The Lattice of Computably Enumerable Vector Spaces

Rumen D. Dimitrov[1] and Valentina Harizanov[2](✉)

[1] Department of Mathematics, Western Illinois University, Macomb, IL 61455, USA
rd-dimitrov@wiu.edu

[2] Department of Mathematics, George Washington University, Washington, DC 20052, USA
harizanv@gwu.edu

Abstract. We survey fundamental notions and results in the study of the lattice of computably enumerable vector spaces and its quotient lattice modulo finite dimension. These lattices were introduced and first studied by Metakides and Nerode in the late 1970s and later extensively investigated by Downey, Remmel and others. First, we focus on the role of the dependence algorithm, the effectiveness of the bases, and the Turing degree-theoretic complexity of the dependence relations. We present a result on the undecidability of the theories of the above lattices. We show the development of various notions of maximality for vector spaces, and role they play in the study of lattice automorphisms and automorphism bases. We establish a new result about the role of supermaximal spaces in the quotient lattice automorphism bases. Finally, we discuss the problem of finding orbits of maximal spaces and the recent progress on this topic.

1 Computable and Computably Enumerable Vector Spaces

Computable model theory uses the tools of computability theory to investigate algorithmic content (effectiveness) of notions, theorems, and constructions in classical mathematics (see [28]). Computably enumerable vector spaces and computability-theoretic complexity of their bases were first considered by Mal'tsev in [40] and Dekker in [4]. Modern study of these spaces including the use of the priority method has been introduced by Metakides and Nerode in [43]. Computably enumerable vector spaces have been further investigated in computable model theory (see Downey and Remmel [26] and Nerode and Remmel [50]). For more recent developments in the study of effective vector spaces, see [9,11]. Many of the results about vector spaces can be generalized to certain

Harizanov was partially supported by the NSF grant 1202328, by the CCFF award and CCAS Dean's research chair award of the George Washington University, as well as by the Simons Foundation Collaboration Grants for Mathematicians.

A. Day et al. (Eds.): Downey Festschrift, LNCS 10010, pp. 366–393, 2017.
DOI: 10.1007/978-3-319-50062-1_23

effective closure systems (see [26]). More recently, effective vector spaces have been also studied in the context of reverse mathematics.

We will now introduce some definitions and state basic facts about computable and c.e. vector spaces. As customary in model theory, for a structure $\mathcal{A}$ we often use A to denote both the structure and its domain.

Definition 1. *Let $(F, +, \cdot)$ be a computable field and $(V, +, \cdot, \equiv)$ a structure, $V \subseteq \omega$, with a partial computable binary operation $+$ defined on $V \times V$ and a partial computable binary operation $\cdot$ defined on $F \times V$, and a congruence relation $\equiv \subseteq V \times V$ such that the quotient structure $\frac{V}{\equiv}$ is a vector space over F with vector addition induced by $+$ and scalar multiplication induced by $\cdot$.*

(i) The structure $\frac{V}{\equiv}$ is a c.e. vector space *given by $(V, +, \cdot, \equiv)$ if V is a c.e. set and $\equiv$ is a c.e. relation.*
(ii) The structure $\frac{V}{\equiv}$ is a computable vector space *given by $(V, +, \cdot, \equiv)$ if V is a c.e. set, $\equiv$ is a c.e. relation, and the relation $(V \times V) - \equiv$ is also c.e.*
(iii) The structure $\frac{V}{\equiv}$ is a normal vector space *given by $(V, +, \cdot, \equiv)$ if V is a c.e. set and the relation $\equiv$ is the equality, $=$.*

We usually do not write the equality explicitly. Every vector space can be thought of as a quotient space with the congruence relation being the equality. A normal vector space $(V, +, \cdot)$ has a c.e. set of vectors V, a partial computable vector addition $+$, and a partial computable scalar multiplication $\cdot$. Furthermore, since the equality is a computable binary relation on ω, both the equality on V and the inequality on V are c.e. relations. Hence every normal vector space is computable.

Example 2. *Let F be a computable field. Define*

$$V_\infty = \{u \in F^\omega : (\exists n_s)(\forall n \geq n_s)[u(n) = 0]\}.$$

Then V_∞ is a (normal) vector space with domain V_∞ and pointwise operations of vector addition and scalar multiplication of vectors. The set of vectors $E = \{\varepsilon_i \in F^\omega : i \in \omega\}$, where

$$\varepsilon_i(n) = \begin{cases} 1 & if\ n = i, \\ 0 & if\ n \neq i, \end{cases}$$

forms a computable basis for V_∞. We will call this basis a standard basis*.*

Thus, V_∞ is an $\aleph_0$-dimensional computable vector space. Its computable first-order language is $\{+, \{\cdot_f\}_{f \in F}\}$. It has a computable basis and hence a dependence algorithm. Intuitively, a *dependence algorithm* is an effective procedure for deciding whether a finite tuple of vectors is linearly dependent.

Lemma 3. *Every c.e. basis of V_∞ is computable.*

Proof. Assume that B is a c.e. basis of V_∞. Let $b_0, b_1, b_2, \ldots$ be a computable enumeration of B. Let $v \in V_\infty$. We effectively find $\lambda_{i_0}, \ldots, \lambda_{i_{n-1}} \in F - \{0\}$ such that

$$v = \lambda_{i_0} b_{i_0} + \cdots + \lambda_{i_{n-1}} b_{i_{n-1}}.$$

Then we have

$$v \in B \Leftrightarrow (n = 1 \wedge v = b_{i_0}).$$

■

For any set $I \subseteq V_\infty$, by $cl(I)$ we denote the smallest (with respect to inclusion) subspace of V_∞ containing I; that is, $cl(I)$ is the linear span of I. A subspace V of V_∞ is c.e. if its domain V is a c.e. subset of V_∞. The set of all c.e. subspaces of V_∞ is denoted by $\mathcal{L}(V_\infty)$.

Example 4. *Let $W \in \mathcal{L}(V_\infty)$. Let the congruence relation $\equiv_W$ on V_∞ be defined by*

$$x \equiv_W y \Leftrightarrow x - y \in W.$$

Clearly, $\equiv_W$ is a c.e. relation because W is a c.e. set. Hence the quotient space $\frac{V_\infty}{W}$ is a c.e. vector space. If W is computable, then $\frac{V_\infty}{W}$ is a computable vector space.

Let $\frac{V}{\equiv}$ and $\frac{V'}{\equiv'}$ be c.e. vector spaces, and let $f : \frac{V}{\equiv} \to \frac{V'}{\equiv'}$ be a vector space isomorphism. Then we say that f is computable if the relation

$$\{(u, v) \in V \times V' : f([u]_\equiv) = [v]_{\equiv'}\}$$

is c.e.

Proposition 5. *Every c.e. vector space $\frac{V}{\equiv}$ is computably isomorphic to $\frac{V_\infty}{W}$ for some $W \in \mathcal{L}(V_\infty)$. If $\frac{V}{\equiv}$ is a computable vector space, then W is computable.*

Proof. Let $v_0, v_1, \ldots$ be a computable enumeration of V. Define $f : V_\infty \to \frac{V}{\equiv}$ by $(\forall i)[f(\varepsilon_i) = [v_i]_\equiv]$ so that f is a linear function from V_∞ to $\frac{V}{\equiv}$. Clearly, f is onto. Let $W =_{def} \ker(f) = \{v \in V_\infty : f(v) = [0]_\equiv\}$. Then W is a c.e. subspace of V_∞. If $\frac{V}{\equiv}$ is computable, then W is also computable. Let an isomorphism $g : \frac{V_\infty}{W} \to \frac{V}{\equiv}$ be defined by

$$g(v + W) = [f(v)]_\equiv.$$

Clearly, g is a computable isomorphism. ■

Lemma 6. *Every computable vector space $\frac{V}{\equiv}$ is computably isomorphic to a normal vector space.*

Proof. Let $\frac{V}{\equiv}$ be a computable vector space given by $(V, +, \cdot, \equiv)$. Assume that $v_0, v_1, v_2, \ldots$ is a computable enumeration of V. Define $W = \{v_i : (\forall j < i)\neg[v_i \equiv v_j]\}$. The set W is a computable subset of V. Clearly, $(W, +, \cdot, \equiv)$ is a normal vector space. Let $f : \frac{V}{\equiv} \to W$ be a linear function given by $f([v_n]_\equiv) = v_i$, where v_i is the unique element in W such that $v_n \equiv v_i$. Then f is a computable isomorphism. ■

We will now discuss the structure on $\mathcal{L}(V_\infty)$. A lattice is a structure L in the language $\{\leq, \vee, \wedge\}$ such that $\leq$ is a partial order, and $\vee$ and $\wedge$ are supremum and infimum, respectively. If a lattice has the greatest element and the least element, then they are denoted by 1 and 0, respectively. If L is a lattice with 1, then $a \in L$ is called a *co-atom* (dual atom) if

$$a < 1 \wedge (\forall b \in L)\,[a < b \Rightarrow b = 1].$$

As usual, by $\mathcal{E}$ we denote the lattice of all c.e. subsets of ω.

Let $U, V \in \mathcal{L}(V_\infty)$. Then $U \cap V$ is the subspace with domain $U \cap V$, and $U + V$ is the subspace with domain

$$U + V = \{u + v : u \in U \wedge v \in V\}.$$

By $Y = U \oplus V$ we denote that $Y = U + V$ and $U \cap V = \{0\}$. We write $U \subseteq V$ if U is a subspace of V. Consider the lattice $(\mathcal{L}(V_\infty), \subseteq, \cap, +, \{0\}, V_\infty)$. The lattice $\mathcal{L}(V_\infty)$ modulo finite dimension is denoted by $\mathcal{L}^*(V_\infty)$.

For $A, B \in \mathcal{E}$ we will use $A =^* B$ to denote that the symmetric difference $A \bigtriangleup B$ is a finite set. Similarly, for $U, V \in \mathcal{L}(V_\infty)$ we write $U =^* V$ if there is a finite-dimensional subspace W such that $U + W = V + W$. This means that $cl(U \cup P) = cl(V \cup Q)$ for some finite sets of vectors P and Q. Hence $\mathcal{E}^* = (\mathcal{E}/_{=^*})$ and $\mathcal{L}^*(V_\infty) = (\mathcal{L}(V_\infty)/_{=^*})$. Clearly, each of the lattices $\mathcal{E}$, $\mathcal{E}^*, \mathcal{L}(V_\infty)$, and $\mathcal{L}^*(V_\infty)$ has both 1 and 0.

The structure and automorphisms of $\mathcal{L}(V_\infty)$ and $\mathcal{L}^*(V_\infty)$ have been studied extensively. The approach, in general, has been modelled upon the study of the distributive lattices $\mathcal{E}$ and $\mathcal{E}^*$ in computability theory. However, the study of $\mathcal{L}(V_\infty)$ and $\mathcal{L}^*(V_\infty)$ follows a more geometric approach because these lattices are modular and nondistributive. For more on lattice theory see [1].

Proposition 7. *The structure $\mathcal{L}(V_\infty)$ is a modular nondistributive lattice.*

Proof. To prove that the lattice $\mathcal{L}(V_\infty)$ is modular, we will show that

$$U \subseteq V \Rightarrow [(W + U) \cap V = (W \cap V) + U].$$

Let $U, V, W \in \mathcal{L}(V_\infty)$, where $U \subseteq V$. It is easy to see that then $(W \cap V) + U \subseteq (W + U) \cap V$. Now, let $v \in (W + U) \cap V$. Then $v = w + u$ for some $w \in W$ and $u \in U$. Hence, $w = v - u$, so, since $U \subseteq V$, $w \in V$. Thus, $w + u \in (W \cap V) + U$, i.e., $v \in (W \cap V) + U$.

To show that $\mathcal{L}(V_\infty)$ is not distributive, choose two (nonzero) independent vectors, u and v. Consider the following three subspaces: $cl(\{u\})$, $cl(\{v\})$ and $cl(\{u + v\})$. Then

$$(cl(\{u\}) + cl(\{v\})) \cap cl(\{u + v\}) = cl(\{u + v\}),$$

but

$$(cl(\{u\}) \cap cl(\{u + v\})) + (cl(\{v\}) \cap cl(\{u + v\})) = \{0\}.$$

■

Let $I_0, I_1, I_2, \ldots$ be a fixed effective enumeration of all *c.e. independent* subsets of V_∞. For $e \in \omega$, let

$$V_e =_{def} cl(I_e).$$

Hence, $V_0, V_1, V_2, \ldots$ is a fixed effective enumeration of all c.e. subspaces of V_∞. For $s \in \omega$, let $V_{e,s} =_{def} cl(I_{e,s})$. Hence $V_e = \bigcup\limits_{s\in\omega} V_{e,s}$.

Proposition 8. *Let V be a c.e. vector space. If V has a c.e. basis, then V has a dependence algorithm.*

Proof. Assume that V has a c.e. basis $b_0, b_1, \ldots$ Let $u_0, \ldots, u_{n-1} \in V$. Effectively find the least $k \in \omega$ and $\alpha_{ij} \in F$, for $i \in \{0, \ldots, n-1\}$ and $j \in \{0, \ldots, k-1\}$, such that $u_i = \sum\limits_{j=0}^{k-1} \alpha_{ij} b_j$. Form a matrix $M = [\alpha_{ij}]_{n\times k}$, and algorithmically find the rank of M. Then $u_0, \ldots, u_{n-1}$ are linearly dependent iff $rank(M) < n$. ■

Theorem 9. *Let V be a c.e. vector space. If V has a dependence algorithm, then V has a computable basis.*

Proof. If V is finite-dimensional, then every basis of V is computable. Therefore, we assume that V is infinite-dimensional. Let $b_0, b_1, b_2, \ldots$ be an effective enumeration of a c.e. basis of V. We will enumerate a computable basis $a_0, a_1, a_2, \ldots$ of V. As usual, assume that $V \subseteq \omega$ with the usual ordering $<$. Inductively, let $a_0, \ldots, a_{2n}$ be defined such that

$a_0, \ldots, a_{2n}$ are linearly independent,
$b_{n-1} \in cl(\{a_0, \ldots, a_{2n}\})$, and
$a_0 < \cdots < a_{2n}$.

We will now extend the sequence $a_0, \ldots, a_{2n}$ by defining a_{2n+1} and a_{2n+2}. We first effectively check whether $b_n \in cl(\{a_0, \ldots, a_{2n}\})$.

If $b_n \in cl(\{a_0, \ldots, a_{2n}\})$, then we choose the least two vectors $b, d \in V$ such that $a_{2n} < b < d$, and $a_0, \ldots, a_{2n}, b, d$ are linearly independent. Let $a_{2n+1} =_{def} b$ and $a_{2n+2} =_{def} d$.

Assume that $b_n \notin cl(\{a_0, \ldots, a_{2n}\})$. Choose the least vector $x \in V$ such that $x > \max\{a_{2n}, a_{2n} - b_n\}$ and $a_0, \ldots, a_{2n}, b_n, x$ are linearly independent. Such x exists because V is infinite-dimensional. Hence, $b_n + x > a_{2n}$ and $a_0, \ldots, a_{2n}, x, b_n + x$ are linearly independent. We define a_{2n+1} and a_{2n+2} such that $\{a_{2n+1}, a_{2n+2}\} = \{x, b_n + x\}$.

If the underlying field for V_∞ is infinite, then there is an easier way to obtain a computable basis for V. Namely, we can choose $k_1, k_2, \ldots \in F$ such that $b_0 < k_1 b_1 < k_2 b_2 < \cdots$. Then $\{b_0, k_1 b_1, k_2 b_2, \ldots\}$ is a computable basis for V. ■

Hence, if $V \in \mathcal{L}(V_\infty)$, then V has a computable basis, as first established by Dekker [4]. Metakides and Nerode further showed that V has a c.e. basis B such that $V \equiv_T B$. As usual, we use $\leq_T$ for Turing reducibility and $\equiv_T$ for Turing equivalence of sets. The Turing degree of X is denoted by $\deg(X) = \mathbf{x}$, the *nth* Turing jump of X by $X^{(n)}$, and $\mathbf{x}^{(n)} = \deg(X^{(n)})$. In particular, $\mathbf{0}'$ denotes the

Turing degree of the halting set $\emptyset'$. The Turing degrees form an upper semilattice. For more on computability theory see [56].

The result of classical mathematics that every independent set of vectors can be extended to a basis of the whole vector space does not effectivize. That is, some independent sets cannot be extended to c.e. independent sets by adding infinitely many vectors.

Let $J \subseteq V_\infty$ be an independent set. The set J is called *nonextendible* if $\dim \frac{V_\infty}{cl(J)} = \infty$ and for every $e \in \omega$:

$$J \subseteq I_e \Rightarrow |I_e - J| < \infty.$$

Otherwise, the independent set J is called *extendible*. Metakides and Nerode [43] showed that there is a c.e. nonextendible independent subset J of V_∞. We say that a c.e. subspace V has a (*fully*) *extendible* basis if some c.e. basis of V can be extended to a c.e. basis of V_∞.

Theorem 10 *(Metakides and Nerode [43]).* *Let V_∞ be over any computable field. Then there is a c.e. subspace space V of V_∞ such that no basis of V is fully extendible.*

2 Dependence Relation and k-Dependence Relations

We have already considered a dependence algorithm. Now, we formally introduce dependence relations. Let $V \subseteq V_\infty$. The *dependence relation over* V, in symbols $D(V)$, is defined by

$$\begin{aligned} D(V) = \{(u_0, \ldots, u_{k-1}) : k \in \omega \wedge u_0, \ldots, u_{k-1} \in V_\infty \wedge \\ (u_0, \ldots, u_{k-1} \text{ are linearly dependent over } V)\}. \end{aligned}$$

Since for $v \in V_\infty$, we have $v \notin V$ iff $v \in D(V)$, it follows that

$$V \leq_T D(V).$$

Hence, if $D(V)$ is computable, then V is computable. The *dependence degree* of V is the Turing degree of $D(V)$, $deg(D(V))$. A space V is called *decidable* if its dependence degree is $\mathbf{0}$, that is, $D(V)$ is a computable set. Equivalently, V is decidable if $\frac{V_\infty}{V}$ has a dependence algorithm.

Proposition 11. *Let V_∞ be a vector space over a finite computable field F. Then, for $V \in \mathcal{L}(V_\infty)$, we have*

$$V \equiv_T D(V).$$

Proof. It is enough to show that $D(V) \leq_T V$. Let $|F| = n$. For any given $v_0, \ldots, v_{k-1} \in V$, there are $(n^k - 1)$ nontrivial linear combinations. To determine whether $v_0, \ldots, v_{k-1}$ are linearly dependent, list all nontrivial linear combinations, and use oracle V to test whether any of them belongs to V. ■

Proposition 12. *Let V, W be vector subspaces of V_∞ such that $V \subseteq W$ and $\dim \frac{W}{V} < \infty$.*

(*i*) *Then*
$$D(W) \leq_T D(V).$$
(*ii*) *If, in addition, $V, W \in \mathcal{L}(V_\infty)$, then*
$$D(V) \leq_T D(W).$$

Proof. (i) Assume that $\dim \frac{W}{V} = k$ and let $w_0 + V, \ldots, w_{k-1} + V$ be a basis for $\frac{W}{V}$. Let $u_0, \ldots, u_{n-1} \in V_\infty$.

We have
$$(u_0, \ldots, u_{n-1}) \in D(W) \text{ iff}$$

$$(\exists \alpha_0, \ldots, \alpha_{n-1} \in F)(\exists w \in W)[\alpha_0 u_0 + \cdots + \alpha_{n-1} u_{n-1} = w] \text{ iff}$$

$$(\exists \alpha_0, \ldots, \alpha_{n-1} \in F)(\exists \beta_0, \ldots, \beta_{k-1} \in F)(\exists v \in V)[\alpha_0 u_0 + \cdots + \alpha_{n-1} u_{n-1} = \beta_0 w_0 + \cdots + \beta_{k-1} w_{k-1} + v] \text{ iff}$$

$$(u_0, \ldots, u_{n-1}, w_0, \ldots, w_{k-1}) \in D(V).$$

Hence $D(W) \leq_T D(V)$. ■

Metakides and Nerode proved that if the (computable) field F for V_∞ is infinite then for an arbitrary c.e. Turing degree $\mathbf{c}$, there is a computable vector subspace V of V_∞ such that
$$\deg(D(V)) = \mathbf{c}.$$

Proposition 8 can be easily generalized to quotient c.e. vector spaces. It can also be relativized. Namely, we have the following proposition.

Proposition 13. *Let $V \in \mathcal{L}(V_\infty)$.*

(*i*) *Then $\frac{V_\infty}{V}$ has a dependence algorithm iff $\frac{V_\infty}{V}$ has a c.e. basis.*
(*ii*) *Let $C \subseteq \omega$. Then $D(V) \leq_T C$ iff $\frac{V_\infty}{V}$ has a basis that is computable in C.*

Let $V \in \mathcal{L}(V_\infty)$. Then we say that V is a *complemented* element of $\mathcal{L}(V_\infty)$ if there exists $W \in L(V_\infty)$ such that $V \oplus W = V_\infty$.

Theorem 14 *(Metakides and Nerode [43])***.** *Let $V \in \mathcal{L}(V_\infty)$. Then the following conditions are equivalent.*

(*i*) *The space V is decidable.*
(*ii*) *Every c.e. basis of V is extendible to a computable basis of V_∞.*
(*iii*) *The space V has a computable basis that is extendible to a computable basis of V_∞.*
(*iv*) *The space V is a complemented element in $\mathcal{L}(V_\infty)$.*

Proof. (i)⇒(ii) Let A be a c.e. basis for V. Assume that V is decidable. Thus $\frac{V_\infty}{V}$ has a dependence algorithm, and hence a c.e. basis. Let $b_0+V, b_1+V, b_2+V, \ldots$ be a computable enumeration of a basis for $\frac{V_\infty}{V}$. Let $B=\{b_0, b_1, b_2, \ldots\}$. Then $A \cup B$ is a c.e. basis, and hence a computable basis of V_∞.

(ii)⇒(iii) Since $V \in \mathcal{L}(V_\infty)$, V has a computable basis. Let B be a computable basis for V. Extend B to a computable basis for V_∞.

(iii)⇒(iv) Assume that V has a computable basis B that is extendible to a computable basis A for V_∞. Let $W=cl(A-B)$. Then $W \in \mathcal{L}(V_\infty)$, $V \cup W{=}V_\infty$ and $V \cap W{=}\{0\}$.

(iv)⇒(i) Assume that $V, W \in \mathcal{L}(V_\infty)$, where $V \oplus W = V_\infty$. Since $W \in \mathcal{L}(V_\infty)$, W has a c.e. basis B. Then $\{b+V : b \in B\}$ is a c.e. basis for $\frac{V_\infty}{V}$. Hence, $\frac{V_\infty}{V}$ has a dependence algorithm. ■

The set of all decidable subspaces of V_∞ is denoted by $\mathcal{S}(V_\infty)$. In the next proposition we will establish that the structure $(\mathcal{S}(V_\infty), \subseteq, \cap, +, \{0\}, V_\infty)$ is a lower semilattice.

Proposition 15. *Let $V_0, V_1 \in \mathcal{S}(V_\infty)$. Then $V_0 \cap V_1 \in \mathcal{S}(V_\infty)$.*

Proof. Let $\overrightarrow{v} = (v_0, \ldots, v_{n-1}) \in (V_\infty)^n$ for some $n \in \omega$. We will present an algorithm that decides whether $\overrightarrow{v}$ is dependent over $V_0 \cap V_1$, equivalently, whether $cl(\overrightarrow{v}) \cap (V_0 \cap V_1) \neq \{0\}$ (where $cl(\overrightarrow{v}) =_{def} cl(rng(\overrightarrow{v}))$). If $cl(\overrightarrow{v}) \cap V_0 = \{0\}$ (that is, $\overrightarrow{v}$ is independent over V_0), then $\overrightarrow{v}$ is independent over $V_0 \cap V_1$. Assume that $cl(\overrightarrow{v}) \cap V_0 \neq \{0\}$. Now, we effectively compute a basis B of $cl(\overrightarrow{v}) \cap V_0$ in the following way. We find the least $z_0 \in V_0 - \{0\}$ such that $z_0 \in cl(\overrightarrow{v})$. Exchange z_0 with the first appropriate v_i. Now check whether $(v_0, \ldots, v_{i-1}, z_0, v_{i+1}, \ldots, v_{n-1})$ is independent over V_0. If it is, we stop. Otherwise, we look for the least $z_1 \in V_0 \cap cl(\overrightarrow{v})$ such that z_1 is independent of z_0 over V_0. We continue until we find the basis $B=\{z_0, \ldots, z_{m-1}\}$. Now, $\overrightarrow{v}$ is dependent over $V_0 \cap V_1$ iff B is dependent over V_1. ■

Theorem 16 *(Ash and Downey [3]).* *Let $U, V, W \in \mathcal{L}(V_\infty)$ be such that $\dim(U) = \infty$ and $U \oplus V = W$. Then there exists $D \in \mathcal{S}(V_\infty)$ such that $U \oplus D = W$.*

As a corollary we obtain that if $U \in \mathcal{S}(V_\infty)$ and $W \in \mathcal{L}(V_\infty)$ are such that $\dim(U) = \infty$ and $U \subseteq W$, then there exists $D \in \mathcal{S}(V_\infty)$ such that $U \oplus D = W$. Furthermore, we have the following result.

Theorem 17 *(Ash and Downey [3]).* *For every $W \in \mathcal{L}(V_\infty)$, there are $D_0, D_1 \in \mathcal{S}(V_\infty)$ such that $D_0 \oplus D_1 = W$.*

Let $A, B \in \mathcal{L}(V_\infty)$ be such that $B \subseteq A$ and $\dim \frac{A}{B} = \infty$. Kalantari defined the space B to be a *major subspace* of A if for every $e \in \omega$:

$$(V_e + A = V_\infty) \Rightarrow (V_e + B =^* V_\infty).$$

Guichard defined the space B to a *supermajor subspace* of A if for every $e \in \omega$:

$$(V_e + A = V_\infty) \Rightarrow (V_e + B = V_\infty).$$

Theorem 18 *(Guichard [31]). Let A be a nondecidable c.e. subspace of V_∞. Then there is a supermajor subspace of A.*

For any $V \subseteq V_\infty$ and $k \geq 1$, let

$$D_k(V) =_{def} \{(x_0, \ldots, x_{k-1}) : x_0, \ldots, x_{k-1} \text{ are linearly dependent over } V\}.$$

The *$k-th$ dependence degree* of V is the Turing degree of $D_k(V)$. Therefore, $D(V) =_{def} \bigcup\limits_{k\geq 1} D_k(V)$. We can easily establish the following facts.

(i) Uniformly in k, $D_k(V) \leq_T D(V)$.
(ii) Assume that $\dim(\frac{V_\infty}{V}) = \infty$. Then $D_k(V) \leq_T D_{k+1}(V)$.
(iii) If $V \in \mathcal{L}(V_\infty)$, then $D_k(V)$ is a c.e. set.

The next lemma will be used to establish the theorem that follows it.

Lemma 19 *(Shore [54]). Assume that V is a finite-dimensional subspace of V_∞. Let $k \in \omega$, and let the vectors $v_0, \ldots, v_k$ be linearly independent over V. Assume that X is a finite set of tuples of vectors of length $\leq k$ such that every tuple from X is independent over V. Then there are scalars $\lambda_0, \ldots, \lambda_k$ such that every tuple from X is still independent over $cl(V \cup \{\lambda_0 v_0 + \cdots + \lambda_k v_k\})$.*

Theorem 20 *(Shore [54]). Let the space V_∞ be over an infinite (computable) field. Assume that $E_1, E_2, E_3, \ldots, E_0$ is a c.e. sequence of c.e. sets such that $E_k \leq_T E_{k+1}$ and $E_k \leq_T E_0$, uniformly in k. Then there is a c.e. subspace V such that for every $k \geq 1$,*

$$D_k(V) \equiv_T E_k \wedge D(V) \equiv_T E_0.$$

Let V be a computable vector space. Its *computable automorphism group,* $Aut_{\mathbf{0}}(V)$, consists of all computable automorphisms of V. An automorphism f of a vector space V is *trivial* if it maps every 1-dimensional subspace of V into itself. That is, $f = f_\alpha$ for some $\alpha \in F - \{0\}$ where

$$(\forall v \in V)[f_\alpha(v) = \alpha v].$$

Hence f also maps every subspace of V into itself. A computable vector space is called *computably rigid* if its computable automorphism group is trivial. Morozov [44] constructed a computable vector space V such that $\frac{V_\infty}{V}$ is computably rigid.

We will now assume that the computable field F is infinite. In [44], Morozov asked whether it is possible to obtain for every $k \geq 2$, a computable vector space V such that $\frac{V_\infty}{V}$ is computably rigid, has the k-dependence algorithm *mod* V, does not have the $(k+1)$-dependence algorithm *mod* V, and its dependence algorithm *mod* V has an arbitrary nonzero c.e. Turing degree. Clearly, if $deg(D(V)) = \mathbf{0}$, then $\frac{V_\infty}{V}$ has a computable basis, and hence the computable automorphism group of $\frac{V_\infty}{V}$ is nontrivial. We have the following lemma for the nontrivial automorphisms of vector spaces.

Lemma 21 *(Dimitrov, Harizanov and Morozov [10]). Let ψ be a total function such that $\psi : V_\infty \to V_\infty$. If ψ does not induce a trivial automorphism of $\frac{V_\infty}{V}$, then one of the following conditions hold:*

(1) *There exist $u, v \in V_\infty$ and $\alpha, \beta \in F$ such that*

$$\psi(\alpha u + \beta v) \neq_{mod\ V} \alpha\psi(u) + \beta\psi(v),$$

(2) *There exists $w \in V_\infty - V$ such that $\psi(w) \in V$,*

(3) *There exists $w \in V_\infty - V$ such that the set $\{w, \psi(w)\}$ is independent mod V.*

In [10], Morozov's question was answered positively by establishing a more general result.

Theorem 22 *(Dimitrov, Harizanov and Morozov [10]). Let E_0 be a noncomputable c.e. set, and let $E_1, E_2, E_3, \ldots$ be a c.e. sequence of c.e. sets such that E_1 is computable, and*

$$E_1 \leq_T \cdots \leq_T E_k \leq_T E_{k+1} \leq_T \cdots \leq_T E_0,$$

uniformly in k. Then there is a computable subspace V of V_∞ such that $\frac{V_\infty}{V}$ is computably rigid, and for $k \geq 1$,

$$D_k(V) \equiv_T E_k \wedge D(V) \equiv_T E_0.$$

3 Maximal Vector Spaces

We now introduce the notion of a maximal vector space, which is analogous to the notion of a maximal set in classical computability theory. Maximal sets have been extensively studied within the lattice $\mathcal{E}$ of c.e. sets. Recall that an infinite set $C \subseteq \omega$ is *cohesive* if for every c.e. set W, either $W \cap C$ or $\overline{W} \cap C$ is finite. A set $M \subseteq \omega$ is *maximal* if M is c.e. and $\overline{M}$ is cohesive. Equivalently, a set $M \in \mathcal{E}$ is *maximal* if $\overline{M}$ is infinite and

$$(\forall E \in \mathcal{E})\,[(M \subseteq E \wedge |E\ -\ M| = \infty) \Rightarrow (E =^* \omega)].$$

For $X \in \mathcal{E}$ as well as for $X \in \mathcal{L}(V_\infty)$ we will use $[X]$ to denote the equivalence class of X modulo the corresponding equivalence relation $=^*$. Hence $[M]$ is a co-atom in $\mathcal{E}^*$. A maximal set was first constructed by Friedberg. Soare established that for any two maximal sets M_1 and M_2, there is an automorphism Φ of $\mathcal{E}$ such that $\Phi(M_1) = M_2$ (see [56]). A set $B \subseteq \omega$ is *quasimaximal* if it is the intersection of finitely many maximal sets, $B = \bigcap_{i=1}^{n} M_i$ where M_i's are maximal. The number n is called the *rank* of B.

Definition 23. *Let $V \in \mathcal{L}(V_\infty)$. The subspace V is maximal if* $\dim(\frac{V_\infty}{V}) = \infty$ *and for every c.e. space W such that $V \subseteq W$, we have that*

$$\dim(\frac{V_\infty}{W}) < \infty \vee \dim(\frac{W}{V}) < \infty.$$

Hence, a subspace $V \in \mathcal{L}(V_\infty)$ is maximal if its equivalence class $[V]$ is a co-atom in $\mathcal{L}^*(V_\infty)$. Metakides and Nerode [43] showed that a maximal space can be constructed by modifying the e-state construction of a maximal set. For $v \in V_\infty$ and $e \in \omega$, the e-state of v is the following string in $\{0,1\}^{e+1}$: $(V_0(v), \ldots, V_e(v))$. If a computable basis of V_∞ is identified with the set ω, then maximal sets generate maximal spaces.

Theorem 24 *(Shore, see [43]). Let M be a maximal subset of a computable basis B of V_∞. Then M^* is a maximal subspace of V_∞.*

There are stronger notions of maximality for vector spaces.

Definition 25. *Let $V \in \mathcal{L}(V_\infty)$.*

(i) The subspace V is supermaximal *if* $\dim(\frac{V_\infty}{V}) = \infty$ *and for every c.e. space W such that $V \subseteq W$, we have that*

$$V_\infty = W \vee \dim(\frac{W}{V}) < \infty.$$

(ii) The subspace V is strongly supermaximal *if* $\dim(\frac{V_\infty}{V}) = \infty$ *and for every c.e. set X contained in $V_\infty - V$, there are $a_0, \ldots, a_{n-1} \in V_\infty$ such that*

$$X \subseteq cl(V \cup \{a_0, \ldots, a_{n-1}\}).$$

Clearly, every supermaximal space is maximal. The existence of a supermaximal space was first established by Kalantari and Retzlaff [36].

Theorem 26 *(Kalantari and Retzlaff [36]). There is a maximal space that is not supermaximal.*

Theorem 27 *(Nerode and Remmel [49]). Let the space V_∞ be over an infinite field. Let $k \geq 1$. Assume that $E_1, E_2, E_3, \ldots, E_0$ is a c.e. sequence of c.e. sets such that E_0 is non-computable, $E_k \leq_T E_{k+1}$ and $E_k \leq_T E_0$. Then there are supermaximal non-automorphic subspaces V and W such that*

$$D(V) \equiv_T D(W) \equiv_T E_0 \text{ and}$$

$$D_k(V) \equiv_T D_k(W) \equiv_T E_k.$$

Let V be a vector space with a basis J. Let $v \in V$. The support of v with respect to J, in symbols $supp_J(v)$, is the set of all vectors appearing in the linear combination of vectors in J, which equals v.

Theorem 28 *(Downey and Hird [19]). There is a strongly supermaximal vector space.*

Proof. Let $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ be an effective enumeration of a computable basis for V_∞. At every stage $s \geq 0$, we will have a finite set J_s of linearly independent vectors and an effective enumeration $b_0^s, b_1^s, b_2^s, \ldots$ of a computable set of linearly independent vectors such that $J_s \cup \{b_0^s, b_1^s, b_2^s, \ldots\}$ is a basis for V_∞. At the end of the construction we will define $J = \bigcup_{s \geq 0} J_s$ and show that J is a basis of a strongly supermaximal vector space V. That is, $V =_{def} cl(J)$. We will satisfy the following requirements for every $e \in \omega$,

$$P_e : (W_e \cap cl(J) = \emptyset) \Rightarrow (W_e \subseteq cl(J \cup \{b_0, \ldots, b_{e-1}\})),$$

$$N_e : b_e = \lim_{s \to \infty} b_e^s \text{ exists.}$$

The positive requirements P_e, $e \in \omega$, ensure that the space V is supermaximal. The negative requirements N_e, $e \in \omega$, ensure that $\dim(\frac{V_\infty}{V})$ is infinite. The priority ordering of the requirements is

$$P_0, N_0, P_1, N_1, \ldots$$

We say that P_e *requires attention* at stage $s+1$ if

$$W_{e,s+1} \cap J_s = \emptyset, \text{ and}$$

$$W_{e,s+1} - cl(J_s \cup \{b_0^s, \ldots, b_{e-1}^s\}) \neq \emptyset.$$

Construction of J.

Stage 0. Let $J_0 = \emptyset$, and $b_i^0 = \varepsilon_i$ for $i \in \omega$.

Stage $s+1$. If no positive requirement requires attention at stage $s+1$, define $J_{s+1} = J_s$ and $b_i^{s+1} = b_i^s$.

Now assume that P_e is the first requirement that requires attention at $s+1$. Let v be the least element such that $v \in W_{e,s+1}$ and $v \notin cl(J_s \cup \{b_0^s, \ldots, b_{e-1}^s\})$. Let

$$J_{s+1} =_{def} J_s \cup \{v\}.$$

Let j be the least number such that $j \geq e$ and

$$b_j^s \in supp_{J_s \cup \{b_0^s, b_1^s, \ldots\}}(v).$$

That is,

$$v = a + k_0^s b_0^s + \cdots + k_{e-1}^s b_{e-1}^s + k_j^s b_j^s + k_{j+1}^s b_{j+1}^s + \cdots,$$

where $a \in J_s$ and $k_j^s \neq 0$. Define

$$b_n^{s+1} = \begin{cases} b_n^s \text{ if } n < j; \\ b_{n+1}^s \text{ if } n \geq j. \end{cases}$$

End of construction. ■

Proposition 29 *(Downey and Hird [19]). Every strongly supermaximal vector space is supermaximal.*

Proof. Assume that V is a strongly supermaximal space, which is not supermaximal. Let a c.e. space W be such that $V \subseteq W$, $V_\infty \neq W$ and $\dim(\frac{W}{V})$ is infinite. Choose $u \in V_\infty - W$, and let $w_0, w_1, w_2, \ldots$ be an effective enumeration of W. For every $i \in \omega$, we have $u + w_i \notin W$, since $u = (u + w_i) - w_i$, and $w_i \in W$ and $u \notin W$. Let $X =_{def} \{u, u + w_0, u + w_1, u + w_2, \ldots\}$. Thus,

$$X \subseteq V_\infty - W \subseteq V_\infty - V.$$

However,

$$W \subseteq cl(X).$$

Note that since X is a c.e. set and V is a strongly supermaximal space, there are $a_0, \ldots, a_{n-1} \in V_\infty$ such that

$$X \subseteq cl(V \cup \{a_0, \ldots, a_{n-1}\}).$$

Hence

$$W \subseteq cl(V \cup \{a_0, \ldots, a_{n-1}\}).$$

Clearly, this implies that

$$\dim(\frac{W}{V}) \leq \dim(\frac{cl(V \cup \{a_0, \ldots, a_{n-1}\})}{V}) \leq n,$$

which contradicts the fact that $\dim(\frac{W}{V})$ is infinite. ■

Theorem 30 *(Hird [33]). There is a supermaximal space that is not strongly supermaximal.*

Hird [32] further introduced a computable model-theoretic notion of a quasi-simple subset of a model. See [2,33] for the appropriate definition. This model-theoretic quasi-simplicity translates as computability-theoretic simplicity in the structure $(\omega, =)$. However, it turns out that a vector subspace of V_∞ is quasi-simple iff it is strongly supermaximal.

The following definition generalizes the notion of a supermaximal space within the class of maximal subspaces of V_∞.

Definition 31 *(Kalantari and Retzlaff [36]). Let $V \in \mathcal{L}(V_\infty)$.*

(i) The subspace V is called 0-thin *if it is supermaximal.*
(ii) Let $k \in \omega - \{0\}$. The subspace V is called k-thin *if* $\dim(\frac{V_\infty}{V}) = \infty$*, there is a c.e. space U such that*

$$\dim(\frac{V_\infty}{U}) = k,$$

and for every c.e. space W such that $V \subseteq W$, we have that

$$\dim(\frac{V_\infty}{W}) \leq k \vee \dim(\frac{W}{V}) < \infty.$$

Kalantari and Retzlaff [36] showed that k-thin spaces exist for all k.

4 Undecidability of the First-Order Theories of $\mathcal{L}(V_\infty)$ and $\mathcal{L}^*(V_\infty)$

The structure of $\mathcal{L}^*(V_\infty)$ is not as well-understood as that of $\mathcal{E}^*$. Both $\mathcal{L}(V_\infty)$ and $\mathcal{L}^*(V_\infty)$ are modular nondistributive lattices. This means that the "diamond" lattice M_5 can be embedded in $\mathcal{L}(V_\infty)$ and $\mathcal{L}^*(V_\infty)$, while the "pentagon" lattice N_5 cannot. The lattice $\mathcal{L}(V_\infty)$ has both atoms and co-atoms. More generally, if V is a finite k-dimensional subspace of V_∞, then the lattice of subspaces of V is an initial segment of the lattice $\mathcal{L}(V_\infty)$ and so it has the structure of the lattice $L(k, F)$ of all subspaces of any k-dimensional vector space over the field F. Also, if $V \in \mathcal{L}(V_\infty)$ is such that $\dim(\frac{V_\infty}{V}) = k$, then the principal filter $\mathcal{L}(V, \uparrow)$ of V in $\mathcal{L}(V_\infty)$ is also isomorphic to $L(k, F)$. These finite-rank initial and final segments collapse to the least and the greatest elements in $\mathcal{L}^*(V_\infty)$, respectively. We know that the lattice $\mathcal{L}^*(V_\infty)$ has co-atoms but does not have atoms. Remmel [52] and Downey [21] showed that every Σ^3_0 Boolean algebra is isomorphic to $\mathcal{L}^*(V, \uparrow)$ for some $V \in \mathcal{L}(V_\infty)$. Downey conjectured that every bounded Σ^3_0 modular lattice is a filter in $\mathcal{L}^*(V_\infty)$. Nerode and Smith established the following key structural result about $\mathcal{L}^*(V_\infty)$.

Theorem 32 *(Nerode and Smith [51]). Every finite distributive lattice is a filter in $\mathcal{L}^*(V_\infty)$.*

The proof is based on an interesting combinatorial construction, which uses Birkhoff's characterization of finite distributive lattices. The construction has requirements similar to those used in the construction of a supermaximal space. The following undecidability results are the main corollaries of the theorem.

Theorem 33 *(Nerode and Smith [51]).*

(*i*) *The first-order theory of $\mathcal{L}^*(V_\infty)$ is undecidable.*
(*ii*) *The first-order theory of $\mathcal{L}(V_\infty)$ is undecidable.*

The first result (i) is a corollary of Theorem 32, and an earlier result by Ershov and Taitslin, which establishes that the theory of distributive lattices is computably inseparable from the set of sentences refutable in some finite distributive lattices. Note that $V \in \mathcal{L}(V_\infty)$ is finite-dimensional if and only if every $W \subseteq V$ is complemented in $\mathcal{L}(V_\infty)$. The second result (ii) then follows from (i) using the definability of $\subseteq^*$ in $\mathcal{L}(V_\infty)$. Later, Galminas and Rosenthal [29] established that the theory of $\mathcal{L}(V_\infty)$ has the same logical complexity as the first-order number theory. The question whether $\forall\exists$-theory of $\mathcal{L}^*(V_\infty)$ is decidable is still open.

In [21], Downey introduced the following important notion.

Definition 34 *(Downey [21]). A c.e. set A has the* lifting property *if A is coinfinite and for every c.e. strong array $\{D_{g(x)} : x \in \omega\}$, for almost all x, $|D_{g(x)} - A| \leq 1$.*

Downey used the lifting property to obtain undecidability results for a large class of lattices of c.e. structures, including $\mathcal{L}^*(V_\infty)$. The lifting property guarantees the "lifting" of principal filters under the closure operation. We will state these results of Downey only for $\mathcal{L}^*(V_\infty)$. In particular, let B is a computable basis of V_∞ and let $A \subseteq B$ have the lifting property. If we identify B with ω, then $\mathcal{E}^*(A, \uparrow) \cong \mathcal{L}^*(cl(A), \uparrow)$. Recall that a set $A \subseteq \omega$ is *semi-low* if $\{e : W_e \cap A \neq \emptyset\} \leq_T \emptyset'$.

Theorem 35 *(Downey [24]). There exists a c.e. set A with the lifting property such that $\overline{A}$ is semi-low.*

The undecidabilty results in [21,24] are then obtained using an earlier result by Soare that for such A we have that $\mathcal{E}^*(A, \uparrow)$ is effectively isomorphic to $\mathcal{E}^*$. Therefore, it follows that the first-order theory of $\mathcal{L}^*(V_\infty)$ is undecidable.

In [21], Downey also established that every Σ_0^3 Boolean algebra is isomorphic to a principal filter for a large class of lattices of c.e. structures. This result stated only for $\mathcal{L}^*(V_\infty)$ is the following.

Theorem 36 *(Downey [21]). Let $\mathfrak{B}$ be a Σ_0^3 Boolean algebra. Then exists a c.e. set A with the lifting property such that $\mathcal{E}^*(A, \uparrow) \cong \mathfrak{B}$.*

Corollary 37 *(Downey [21]). Every Σ_0^3 Boolean algebra is a filter in $\mathcal{L}^*(V_\infty)$.*

5 The Co-atoms Form an Automorphism Basis for $\mathcal{L}^*(V_\infty)$

Recall that for $X \in \mathcal{E}$ (or $X \in \mathcal{L}(V_\infty)$), we use $[X]$ to denote the equivalence class of X modulo the corresponding equivalence relation $=^*$. If S and T are arbitrary sets of vectors, then

$$\dim(S \ mod\ T) =_{def} \dim(\frac{cl(S \cup T)}{cl(T)}).$$

By $\mathfrak{M}^*$ and $\mathfrak{R}^*$ we denote the classes of maximal and computable sets modulo $=^*$, respectively. Clearly, the computable, as well as the maximal sets are closed under $=^*$. Note that $\mathfrak{M}^*$ can also be described as the set of the co-atoms in $\mathcal{E}^*$, while $\mathfrak{R}^*$ is the set of the complemented elements of $\mathcal{E}^*$. Nerode asked the following questions.

(1) Is every automorphism of $\mathcal{E}^*$ uniquely determined by its action on $\mathfrak{R}^*$?
(2) Does every automorphism of $\mathfrak{R}^*$ extend to an automorphism of $\mathcal{E}^*$?

In [54], Shore answered the first question positively and the second question negatively. In particular, he established the following results.

Proposition 38 *(Shore [54]). Assume that Φ_1 and Φ_2 are automorphisms of $\mathcal{E}^*$.*

(i) *If* Φ_1 *and* Φ_2 *agree on the low sets, then* $\Phi_1 = \Phi_2$.
(ii) *If* Φ_1 *and* Φ_2 *agree on* $\mathfrak{M}^*$, *then* $\Phi_1 = \Phi_2$.
(iii) *If* Φ_1 *and* Φ_2 *agree on* $\mathfrak{R}^*$, *then* $\Phi_1 = \Phi_2$.

For (i) Shore used Sacks splitting theorem that every c.e. set is the union of two disjoint low sets (see Theorem 3.2 in [56]). Then the proof of (ii) uses (i) and results from Lachlan [38], while the proof of (iii) uses (ii).

Theorem 39 *(Shore [54]).* *Let* $\mathfrak{C}^*$ *be any nontrivial class of c.e. sets (i.e., none of* $\emptyset$, $\{0\}$, $\{N\}$*), modulo finite sets, closed under computable isomorphism. If* Φ_1 *and* Φ_2 *agree on* $\mathfrak{C}^*$, *then* $\Phi_1 = \Phi_2$.

The proof of Theorem 39 uses Proposition 38 (iii). In a later paper, Shore proved that nowhere simple sets generate $\mathcal{E}$, thus improving Theorem 39.

It is natural to ask which natural classes of c.e. vector spaces form automorphism bases in the lattices $\mathcal{L}(V_\infty)$ and $\mathcal{L}^*(V_\infty)$. Currently, we do not know of any analogue of Proposition 38 (i) for the lattices $\mathcal{L}(V_\infty)$ or $\mathcal{L}^*(V_\infty)$. Ash and Downey established an analogue of Proposition 38 (iii) for the lattice $\mathcal{L}(V_\infty)$ (see Corollary 40 below). The result easily extends to $\mathcal{L}^*(V_\infty)$ and we will later give a short proof of this fact. We will also give a direct proof of an analogue of Proposition 38 (ii) for $\mathcal{L}^*(V_\infty)$ (see Theorem 44 below). An analogue of Theorem 39 for $\mathcal{L}(V_\infty)$ has been given by Nerode and Remmel in [48]. An analogue of Theorem 39 for $\mathcal{L}^*(V_\infty)$ has been given by Downey and Remmel in [27]. The following result follows immediately from Theorem 17.

Corollary 40. (i) *The lattice* $\mathcal{L}(V_\infty)$ *is generated, under* $\oplus$, *by the decidable subspaces of* V_∞.
(ii) *Each automorphism of* $\mathcal{L}(V_\infty)$ *is uniquely determined by its action on the decidable subspaces.*

It is known that this result of Ash and Downey extends to $\mathcal{L}^*(V_\infty)$ as follows.

(a) The lattice $\mathcal{L}^*(V_\infty)$ is generated, under $\vee$, by the equivalence classes of the decidable subspaces of V_∞.

(b) Every automorphism of $\mathcal{L}^*(V_\infty)$ is uniquely determined by its action on the complemented elements of $\mathcal{L}^*(V_\infty)$.

Before we give proofs for these statements we will establish the following result.

Proposition 41. *If* $V, W \in \mathcal{L}(V_\infty)$ *are such that* $[V] = [W]$, *then*

$$D(V) \equiv_T D(W).$$

Proof. Suppose that $A = \{a_1, \ldots, a_p\}$ and $B = \{b_1, \ldots, b_q\}$ are sets of vectors that are independent modulo V and W, respectively, such that $cl(V \cup A) = cl(W \cup B)$. We claim that

$$D(V) \equiv_T D(cl(V \cup A)) = D(cl(W \cup B)) \equiv_T D(W).$$

We will only prove $D(V) \equiv_T D(cl(V \cup A))$. (The proof that $D(cl(W \cup B)) \equiv_T D(W)$ is identical.)

To prove that $D(V) \leq_T D(cl(V \cup A))$, fix arbitrary $x_1, \ldots, x_n \in V_\infty$ and use oracle $D(cl(V \cup A))$ to decide whether $(x_1, \ldots, x_n) \in D(cl(V \cup A))$.
Case (1). Let $(x_1, \ldots, x_n) \notin D(cl(V \cup A))$. Clearly, $(x_1, ..., x_n) \notin D(V)$.
Case (2). Let $(x_1, \ldots, x_n) \in D(cl(V \cup A))$. Suppose that I_1 is a computable basis of V. (Recall that such a basis exists.) Using oracle $D(cl(V \cup A))$, we construct a $D(cl(V \cup A))$-computable basis I_2 of $(V_\infty \ mod \ cl(V \cup A))$. Then $I_1 \cup A \cup I_2$ is a $D(cl(V \cup A))$-computable basis of V_∞. Representing each element in the sequence $x_1, \ldots, x_n$ as a linear combination in the basis $I_1 \cup A \cup I_2$ and using standard linear algebra we can decide whether the set $\{x_1, \ldots, x_n\} \cup I_1$ is dependent. Therefore, $D(V) \leq_T D(cl(V \cup A))$.

To prove that $D(cl(V \cup A)) \leq_T D(V)$, we will use oracle $D(V)$ to decide whether $(x_1, \ldots, x_n) \in D(cl(V \cup A))$. We check whether $(x_1, \ldots, x_n, a_1, \ldots, a_p) \in D(V)$. If the answer is positive, then $(x_1, \ldots, x_n) \in D(cl(V \cup A))$. Otherwise, $(x_1, \ldots, x_n) \notin D(cl(V \cup A))$. Therefore, $D(cl(V \cup A)) \leq_T D(V)$. ■

We will use the following notation for the co-atoms and the complemented elements in $\mathcal{L}^*(V_\infty)$.

$\mathcal{M}^* = \{[M] : M$ is a maximal subspace of $V_\infty\}$

$\mathcal{S}^*(V_\infty) = \{[D] : D$ is a decidable subspace of $V_\infty\}$

Note that $\mathcal{S}^*(V_\infty)$ is well-defined by Proposition 41. It is immediate that if M_1 is a maximal subspace of V_∞ and $M_1 =^* M_2$, then the space M_2 is also maximal. Therefore, $\mathcal{M}^*$ is also well-defined.

Corollary 42

(i) $\mathcal{L}^*(V_\infty)$ *is generated, under* $\vee$, *by* $\mathcal{S}^*(V_\infty)$.
(ii) Each automorphism of $\mathcal{L}^*(V_\infty)$ *is uniquely determined by its action on* $\mathcal{S}^*(V_\infty)$.

Proof. (i) Let $[V] \in \mathcal{L}^*(V_\infty)$. By Corollary 17, there are decidable spaces $D_1, D_2 \in \mathcal{L}(V_\infty)$ such that $V = D_1 \oplus D_2$. Then $[V] = [D_1] \vee [D_2]$. ■

An analogue of Theorem 39 has been given by Nerode and Remmel in [48] and by Downey and Remmel in [27]. The result by Downey and Remmel for the lattice $\mathcal{L}^*(V_\infty)$ is as follows.

Theorem 43 *(Downey and Remmel [27]). Let* $\mathfrak{C}^*$ *be any nontrivial class of elements of* $\mathcal{L}^*(V_\infty)$ *(i.e., none of* $\emptyset$, $\{[0]\}$, $\{[V_\infty]\}$, $\{[0], [V_\infty]\}$*), which is closed under automorphisms of* $\mathcal{L}^*(V_\infty)$ *that are generated by invertible computable linear transformations. Then, if* Φ *is an automorphism of* $\mathcal{L}^*(V_\infty)$ *such that* $\Phi\restriction_{\mathcal{C}^*} = id\restriction_{\mathcal{C}^*}$, *then* $\Phi\restriction_{\mathcal{L}^*(V_\infty)} = id$.

Proof. Suppose that $\Phi\restriction_{\mathcal{L}^*(V_\infty)} \neq id$, and let $[D] \in \mathcal{S}^*(V_\infty)$ be such that $\Phi([D]) \neq [D]$. Since $\Phi([D])$ is complemented, without loss of generality, assume that $D_1 \in \Phi([D])$ and $dim(D_1 \ mod \ D) = \infty$.

Let A be a computable basis of D. Extend A to a computable basis $A \cup B \cup C$ of V_∞ such that $B \subseteq D_1$ is an infinite independent set modulo D, and C is a c.e. set. Let $[V] \in \mathfrak{C}^*$ be such that $[V] \neq [0]$ and $[V] \neq [V_\infty]$. Then V has an infinite-dimensional subspace R such that $[R] \in \mathcal{S}^*(V_\infty)$. Let S_1 be a computable basis of R, and let S_2 be a computable independent set such that $S_1 \cup S_2$ is a basis of V_∞. Let f be the computable invertible linear transformation such that $f(S_1) = A \cup C$ and $f(S_2) = B$. Let $[W] = [f(V)]$ and note that $[W] \in \mathfrak{C}^*$, so $\Phi([W]) = [W]$.

Then $S_1 \subseteq V$ and hence $[cl(f(S_1))] = [cl(A \cup C)] \leq [W]$. Thus,

$$[V_\infty] = [cl(A \cup C)] \vee [cl(B)] \leq [W] \vee [cl(B)],$$

and so

$$\Phi^{-1}([W]) \vee \Phi^{-1}([cl(B)]) = [V_\infty].$$

However, $\Phi^{-1}([cl(B)]) \leq \Phi^{-1}([D_1]) = [D] = [cl(A)] \leq [cl(A \cup C)] \leq [W]$, and so

$$[W] \vee \Phi^{-1}([cl(B)]) = [W].$$

This implies that $[W] \neq \Phi^{-1}([W])$, which is a contradiction. ■

The analogue of Proposition 38 (ii) for $\mathcal{L}^*(V_\infty)$ follows from Downey and Remmel's result. It will also follow from the following theorem, where we construct a certain supermaximal space.

Theorem 44. *Let Φ_1 and Φ_2 be automorphisms of the lattice $\mathcal{L}^*(V_\infty)$ such that for some $[W] \in \mathcal{L}^*(V_\infty)$ we have*

$$\Phi_1([W]) \neq \Phi_2([W]).$$

Then there is a supermaximal space M such that $\Phi_1^{-1}([M]) \neq \Phi_2^{-1}([M])$.

Proof. By Corollary 42 (ii), there is a decidable space D such that $\Phi_1([D]) \neq \Phi_2([D])$. Note that $\Phi_1([V_\infty]) = [V_\infty] = \Phi_2([V_\infty])$ since every automorphism of $\mathcal{L}^*(V_\infty)$ fixes its greatest element. Therefore, $[D] \neq [V_\infty]$. Suppose that $U, V \in \mathcal{L}(V_\infty)$ are such that

$$[U] = \Phi_1([D]) \neq \Phi_2([D]) = [V].$$

Assume also that $dim(V \ mod \ U) = \infty$. We will construct a supermaximal space M such that $\Phi_1^{-1}([M]) \neq \Phi_2^{-1}([M])$. The space M will be such that $U \subseteq M$, $dim(M \ mod \ U) = \infty$, and $dim(V \ mod \ M) = \infty$ (see Fig. 1). ■

In the language of lattices $\{\leq, \vee, \wedge\}$ these conditions are:
$[U] \lneq [M]$ ($U \subseteq M$, and $[U] \neq [M]$ since $dim(M \ mod \ U) = \infty$), and
$[V] \nleq [M]$ (because $dim(V \ mod \ M) = \infty$).

Before we proceed with the construction of M we will prove that these requirements guarantee that

$$\Phi_1^{-1}([M]) \neq \Phi_2^{-1}([M]).$$

To see this, note that in the lattice $\mathcal{L}^*(V_\infty)$ we have:

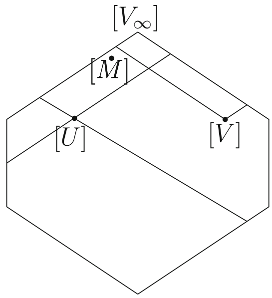

Fig. 1. Assume $[V] = \Phi_2([D])$ is not in the lower cone of $[U] = \Phi_1([D])$ in $\mathcal{L}^*(V_\infty)$. We construct a maximal space M such that $[M]$ is in the upper cone of $[U]$ while avoiding the upper cone of $[V]$. Note that we do not require that [V] avoids the upper cone of [U] despite our choice to draw it this way in the diagram.

(i) $[M] \vee [V] = [V_\infty]$ since $[M]$ is a co-atom in $\mathcal{L}^*(V_\infty)$ and $[V] \not\leq [M]$,

(ii) $\Phi_2^{-1}([M]) \vee \Phi_2^{-1}([V]) = \Phi_2^{-1}([M] \vee [V]) = \Phi_2^{-1}([V_\infty]) = [V_\infty]$,

(iii) $\Phi_1^{-1}([M]) \vee \Phi_1^{-1}([U]) = \Phi_1^{-1}([M] \vee [U]) = \Phi_1^{-1}([M])$ since $[U] \lneq [M]$,

(iv) $\Phi_2^{-1}([M]) \vee \Phi_2^{-1}([V]) \neq \Phi_1^{-1}([M]) \vee \Phi_1^{-1}([U])$ by (ii) and (iii).
By substituting $\Phi_2^{-1}([V]) = [D]$ and $\Phi_1^{-1}([U]) = [D]$ in (iv) we obtain:

(v) $\Phi_2^{-1}([M]) \vee [D] \neq \Phi_1^{-1}([M]) \vee [D]$, and therefore,

(vi) $\Phi_1^{-1}([M]) \neq \Phi_2^{-1}([M])$.

We will now construct a supermaximal space the M. Note that both $[U]$ and $[V]$ are complemented in $\mathcal{L}^*(V_\infty)$ because they are images of the complemented $[D]$ under the automorphisms Φ_1, Φ_2, respectively. Therefore, U and V are decidable spaces. We can find computable bases A, B, and C of V, U, and $(V_\infty \ mod \ U)$, respectively. Let $A = \{a_0, a_1, \ldots\}, B = \{b_0, b_1, \ldots\}$, and $C = \{c_0, c_1, \ldots\}$ be fixed computable enumerations of these bases. We can regard C as a computable subset of V_∞. Thus, $B \cup C$ is a computable basis of V_∞, which extends the basis B of U. A space M will be constructed in stages. By M^s we will denote the approximation of M at the end of stage s.

At every stage s, the set B^s will be a computable basis for M^s. At stage 0, we will let $B^0 = B$ (and, therefore, $M^0 = U$). At stage $s > 0$, we will enumerate at most one vector $v \notin M^{s-1}$ into B^s, and then let $M^s = cl(B^s)$. Hence $dim(M^s \ mod \ M^0) < \infty$ and, therefore, M^s will be a decidable space, uniformly in s, for every $s \geq 0$.

Recall that V_e is the e-th c.e. subspace of V_∞. In the construction of M we will satisfy the following requirements for every $e \geq 0$:

$$R_e : \text{If } dim((V_e \vee M) \ \ mod \ M) = \infty, \text{ then } V_e \vee M = V_\infty.$$

Every R_e will be satisfied by satisfying the following sub-requirements for every $k \geq 0$:

$$R_{\langle e,k \rangle} : \text{If } dim((V_e \vee M) \ \ mod \ M) = \infty, \ \text{ then } c_k \in V_e \vee M.$$

We will also satisfy the following negative requirements for every $e \geq 0$:

$$N_e : dim(V \ mod \ M) > e.$$

Note that the satisfaction of $R_{\langle e,k \rangle}$ and N_e for each $e, k \geq 0$ will guarantee that M is a supermaximal subspace of V_∞ with the desired properties. To see this, note that if $M \subseteq V_{e_1}$ and $dim(V_{e_1} \ mod \ M) = \infty$ for some $e_1 \in \omega$, then $V_{e_1} = V_e \vee M$ for some $e \in \omega$. By construction, $B \subseteq U \subseteq M \subseteq V_{e_1}$. The satisfaction of the requirements $R_{\langle e,k \rangle}$ for all $e, k \geq 0$ will guarantee that $C \subseteq V_{e_1}$. Since $cl(B \cup C) = V_\infty$, we conclude that $V_{e_1} = V_\infty$.

At stage s, each requirement N_e will place a marker Γ_e on the first element $a_n \in A$ such that

$$\dim(\{a_0, \ldots, a_n\} \ mod \ M^s) = e + 1.$$

For all $e, k \geq 0$ the requirements N_m for $m \leq \langle e, k \rangle$ will have higher priority than the requirement $R_{\langle e,k \rangle}$. The requirement $R_{\langle e,k \rangle}$ will respect the higher priority requirements N_m by not allowing markers $\Gamma_0, \ldots, \Gamma_m$ to be moved.

The requirement $R_{\langle e,k \rangle}$ *requires attention at stage* $s + 1$ if:

(1) $R_{\langle e,k \rangle}$ has not been satisfied, and

(2) there is $y \in V_e^s$ with $y \leq s$ such that the following conditions are satisfied:

(i) $y + c_k \notin M^s$,

(ii) if a_{n_j} is the element of A marked by the marker Γ_j at stage s, then

$$\dim(\{a_{n_0}, \ldots, a_{n_{\langle e,k \rangle}}\} \ mod \ M^s) =$$

$$\dim(\{a_{n_0}, \ldots, a_{n_{\langle e,k \rangle}}\} \ mod \ cl(M^s \cup \{y + c_k\})).$$

If such y exists, then we say that $R_{\langle e,k \rangle}$ requires attention via y at stage $s + 1$.

Construction

Stage 0. Let $B^0 = B$ and $M^0 = cl(B^0)$. For each $i \geq 0$, place the marker Γ_i on the first element $a_n \in A$ such that

$$\dim(\{a_0, \ldots, a_n\} \ mod \ M^0) = i + 1.$$

Stage $s + 1$. Check if some requirement $R_{\langle e_1,k_1 \rangle}$, where $\langle e_1, k_1 \rangle \leq s + 1$, requires attention at stage $s + 1$. If there is no such requirement, let $B^{s+1} = B^s$, $M^{s+1} = cl(B^{s+1})$, and go to the next stage. Otherwise, let $\langle e, k \rangle$ be the least such that $R_{\langle e,k \rangle}$ requires attention, and let y be the least such that $R_{\langle e,k \rangle}$ requires attention via y at stage $s + 1$. Let $x =_{def} y + c_k$. Then

(a) let $M^{s+1} = cl(B^{s+1})$,

(b) for every $i \geq 0$ place the marker Γ_i on the first element $a_n \in A$ such that

$$\dim(\{a_0, \ldots, a_n\} \ mod \ M^{s+1}) = i + 1.$$

We say that $R_{\langle e,k \rangle}$ received attention. Note that the condition above can be checked effectively since M^{s+1} is a decidable space. Note also that, because of the condition (2)(ii), only the markers $\Gamma_{\langle e,k\rangle+1}, \Gamma_{\langle e,k\rangle+2}, \ldots$ are moved from the elements they marked at the previous stage.

End of Construction

In the following lemmas we will prove that the space M is supermaximal. Lemma 46 will imply that $\dim(V \ mod \ M) = \infty$. Hence $[M]$ avoids the upper cone of $[V]$ and, therefore, $\dim(V_\infty \ mod \ M) = \infty$. Lemma 47 will imply that if $dim((V_e \vee M) \ mod \ M) = \infty$, then $V_e \vee M = V_\infty$.

Lemma 45. *Each requirement $R_{\langle e,k\rangle}$ receives attention at most once.*

Proof. If $R_{\langle e,k\rangle}$ receives attention at stage $s+1$ via $y \in V_e^s$, then $x = y + c_k$ is enumerated into M^{s+1}. Then $c_k = (y + c_k) - y \in M^{s+1} \vee V_e^{s+1}$, and, therefore, $R_{\langle e,k\rangle}$ will be satisfied at stage $s+1$ and will not require attention at any later stage. ∎

Lemma 46. *Each marker Γ_m moves finitely often.*

Proof. Let s be a stage such that no $R_{\langle e,k\rangle}$ for $\langle e,k\rangle \leq m$ requires attention after stage s. Then the construction guarantees that Γ_m will not be moved after s. ∎

Lemma 47. *Each requirement $R_{\langle e,k\rangle}$ is satisfied.*

Proof. Suppose that $\langle e,k\rangle$ is the least number such that $R_{\langle e,k\rangle}$ is not satisfied. That means that $dim((V_e \vee M) \ mod \ M) = \infty$, but $c_k \notin M \vee V_e$. Suppose that s is the least stage such that no $R_{\langle e_1,k_1\rangle}$ for $\langle e_1,k_1\rangle < \langle e,k\rangle$ requires attention after s. This means that no marker Γ_j for $j \leq \langle e,k\rangle$ is moved after stage s. Suppose that a_{n_j} is the element marked by the marker Γ_j for $j = 0, \ldots, \langle e,k\rangle$. Since $dim((V_e \vee M) \ mod \ M) = \infty$, we also have

$$dim((V_e \vee M) \ mod \ cl(M \cup \{a_{n_0}, \ldots, a_{n_{\langle e,k\rangle}}, c_k\})) = \infty.$$

Therefore, there are a stage $s_1 > s$ and $y \in V_e^{s_1}$ such that

$$y \notin cl(M^{s_1} \cup \{a_{n_0}, \ldots, a_{n_{\langle e,k\rangle}}, c_k\}).$$

Then $y + c_k \notin cl(M^{s_1} \cup \{a_{n_0}, \ldots, a_{n_{\langle e,k\rangle}}\})$. The requirement $R_{\langle e,k\rangle}$ will receive attention via y at stage s_1, and will then remain satisfied. ∎

6 Automorphisms of the Lattices of Vector Spaces

The study of automorphisms of structures of importance in computable model theory connects computability theory with classical group theory. Let $\mathbf{d}$ be a Turing degree. For an infinite computable structure A, we define $Aut_{\mathbf{d}}(A)$ to be the set of all automorphisms of A, which are computable in $\mathbf{d}$. The set $Aut_{\mathbf{d}}(A)$ forms a group under composition and it is a subgroup of the group $Aut(A)$ of all automorphisms of A. It is natural to ask questions about computability-theoretic properties of this group and its subgroups. When the structure A is

ω with equality, then its automorphism group $Aut(A)$ is usually denoted by $Sym(\omega)$, the symmetric group of ω. Hence we have

$$Sym_{\mathbf{d}}(\omega) = \{f \in Sym(\omega) : \deg(f) \leq \mathbf{d}\}.$$

Lachlan showed that there are $2^{\aleph_0}$ automorphisms of $\mathcal{E}^*$. Every automorphism of $\mathcal{E}$ induces an automorphism of $\mathcal{E}^*$. Every computable permutation of ω induces an automorphism of $\mathcal{E}$, and hence of $\mathcal{E}^*$. Every automorphism of $\mathcal{E}^*$ is induced by some permutation of ω, which is not necessarily computable. Hence, since every automorphism of $\mathcal{E}^*$ is induced by some automorphism of $\mathcal{E}$, there are $2^{\aleph_0}$ automorphisms of $\mathcal{E}$.

By $\mathcal{L}$ we denote the lattice of all subspaces of V_∞. For a Turing degree $\mathbf{d}$, by $\mathcal{L}_{\mathbf{d}}(V_\infty)$ we denote the following sublattice of $\mathcal{L}$:

$$\mathcal{L}_{\mathbf{d}}(V_\infty) = \{V \in \mathcal{L} : V \text{ is } \mathbf{d}\text{-computably enumerable}\}.$$

Note that $\mathcal{L}_{\mathbf{0}}(V_\infty)$ is the same as $\mathcal{L}(V_\infty)$. The problem of finding the number of automorphisms of $\mathcal{L}^*(V_\infty)$ is still open. However, Guichard [30] established that there are countably many automorphisms of $\mathcal{L}(V_\infty)$ by showing that each computable automorphism is generated by a $1-1$ and onto computable semilinear transformation of V_∞.

Recall that a pair (μ, σ) is a *semilinear* transformation of V_∞ if $\mu : V_\infty \to V_\infty$ and σ is an automorphism of F such that

$$\mu(\alpha u + \beta v) = \sigma(\alpha)\mu(u) + \sigma(\beta)\mu(v)$$

for every $u, v \in V_\infty$ and every $\alpha, \beta \in F$. By $GSL_{\mathbf{d}}$ we will denote the group of $1-1$ and onto semilinear transformations (μ, σ) such that $deg(\mu) \leq \mathbf{d}$ and $deg(\sigma) \leq \mathbf{d}.$ Thus, Guichard proved that every element of $Aut(\mathcal{L}_{\mathbf{0}}(V_\infty))$ is generated by an element of $GSL_{\mathbf{0}}$. It is easy to show that this result can be relativized to an arbitrary Turing degree $\mathbf{d}$.

Theorem 48. *Every $\Phi \in Aut(\mathcal{L}_{\mathbf{d}}(V_\infty))$ is generated by some $(\mu, \sigma) \in GSL_{\mathbf{d}}$. Moreover, if Φ is also generated by some other $(\mu_1, \sigma_1) \in GSL_{\mathbf{d}}$, then there is $\gamma \in F$ such that*

$$(\forall v \in V_\infty)\,[\mu(v) = \gamma\mu_1(v)].$$

Proof. Note that each automorphism Φ of $\mathcal{L}_{\mathbf{d}}(V_\infty)$ acts on the one-dimensional subspaces of V_∞ and hence generates a unique automorphism $\overline{\Phi}$ of $\mathcal{L}$. By the fundamental theorem of projective geometry applied to the lattice $\mathcal{L}$, since $\overline{\Phi}$ is in $Aut(\mathcal{L})$, it follows that it is generated by a semilinear transformation (μ, σ). Note that (μ, σ) also generates Φ. We will now show that $deg(\mu) \leq \mathbf{d}$ and $deg(\sigma) \leq \mathbf{d}.$

Let $\alpha_0, \alpha_1, \alpha_2, \ldots$ be a fixed computable enumeration of the elements of the field F. Assume that $v_0, v_1, v_2, ...$ is a computable enumeration of a computable basis of V_∞. Define the following computable subspaces of V_∞:

$U_1 = cl(\{v_0, v_2, v_4, \ldots\})$,
$U_2 = cl(\{v_1, v_3, v_5, \ldots\})$,
$U_3 = cl(\{v_0 + v_1, v_2 + v_3, v_4 + v_5, \ldots\})$,

$U_4 = cl(\{v_1 + v_2, v_3 + v_4, v_5 + v_6, \ldots\})$,
$U_5 = cl(\{v_0 + \alpha_0 v_1, v_2 + \alpha_1 v_3, v_4 + \alpha_2 v_5, \ldots\})$.
Suppose that $\Phi(U_i) = Y_i$ for $i = 1, \ldots, 5$, and note that $Y_i \in \mathcal{L}_{\mathbf{d}}(V_\infty)$ since $U_i \in \mathcal{L}_{\mathbf{d}}(V_\infty)$.

To prove that $deg(\mu) \leq \mathbf{d}$, suppose that $\mu(v_0) = w_0$ for some fixed w_0. Assume inductively that $\mu(v_{2i}) = w_{2i}$ has been found **d**-computably. To find **d**-computably $\mu(v_{2i+1})$, we let w_{2i+1} be the least $y \in Y_2$ such that $w_{2i} + y \in Y_3$. Then we have $\mu(v_{2i+1}) = w_{2i+1}$. Next, to find **d**-computably $\mu(v_{2i+2})$, we let w_{2i+2} be the least $y \in Y_1$ such that $w_{2i+1} + y \in Y_4$. Then we have $\mu(v_{2i+2}) = w_{2i+2}$.

Finally, to find **d**-computably $\sigma(\alpha_i)$, we look for the least $w \in Y_5$ and $\beta \in F$ such that $w = w_{2i} + \beta w_{2i+1}$ and note that $\sigma(\alpha_i) = \beta$. It is not difficult to prove that if our choice for $\mu(v_0)$ is a scalar multiple of the original w_0, namely, $\mu(v_0) = \gamma w_0$, then $\mu(v_i) = \gamma w_i$ for every $i \geq 1$. ■

The *Turing degree spectrum* of a countable structure A is

$$DgSp(A) = \{\deg(B) : B \cong A\},$$

where $\deg(B)$ is the Turing degree of the atomic diagram of B. Knight [37] proved that the degree spectrum of any structure is either a singleton or is upward closed. Jockusch and Richter (see [53]) defined the *degree of the isomorphism type* of a structure, if it exists, to be the least Turing degree in its Turing degree spectrum. Morozov [47] established that the degree of the isomorphism type of the group $Sym_{\mathbf{d}}(\omega)$ is $\mathbf{d}''$.

Theorem 49 *(Dimitrov, Harizanov and Morozov [12])***.** *The degree of the isomorphisms type of the group $GSL_{\mathbf{d}}$ is $\mathbf{d}''$.*

In 1998, Downey and Remmel [26] raised the question of finding meaningful orbits in $\mathcal{L}^*(V_\infty)$. Recently, Dimitrov and Harizanov [9] gave a necessary and sufficient condition for quasimaximal vector spaces with extendible bases to be in the same orbit of $\mathcal{L}^*(V_\infty)$. The condition is stated in terms of m-degrees.

Unlike for the principal filters in $\mathcal{E}^*$ determined by quasimaximal sets of a fixed rank, there are several possibilities for the principal filters in $\mathcal{L}^*(V_\infty)$ determined by the closures of quasimaximal subsets of a computable basis. More precisely, Dimitrov [5,6] gave a description of all possible isomorphism types of $\mathcal{L}^*(cl(B), \uparrow)$ when B is a quasimaximal subset of rank n of any computable basis of V_∞. He proved that $\mathcal{L}^*(cl(B), \uparrow)$ is isomorphic to either:

(1) Boolean algebra $\mathbf{B_n}$ (which has 2^n elements),

(2) the lattice $L(n, \prod_C F)$ of all subspaces of an n -dimensional vector space over a certain extension $\prod_C F$ of the underlying field F, or

(3) a finite product of structures from the previous two cases.

Note that the Boolean algebra $\mathbf{B_n}$ in (1) can also be viewed as a product of n copies of the Boolean algebra $\mathbf{B_1}$. The extensions $\prod_C F$ of F mentioned in (2) are *cohesive powers* (see the definition below) of the field F over various cohesive sets C. Using the results in [11] it follows that these principal filters fall into infinitely many non-isomorphic classes, even when the filters are isomorphic to the lattices of subspaces of the vector spaces of the same dimension. Cohesive power is related to the versions of effective ultraproducts previously used by Hirschfeld, Wheeler, and McLaughlin [34,35,41,42] in their study of models of various fragments of arithmetic. As usual, we will denote the equality of partial functions by $\simeq$.

Definition 50. *Let $\mathcal{A}$ be a computable structure with domain A in a computable language S, and let $C \subseteq \omega$ be a cohesive set. The* cohesive power of $\mathcal{A}$ over C, *denoted by $\prod_C \mathcal{A}$, is a structure $\mathcal{B}$ for S with domain B defined as follows.*

(1) *The set B is $D/(=_C)$, where $D = \{\varphi \mid \varphi : \omega \to A$ is a partial computable function, and $C \subseteq^* dom(\varphi)\}$.*
For $\varphi_1, \varphi_2 \in D$, we have

$$\varphi_1 =_C \varphi_2 \quad iff \quad C \subseteq^* \{x : \varphi_1(x) \downarrow = \varphi_2(x) \downarrow\}.$$

The equivalence class of φ with respect to $=_C$ will be denoted by $[\varphi]_C$, or simply by $[\varphi]$ (when the reference to C is clear from the context).

(2) *If $f \in S$ is an n-ary function symbol, then $f^{\mathcal{B}}$ is an n-ary function on B such that for every $[\varphi_1], \ldots, [\varphi_n] \in B$, we have $f^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_n]) = [\varphi]$, where for every $x \in \omega$,*

$$\varphi(x) \simeq f^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_n(x)).$$

If $P \in S$ is an m-ary predicate symbol, then $P^{\mathcal{B}}$ is an m-ary relation on B such that for every $[\varphi_1], \ldots, [\varphi_m] \in B$,

$$P^{\mathcal{B}}([\varphi_1], \ldots, [\varphi_m]) \quad \text{iff} \quad C \subseteq^* \{x \in \omega \mid P^{\mathcal{A}}(\varphi_1(x), \ldots, \varphi_m(x))\}.$$

If $c \in S$ is a constant symbol, then $c^{\mathcal{B}}$ is the equivalence class of the (total) computable function on A with constant value $c^{\mathcal{A}}$.

In the context of c.e. vector spaces, the most interesting cases occur when F is finite or $F = \mathbb{Q}$. For finite F, we have $\prod_C F \cong F$. Various results about the cohesive powers of $\mathbb{Q}$ have been established in [7,11]. These results, together with the above classification of the possible isomorphism types of $\mathcal{L}^*(cl(B), \uparrow)$, were used in the proof of the result discussed in the next paragraph.

To state the theorem, we introduced the notion of an *m-degree type* of a quasimaximal set $E = \bigcap_{i=1}^{n} M_i$ of rank n, denoted by *type*(E). This notion captures the number and the m-degrees of the maximal sets M_i's. For $i = 1, 2$, let $E_i \subseteq A_i$

be a quasimaximal subset of a computable basis A_i. Dimitrov and Harizanov [9] proved that, assuming that the field is $\mathbb{Q}$, there is an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that $\Phi([E_1]) = [E_2]$ if and only if $type_{A_1}(E_1) = type_{A_2}(E_2)$. Since maximal sets are also quasimaximal, we have the following corollary.

Theorem 51 *(Dimitrov and Harizanov [9]).* *Assume that the underlying field is $\mathbb{Q}$. Let M_1 and M_2 be maximal subsets of computable bases B_1 and B_2 of V_∞, respectively. Then the following are equivalent:*

(1) There is an automorphism Φ of $\mathcal{L}^*(V_\infty)$ such that

$$\Phi([M_1]) = [M_2],$$

(2) $deg_m(M_1) = deg_m(M_2)$.

In some cases, it is also possible to connect the embeddability of the subgroups with Turing degree complexity. Morozov showed that the correspondence $\mathbf{d} \to Sym_{\mathbf{d}}(\omega)$ can be used to substitute Turing reducibility with group-theoretic embedding. More precisely, Morozov [45] established that for every pair $\mathbf{d}$, $\mathbf{s}$ of Turing degrees, we have

$$(Sym_{\mathbf{d}}(\omega) \hookrightarrow Sym_{\mathbf{s}}(\omega)) \Leftrightarrow (\mathbf{d} \leq \mathbf{s}).$$

It follows from this result that $\mathbf{d} = \mathbf{s}$ if and only if $Sym_{\mathbf{d}}(\omega) \cong Sym_{\mathbf{s}}(\omega)$. In [12], we established a similar result for the subgroups of the group of automorphisms of the lattice of the subspaces of V_∞.

Theorem 52 *(Dimitrov, Harizanov and Morozov [12]).* *For any pair of Turing degrees $\mathbf{d}, \mathbf{s}$ we have*

$$(Aut(\mathcal{L}_{\mathbf{d}}(V_\infty)) \hookrightarrow Aut(\mathcal{L}_{\mathbf{s}}(V_\infty))) \Leftrightarrow \mathbf{d} \leq \mathbf{s}.$$

References

1. Aigner, M.: Combinatorial Theory. Springer, Berlin (1997)
2. Ash, C.J., Knight, J.F., Remmel, J.B.: Quasi-simple relations in copies of a given recursive structure. Ann. Pure Appl. Logic **86**, 203–218 (1997)
3. Ash, C.J., Downey, R.G.: Decidable subspaces and recursively enumerable subspaces. J. Symbolic Logic **49**, 1137–1145 (1984)
4. Dekker, J.C.E.: Countable vector spaces with recursive operations, part I. J. Symbolic Logic **34**, 363–387 (1969)
5. Dimitrov, R.: Quasimaximality and principal filters isomorphism between $\mathcal{E}^*$ and $\mathcal{L}^*(V_\infty)$. Arch. Math. Logic **43**, 415–424 (2004)
6. Dimitrov, R.: A class of Σ^0_3 modular lattices embeddable as principal filters in $\mathcal{L}^*(V_\infty)$. Arch. Math. Logic **47**, 111–132 (2008)
7. Dimitrov, R.: Cohesive powers of computable structures. Annuaire de l'Université de Sofia "St. Kliment Ohridski", Faculté de Mathématiques et Informatique, vol. 99, pp. 193–201 (2009)

8. Dimitrov, R.: Extensions of certain partial automorphisms of $\mathcal{L}^*(V_\infty)$. Annuaire de l'Université de Sofia "St. Kliment Ohridski", Faculté de Mathématiques et Informatique, vol. 99, pp. 183–191 (2009)
9. Dimitrov, R.D., Harizanov, V.: Orbits of maximal vector spaces. Algebra and Logic **54**, 680–732 (2015) (in Russian). 440–477 (in English) (2016)
10. Dimitrov, R.D., Harizanov, V.S., Morozov, A.S.: Dependence relations in computably rigid computable vector spaces. Ann. Pure Appl. Logic **132**, 97–108 (2005)
11. Dimitrov, R., Harizanov, V., Miller, R., Mourad, K.J.: Isomorphisms of nonstandard fields and Ash's conjecture. In: Beckmann, A., Csuhaj-Varjú, E., Meer, K. (eds.) CiE 2014. LNCS, vol. 8493, pp. 143–152. Springer, Heidelberg (2014). doi:10.1007/978-3-319-08019-2_15
12. Dimitrov, R., Harizanov, V., Morozov, A.: Automorphism groups of substructure lattices of vector spaces in computable algebra. In: Beckmann, A., Bienvenu, L., Jonoska, N. (eds.) CiE 2016. LNCS, vol. 9709, pp. 251–260. Springer, Heidelberg (2016). doi:10.1007/978-3-319-40189-8_26
13. Downey, R.: On a question of A. Retzlaff. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **29**, 379–384 (1983)
14. Downey, R.G.: Nowhere simplicity in matroids. J. Aust. Math. Soc. Ser. A. Pure Math. Stat. **35**, 28–45 (1983)
15. Downey, R.: Bases of supermaximal subspaces and Steinitz systems, part I. J. Symbolic Logic **49**, 1146–1159 (1984)
16. Downey, R.G.: A note on decompositions of recursively enumerable subspaces. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **30**, 465–470 (1984)
17. Downey, R.: Co-immune subspaces and complementation in V_∞. J. Symbolic Logic **49**, 528–538 (1984)
18. Downey, R.G.: The degrees of r.e. sets without the universal splitting property. Trans. Am. Math. Soc. **291**, 337–351 (1985)
19. Downey, R.G., Hird, G.R.: Automorphisms of supermaximal subspaces. J. Symbolic Logic **50**, 1–9 (1985)
20. Downey, R.G.: Bases of supermaximal subspaces and Steinitz systems, part II. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **32**, 203–210 (1986)
21. Downey, R.G.: Undecidability of $L(F_\infty)$ and other lattices of r.e. substructures. Ann. Pure Appl. Logic **32**, 17–26 (1986)
22. Downey, R.G.: Maximal theories. Ann. Pure Appl. Logic **33**, 245–282 (1987)
23. Downey, R.G.: Orbits of creative subspaces. Proc. Am. Math. Soc. **99**, 163–170 (1987)
24. Downey, R.: Correction to "Undecidability of $L(F_\infty)$ and other lattices of r.e. substructures". Ann. Pure Appl. Logic **48**, 299–301 (1990)
25. Downey, R.: On the universal splitting property. Math. Logic Q. **43**, 311–320 (1997)
26. Downey, R.G., Remmel, J.B.: Computable algebras, closure systems: coding properties. In: Ershov, Y.L., Goncharov, S.S., Nerode, A., Remmel, J.B., (ed.) Handbook of Recursive Mathematics. Studies in Logic and the Foundations of Mathematics 139, vol. 2, pp. 997–1039 (1998)
27. Downey, R.G., Remmel, J.B.: Automorphisms and recursive structures. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik **33**, 339–345 (1987)
28. Fokina, E., Harizanov, V., Melnikov, A.: Computable model theory. In: Downey, R. (ed.) Turing's Legacy: Developments from Turing Ideas in Logic. Lecture Notes in Logic, vol. 42, pp. 124–194. Cambridge University Press/ASL, Cambridge (2014)

29. Galminas, L.R., Rosenthal, J.W.: More undecidable lattices of Steinitz exchange systems. J. Symbolic Logic **67**, 859–878 (2002)
30. Guichard, D.R.: Automorphisms of substructure lattices in recursive algebra. Ann. Pure Appl. Logic **25**, 47–58 (1983)
31. Guichard, D.R.: A note on r-maximal subspaces of V_∞. Ann. Pure Appl. Logic **26**, 1–9 (1984)
32. Hird, G.R.: Recursive properties of relations on models. Ann. Pure Appl. Logic **63**, 241–269 (1993)
33. Hird, G.R.: Recursive properties of relations on models. Ph.D. dissertation, Monash University, Melbourne, Australia (1984)
34. Hirschfeld, J.: Models of arithmetic and recursive functions. Isr. J. Math. **20**, 111–126 (1975)
35. Hirschfeld, J., Wheeler, W.H.: Forcing, Arithmetic, Division Rings. Lecture Notes in Mathematics, vol. 454. Springer, Heidelberg (1975)
36. Kalantari, I., Retzlaff, A.: Maximal vector spaces under automorphisms of the lattice of recursively enumerable spaces. J. Symbolic Logic **42**, 481–491 (1977)
37. Knight, J.F.: Degrees coded in jumps of orderings. J. Symbolic Logic **51**, 1034–1042 (1986)
38. Lachlan, A.H.: Degrees of recursively enumerable sets which have no maximal supersets. J. Symbolic Logic **33**, 431–443 (1968)
39. Lian, W.S.: Automorphisms of the lattice of recursively enumerable vector spaces and embeddings of $\mathcal{L}^*(V_\infty)$. Senior thesis, National University of Singapore (1985)
40. Mal'tsev, A.I.: On recursive Abelian groups. Doklady Akademii Nauk SSSR **146**, 1009–1012 (1962) (in Russian). 1431–1434 (in English)
41. McLaughlin, T.G.: Sub-arithmetical ultrapowers: a survey. Ann. Pure Appl. Logic **49**, 143–191 (1990)
42. McLaughlin, T.G.: Δ_1 ultrapowers are totally rigid. Arch. Math. Logic **46**, 379–384 (2007)
43. Metakides, G., Nerode, A.: Recursively enumerable vector spaces. Ann. Math. Logic **11**, 147–171 (1977)
44. Morozov, A.S.: Rigid constructive modules. Algebra and Logic **28**, 570–583 (1989) (in Russian). 379–387 (1990) (in English)
45. Morozov, A.S.: Turing reducibility as algebraic embeddability. Siberian Math. J. **38**, 362–364 (1997) (in Russian). 312–313 (in English)
46. Morozov, A.S.: Groups of computable automorphisms. In: Ershov, Y.L., Goncharov, S.S., Nerode, A., Remmel, J.B., (eds.) Handbook of Recursive Mathematics, Studies in Logic and the Foundations of Mathematics 139, vol. 1, pp. 311–345 (1998)
47. Morozov, A.S.: Permutations and implicit definability. Algebra and Logic **27**, 19–36 (1988) (in Russian). 12–24 (in English)
48. Nerode, A., Remmel, J.: Recursion theory on matroids. In: Metakides, G., (ed.) Patras Logic Symposium (Patras, 1980), Studies in Logic and the Foundations of Mathematics, vol. 109, pp. 41–65 (1982)
49. Nerode, A., Remmel, J.B.: Recursion theory on matroids, part II. In: Chong, C.-T., Wicks, M.J., (eds.) Southeast Asian Conference on Logic (Singapore, 1981), Studies in Logic and the Foundations of Mathematics, vol. 111, pp. 133–184 (1983)
50. Nerode, A., Remmel, J.: A survey of lattices of r.e. substructures. In: Nerode, A., Shore, R. (eds.) Recursion Theory, Proceedings of Symposia in Pure Mathematics, vol. 42, pp. 323–375. American Mathematical Society, Providence (1985)

51. Nerode, A., Smith, R.L.: The undecidability of the lattice of recursively enumerable subspaces. In: Arruda, A.I., da Costa, N.C.A., Settepp, A.M. (eds.) Proceedings of the Third Brazilian Conference on Mathematical Logic (Federal University of Pernambuco, Recife, 1979), pp. 245–252. Sociedade Brasileira de Lógica, São Paulo (1980)
52. Remmel, J.B.: On the lattice of r.e. superspaces of an r.e. vector space. Notices of the American Mathematical Society **24**, abstract #77T–E26 (1977)
53. Richter, L.J.: Degrees of unsolvability of models. Ph.D. dissertation, University of Illinois at Urbana-Champaign (1977)
54. Shore, R.A.: Determining automorphisms of the recursively enumerable sets. Proc. Am. Math. Soc. **65**, 318–325 (1977)
55. Soare, R.I.: Automorphisms of the lattice of recursively enumerable sets, part I: maximal sets. Ann. Math. **100**, 80–120 (1974)
56. Soare, R.I.: Recursively Enumerable Sets and Degrees. A Study of Computable Functions and Computably Generated Sets. Springer, Berlin (1987)