis Theorem implies that every computably axiomatizable theory has a model of low Turing degree.

Knight [112] has shown that for a model $\mathcal{A}$, either there is a finite set $S \subseteq A$ such that all bijections of A that fix S are automorphisms of $\mathcal{A}$; or for every Turing degree $\mathbf{d} \geq \deg(\mathcal{A})$, there is a model $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that $\deg(\mathcal{A}) = \mathbf{d}$. Wehner [212] and Slaman [195] have independently found a countable model $\mathcal{A}$ such that the Turing degrees of models isomorphic to $\mathcal{A}$ are exactly the noncomputable degrees.

Since the degree of a model is not invariant under isomorphisms, Jockusch has introduced the following complexity measure of the isomorphism type of a model. The isomorphism type of a model $\mathcal{A}$ is the set of all models isomorphic to $\mathcal{A}$.

Definition 15.4. (Richter [187]) The degree of the isomorphism class of $\mathcal{A}$, if it exists, is the least Turing degree in $\{\deg(\mathcal{B}) : \mathcal{B} \cong \mathcal{A}\}$.

The following theorem establishes that the degree of the isomorphism class of a model satisfying certain general computable condition cannot be different from $\mathbf{0}$.

Theorem 15.5. (*Richter [186]*) *Assume that a model $\mathcal{A}$ satisfies the following computable embeddability condition.*

For every finite model $\mathcal{C}$ isomorphic to a submodel of $\mathcal{A}$ and every embedding f of $\mathcal{C}$ into $\mathcal{A}$, there is an algorithm which determines whether a given finite model $\mathcal{D}$ extending $\mathcal{C}$ can be embedded into $\mathcal{A}$ by an embedding extending f.

Then if the degree of the isomorphism class of $\mathcal{A}$ exists, it must be $\mathbf{0}$.

Proof. If $\mathcal{A}$ is a computable model, then the statement follows immediately. Assume that $\mathcal{A}$ is not computable. We will prove that there is model $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that $\deg(\mathcal{A})$ and $\deg(\mathcal{B})$ form a minimal pair. Hence $\mathbf{0}$ will be the only possible degree of the isomorphism class of $\mathcal{A}$. A model $\mathcal{B}$ and an isomorphism h from $\mathcal{B}$ onto $\mathcal{A}$ will be constructed in stages by finite extension. Let L be the language of $\mathcal{A}$.

Construction

Stage 0: Let $\mathcal{B}_0 = \emptyset$ and $h_0 = \emptyset$.

Stage $s = 2e+1$: First assume that there is a finite model $\mathcal{C}$ for L extending $\mathcal{B}_s$ and an embedding g of $\mathcal{C}$ into $\mathcal{A}$ extending h_s, such that for some $n \in \omega$, both $\{e\}^{\mathcal{C}}(n)$ and $\{e\}^{\mathcal{A}}(n)$ are defined and

$$\{e\}^{\mathcal{C}}(n) \neq \{e\}^{\mathcal{A}}(n).$$

In this case, for some such $\mathcal{C}$ and g, let $\mathcal{B}_{s+1} =_{def} \mathcal{C}$ and $h_{s+1} =_{def} g$. Otherwise, let $\mathcal{B}_{s+1} =_{def} \mathcal{B}_s$ and $h_{s+1} =_{def} h_s$.

Stage $s = 2e+2$: If $e \in A - rng(h_s)$, let $h_{s+1} =_{def} h_s \cup \{(u,e)\}$, where $u \in \omega$ is the least number such that $u \notin dom(h_s)$. Otherwise, let $h_{s+1} =_{def} h_s$. In both cases, extend $\mathcal{B}_s$ to $\mathcal{B}_{s+1}$ such that h_{s+1} is an embedding of $\mathcal{B}_{s+1}$ into $\mathcal{A}$. End of the construction. Let $\mathcal{B} =_{def} \bigcup_{s\in\omega} \mathcal{B}_s$ and $h =_{def} \bigcup_{s\in\omega} h_s$. Clearly, h is an isomorphism from $\mathcal{B}$ to $\mathcal{A}$. Now, let us prove that $\deg(\mathcal{A})$ and $\deg(\mathcal{B})$ form a minimal pair. Since $\mathcal{A}$ is not computable, by Posner's Lemma, it is enough to prove that for every $e \in \omega$:

$$\{e\}^{\mathcal{A}} = \{e\}^{\mathcal{B}} = f \ \text{ total } \implies f \ \text{ is computable.}$$

Thus, assume $\{e\}^{\mathcal{A}} = \{e\}^{\mathcal{B}} = f$, where f is total. By construction, there is a stage s such that for every finite extension $\mathcal{C}$ of $\mathcal{B}_s$ which can be embedded into $\mathcal{A}$, and every $n \in \omega$ such that $\{e\}^{\mathcal{C}}(n)$ is defined, we have $\{e\}^{\mathcal{C}}(n) = \{e\}^{\mathcal{A}}(n)$. Hence $f(n) = \{e\}^{\mathcal{A}}(n)$. By the computable embeddability condition, f must be computable. □

The previous theorem can be applied to show that the isomorphism class of a countable tree which is not isomorphic to a computable tree, does not have a degree. Hence, the isomorphism class of a countable linear ordering which is not isomorphic to a computable linear ordering does not have a degree.

Theorem 15.6. (*Richter [186]*) *Let S be a theory in a finite language L such that there is a computable sequence $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2, \ldots$ of finite models for L which* *are pairwise non-embeddable. Assume that for every $X \subseteq \omega$, there is a countable model $\mathcal{A}_X$ of S which is computable in X and*

$$(\forall i)[\mathcal{A}_i \textit{ is embeddable in } \mathcal{A}_X \Leftrightarrow i \in X].$$

Then for every Turing degree $\mathbf{d}$*, there is a countable model of S whose isomorphism class has degree* $\mathbf{d}$.

Proof. Let $\mathbf{d}$ be a Turing degree and $D \subseteq \omega$ be such that $\deg(D) = \mathbf{d}$. Let $X =_{def} D \oplus \overline{D}$. We will show that $\mathcal{A}_X$ is a countable model of S whose isomorphism class has degree $\mathbf{d}$. Clearly,

$$\mathcal{A}_{D\oplus\overline{D}} \leq_T D \oplus \overline{D} \leq_T D,$$

so $\deg(\mathcal{A}_X) \leq \mathbf{d}$. Let $\mathcal{B}$ be a model isomorphic to $\mathcal{A}_X$. It is enough to prove that $\deg(\mathcal{B}) \geq \mathbf{d}$. This follows from the fact that

$$(i \in D \Leftrightarrow \mathcal{A}_{2i} \text{ is embeddable in } \mathcal{B}) \wedge$$
$$(i \notin D \Leftrightarrow \mathcal{A}_{2i+1} \text{ is embeddable in } \mathcal{B}).$$

□

The previous theorem can be used to show that for every Turing degree $\mathbf{d}$, there is a countable abelian group whose isomorphism class has degree $\mathbf{d}$. A corresponding sequence of finite models consists of cyclic groups of every prime order, and the abelian group assigned to an arbitrary set of natural numbers is obtained by forming countable direct sums.

Theorem 15.3 implies that there is a nonstandard model of Peano arithmetic of low degree. McAloon has asked whether there is a nonstandard model $\mathcal{A}$ of Peano arithmetic such that the theory of $\mathcal{A}$ is not arithmetic and the degree of $\mathcal{A}$ is arithmetic. Harrington has given the answer by establishing the following result.

Theorem 15.7. (*Harrington*) *There is a nonstandard model $\mathcal{A}$ of Peano arithmetic such that the theory of $\mathcal{A}$ has degree $\mathbf{0}^{(\omega)}$ and the degree of $\mathcal{A}$ is $\leq \mathbf{0}'$.*

The construction uses Harrington's *worker method* with infinitely many workers. The n–th worker produces the Σ_n–part of the complete diagram of the model, using $\emptyset^{(n)}$ as an oracle. To assure coherence, every n–th worker constantly guesses what the $(n+1)$–st worker has done. In [113], Knight has improved Harrington's result by showing that there is a nonstandard model $\mathcal{A}$ of Peano arithmetic such that the theory of $\mathcal{A}$ has degree $\mathbf{0}^{(\omega)}$, and the degree of $\mathcal{A}$ is low. This result follows from a general theorem of Knight [113] for which she has used Harrington's worker method with infinitely many workers to produce a model of a theory T, which realizes a certain set of types of bounded complexity.

Feferman [59] has stated that every arithmetic set is computable in the degree of every nonstandard model of true arithmetic. In fact, his proof yields a stronger result. First we need the following definition.

Definition 15.8. A Turing degree d is a *subuniform upper bound* for the arithmetic sets if there is $X \subseteq \omega$ such that $\deg(X) \leq d$ and

$$(\forall n)(\exists i)[X^{[i]} = \emptyset^{(n)}].$$

Theorem 15.9. (*Feferman [59]*) *If $\mathcal{A}$ is a nonstandard model of true arithmetic of degree* $\mathbf{d}$*, then* $\mathbf{d}$ *is a subuniform upper bound for the arithmetic sets.*

As Marker has pointed out, certain results on the degrees of nonstandard models of true arithmetic are analogous to the results on the degrees of nonstandard models of Peano arithmetic. From the fact that the degree of true arithmetic is $\mathbf{0}^{(\omega)}$, it follows that there is a nonstandard model of true arithmetic of degree $\leq \mathbf{0}^{(\omega)}$. Knight has shown that there is such a model of degree $< \mathbf{0}^{(\omega)}$. Marker has used a modification of Harrington's worker method with three workers to obtain the following result.

Theorem 15.10. (*Marker [128]*) *Let* $\mathbf{d}$ *be a Turing degree such that for every* $n \geq 0$*,* $\mathbf{d} > \mathbf{0}^{(n)}$*. Then there is a nonstandard model $\mathcal{A}$ of true arithmetic such that* $\deg(\mathcal{A}) \leq \mathbf{d}'$*.*

It follows from the previous theorem that there is a nonstandard model of true arithmetic whose degree $\mathbf{d}$ is such that $\mathbf{d}' = \mathbf{0}^{(\omega)}$. Marker has also shown that for a nonstandard model $\mathcal{A}$ of Peano arithmetic, the set of degrees of all models isomorphic to $\mathcal{A}$ is closed upward. In particular, the set of degrees of all nonstandard models of true arithmetic is closed upward.

Knight, Lachlan and Soare [110] have strengthened Theorem 15.10 by showing that, given $\mathbf{d}$ as in Theorem 15.10, there is a nonstandard model $\mathcal{A}$ of true arithmetic such that $(\deg(\mathcal{A}))' \leq \mathbf{d}'$. As a consequence of their result, they have obtained

Corollary 15.11. (*Knight-Lachlan-Soare [110]*) *There is a nonstandard model of true arithmetic of degree* $\mathbf{d}$ *such that* $\mathbf{d}'' = \mathbf{0}^{(\omega)}$*.*

Proof. By a result of Sacks, there is a Turing degree $\mathbf{d}$ such that $\mathbf{d}'' = \mathbf{0}^{(\omega)}$, and for every $n \in \omega$, $\mathbf{d} > \mathbf{0}^{(n)}$. Fix such $\mathbf{d}$. Let $\mathcal{A}$ be a nonstandard model of true arithmetic such that $(\deg(\mathcal{A}))' \leq \mathbf{d}'$. Hence $(\deg(\mathcal{A}))'' \leq \mathbf{0}^{(\omega)}$. □

Knight attempted to answer Jockusch's question about a characterization of the degrees of nonstandard models of true arithmetic, by conjecturing that if a degree $\mathbf{d}$ is such that $(\forall n \geq 1)[\mathbf{d} > \mathbf{0}^{(n)}]$, then $\mathbf{d}$ is the degree of a model of true arithmetic. This conjecture is refuted by the following theorem.

Theorem 15.12. (*Knight-Lachlan-Soare [110]*) *There is a Turing degree* $\mathbf{d}$ *which is not a subuniform upper bound for the arithmetic sets, such that* $(\forall n \geq 1)[\mathbf{d} > \mathbf{0}^{(n)}]$*. In addition,* $\mathbf{d}'' = \mathbf{0}^{(\omega)}$*.*

If **d** is as in Theorem 15.12, then, by Theorem 15.9, **d** is not the degree of a model of true arithmetic.

In the 1984 *Logic Colloquium* material, Solovay gave a characterization of the degrees of nonstandard model of true arithmetic. Solovay's characterization is in terms of the effective enumerations of families of the so-called Scott sets.

Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be a computable enumeration without repetition of all nodes in $2^{<\omega}$.

Definition 15.13. A set $S \subseteq P(\omega)$ is called a Scott set if it satisfies the following conditions for all $X, Y \subseteq \omega$:

(1) $(X \in S \wedge Y \leq_T X) \Rightarrow Y \in S$;

(2) $(X \in S \wedge Y \in S) \Rightarrow X \oplus Y \in S$;

(3) $[X \in S \wedge (T = \{\alpha_n : n \in X\}$ is an infinite tree$)]$
$\Longrightarrow (\exists Z \in S)[\{\alpha_n : n \in Z\}$ is an infinite branch of $T]$.

For $n \in \omega$, let $\theta_n(x)$ be a formula in the language of Peano arithmetic which expresses that "x is divisible by the n–th prime number". If $\mathcal{A}$ is a nonstandard model of Peano arithmetic, then

$$\{\{n : \mathcal{A} \models \theta_n(x)[a]\} \mid a \in A\}$$

is a Scott set. It is called *the Scott set of* $\mathcal{A}$ and is denoted by $Scott(\mathcal{A})$.

Definition 15.14. Let T be a complete extension of Peano arithmetic and let $X \subseteq \omega$. X is *representable* with respect to T if for some formula $\theta(x)$ of $L(T)$ and every $n \in \omega$:

$$[T \vdash \theta(\mathbf{n})] \Leftrightarrow n \in X.$$

Scott [192] has proven that the family of all Scott sets coincides with the family of sets which are representable with respect to some complete extension of Peano arithmetic.

An *enumeration* of a countable family $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is a binary relation ν such that $\mathcal{S} = \{\nu_0, \nu_1, \nu_2, \ldots\}$, where for every $i \in \omega$, $\nu_i =_{def} \{n : (i, n) \in \nu\}$. By "effectivizing" conditions (1) – (3) in Definition 15.13, we obtain the notion of an effective enumeration of a Scott set.

Definition 15.15. Let S be a countable Scott set. An enumeration ν of S is an effective enumeration if there are computable functions $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ such that the following conditions are satisfied for all $i,j \in \omega$:

(1) $(\nu_i = X \wedge Y = \{e\}^X) \Rightarrow Y = \nu_{f(i,e)}$;

(2) $\nu_i \oplus \nu_j = \nu_{g(i,j)}$;

(3) $[\nu_i = X \wedge (T = \{\alpha_n : n \in X\}$ is an infinite tree$) \wedge \nu_{h(i)} = Y]$
$\Longrightarrow [\{\alpha_n : n \in Y\}$ is an infinite branch of $T]$.

Theorem 15.16. (*Solovay*) *Let* **d** *be a Turing degree.*

(*i*) **d** *is the degree of a nonstandard model of true arithmetic* $\Leftrightarrow$ **d** *is the degree of an effective enumeration of a*
countable Scott set which contains all arithmetic sets.

(*ii*) *Let $\mathcal{S}$ be a countable Scott set.* $\mathbf{d}$ *is the degree of a nonstandard model of true*
arithmetic with the Scott set $\mathcal{S} \Leftrightarrow \mathbf{d}$ *is the degree of an effective enumeration of* $\mathcal{S}$.

In the following theorem, Knight has established a general sufficient condition for a Turing degree to be the degree of a model representing a given Scott set. We will use the following notation in the theorem. For a theory T and $n \in \omega$, we define T_n to be the set of Gődel numbers of all Σ_n sentences in T.

Theorem 15.17. (*Knight [115]*) *Let ν be an effective enumeration of a (countable) Scott set $\mathcal{S}$, and let T be a complete theory such that for every $n \in \omega$, $T_n \in \mathcal{S}$. Assume that there is an algorithm which on every input $n \in \omega$, using the n–th jump of ν, outputs $i \in \omega$ such that $\nu_i = T_{n+1}$. Then there is a model $\mathcal{A}$ of T which represents $\mathcal{S}$, such that the atomic diagram of $\mathcal{A}$ is Turing reducible to ν.*

This theorem gives Theorem 15.16 as a corollary. Another corollary is the following strengthening of Theorem 15.7.

Theorem 15.18. (*Knight [115]*) *Let ν be an effective enumeration of a (countable) Scott set $\mathcal{S}$. Let* $\mathbf{d}$ *be the Turing degree of ν. There is a nonstandard model $\mathcal{A}$ of Peano arithmetic with the Scott set $\mathcal{S}$ such that the theory of $\mathcal{A}$ has degree* $\geq \mathbf{d}^{(\omega)}$ *and the degree of $\mathcal{A}$ is* $\leq \mathbf{d}$.

Since many models do not have the degree of their isomorphism class, Jockusch has introduced another measure of model complexity which is invariant under isomorphisms. This measure uses jumps of the degrees of models.

Definition 15.19. Let α be a computable ordinal. The α–th jump degree of a model $\mathcal{A}$ is, if it exists, the least Turing degree among $\{\deg(\mathcal{B})^{(\alpha)} : \mathcal{B} \cong \mathcal{A}\}$.

Obviously, the notion of the 0–th jump degree of $\mathcal{A}$ coincides with the notion of the degree of the isomorphism class of $\mathcal{A}$. While Richter [186] has shown that the only possible 0–th jump degree of a linear ordering is $\mathbf{0}$, Knight [112] has shown that the only possible first jump degree of a linear ordering is $\mathbf{0}'$. No nonstandard model of Peano arithmetic has 0–th jump degree. There is a nonstandard model of Peano arithmetic with a 1–st jump degree. We have the following general results for jump degrees of linear orderings and Boolean algebras.

Theorem 15.20. (*Knight [112], Ash-Knight [10], Jockusch-Soare [96], Ash-Jockusch-Knight [9], Downey-Knight [46]*) *Let $\alpha \geq 1$ be a computable ordinal and let* $\mathbf{d}$ *be a Turing degree such that* $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$. *Then there is a linear ordering $\mathcal{A}$ whose α–th jump degree is* $\mathbf{d}$ *and such that $\mathcal{A}$ does not have β–th jump degree for any $\beta < \alpha$.*

Theorem 15.21. (*Jockusch-Soare [97]*)
(*i*) *Let* $\mathbf{d}$ *be a Turing degree such that* $\mathbf{d} \geq \mathbf{0}^{(\omega)}$. *Then there is a Boolean algebra $\mathcal{A}$ whose ω–th jump degree is* $\mathbf{d}$.

(*ii*) *Let* $n \in \omega$, *and* $\mathbf{d}$ *be a Turing degree such that* $\mathbf{d} > \mathbf{0}^{(n)}$. *Then there is no Boolean algebra* $\mathcal{A}$ *whose* n*–th jump degree is* $\mathbf{d}$.

Result (i) of Theorem 15.21 is a straightforward application of a method by Feiner, see [97].

16 Automorphisms and Computable Models

One of the important and interesting questions in computable model theory is how a specific aspect of a computable model may change if the model is isomorphically transformed so that it remains computable. A model $\mathcal{B}$ isomorphic to a computable model $\mathcal{A}$ is not necessarily computable. However, even if $\mathcal{B}$ is computable, it can still lose many of the computable properties of $\mathcal{A}$.

A computable property of a computable model $\mathcal{A}$ which Ash and Nerode have considered is an additional computable relation R on the domain of $\mathcal{A}$ (that is, R is not named in the language of $\mathcal{A}$). For example, Ash and Nerode have studied conditions under which the image of R under any isomorphism from $\mathcal{A}$ to another computable model is necessarily a computable or a c.e. relation.

Definition 16.1. Let R be an additional relation on the domain of a computable model $\mathcal{A}$.

(i) (Ash-Nerode [21]) R is *intrinsically c.e.* on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ to a computable model is c.e.

(ii) Let $\mathcal{P}$ be a certain class of relations. R is called *intrinsically* $\mathcal{P}$ on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ to a computable model belongs to $\mathcal{P}$.

For example, Moses [156] has established that relations which are intrinsically computable on a computable linear order $\mathcal{A}$ are precisely those that are equivalent in $\mathcal{A}$ to quantifier-free formulae with finitely many parameters. Let $\mathcal{A}$ be a computable Boolean algebra and let R be a computable subalgebra of $\mathcal{A}$. Odintsov [162] has established that R is intrinsically c.e. if and only if R is generated by a finite set of elements and a finite set of principal ideals of $\mathcal{A}$. This characterization implies that if R is intrinsically c.e. then R is intrinsically computable. However, it is easy to see that there are intrinsically c.e. relations which are not intrinsically computable.

Ash and Nerode have introduced a computable syntactic condition for a new relation on the domain of a computable model, to be called a formally c.e. relation.

Definition 16.2. (i) An $L_{\omega_1\omega}$ formula with free variables among $\overline{x}$ is a (computable) Σ_1 formula if it is equivalent to a formula of the form

$$\bigvee_{n\in\omega} \exists \overline{y}_n \theta_n(\overline{x}, \overline{y}_n),$$

where $(\theta_n(\overline{x}, \overline{y}_n))_{n\in\omega}$ is a (computable) sequence of quantifier-free formulae.

(ii) (Ash-Nerode [21]) Let R be an additional m–ary computable relation on the domain of a computable model $\mathcal{A}$. R is *formally c.e.* on $\mathcal{A}$ if and only if there is a finite sequence $(b_0, \dots, b_{k-1})$ of elements in A and a computable Σ_1 formula $F(x_0, \dots, x_{m-1}, b_0, \dots, b_{k-1})$ such that the following equivalence holds for every $a_0, \dots, a_{m-1} \in A$:

$$R(a_0, \dots, a_{m-1}) \Leftrightarrow \mathcal{A}_A \models \mathcal{F}(\mathbf{a}_0, \dots, \mathbf{a}_{m-1}, \mathbf{b}_0, \dots, \mathbf{b}_{k-1}).$$

R is formally computable on $\mathcal{A}$ if both R and its complement are formally c.e. on $\mathcal{A}$.

That is, R is formally c.e. on $\mathcal{A}$ if and only if R is equivalent to an infinite disjunction of a computable sequence of existential formulae with finitely many fixed parameters from A. A formally c.e. relation is also called a *formally* Σ_1 relation.

Clearly, every formally c.e. relation on a computable model is intrinsically c.e. Ash and Nerode have proven, under a certain decidability condition (D), the converse, thus establishing the equivalence of a syntactic and a semantic condition. For an m–ary relation R on a model $\mathcal{A}$, the condition (D) is:

There is an algorithm which determines for $k \in \omega$, for an existential formula $\psi(x_0, \dots, x_{m-1}, y_0, \dots, y_{k-1})$ and a sequence $(b_0, \dots, b_{k-1})$ of elements of A, whether the following implication holds for every $a_0, \dots, a_{m-1} \in A$:

$$[\mathcal{A}_A \models \psi(\mathbf{a}_0, \dots, \mathbf{a}_{m-1}, \mathbf{b}_0, \dots, \mathbf{b}_{k-1})] \Longrightarrow R(a_0, \dots, a_{m-1}).$$

Condition (D) implies that R is a computable relation. It also implies that $\mathcal{A}$ is 1–computable, which is a property of a model defined as follows.

Definition 16.3. A model $\mathcal{A}$ is 1–*computable* if there is an algorithm which determines for every existential formula $\psi(x_0, \dots, x_{n-1})$ and every sequence $(a_0, \dots, a_{n-1})$ of elements of A, whether $\psi(a_0, \dots, a_{n-1})$ is true in $\mathcal{A}_A$.

Let $\mathcal{P}$ be a class of formulae. Define the $\mathcal{P}$–*diagram* of a model $\mathcal{A}$ for language L to be the set of all $\mathcal{P}$–sentences in L_A which are true in $\mathcal{A}_A$. Thus, a model is 1–computable if its existential diagram is computable or, equivalently, its universal diagram is computable.

Theorem 16.4. (*Ash-Nerode [21]*) *Let R be an additional m–ary relation on the domain of a model $\mathcal{A}$, satisfying the decidability condition (D). Then*

$$R \textit{ is intrinsically c.e. on } \mathcal{A} \Leftrightarrow R \textit{ is formally c.e. on } \mathcal{A}.$$

Proof. $(\Leftarrow)$ Always true for a relation R on the domain of any computable model. $(\Rightarrow)$ Without loss of generality, let R be a unary relation. Assume that R is not formally c.e. We assume that ω is the domain of all considered computable models. We will construct a computable model $\mathcal{B}$ and an isomorphism $f : B \to A$ such that $f^{-1}(R)$ is not c.e. Let s be an arbitrary stage of the construction. We will define a finite set Ψ^s of formulae of the open diagram of $\mathcal{B}$, and a finite partial isomorphism f_s from $\mathcal{B}$ to $\mathcal{A}$. By a finite partial isomorphism from $\mathcal{B}$

to $\mathcal{A}$ at stage s, we understand an injective function g with a finite domain such that for every $\theta \in \Psi^s$, if $\theta = \theta(\mathbf{b}_0, \ldots, \mathbf{b}_{n-1})$ for some $b_0, \ldots, b_{n-1} \in \omega$, then $g(b_0) \downarrow, \ldots, g(b_{n-1}) \downarrow$ and $\mathcal{A} \models \theta[g(b_0), \ldots, g(b_{n-1})]$. Define $\Psi^{-1} = \emptyset$ and $f_{-1} = \emptyset$. Let $X_s = f_s^{-1}(R)$ for $s \in \omega$. At the end of the construction, we will have that $f = \lim_s f_s$ exists. Let $X =_{def} f^{-1}(R)$ and $\Psi = {}_{s \geq -1}\Psi^s$. The construction will ensure that X is not c.e. Let $(\theta_e)_{e \in \omega}$ be an effective list of all atomic and negated atomic formulae in the language of $\mathcal{A}$, augmented with the constants for the elements of ω. The construction will meet the following requirements for every $e \geq 0$,

$$\begin{array}{ll} P_e^0: & \theta_e \in \Psi \text{ or } \neg\theta_e \in \Psi; \\ P_e^1: & e \in dom(f); \\ P_e^2: & e \in rng(f); \\ Q_e: & X \neq W_e. \end{array}$$

The strategy for meeting a single requirement Q_e is to wait for a stage s such that for some $b \in \omega$, $b \in W_{e,s}$. Define $f_s(b)$ such that $f_s(b) \notin R$. Hence $b \notin X_s$. Now, let $n_e^s =_{def} b$. Let n_e^{-1} be undefined for every $e \in \omega$. We say that at stage s,

P_e^0 *requires attention* if $\theta_e \notin \Psi^{s-1}$, $\neg\theta_e \notin \Psi^{s-1}$ and all elements of ω occurring in θ_e are in the domain of f_{s-1};

P_e^1 (P_e^2) *requires attention* if $e \notin dom(f_{s-1})$ ($e \notin rng(f_{s-1})$);

Q_e *requires attention* if n_e^{s-1} is undefined;

P_e^1 (P_e^2) *is injured* if $f_s(e) \neq f_{s-1}(e)$ ($f_s^{-1}(e) \neq f_{s-1}^{-1}(e)$);

Q_e *is injured* if n_e^{s-1} is defined and $f_s(n_e^{s-1}) \neq f_{s-1}(n_e^{s-1})$.

Construction

Stage s: For a requirement Req, we have the following clauses in the definition of Req *is attacked* at stage s.

$Req = P_e^0$ Let $\theta_e = \theta_e(\mathbf{b}_0, \ldots, \mathbf{b}_{n-1})$ for some $b_0, \ldots, b_{n-1} \in \omega$. Define $\Psi^s = \Psi^{s-1} \cup \{\theta_e^k\}$, where $k \in \{0, 1\}$ is such that

$$\mathcal{A} \models \theta_e^k[f_{s-1}(b_0), \ldots, f_{s-1}(b_{n-1})].$$

Let $f_s =_{def} f_{s-1}$.

$Req = P_e^1$ Define $\Psi^s = \Psi^{s-1}$, and $f_s = f_{s-1} \cup \{(e, a)\}$, where $a \in \omega$ is the least new element at stage s.

$Req = P_e^2$ Define $\Psi^s = \Psi^{s-1}$, and $f_s = f_{s-1} \cup \{(b, e)\}$, where $b \in \omega$ is the least new element at stage s.

$Req = Q_e$ Let $\Psi^s =_{def} \Psi^{s-1}$. There exists $b \in W_{e,s}$ and a partial isomorphism from $\mathcal{B}$ to $\mathcal{A}$ at stage s which maps b into an element from $(\omega - R)$. Choose the least such b, and then define f_s to be the least corresponding partial isomorphism (in some effective ordering of all finite functions on ω). Hence $b \notin X_s$. Define $n_e^s = b$.

Attack the highest priority requirement Req which requires attention at stage s, and which can be attacked without injuring any requirement of a higher priority. Whether this can be done for a Q–requirement can be checked effectively because the decidability condition (D) holds. If some lower priority requirement Q_i is injured at s, then n_i^s becomes undefined. End of the construction. It is not difficult to show that each requirement is attacked and injured only finitely often, and that all P–requirements are met. Thus, we have a computable model $\mathcal{B}$ and an isomorphism f.

Lemma 16.5. *Every Q–requirement is satisfied.*

Proof. Assume otherwise. For example, let Q_e be the requirement of the highest priority which is not satisfied. Then $X = W_e$. Let s_0 be a stage by which all requirements of higher priority than Q_e have been attacked for the last time, and at which the sequences of numbers coming from the higher priority requirements have reached their final values, $\overline{d}$ and $f(\overline{d})$. Let $b_0, b_1, b_2, \ldots$ be a computable enumeration of W_e. Consider an arbitrary b_k. Find the least corresponding stage s. Let $\psi_{_k}(x, \overline{\mathbf{d}})$ be the corresponding existential formula. That is, $\psi_{_k}(x, \overline{\mathbf{d}}) = (\exists \overline{y})\delta(x, \overline{\mathbf{d}}, \overline{y})$, where $\delta(b_k, \overline{\mathbf{d}}, \overline{\mathbf{d}}')$ is the conjunction of all formulae of Ψ^{s-1}, and $lh(\overline{y}) = lh(\overline{d}')$. Clearly, $\mathcal{B} \models \psi_{_k}[b_k, \overline{d}]$, so $\mathcal{A} \models \psi_{_k}[f(b_k), f(\overline{d})]$. Let $f(\overline{d}) = (a_0, \ldots, a_{n-1})$. Since Q_e is not attacked at s, we have for every $a \in A$

$$[\mathcal{A}_A \models \psi_{_k}(\mathbf{a}, \mathbf{a}_0, \ldots, \mathbf{a}_{n-1})] \implies R(a).$$

Conversely, for every $a \in R$, there is $k \in \omega$ such that $a = f(b_k)$. Thus, the following equivalence holds for every $a \in A$

$$[A_A \models \bigvee_{k \in \omega} \psi_k(\mathbf{a}, \mathbf{a}_0, \ldots, \mathbf{a}_{n-1})] \iff R(a).$$

This is a contradiction since R is not formally c.e. on $\mathcal{A}$.

□

As an immediate consequence, we have that if both R and its complement satisfy the decidability condition (D), then

$$R \text{ is intrinsically computable on } \mathcal{A} \iff R \text{ is formally computable on } A.$$

The decidability condition (D) cannot be omitted from the previous theorem. Goncharov [70] and Manasse [127] have shown that there are computable models with intrinsically c.e. relations which are not formally c.e. Chisholm [33] has established the best possible result on the definability of intrinsically c.e. relations on 1–computable models.

Definition 16.6. Let F be an $L_{\omega_1 \omega}$ formula with free variables among $\overline{x}$. F is a (computable) Σ_2 formula if it is equivalent to a formula of the form

$$\bigvee_{n \in \omega} \exists \overline{y}_n \bigvee_{m \in \omega} \forall \overline{z}_{mn} \theta_{mn}(\overline{x}, \overline{y}_n, \overline{z}_{mn})$$

and F is a (computable) Π_2 formula if it is equivalent to a formula of the form

$$\bigvee_{n\in\omega} \forall \overline{y}_n \bigvee_{m\in\omega} \exists \overline{z}_{mn} \theta_{mn}(\overline{x}, \overline{y}_n, \overline{z}_{mn}),$$

where $(\theta_{mn}(\overline{x}, \overline{y}_n, \overline{z}_{mn}))_{n,m\in\omega}$ is a (computable) sequence of quantifier-free formulae.

This definition has been extended by Ash [2] to all (computable) Σ_α and Π_α formulae, where α is a computable ordinal.

Definition 16.7. Let R be an additional m–ary computable relation on the domain of a computable model $\mathcal{A}$. R is formally Σ_2^0 (Π_2^0, respectively) on $\mathcal{A}$ if and only if there is a finite sequence $(b_0, \ldots, b_{k-1})$ of elements in A and a computable Σ_2 (Π_2, respectively) formula $F(x_0, \ldots, x_{m-1}, b_0, \ldots, b_{k-1})$ such that the following equivalence holds for every $a_0, \ldots, a_{m-1} \in A$.

$$[\mathcal{A}_A \models \mathcal{F}(\mathbf{a}_0, \ldots, \mathbf{a}_{m-1}, \mathbf{b}_0, \ldots, \mathbf{b}_{k-1})] \iff R(a_0, \ldots, a_{m-1}).$$

R is formally Δ_2 on $\mathcal{A}$ if R is both formally Σ_2^0 and formally Π_2^0 on $\mathcal{A}$.

Theorem 16.8. (*Chisholm [33]*)
(i) *Let R be an additional relation on the domain of the* 1*–computable model $\mathcal{A}$. Then*

$$R \text{ is intrinsically c.e. on } \mathcal{A} \implies R \text{ is formally } \Pi_2^0 \text{ on } \mathcal{A}.$$

(ii) *There is a decidable model $\mathcal{A}$ and an additional relation R on its domain, such that R is intrinsically c.e. and not formally Σ_2^0 on $\mathcal{A}$. Moreover, R is not definable by any Σ_2 formula.*

Barker [24] has extended Theorem 16.4 to Σ_2^0 relations. He has proved that if certain extra decidability conditions are satisfied, then R is intrinsically Σ_2^0 if and only if R is formally Σ_2^0. Barker [25] has further proved an analogous result for all Σ_α^0 relations, where α is a computable ordinal.

Let $\mathcal{A}$ be a computable model. Davey [40] has considered two additional, disjoint, computable relations, R_1 and R_2, on the domain A. He has studied conditions under which there is a computable model $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that the corresponding isomorphic images of R_1 and R_2 are Δ_α^0–inseparable. For example, let R_1 and R_2 be infinite, disjoint, computable subsets of ω such that $R_1 \cup R_2$ is coinfinite. Then, there is a computable model isomorphic to $(\omega, <)$ such that the images of R_1 and R_2 are computably inseparable.

While all the previous results address only levels of the arithmetic or hyperarithmetic hierarchy, Harizanov has also considered Turing degrees of the images of a computable relation on the domain of a computable model $\mathcal{A}$, under all isomorphisms from $\mathcal{A}$ to computable models.

Definition 16.9. (Harizanov [83]) Let R be an additional relation on the domain of a computable model $\mathcal{A}$. The (Turing) degree spectrum of R on $\mathcal{A}$, in symbols $Dg_{\mathcal{A}}(R)$, is the set of Turing degrees of the images of R under all isomorphisms from $\mathcal{A}$ to computable models.

For a computable model $\mathcal{B}$ isomorphic to $\mathcal{A}$, the (Turing) degree spectrum of R on $\mathcal{A}$ with respect to $\mathcal{B}$, in symbols $Dg_{\mathcal{A},\mathcal{B}}(R)$, is the set of Turing degrees of the images of R under all isomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

Harizanov has studied various aspects of degree spectra, such as: the structure of uncountable degree spectra, the effect of decidability condition (D) on the cardinality of a degree spectrum, realizing c.e. degrees in a degree spectrum via c.e. and, in general, via Δ^0_2 isomorphic images of R, and finite degree spectra.

To state results about uncountable degree spectra we assume, without loss of generality, that R is unary. Let $\mathcal{B}$ be a computable model isomorphic to $\mathcal{A}$. By $\mathcal{I}(\mathcal{A},\mathcal{B})$ we denote the set of all isomorphisms from $\mathcal{A}$ to $\mathcal{B}$. We say that a partial function p from A to B is a *finite isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if p is one-to-one, $dom(p)$ is finite and for every atomic formula $\alpha = \alpha(x_0, \dots, x_{n-1})$ in $L(\mathcal{A})$, and every $a_0, \dots, a_{n-1} \in dom(p)$, we have

$$\mathcal{A} \models \alpha[a_0, \dots, a_{n-1}] \iff \mathcal{B} \models \alpha[p(a_0), \dots, p(a_{n-1})].$$

By $\mathcal{I}_{\text{fin}}(\mathcal{A},\mathcal{B})$ we denote the set of all finite isomorphisms from $\mathcal{A}$ to $\mathcal{B}$. We define the R–equivalence relation $\sim_R$ on $\mathcal{I}_{\text{fin}}(\mathcal{A},\mathcal{B})$ as follows:

$$q \sim_R r \iff (\forall b \in ran(q) \cap ran(r))[q^{-1}(b) \in R \Leftrightarrow r^{-1}(b) \in R].$$

Theorem 16.10. (*Harizanov [85]*)

(i) *The following are equivalent:*

(*0*) *$Dg_{\mathcal{A}}(R)$ is uncountable.*

(*1*) *$Dg_{\mathcal{A},\mathcal{B}}(R)$ is uncountable.*

(*2*) *$Dg_{\mathcal{A},\mathcal{B}}(R)$ has cardinality 2^ω.*

(*3*) *There is a nonempty set $\mathbb{S} \subseteq \mathcal{I}_{fin}(\mathcal{A},\mathcal{B})$ such that the following two conditions are satisfied:*

$$(A) \quad (\forall p \in \mathbb{S})(\forall a \in A)(\forall b \in B)(\exists q \in \mathbb{S})$$
$$[(q \supseteq p) \wedge (a \in dom(q)) \wedge (b \in rng(q))];$$

$$(B) \quad (\forall p \in \mathbb{S})(\exists q, r \in \mathbb{S})[(q \supseteq p) \wedge (r \supseteq p) \wedge \neg(q \sim_R r)].$$

(ii) *Let $\mathbb{S}$ be as in (i)(3). Then for every set $C \geq_T \mathbb{S}$, there is an isomorphism f from$\mathcal{A}$ to$\mathcal{B}$ such that*

$$C \equiv_T f(R) \oplus \mathbb{S} \equiv_T f \oplus \mathbb{S}.$$

In particular, if $\mathbb{S}$ is computable, then $Dg_{\mathcal{A},\mathcal{B}}(R) = \mathcal{D}$ and, moreover, for every set $C \subseteq \omega$, there is an isomorphism f from $\mathcal{A}$ to $\mathcal{B}$ such that

$$C \equiv_T f(R) \equiv_T f.$$

Theorem 16.11. (*Harizanov [87]; Ash, Cholak and Knight [6]*) *The following are equivalent:*

(*1*) $Dg_{\mathcal{A},\mathcal{B}}(R) = \mathcal{D}$ *and, moreover, for every set* $C \subseteq \omega$, *there is an isomorphism* f *from* $\mathcal{A}$ *to* $\mathcal{B}$ *such that* $C \equiv_T f(R) \equiv_T f$.

(*2*) *There is* $e \in \omega$ *and* $p \in 2^{<\omega}$ *such that the set*

$$\mathbb{S}_{e,p} =_{def} \{\phi_e^q : q \in 2^{<\omega} \wedge q \supseteq p\}$$

has the following properties:

$\mathbb{S}_{e,p} \subseteq \mathcal{I}_{fin}(\mathcal{A},\mathcal{B})$,

Condition (*3*)(*A*) *from Theorem 16.10 is satisfied for* $\mathbb{S} = \mathbb{S}_{e,p}$, *and*

$(\exists i \in \omega)(\forall q \supseteq p)(\forall a \in dom(q))[\phi_i^{\phi_e^q(R)}(a) \downarrow = q(a)]$.

(*3*) *There is a nonempty computable set* $\mathbb{S} \subseteq \mathcal{I}_{fin}(\mathcal{A},\mathcal{B})$ *such that the conditions* (*A*) *and* (*B*) *from Theorem 16.10 are satisfied.*

In the proof of $\neg(2) \Rightarrow \neg(1)$ for Theorem 16.11 in [87], the construction of C can be done computably in $\emptyset''$. Hence $C \in \Delta_3^0$. Thus, if not every Turing degree is obtained in a degree spectrum $Dg_{\mathcal{A},\mathcal{B}}(R)$ via an isomorphism of the same Turing degree, then there is such a Δ_3^0 degree. This conclusion also follows from the proof in [6] since there is a generic Δ_3^0 set.

In [84], the priority method has been used to establish how the Ash-Nerode decidability condition affects the cardinality of the degree spectrum.

Theorem 16.12. (*Harizanov [84]*)

(*i*) *If the Ash-Nerode decidability condition* (*D*) *holds for a non-intrinsically c.e. relation* R *on a model* $\mathcal{A}$, *then the degree spectrum of* R *on* $\mathcal{A}$ *is infinite.*

(*ii*) *There is a computable non-intrinsically c.e. relation* R *on a computable model* $\mathcal{A}$ *such that the degree spectrum of* R *on* $\mathcal{A}$ *has exactly two degrees.*

Also, in [84] some new computable syntactic conditions have been introduced, which have allowed the use of the permitting method to obtain every c.e. degree in the degree spectrum. Ash, Cholak and Knight [6] have generalized this result to include in the degree spectrum all α–c.e. degrees in Ershov's hierarchy of Δ_2^0 degrees, see [52, 53, 54]. For a computable ordinal α, a Turing degree is α–c.e. if it contains an α–c.e. set. A set $C \subseteq \omega$ is α–c.e. if there exists a computable function $f : \omega^2 \to \{0, 1\}$ and a computable function $o : \omega^2 \to \alpha + 1$ with the following properties:

$(\forall x)[\lim_{s\to\infty} f(x, s) = C(x) \wedge f(x, 0) = 0]$,

$(\forall x)(\forall s)[o(x, s + 1) \leq o(x, s) \wedge o(x, 0) = \alpha]$, and

$(\forall x)(\forall s)[f(x, s + 1) \neq f(x, s) \Rightarrow o(x, s + 1) < o(x, s)]$.

In particular, 1–c.e. sets are c.e. sets, and 2–c.e. sets are d–c.e. sets. For other characterizations of α–c.e. sets, also see [51, 16]. In [16], Ash and Knight have studied intrinsically α–c.e. relations. For other generalizations of a syntactic condition in [84], see [15, 17].

In [86], Goncharov's infinite injury method has been modified to construct a computable non-intrinsically c.e. relation with a two-element degree spectrum whose nonzero degree is $\leq \mathbf{0}'$. First, a family $\mathcal{S}$ of c.e. sets and a computable set P, which have certain required properties, have been constructed. A function ν from ω onto $\mathcal{S}$ is called a *computable enumeration* of $\mathcal{S}$ if there is a uniformly computable sequence $\{\nu_t\}_{t\in\omega}$ of functions from ω to the set of finite subsets of ω such that for every $n \in \omega$, $\nu(n) = \cup\{\nu_t(n) : t \in \omega\}$. The family $\mathcal{S}$ constructed has two injective computable enumerations, ν and μ, such that every other injective computable enumeration λ of $\mathcal{S}$ is computably equivalent to ν or μ. Here, λ is *computably equivalent* to ν if the function $f : \omega \to \omega$ such that $\nu = \lambda f$ is recursive. The set Y defined by $Y = \{n \in \omega : (\exists m \in P)[\nu(m) = \mu(n)]\}$ is a non-c.e. Δ^0_2 set. The enumeration ν has then been encoded into a rigid computable model $\mathcal{A}$. The category of injective computable enumerations of $\mathcal{S}$, whose morphisms are equivalences (computable equivalencies, respectively) of the enumerations, is equivalent to the category of computable models isomorphic to $\mathcal{A}$ whose morphisms are isomorphisms (computable isomorphisms, respectively) of the models. The set R which encodes P in $\mathcal{A}$ is computable and its degree spectrum on $\mathcal{A}$ has the required property.

The ideas described in the previous paragraph have originated in Goncharov's work [69, 70] on the dimension of a computable model (see Theorem 16.15). Similar ideas have also been used by Ventsov [206, 207, 209], as well as by Cholak, Goncharov, Khoussainov and Shore [36].

Definition 16.13. Let $\mathcal{P}$ be a certain class of functions. A computable model $\mathcal{A}$ is $\mathcal{P}$–categorical if for every computable model $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, which belongs to P.

An example of a computably categorical model is the ordered set of rationals. In general, a computable linear ordering is computably categorical if and only it has only finitely many elements with an immediate successor [77, 183]. A computable Boolean algebra is computably categorical if and only if it has finitely many atoms ([182], also see Theorem 1 in [77]). For more examples of computably categorical models see [39].

Ash [4] has established for every ordinal $\alpha < \omega_1^{CK}$, under certain extra decidability assumptions, a necessary and sufficient condition for a computable model $\mathcal{A}$ to be Δ^0_α–categorical, termed $\mathcal{A}$ has a Σ^0_α *Scott family.* (The extra decidability assumptions are needed only for establishing the necessary condition.) For $\alpha = 1$, this result has been first obtained by Goncharov [65].

Definition 16.14. A computable model $\mathcal{A}$ has a Σ^0_1 *Scott family* if there is a finite sequence $(b_0, \dots, b_{k-1})$ of elements in A and a computable sequence $(F_n(x_0, \dots, x_{m-1}, b_0, \dots, b_{k-1}))_{\ n\in\omega}$ of existential formulae satisfiable in $\mathcal{A}_A$ such that the following two conditions hold.

(1) For every $a_0, \dots, a_{m-1} \in A$, there is $n \in \omega$ such that

$$\mathcal{A}_A \models \mathcal{F}_n(\mathbf{a}_0, \dots, \mathbf{a}_{m-1}, \mathbf{b}_0, \dots, \mathbf{b}_{k-1}).$$

(2) For every $n \in \omega$ and every two sequences $(a_0, \ldots, a_{m-1}) \in A^m$ and $(d_0, \ldots, d_{m-1}) \in A^m$,

$$\text{if } \mathcal{A}_A \models \mathcal{F}_n(\mathbf{a}_0, \ldots, \mathbf{a}_{m-1}, \mathbf{b}_0, \ldots, \mathbf{b}_{k-1})$$
$$\text{and } \mathcal{A}_A \models \mathcal{F}_n(\mathbf{d}_0, \ldots, \mathbf{d}_{m-1}, \mathbf{b}_0, \ldots, \mathbf{b}_{k-1}),$$
$$\text{then } (\mathcal{A}, a_0, \ldots, a_{m-1}) \cong (\mathcal{A}, d_0, \ldots, d_{m-1}).$$

Khoussainov and Shore [102] and Kudinov [120] have shown that there is a computably categorical model $\mathcal{A}$ without a Σ_1^0 Scott family. Moreover, they proved that there is such a model with the additional property that every expansion by finitely many constants is computably categorical.

The notion of a dimension of a computable model originates in Mal'cev's work on computable algebras in early 1960's. We say that two computable models $\mathcal{A}$ and $\mathcal{B}$ have the same computable isomorphism type if there is a computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$. The *dimension* of a computable model $\mathcal{A}$ is the number of computable isomorphism types of computable models which are isomorphic to $\mathcal{A}$. Clearly, the dimension of a computable model is $\leq \omega$, and a computable model is computably categorical if its dimension is 1. It has been shown that for many classes of computable models, the dimension of the models is either 1 or ω, see [65, 67, 74, 77, 78, 208, 210].

Theorem 16.15. (*Goncharov [70, 71]*) *For every natural number $n \geq 2$, there is a rigid computable partial ordering with dimension n.*

In the following theorem, Millar, extending an earlier result of Goncharov, has proved that a small amount of decidability for a computably categorical model is sufficient to preserve computable categoricity under expansions by finitely many constants.

Theorem 16.16. (*Millar [148]*) *Let $\mathcal{A}$ be a 1–computable and computably categorical model. For every finite sequence of elements $a_0, \ldots, a_{n-1}$ from A, the model $(\mathcal{A}, a_0, \ldots, a_{n-1})$ is computably categorical.*

The question then remains whether the condition of 1–computability in the previous theorem can be removed. Cholak, Goncharov, Khoussainov and Shore have answered negatively by establishing the following stronger result.

Theorem 16.17. (*Cholak-Goncharov-Khoussainov-Shore [36]*) *Let $n \in \omega$. There exists a computably categorical model $\mathcal{A}$ such that for every element $a \in A$, the expanded structure $(\mathcal{A}, a)$ has dimension n.*

It is not known whether the previous result holds for $n = \omega$.

Khoussainov [105] has also studied a generalization of the notion of a dimension of a computable model by allowing homomorphic images. Other types of algorithmic dimensions of computable models, such as program dimension and uniform dimension, have also been studied [104, 106].

In [160], Nurtazin gave several characterizations of a decidable model $\mathcal{A}$ which is computably isomorphic to every other isomorphic decidable model. One of the characterizations is that there is an expansion $(\mathcal{A}, a_0, \ldots, a_{n-1})$ of $\mathcal{A}$ (by finitely many constants) such that the set of atoms of the Lindenbaum algebra of $Th(\mathcal{A}, a_0, \ldots, a_{n-1})$ is computable and $(\mathcal{A}, a_0, \ldots, a_{n-1})$ is the prime model of $Th(\mathcal{A}, a_0, \ldots, a_{n-1})$.

Ash and Nerode [21] and Goncharov [65] have also studied the class of the so-called computably stable models.

Definition 16.18. (i) A computable model $\mathcal{A}$ is *computably stable* if every isomorphism from $\mathcal{A}$ to a computable model is computable.

(ii) Let P be a certain class of functions. A computable model $\mathcal{A}$ is *P–stable* if every isomorphism from $\mathcal{A}$ to a computable model belongs to P.

Thus, computably stable is the same as Δ^0_1–stable. It is easy to see that $\mathcal{A}$ is computably stable if and only if all computable relations on the domain A are intrinsically computable. Ash and Nerode, and Goncharov have given a computable syntactic condition for $\mathcal{A}$ which is equivalent to $\mathcal{A}$ being computably stable, under the assumption that A is 1–computable.

Theorem 16.19. (*Ash-Nerode [21], Goncharov [65]*) *Let $\mathcal{A}$ be a 1–computable model. Then $\mathcal{A}$ is computably stable if and only if there is a sequence of elements $a_0, \ldots, a_{m-1}$ from A and a computable sequence $\psi_0, \psi_1, \psi_2, \ldots$ of existential formulae in free variables $x, x_0, \ldots, x_{m-1}$ such that the sets*

$$\{a \in A : \mathcal{A}_A \models \psi_n(\mathbf{a}, \mathbf{a}_0, \ldots, \mathbf{a}_{m-1})\}$$

form a family of singletons whose union is A.

Ash [3] has generalized this result to Δ^0_n–*stable* models for every $n > 1$. He has established a syntactic condition, termed $\mathcal{A}$ *has a formally Δ^0_n–enumeration,* which is, under certain additional decidability conditions, equivalent to $\mathcal{A}$ being Δ^0_n–stable. Ash [2] has also established a similar result for all hyperarithmetic degrees. For example, for every computable ordinal α, no infinite reduced abelian p–group is Δ^0_α–stable, as shown by Barker [26].

Ash and Goncharov [8] have also introduced and studied the notions of *strong Δ^0_2–stability* and *strong Δ^0_2–categoricity.*

Definition 16.20. (Ash-Knight [10]) Let $(\mathcal{A}, \mathcal{B})$ be a pair of computable models (c.e. models, respectively), and let $X \subseteq \omega$. We say that $(\mathcal{A}, \mathcal{B})$ codes X via a computable sequence $(\mathcal{D}_n)_{n\in\omega}$ of computable models (c.e. models, respectively) if the following isomorphism condition is satisfied:

$$[\mathcal{D}_n \cong \mathcal{A} \text{ if } n \in X] \qquad \text{and} \qquad [\mathcal{D}_n \cong \mathcal{B} \text{ if } n \notin X].$$

For example, if X is a Π^0_3 set, then there is a computable sequence $(\mathcal{D}_n)_{n\in\omega}$ of computable linear orders such that $\mathcal{D}_n$ is isomorphic to $\omega + \omega^*$ if $n \in X$, and $\mathcal{D}_n$ is isomorphic to $\omega + 1 + \omega^*$ if $n \notin X$. Ash and Knight [10] have obtained some general computable syntactic conditions on computable models $\mathcal{A}$ and $\mathcal{B}$ and a computable ordinal α, so that $(\mathcal{A}, \mathcal{B})$ codes every Π^0_α set X via a computable sequence of computable models. A necessary condition is that all computable infinitary Σ_α sentences true in $\mathcal{A}$ are also true in $\mathcal{B}$. Ash and Knight have shown that if certain "useful relations" which give information about $\mathcal{A}$ and $\mathcal{B}$ are c.e., then this necessary condition is also sufficient. For every computable ordinal α, Knight [116, 117] has also established a different set of sufficient conditions for a pair $(\mathcal{A}, \mathcal{B})$ of computable models to code every Π^0_α set via a computable sequence of computable models. Here, not all "useful" relations have to be c.e., but $\mathcal{A}$ and $\mathcal{B}$ must be "more alike."

Ash [5] has also established, under certain assumptions, a necessary and sufficient condition on c.e. models $\mathcal{A}$ and $\mathcal{B}$ and a computable ordinal α, which allows $(\mathcal{A}, \mathcal{B})$ to code every Π^0_α set X via a computable sequence of c.e. models. The method used is an extension of Ash's method of the so-called α–systems, introduced in [2]. For further extensions of this method see [13, 14].

Ash, Knight, Manasse and Slaman [1], and Chisholm [35, 34] have also considered a different approach to studying the effectiveness of model theory. While Ash, Knight, Manasse and Slaman call this approach *relatively computable model theory,* Chisholm calls it *effective model theory.* The basic idea is to allow, instead of computable models, arbitrary models, and to require all notions to be relativized to the complexity of the corresponding models. One of the advantages of this approach is the elimination of certain "pathological" situations. For example, the notion of intrinsically c.e. is replaced by the following notion of *relatively c.e.*

Definition 16.21. Let R be an additional relation on the domain of a computable model $\mathcal{A}$. R is called *relatively c.e.* on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ to any model $\mathcal{B}$ is c.e. in the atomic diagram of $\mathcal{B}$.

Now a forcing method has been used to obtain the following analogue of Theorem 16.4, thus establishing the equivalence of a semantic and a syntactic notion, without an extra decidability condition. (See a related paper [12], which involves a new classification of computable infinitary formulae.)

Theorem 16.22. (*Ash-Knight-Manasse-Slaman [1], Chisholm [34]*) *Let R be an additional relation on the domain of a computable model $\mathcal{A}$. Then*

$$R \text{ is relatively c.e. on } \mathcal{A} \Longleftrightarrow R \text{ is formally c.e. on } \mathcal{A}.$$

Using forcing, a similar result has been obtained for the new notion of *relative categoricity.* This line of investigation has been continued by Soskov [197, 198] to intrinsically Π^1_1 relations and to hyperarithmetic relations.

Let $\mathcal{A}$ be a computable model and let $\sigma(\mathbf{R})$ be a computable sentence true in an expansion of $\mathcal{A}$ by a computable relation R. Ash, Knight and Slaman [11, 19] have investigated the conditions under which there is a computable model $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that no relation on B satisfies $\sigma(\mathbf{R})$ and is computable relative to $\mathcal{B}$.

Vlach [211] has studied the degrees of algebraically independent sets on computable models. Hird [91] and Ash, Knight and Remmel [18] have investigated the existence and the degrees of the so-called quasi-simple relations on computable models.

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