COMPUTABILITY-THEORETIC CATEGORICITY AND SCOTT FAMILIES

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Abstract. Computability-theoretic investigation of algorithmic complexity of isomorphisms between countable structures is a key topic in computable structure theory since Fröhlich and Shepherdson, Mal'cev, and Metakides and Nerode. A computable structure $\mathcal{A}$ is called computably categorical if for every computable isomorphic $\mathcal{B}$, there is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. By relativizing the notion of computable categoricity to a Turing degree $\mathbf{d}$, we obtain the notion of $\mathbf{d}$-computable categoricity. For the case when $\mathbf{d}$ is $\mathbf{0}^{(n-1)}$, we also speak about Δ^0_n-categoricity, for $n \geq 1$. More generally, $\mathcal{A}$ is relatively Δ^0_n-categorical if for every isomorphic $\mathcal{B}$, there is an isomorphism that is Δ^0_n relative to the atomic diagram of $\mathcal{B}$. Equivalently, $\mathcal{A}$ is relatively Δ^0_n-categorical if and only if $\mathcal{A}$ has a computably enumerable Scott family of computable (infinitary) Σ_n formulas. Relative Δ^0_n-categoricity implies Δ^0_n-categoricity, but not *vice versa*.

In this paper, we present an example of a computable Fraïssé limit that is computably categorical (that is, Δ^0_1-categorical) but not relatively computably categorical. We also present examples of Δ^0_2-categorical but not relatively Δ^0_2-categorical structures in natural classes such as trees of finite and infinite heights, and homogenous, completely decomposable, abelian groups. It is known that for structures from these classes computable categoricity and relative computable categoricity coincide.

The categoricity spectrum of a computable structure $\mathcal{M}$ is the set of all Turing degrees $\mathbf{d}$ such that $\mathcal{M}$ is $\mathbf{d}$-computably categorical. The degree of categoricity of $\mathcal{M}$ is the least degree in the categoricity spectrum of $\mathcal{M}$, if such a degree exists. It provides the exact level of categoricity of the structure. In this paper, we compute degrees of categoricity for relatively Δ^0_2-categorical abelian p-groups and for relatively Δ^0_3-categorical Boolean algebras.

1. Introduction and preliminaries

In computable model theory we use the tools and techniques of computability theory to investigate algorithmic content of notions and constructions in classical mathematics. Since isomorphisms may not transfer computability-theoretic properties of computable structures, computability theorists are interested in computable isomorphisms. This investigation has been one of the main topics in computable model theory. It dates back to Fröhlich and Shepherdson [19] who produced examples of isomorphic computable fields that are not computably isomorphic. We

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consider only countable structures for computable (often finite) languages. All finite structures are computable. An infinite structure $\mathcal{A}$ is *computable* if its universe can be identified with the set ω of natural numbers in such a way that the relations and operations of $\mathcal{A}$ are uniformly computable; that is, its atomic diagram is computable. We say that a computable structure $\mathcal{A}$ is *computably categorical* if for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists a computable isomorphism from $\mathcal{A}$ onto $\mathcal{B}$.

The notion of computable categoricity has been extended to higher level of hyperarithmetic hierarchy. A computable structure $\mathcal{A}$ is Δ^0_α-*categorical*, where α be a computable ordinal, if for every computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there exists a Δ^0_α isomorphism. For example, Barker [6] proved that for every computable ordinal α, there is a $\Delta^0_{2\alpha+2}$-categorical but not $\Delta^0_{2\alpha+1}$-categorical abelian p-group. More generally, a computable structure $\mathcal{A}$ is *relatively* Δ^0_α-*categorical* if for every $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$, which is Δ^0_α relative to (the atomic diagram of)c $\mathcal{B}$. For example, every computable equivalence structure is relatively Δ^0_3-categorical. Clearly, a relatively Δ^0_α-categorical structure is Δ^0_α-categorical.

There is a powerful syntactic condition that involves the existence of certain Scott families, which implies Δ^0_α-categoricity and is equivalent to relative Δ^0_α-categoricity. This connection between computability and definability is one of the main themes in computable model theory. A *Scott family* for a structure $\mathcal{A}$ is a countable family Φ of $L_{\omega_1\omega}$-formulas, with finitely many fixed parameters from A, such that:

(i) Each finite tuple in $\mathcal{A}$ satisfies some $\psi \in \Phi$;

(ii) If $\overline{a}$, $\overline{b}$ are tuples in $\mathcal{A}$, of the same length, satisfying the same formulas in Φ, then there is an automorphism of $\mathcal{A}$, which maps $\overline{a}$ to $\overline{b}$.

Ash [3] defined computable Σ_α and Π_α formulas of $L_{\omega_1\omega}$, where α is a computable ordinal, recursively and simultaneously and together with their Gödel numbers. The computable Σ_0 and Π_0 formulas are the finitary quantifier-free formulas. The computable $\Sigma_{\alpha+1}$ formulas are of the form

$$\bigvee_{n\in W_e} \exists\overline{y}_n \psi_n(\overline{x}, \overline{y}_n),$$

where for $n \in W_e$, ψ_n is a Π_α formula indexed by its Gödel number n, and $\exists\overline{y}_n$ is a finite block of existential quantifiers. Similarly, $\Pi_{\alpha+1}$ formulas are c.e. conjunctions of $\forall\Sigma_\alpha$ formulas. If α is a limit ordinal, then Σ_α (Π_α, respectively) formulas are of the form $\bigvee\limits_{n\in W_e} \psi_n$ ($\bigwedge\limits_{n\in W_e} \psi_n$, respectively), such that there is a sequence $(\alpha_n)_{n\in W_e}$ of ordinals less than α, given by the ordinal notation for α, and every ψ_n is a Σ_{α_n} (Π_{α_n}, respectively) formula. For a more precise definition see [3]. A *formally* Σ^0_α *Scott family* is a Σ^0_α Scott family of computable Σ_α formulas.

The following equivalence (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) for a computable structure $\mathcal{A}$ was established by Goncharov [23] for $\alpha = 1$, and by Ash, Knight, Manasse, and Slaman [4] and independently by Chisholm [11] for any computable ordinal α.

(i) The structure $\mathcal{A}$ is relatively Δ^0_α-categorical.

(ii) The structure $\mathcal{A}$ has a formally Σ^0_α Scott family.

(iii) The structure $\mathcal{A}$ has a c.e. Scott family consisting of computable Σ_α formulas.

It follows that a computable structure is relatively computably categorical if and only if it has a c.e. Scott family of (finitary) existential formulas.

Goncharov [22] was the first to show that computable categoricity does not always coincide with relative computable categoricity. For his example, Goncharov used a family of sets with special enumeration properties constructed by Selivanov [42]. These structures are somewhat pathological and, for example, cannot be found in the classes of linear orderings [24, 40], Boolean algebras [24, 41, 33], trees of finite height [34], abelian p-groups [21, 43, 7], equivalence structures [9], injection structures [10], algebraic fields with splitting algorithms [39]. However, Hirschfeldt, Khoussainov, Shore, and Slinko [28] established that there are computably categorical but not relatively computably categorical structures in the following classes: partial orders, lattices, 2-step nilpotent groups, commutative semigroups, and integral domains of arbitrary characteristic. Hirschfeldt, Kramer, R. Miller, and Shlapentokh [27] showed that there is a computably categorical algebraic field, which is not relatively computably categorical. In this paper, we further investigate effectively categorical structures for which computablility-theoretic properties of isomorphisms do not come from corresponding definability.

In Section 2, we construct a computable structure $\mathcal{A}$ that is a Fraïssé limit, which is computably categorical but not relatively computably categorical. The language of $\mathcal{A}$ can be finite or it can be relational, but not both. Moreover, the structure $\mathcal{A}$ is 1-decidable, that is, its $\exists$-diagram (equivalently, $\forall$-diagram) is decidable. This is optimal since Goncharov [23] proved that every 2-decidable computably categorical structure must be relatively computably categorical.

In [25] Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon lifted Goncharov's result to higher levels in the hyperarithmetic hierarchy by showing that for every computable successor ordinal $\alpha > 1$, there is a Δ^0_α-categorical but not relatively Δ^0_α-categorical structure. Chisholm, Fokina, Goncharov, Harizanov, Knight, and Quinn [12] established a similar result for computable limit ordinals. The structures we constructed in [25, 12] are very complicated and unnatural. Thus, our goal in this paper is to find such structures in natural classes. We focus on $\alpha = 2$. That is, we present some new examples of structures in natural classes, which are Δ^0_2-categorical but not relatively Δ^0_2-categorical. For some natural classes, there are no such examples. For example, Cenzer, Harizanov, and Remmel [10] showed that every Δ^0_2-categorical injection structure is relatively Δ^0_2-categorical. Bazhenov [8] and Harris [26] independently showed that for Boolean algebras the notions of Δ^0_2-categoricity and relative Δ^0_2-categoricity coincide.

More specifically, in Section 3, we build Δ^0_2-categorical but not relatively Δ^0_2-categorical trees of finite and infinite heights. Here, a tree can be viewed both as a partial order and as a directed graph. In Section 4, we prove that there is a homogenous completely decomposable abelian group, which is Δ^0_2-categorical but not relatively Δ^0_2-categorical. Our results complement the following results. Kach and Turetsky [30] showed that there exists a Δ^0_2-categorical equivalence structure $\mathcal{M}$, which is not relatively Δ^0_2-categorical. Downey, Melnikov and Ng [16] built examples of abelian p-groups that show that the notions of Δ^0_2-categoricity and relative Δ^0_2-categoricity do not coincide for these groups. However, it still remains open whether there is a Δ^0_2-categorical linear order, which is not relatively Δ^0_2-categorical, although McCoy [35] characterized relatively Δ^0_2-categorical linear orders. Frolov [20] announced that there is a Δ^0_3-categorical linear order that is not relatively Δ^0_3-categorical.

In this paper, we also investigate the degrees of categoricity of structures in natural classes. This notion was introduced in computable model theory relatively recently by Fokina, Kalimullin, and R. Miller [18], and tries to capture the least degree in the set of all Turing degrees capable of computing isomorphisms between computable isomorphic copies of structures. More precisely, the *categoricity spectrum* of a computable structure $\mathcal{A}$ is the following set of Turing degrees:

$$CatSpec(\mathcal{A}) = \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-computably categorical}\}.$$

The *degree of categoricity* of $\mathcal{A}$, if it exists, is the least Turing degree in CatSpec($\mathcal{A}$). Not every computable structure has the degree of categoricity, as shown by R. Miller [37] and by Fokina, Frolov, and Kalimullin [17].

Fokina, Kalimullin, and R. Miller [18] investigated which arithmetic degrees can be degrees of categoricity of computable structures. Csima, Franklin, and Shore [13] extended their results to hyperarithmetic degrees. For sets X and Y, we say that Y is *c.e. in and above* (c.e.a. in) X if Y is c.e. relative to X, and $X \leq_T Y$. Csima, Franklin, and Shore [13] proved that for every computable ordinal α, $\mathbf{0}^{(\alpha)}$ is the degree of categoricity. They also established that for a computable successor ordinal α, every degree $\mathbf{d}$ that is c.e.a. in $\mathbf{0}^{(\alpha)}$ is the degree of categoricity. There are also negative results in [18, 13]. If $\mathbf{d}$ is a non-hyperarithmetic degree, then $\mathbf{d}$ cannot be the degree of categoricity, so there are only countably many degrees of categoricity. Anderson and Csima [1] showed that there is a Σ^0_2 degree that is not a degree of categoricity, while is not known whether there is such Δ^0_2 degree. They also proved that there is a noncomputable degree $\mathbf{d} \leq \mathbf{0}''$ such that if two computable structures are $\mathbf{d}$-computably isomorphic, then they are computably isomorphic.

In Section 5, we compute the degrees of categoricity for relatively Δ^0_2-categorical abelian p-groups and for relatively Δ^0_3-categorical Boolean algebras. Both results are as expected and show typical behavior of natural structures with regard to the degrees of categoricity. Our result about Boolean algebras extends Bazhenov's investigation in [8] where he computed the degrees of categoricity for relatively Δ^0_2-categorical Boolean algebras.

2. Computably categorical but not relatively computably categorical Fraïssé limits

For a computable ordinal α, the notions of Δ^0_α-categoricity and relative Δ^0_α-categoricity of a computable structure $\mathcal{A}$ coincide if $\mathcal{A}$ satisfies certain extra decidability conditions (see Goncharov [23] and Ash [2]). A structure $\mathcal{A}$ is called *n-decidable*, for $n \geq 1$, if the Σ_n-diagram of $\mathcal{A}$ is decidable. Goncharov [23] proved that if $\mathcal{A}$ is 2-decidable, then computable categoricity and relative computable categoricity of $\mathcal{A}$ coincide. Kudinov [32] showed that Goncharov's assumption of 2-decidability cannot be weakened to 1-decidability, by giving an example of 1-decidable and computably categorical structure, which is not relatively computably categorical. On the other hand, Downey, Kach, Lempp, and Turetsky [14] showed that any 1-decidable computably categorical structure is relatively Δ^0_2-categorical.

The proofs by Goncharov and by Downey, Kach, Lempp, and Turetsky use the decidability of the structure to determine if certain finitely generated substructures can be extended to various larger finitely generated substructures. Because of the special properties of a Fraïssé limit, one might expect that all such questions would be trivial to determine, and so the decidability condition could be weakened or dropped entirely for such structures. However, this is not the case. Here, we give an example of 1-decidable and computably categorical Fraïssé limit, which is not relatively computably categorical.

Let us recall the definition of a Fraïssé limit (see [29, Chapter 6]). The *age* of a structure $\mathcal{M}$ is the class of all finitely generated structures that can be embedded in $\mathcal{M}$. Fraïssé showed that a (nonempty) finite or countable class $\mathbb{K}$ of finitely generated structures is the age of a finite or a countable structure if and only if $\mathbb{K}$ has the hereditary property and the joint embedding property. A class $\mathbb{K}$ has the *hereditary property* if whenever $\mathcal{C} \in \mathbb{K}$ and $\mathcal{S}$ is a finitely generated substructure of $\mathcal{C}$, then $\mathcal{S}$ is isomorphic to some structure in $\mathbb{K}$. A class $\mathbb{K}$ has the *joint embedding property* if for every $\mathcal{B}, \mathcal{C} \in \mathbb{K}$ there is $\mathcal{D} \in \mathbb{K}$ such that $\mathcal{B}$ and $\mathcal{C}$ embed into $\mathcal{D}$. A structure $\mathcal{U}$ is *ultrahomogeneous* if every isomorphism between finitely generated substructures of $\mathcal{U}$ extends to an automorphism of $\mathcal{U}$.

Definition 1. (see [29, Chapter 6]) A structure $\mathcal{A}$ is a *Fraïssé limit* of a class of finitely generated structures $\mathbb{K}$ if $\mathcal{A}$ is countable, ultrahomogeneous, and has age $\mathbb{K}$.

Fraïssé proved that the Fraïssé limit of a class of finitely generated structures is unique up to isomorphism. We say that a structure $\mathcal{A}$ is a Fraïssé limit if for some class $\mathbb{K}$, $\mathcal{A}$ is the Fraïssé limit of $\mathbb{K}$. First we show that every Fraïssé limit is relatively Δ^0_2-categorical.

Theorem 1. *Let $\mathcal{A}$ be a computable structure, which is a Fraïssé limit. Then $\mathcal{A}$ is relatively Δ^0_2-categorical.*

Proof. Because of ultrahomogeneity, we can construct isomorphisms between $\mathcal{A}$ and an isomorphic structure $\mathcal{B}$ using a back-and-forth argument, as long as we can determine for every $\overline{a} \in \mathcal{A}$ and $\overline{b} \in \mathcal{B}$, whether there is an isomorphism from the structure generated by $\overline{a}$ to the structure generated by $\overline{b}$, which maps $\overline{a}$ to $\overline{b}$ in order. This can be determined by $(\mathcal{B})'$, since there is such an isomorphism precisely if there is no atomic formula ϕ with $\mathcal{A} \models \phi(\overline{a})$ and $\mathcal{B} \not\models \phi(\overline{b})$. This is a Π^0_1 condition relative to $\mathcal{A} \oplus \mathcal{B} \equiv_T \mathcal{B}$.

Therefore, we can use $(\mathcal{B})'$ as an oracle to perform the back-and-forth construction of an isomorphism, and so there is an isomorphism that is Δ^0_2 relative to $\mathcal{B}$. □

Remark. Note that if the language of $\mathcal{A}$ is finite and relational, then there are only finitely many atomic formulas ϕ to consider, and the set of such formulas can be effectively determined. Hence, if the language is finite and relational, then a Fraïssé limit is necessarily relatively computably categorical.

Theorem 2. *There is a 1-decidable structure $\mathcal{F}$ that is a Fraïssé limit and computably categorical, but not relatively computably categorical. Moreover, the language for such $\mathcal{F}$ can be finite or it can be relational.*

Proof. The construction we present here is a modification of the first construction in Theorem 3.3 by Downey, Kach, Lempp, and Turetsky [14], where the structure they build is, in particular, 1-decidable, computably categorical but not relatively computably categorical. The only new ingredient we add is to make the resulting structure a Fraïssé limit. We sketch the original construction and explain the modifications we must make to ensure that the resulting structure is a Fraïssé limit. A more formal proof can be easily recovered from the original proof in [14].

The original construction is an undirected graph. To assure that the structure is not relatively computably categorical, we diagonalize agains all potential c.e. Scott families of Σ_1 formulas, with finitely many parameters. This is done by creating infinitely many connected components that are all accumulation points in the Σ_1 type space (details follow); this is similar to the technique used in Kudinov's construction in [32]. Then for any potential c.e. Scott family of Σ_1 formulas, there must be some accumulation point in a component disjoint from the finitely many parameters of the family with the following property. Any Σ_1 formula from the Scott family, which holds of the accumulation point would also need to hold of any other point that is "sufficiently close" in the type space, contradicting the definition of a Scott family.

The original construction created these accumulation points as vertices with loops of various sizes coming out of them. For each accumulation point, there would be a pair of computable sequences $\{n_k\}_{k\in\omega}$ and $\{m_k\}_{k\in\omega}$, chosen exclusively for this accumulation point. For every k, there would be a vertex v_k with attached loops of sizes $n_0, \ldots, n_k$ and a loop of size m_k. The loop of size m_k is meant to identify the component corresponding to v_k, so loops of this size are not used in any other component of the construction. There would also be a vertex v_∞ with attached loops $n_0, n_1, \ldots$ Each v_k and v_∞ would also have infinitely many *rays* – non-branching infinite paths originating from the vertex. The Σ_1 type of v_∞ is then the limit of the Σ_1 types of the v_k.

The construction took place on a tree of strategies, where each accumulation point was created by an individual strategy. Because a strategy might be visited only finitely many times in the construction, not all strategies would create the full set of vertices described above. Each time a strategy was visited, it performed one of the following steps, in alternation:

- Increment k, choose n_{k+1} and attach a loop of size n_{k+1} to v_∞;
- Choose m_k. Create the full v_k component.

Thus, if a strategy was only visited finitely many times, the v_∞-component would have loops of sizes $n_0, \ldots, n_{k+1}$, and the components $v_0, \ldots, v_{k-1}$ would have all been created, and possibly v_k as well. Numbers n_k and m_k are always chosen larger than the current stage, and two distinct strategies choose completely distinct numbers n_k and m_k. That is, any number is chosen by at most one strategy.

Notice that each time the strategy first chooses a sufficiently large new n_{k+1} and attaches a corresponding loop to v_∞. Only after that it chooses a new m_k and creates the v_k component. This ensures that the resulting structure is computably categorical. The fact that each component has infinitely many infinite rays makes the structure 1-decidable. Finally, the structure is not relatively computably categorical, as the construction destroys any potential Scott family.

We describe now two ways of modifying this construction so that the structure becomes a Fraïssé limit while still being computably categorical, 1-decidable and not relatively computably categorical. The first uses a finite language with function symbols, while the second uses an infinite relational language. Let

$$\mathcal{L}_1 = \{E, f, g, h\},$$

where E is a binary relation symbol and f, g and h are unary function symbols. Let

$$\mathcal{L}_\infty = \{E\} \cup \{U_{i,j} : j < i \wedge i, j \in \omega\} \cup \{V_{i,j} : j \leq i \wedge i, j \in \omega\} \cup \{R_i : i \in \omega\} \cup \{S_i : i \in \omega\},$$

where E is a binary relation symbol and each $U_{i,j}$, $V_{i,j}$, R_i and S_i is a unary relation symbol.

The intention is that E is the edge relation of the graph from the original construction. That is, in both cases, the reduct of the structures we make to the language $\{E\}$ will be the original structure in [14]. We will now describe the new functions and relations on the structure.

Suppose that v is one of the v_k or v_∞, and $a_0, \dots, a_{n_k-2}$ are vertices with vEa_0, a_iEa_{i+1} for all $i < n_k - 2$, and $a_{n_k-2}Ev$; that is, $v, a_0, \dots, a_{n_k-2}$ is the loop of size n_k attached to v. Suppose also that a_0 has lower Gödel number than a_{n_k-2}, so that we have chosen a particular orientation of the loop. Then we define $f(a_i) = a_{i+1}$, and $f(a_{n_k-2}) = v$. We also define $g(a_{i+1}) = a_i$ and $g(a_0) = v$. So f "walks" along the loop in one direction, and g "walks" along it in the other direction. We also define $U_{n_k,i}(a_i)$ to hold for every $i < n_k$, while $U_{n_k,i}(x)$ fails to hold for any other x.

For v_k, suppose that $a_0, \dots, a_{m_k-2}$ are vertices as above, so that $v_k, a_0, \dots, a_{m_k-2}$ is the loop of size m_k attached to v_k, again with a chosen orientation. Then we define $f(a_i) = a_{i+1}$, $f(a_{m_k-2}) = v_k$ and $f(v_k) = a_0$. We also define $g(a_{i+1}) = a_i$, $g(a_0) = v_k$ and $g(v_k) = a_{m_k-2}$. So, again, f and g walk along the loop in the opposite directions, but the walks continue through v_k. We also define $V_{m_k,i}(a_i)$ to hold, and $V_{m_k,i}(x)$ fails to hold for any other x, for every $i < m_k$. Finally, we define $V_{m_k,m_k}(z)$ to hold for every vertex z in the same component as v_k.

Suppose that v is one of the v_k's or v_∞, and consider a ray of the form $a_0, a_1, \dots$ with vEa_0 and a_iEa_{i+1} for all $i \in \omega$. For infinitely many of these rays, we define $f(a_i) = a_{i+1}$, $g(a_{i+1}) = a_i$ and $g(a_0) = v$, and for infinitely many rays, we define $g(a_i) = a_{i+1}$, $f(a_{i+1}) = a_i$ and $f(a_0) = v$. So for infinitely many rays, f walks away from v, while g walks towards v, and for infinitely many rays the reverse holds. For every ray, we define $R_i(a_i)$ to hold.

For v_∞, we choose some a_0 from some ray with $g(a_0) = v_\infty$ and define $f(v_\infty) = a_0$. We choose some b_0 from some ray with $f(b_0) = v_\infty$ and define $g(v_\infty) = b_0$.

Suppose that v is one of the v_k's or v_∞, and a is part of the loop of size n_0 with $g(a) = v$. Then we define $h(v) = a$. For every other x, we define $h(x) = f(x)$.

For every vertex x in every component created by strategy i from the priority tree, we define $S_i(x)$ to hold.

Claim 1. *In both $\mathcal{L}_1$ and $\mathcal{L}_\infty$, if $\overline{x}$ and $\overline{y}$ generate substructures that are isomorphic* via *an isomorphism mapping $\overline{x}$ to $\overline{y}$, then there is an automorphism of the full structure $\mathcal{F}$ mapping $\overline{x}$ to $\overline{y}$.*

Proof. We prove the result for singletons x and y. The general case proceeds similarly. The point is that if $x \neq y$, then they must both be vertices from loops/rays within the same component, and they must be the same length along those loops/rays. Then, loops are identified uniquely and for any two rays, there is an automorphism switching those rays and fixing the remainder of the structure. The argument is slightly longer for $\mathcal{L}_\infty$, because rays come in two sorts, and there are two distinguished rays in the component of v_∞.

In $\mathcal{L}_1$, through f or g, the substructure generated by x contains some vertex v_k or v_∞. The same is true for y. Through h, the substructure also contains the entire loop of size n_0. Since n_0 is unique to some strategy from the priority tree, x and y are both placed by the same strategy.

In $\mathcal{L}_\infty$, there is some i such that $S_i(x)$ and $S_i(y)$ hold. Hence x and y must again both be placed by the same strategy.

In $\mathcal{L}_1$, if the substructure generated by x contains v_k, then through $f(v_k)$ it also contains the loop of size m_k. If the substructure contains v_∞, then through $f(v_\infty)$ it also contains an infinite ray with $f(v_\infty) = a_0$. The same holds for y. This loop or ray uniquely characterizes the component, so x and y must be part of the same component.

In $\mathcal{L}_\infty$, if the component of x contains v_k, then $V_{m_k,m_k}(x)$ holds. If, instead, it contains v_∞, then no $V_{m_k,m_k}(x)$ holds for any k. The same is true for y. Hence x and y must be part of the same component.

In $\mathcal{L}_1$, there are four possibilities: $f^i(x) = v$ and $g^j(x) = v$ for some i and j; $f^i(x) = v$ for some i but $g^j(x) \neq v$ for all j; $g^j(x) = v$ for some j but $f^i(x) \neq v$ for all i; or $x = v$. Note that v is uniquely characterized by having degree greater than 2, even in the substructures generated by x or y. In the first case, x must be a_{j-1} from the loop of size $i + j$. In the second case, x must be a_{i-1} from one of the rays in which f walks towards v. In the third case, x must be a_{j-1} from one of the rays in which g walks towards v. The same holds for y. The first case is unique in the component, so in this case we know that $x = y$. If $v \neq v_\infty$, there is a single orbit containing every instance of the second case, and another orbit containing every instance of the third case, so there must be an automorphism mapping x to y. If $v = v_\infty$, then the second case breaks into two subcases: $g(v) = f^{i-1}(x)$, and $g(v) \neq f^{i-1}(x)$. The first subcase is unique in the component, so $x = y$, while the second subcase again comprises a single orbit. We reason similarly in the third case. The fourth case is again unique in the component.

In $\mathcal{L}_\infty$, if x is part of some loop, then there is some $U_{i,j}$, or $V_{i,j}$, which holds of x and no other point. Hence $x = y$. If x is part of some ray, then there is some R_i that holds of x and only of the points on rays, which are distance i from v. Hence y is also a point on a ray, which is distance i from v. Thus, there is an automorphism of the structure switching those two rays, and, in particular, sending x to y.

In $\mathcal{L}_\infty$, v_k is uniquely characterized by $V_{m_k,m_k}(v_k)$ holding, some $S_i(v_k)$ holding, and no other unary relation holding. Hence if $x = v_k$, then $y = v_k$. Also, v_∞ is uniquely characterized by some $S_i(v_\infty)$ holding and no other unary relation holding. Hence if $x = v_\infty$, then $y = v_\infty$. □

It follows that the structures we have just described are Fraïssé limits. Observe that they are defined in a computable fashion. Furthermore, our expanded language does not provide an obstacle to 1-decidability, since n_k and m_k are always chosen larger than the current stage. Thus, any statement about $f^s(x)$, $g^s(x)$, $h^s(x)$, $U_{s,j}(x)$, $V_{s,j}(x)$, $R_s(x)$ or $S_s(x)$ can be decided by considering the construction up through stage s. From the definition of the additional functions and relations it also follows that the expanded structure is still computably categorical but not relatively computably categorical (as the vertices v_∞ are still accumulation points in the Σ_1 space, allowing us to diagonalize against Scott families). □

3. Δ^0_2-categorical but not relatively Δ^0_2-categorical trees

We consider trees as partial orders. R. Miller [38] established that no well-founded tree of infinite height is computably categorical. Lempp, McCoy, R. Miller, and Solomon [34] characterized computably categorical trees of finite height, and established that for these structures, computable categoricity coincides with relative computable categoricity. There is no known characterization of Δ^0_2-categoricity or of higher level categoricity for trees of finite height. Lempp, McCoy, R. Miller, and Solomon [34] proved that for every $n \geq 1$, there is a computable tree of finite height, which is Δ^0_{n+1}-categorical but not Δ^0_n-categorical. We will establish the following result, which also holds when a tree is presented as a directed graph.

Theorem 3. *There is a computable Δ^0_2-categorical tree of finite height, which is not relatively Δ^0_2-categorical.*

Proof. While building a computable tree $\mathcal{T}$ (with domain ω), we diagonalize against all potential c.e. Scott families of computable Σ_2 formulas, with finitely many fixed parameters. Thus, we consider all pairs $(\mathcal{X}, \overline{p})$, where $\mathcal{X}$ is a c.e. family of computable Σ_2 formulas and $\overline{p}$ is a finite tuple of elements from the domain of $\mathcal{T}$. We must ensure that for each pair $(\mathcal{X}, \overline{p})$, $\mathcal{X}$ with parameters $\overline{p}$ is not a Scott family for $\mathcal{T}$. At the same time, we have to assure that every isomorphic computable tree is $\mathbf{0}'$ - isomorphic to $\mathcal{T}$. The construction will be an infinite injury priority construction where strategies are arranged on a priority tree with the true path defined as usual.

We build the tree $\mathcal{T}$ by stages. The root of $\mathcal{T}$ will have infinitely many "children," which we label $c_0, c_1, c_2, \ldots$ Each c_e will have 3 children, a_e, b_e and m_e. The purpose of m_e is to uniquely identify c_e. The node m_e will have a child n_e, and n_e will have $e+1$ many children. When the subtree above c_e first appears in the tree, a_e will have 2 children and b_e will have no children. See the diagram.

Through the action of some strategy, more children may be added to a_e and b_e at later stages. As usually, we denote by $\mathcal{T}_s$ the tree at stage s.

Let $(\mathcal{X}_i, \overline{p}_i)_{i \in \omega}$ be an enumeration of pairs, where $\mathcal{X}_i$ is a c.e. family of computable Σ_2 formulas, and $\overline{p}_i$ is a tuple drawn from ω, the domain of $\mathcal{T}$. We must meet the following categoricity and isomorphism requirements. Let $M_0, M_1, \ldots$ be an effective enumeration of all computable structures in the language of the trees.

R_i : $\mathcal{X}_i$ with parameters $\overline{p}_i$ is not a Scott family for $\mathcal{T}$.

Q_j : If $M_j \cong \mathcal{T}$, then there is a $\mathbf{0}'$-computable isomorphism between M_j and $\mathcal{T}$.

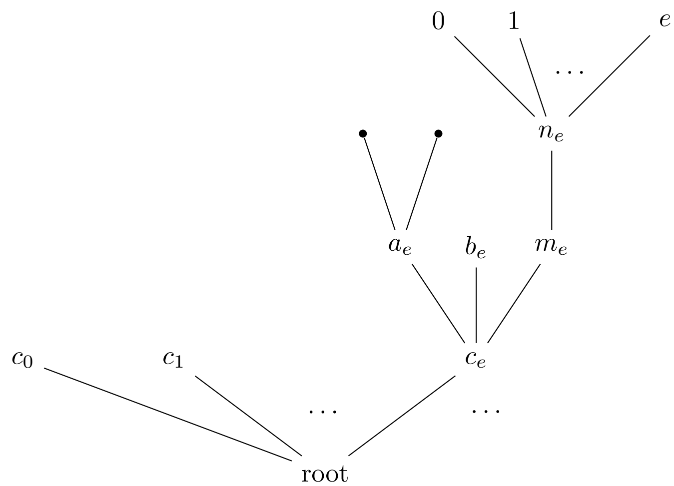

FIGURE 1. Construction of the tree $\mathcal{T}$.

Strategy for R_i

Our strategy will appear on a priority tree. When the strategy is visited, s is always the current stage, and $t < s$ is the last stage at which the strategy took outcome ∞ (or $t = 0$ if the strategy has never before taken outcome ∞). The first time the strategy is visited, we choose a large e to work with. In particular, a_e and b_e must not occur in $\overline{p}_i$, and $e > s$.

Let $\phi(\overline{x}) = \bigvee_{n\in W_e} \exists\overline{y}_n\psi_n(\overline{x}, \overline{y}_n)$ be a Σ_2 formula, where each $\psi_n(\overline{x}, \overline{y}_n)$ is a computable Π_1 formula. For each such ψ_n, denote by ψ_n^r the finite formula obtained from ψ_n by restricting the c.e. conjunction to the first r elements enumerated into the corresponding c.e. set.

We will make use of the fact that if $\phi(\overline{x})$ is a computable Σ_2 formula and $\overline{a} \in \mathcal{T}$, then $\mathcal{T} \models \phi(\overline{a})$ if and only if for some u, and some $\overline{c} \in |\mathcal{T}|, \mathcal{T}_r \models \psi_u^r(\overline{a}, \overline{c})$ for all sufficiently large stages r (such that, in particular, $\overline{c} \in |\mathcal{T}_r|$).

In what follows, without loss of generality we assume that $\mathcal{T}_s \not\models \phi(\overline{a})$ for any $\overline{a}$ if $\phi(\overline{x})$ is not one of the first s elements of $\mathcal{X}_i$.

We proceed as follows. Suppose s is a stage at which the strategy is visited. We perform at most one step according to the following.

(1) Among the first s elements of $\mathcal{X}_i$, locate the $\phi(\overline{x})$ that minimizes the u such that for some $\overline{c} \in |\mathcal{T}_u|$, we have $\mathcal{T}_r \models \psi_u^r(a_e, \overline{c}, \overline{p}_i)$ and $\mathcal{T}_r \models \psi_u^r(b_e, \overline{c}, \overline{p}_i)$ for every $r \in (u, s]$. Note that $u = s$ always works. Decide ties by favoring earlier elements of $\mathcal{X}_i$. If we have found such a ϕ, move to (2) next time the strategy is visited.

(2) Check whether there is an $r \in (t, s]$ with $\mathcal{T}_r \not\models \psi_u^r(a_e, \overline{c}, \overline{p}_i) \wedge \psi_u^r(b_e, \overline{c}, \overline{p}_i)$. If so, move to (3) next time the strategy is visited. Otherwise, stay at (2).

(3) Add a child to both a_e and b_e, ensuring that these children are not elements of $\overline{p}_i$ and move to (4).

(4) Return to Step (1).

As each time the strategy is visited we perform at most one step, we never add more than 1 child to a_e at a single stage. This will be important for interactions with higher priority categoricity requirements. Note also that at every stage, a_e has exactly 2 more children than b_e.

The strategy has infinitely many outcomes: ∞ and $\texttt{fin}_k$ for $k \in \omega$. Every time we reach Step (4), we take outcome ∞ for a single stage. At all other stages, we take outcome $\texttt{fin}_k$, where k is the number of previous stages at which we have taken outcome ∞.

Strategy for Q_j

Suppose σ is a strategy for Q_j. This strategy will also appear on the priority tree. When σ is visited, s is always the current stage and $t < s$ is the last stage at which the strategy took outcome ∞ (or $t = 0$ if σ has never before taken outcome ∞). Throughout the construction, we maintain a parameter d_σ for σ: if d_σ is undefined at the beginning of the stage s, set $d_\sigma = s - 1$ and keep it until the construction sets d_σ to be undefined again.

We construct the isomorphism on c_e and its descendants independently of the isomorphism for all the other $c_{e'}$'s. We begin by searching for a tuple $(r, c, m, n) \in M_j$ with

$$r \lhd_{M_j} c \lhd_{M_j} m \lhd_{M_j} n,$$

and n having $e+1$ many children. When we find such a tuple, we map c_e to c; m_e to m; n_e to n; and the children of n_e to the children of n. Of course, we may later see that the $(e+2)$nd child of n_e appear, in which case we have made a mistake. If this happens, we will discard our mapping and begin again. If $M_j \cong \mathcal{T}$, eventually the tuple in M_j that respects the isomorphism is the Gödel least satisfying the above, and so we will define the correct mapping. The oracle $\mathbf{0}'$ will be able to predict our mistakes, and so can ignore all mappings before the correct one.

Under the assumption that we have correctly mapped c_e, we must map a_e and b_e. This part will not rely on the oracle. To map a_e and b_e to elements a, b, first wait for such a stage s for σ, where $d_\sigma \geq e$. At each such stage s, make one step trying to find a and b as described below.

If e has not been chosen by an R_i-strategy by this point, we know by construction that it will be never chosen. In this case, we search for an $a \rhd_{M_j} c$ such that a has two children and map a_e to a. We then search for any child $b \rhd_{M_j} c$ incomparable with m or a, and map b_e to b.

If e has been chosen by an R_i-strategy, and that strategy is incomparable with σ on the tree, then, under the assumption that σ is along the true path, the strategy that chose e will never be visited again. So, let p^e be the number of children of a_e. We search for an $a \rhd_{M_j} c$ such that a has p^e children, and map a_e to a. We then search for any $b \rhd_{M_j} c$, which is incomparable with m and a, and, in case $p^e > 2$, itself has children, and map b_e to b.

If e has been chosen by an R_i-strategy τ with $\tau\widehat{\ }\infty \subseteq \sigma$, then, under the assumption that σ is along the true path, a_e and b_e are automorphic. So, we search for any $a, b \rhd_{M_j} c$ incomparable with m and having children, and map a_e to a and b_e to b.

If e has been chosen by an R_i-strategy τ with $\tau\widehat{\ }\texttt{fin}_k \subseteq \sigma$, then, under the assumption that σ is along the true path, a_e and b_e will never gain any more children. So let p^e be the number of children of a_e. We search for an $a \rhd_{M_j} c$ such that a has p^e children, and map a_e to a. We then search for any $b \rhd_{M_j} c$ which is incomparable with m, a, and, in case $p^e > 2$, itself has children, and map b_e to b.

If e has been chosen by an R_i-strategy τ with $\sigma^\frown \mathtt{fin}_k \subseteq \tau$, then we wait until a stage t when σ is accessible and $t > e$. At this stage, we know that τ will never again be accessible (since τ was visited before t, the node σ had taken the outcome ∞ at least k times strictly before t, so at least $k+1$ times by any stage after t, so any future outcomes of σ must be ∞ or $\mathtt{fin}_{k'}$ for $k' > k$). So let p^e be the number of children of a_e. We search for an $a \rhd_{M_j} c$ such that a has p^e children, and map a_e to a. We then search for any $b \rhd_{M_j} c$, which is incomparable with m, a, and, in case $p^e > 2$, has children, and map b_e to b.

If e has been chosen by an R_i-strategy τ with $\sigma^\frown\infty \subseteq \tau$, then let p^e_s be the number of children on a_e at the beginning of stage s. We search for an $a \rhd_{M_j} c$ such that a has p^e_s children, and map a_e to a. We then search for any $b \rhd_{M_j} c$, which is incomparable with m, a, and, in case $p^e_s > 2$, has children, and map b_e to b. Note that, unlike in the other cases, p^e_s may change, which is why we have subscripted it with the stage number.

The strategy has infinitely many outcomes: ∞ and $\mathtt{fin}_k$ for $k \in \omega$. At stage s, if for every $e < d_\sigma$, the isomorphism is defined on a_e, which has been chosen by a τ extending $\sigma^\frown\infty$, and further the image of a_e in M_j has p^e_s many children for every such e, then we take outcome ∞ and make d_σ undefined. Otherwise, we take outcome $\mathtt{fin}_k$, where k is the number of previous stages at which we have taken outcome ∞. We also keep the value for d_σ.

Construction

Arrange the strategies on a tree in some effective fashion, and at every stage, allow strategies to be visited according to the outcomes of the previous strategies at that stage, in the usual fashion.

Verification

Define the true path in the usual fashion for a $\mathbf{0}''$ priority construction.

Lemma 1. *Suppose that τ is an R_i-strategy along the true path. Then τ ensures that R_i is satisfied.*

Proof. Since τ is along the true path, it is visited infinitely often. We have 2 cases to consider.

Case 1. There is some $\phi(\overline{x}) \in \mathcal{X}_i$ such that $\mathcal{T} \models \phi(a_e, \overline{p}_i) \wedge \phi(b_e, \overline{p}_i)$. It means there is some u and some $c \in |\mathcal{T}|$ such that $\mathcal{T} \models \psi_u(a_e, \overline{c}, \overline{p}_i) \wedge \psi_u(b_e, \overline{c}, \overline{p}_i)$. Choose the least $\phi(\overline{x})$ satisfying this property for the least such u. Then for any $\theta(\overline{x}) \in \mathcal{X}_i$, which is not one of the first $u+1$ elements of $\mathcal{X}_i$, we know that τ will never choose $\theta(\overline{x})$ because it will always prefer $\phi(\overline{x})$.

Thus, if τ were to take outcome ∞ infinitely many times, by the pigeonhole principle, it would choose one of the first $u+1$ elements of $\mathcal{X}_i$ infinitely many times. However, if there are infinitely many r with $\mathcal{T}_r \not\models \theta(a_e, \overline{p}_i) \wedge \theta(b_e, \overline{p})$, then eventually τ will prefer ϕ over θ, and so will stop choosing θ. Since ϕ was chosen to be the least such, it will eventually be preferred to every other formula, but then once that occurs, we will never again reach Stage 4. Therefore, τ cannot have outcome ∞ infinitely often. Hence τ has true outcome $\mathtt{fin}_k$ for some k, and a_e and b_e have different finite numbers of children. This means that a_e and b_e are not automorphic, so ϕ witnesses the failure of $(\mathcal{X}_i, \overline{p}_i)$ as a Scott family.

Case 2. There is no $\phi(\overline{x}) \in \mathcal{X}_i$ such that $\mathcal{T} \models \phi(a_e, \overline{p}_i) \wedge \phi(b_e, \overline{p}_i)$. Then for any ϕ, for any u and any $\overline{c}$, there always exists a stage $r > u$ such that $\mathcal{T}_r \not\models \psi^r_u(a_e, \overline{c}, \overline{p}_i) \wedge \psi^r_u(b_e, \overline{c}, \overline{p}_i)$. Thus, with any chosen ϕ we eventually reach Step (3), and a_e and b_e have infinitely many children. Hence, a_e and b_e will be automorphic, and, in particular, there will be an automorphism permuting a_e and b_e and fixing $\overline{p}_i$ pointwise. Thus, for any ϕ with $\mathcal{T} \models \phi(a_e, \overline{p}_i)$, we know that $\mathcal{T} \models \phi(b_e, \overline{p}_i)$. Hence there can be no $\phi \in \mathcal{X}_i$, so that $\mathcal{T} \models \phi(a_e, \overline{p}_i)$, and hence $\mathcal{X}_i$ fails to be a Scott family. $\square$

Lemma 2. *Suppose that σ is a Q_j-strategy along the true path, that $M_j \cong \mathcal{T}$, and e is chosen by some $\tau \supseteq \sigma^\frown\infty$. Then σ eventually correctly maps a_e and b_e.*

Proof. Certainly, σ eventually correctly maps c_e and m_e, and defines some map for a_e and b_e. If τ has true outcome ∞, then a_e and b_e are automorphic, so this is a correct map.

Suppose, instead, that τ has true outcome $\mathtt{fin}_k$ (thus a_e has $k+2$ children, and b_e has k children). Let s_0 be the stage at which σ correctly maps c_e, and let t_0 be the final stage at which τ takes outcome ∞. Suppose that $s_0 > t_0$. Then at the stage s_0, the node σ searches for an $a \rhd_{M_j} c$ with $p^e_{s_0} = k+2$ children, and maps a_e to a. By assumption, a_e never gains any more children, so, since $M_j \cong \mathcal{T}$, the correct image of a_e is the only such child of c. The element b_e is correctly mapped by elimination.

If, instead, $s_0 \leq t_0$, then let a be the element to which σ has mapped a_e at the stage t_0. (Such an element must exist because σ must have taken outcome ∞ at the stage t_0.) Since a_e can gain at most one child during the stage t_0, and will gain no children after the stage t_0, it has at least $k+1$ children at the start of the stage t_0. Since σ has outcome ∞ at the stage t_0, a has at least $p^e_{t_0} = k+1$ children. Since $M_j \cong \mathcal{T}$, the correct image of a_e is the only child of c with at least $k+1$ children, so a_e is correctly mapped. The element b_e is correctly mapped by elimination. $\square$

Lemma 3. *Suppose that σ is a Q_j-strategy along the true path, and that $M_j \cong \mathcal{T}$. Then σ has true outcome ∞.*

Proof. Suppose otherwise. Let t_0 be the final stage at which σ takes the outcome ∞. Then there are only finitely many e that are chosen by strategies extending $\sigma^\frown\infty$, and, by Lemma 2, σ eventually correctly maps a_e for each of these e's. Since $M_j \cong \mathcal{T}$, σ eventually sees $p^e_{t_0}$ many children below the target of a_e for each e, and so σ will take outcome ∞ at some stage after t_0, contrary to our assumption. $\square$

Lemma 4. *If $M_j \cong \mathcal{T}$, then there is a Δ^0_2 isomorphism between M_j and $\mathcal{T}$.*

Proof. Non-uniformly fix σ that is the Q_j-strategy along the true path. As argued before, σ eventually correctly maps every c_e and m_e, and $\mathbf{0}'$ can determine when this occurs. By Lemma 2, or by the description of σ's action, σ correctly maps

a_e and b_e, once c_e has been correctly mapped. The only new ingredient is the observation that since σ has true outcome ∞, there is eventually a stage s with $t > e$, thus treating those e's chosen by strategies extending $\sigma^\frown\mathtt{fin}_k$.

Once a_e and b_e are mapped, their children can be mapped by a simple back-and-forth argument. Thus $\mathbf{0}'$ can build an isomorphism. □

This completes the proof. Note that every step we have described above can be performed equally well for partial orders and directed graphs. □

We can modify the construction in the proof of the previous theorem to make the tree have infinite height by extending every child of a_e, b_e and n_e to an infinite non-branching path. Once a_e, b_e and n_e are correctly mapped, we then need to use the $\mathbf{0}'$-oracle to correctly map their descendants. Hence we have the following result.

Theorem 4. *There is a computable Δ^0_2-categorical tree of infinite height, which is not relatively Δ^0_2-categorical.*

4. Δ^0_2-categorical but not relatively Δ^0_2-categorical abelian groups

We will now consider certain torsion-free abelian groups. A *homogenous, completely decomposable, abelian group* is a group of the form $\bigoplus_{i\in\kappa} H$, where H is a subgroup of the additive group of the rationals, $(\mathbb{Q},+)$. Note that we have only a single H in the sum – any two summands are isomorphic. It is well known that such a group is computably categorical if and only if κ is finite; the proof is similar to the analogous result that a computable vector space is computably categorical if and only if it has finite dimension. In the remainder of this section, we will restrict our attention to groups of infinite rank κ.

For P a set of primes, define $Q^{(P)}$ to be the subgroup of $(\mathbb{Q},+)$ generated by $\{\frac{1}{p^k} : p \in P \wedge k \in \omega\}$. Downey and Melnikov [15] showed that a computable, homogenous, completely decomposable, abelian group of infinite rank is Δ^0_2-categorical if and only if it is isomorphic to $\bigoplus_\omega Q^{(P)}$, where P is c.e. and the set (Primes $-$ P) is semi-low. Recall that a set $S \subseteq \omega$ is *semi-low* if the set $H_S = \{e : W_e \cap S \neq \emptyset\}$ is computable from $\emptyset'$. Here, we will first fully characterize the computable, relatively Δ^0_2-categorical, homogenous, completely decomposable, abelian groups of infinite rank.

Theorem 5. *A computable, homogenous, completely decomposable, abelian group of infinite rank is relatively Δ^0_2-categorical if and only if it is isomorphic to $\bigoplus_\omega Q^{(P)}$, where P is a computable set of primes.*

Proof. Suppose that G is relatively Δ^0_2-categorical. Since this implies that G is Δ^0_2-categorical, by the above mentioned result of Downey and Melnikov, we know that $G \cong \bigoplus_\omega Q^{(P)}$ for P a c.e. set of primes. We will show that P is also co-c.e.

Fix $\mathcal{X}$, a c.e. Scott family of computable Σ_2 formulas for G, with parameters $\overline{a} \in G^{<\omega}$. By definition, any element of G has all but finitely many coordinates equal to 0. Choose $l \in \omega$ to be a coordinate which equals to 0 for every element of $\overline{a}$. Fix an element $b \in G$ such that the only non-zero coordinate of b is l. Then b is independent of $\overline{a}$. The map $b \mapsto p \cdot b$ can be extended to an automorphism of G fixing $\overline{a}$ if and only if $p \in P$. Fix some formula $\exists \overline{x}\, \theta(\overline{z}, \overline{x}, y) \in \mathcal{X}$, where θ is a computable Π_1 formula and $G \models \exists \overline{x}\, \theta(\overline{a}, \overline{x}, b)$. Fix some tuple $\overline{c} \in G$ such that $G \models \theta(\overline{a}, \overline{c}, b)$.

Now, decompose the elements of $\overline{c}$ as $c_i = d_i + e_i$, where d_i is a rational multiple of b, and b is independent of $\{\overline{a}, \overline{e}\}$. Observe that the map $b \mapsto p \cdot b$ can be extended to an automorphism of G fixing $\overline{a}$ and $\overline{e}$ if and only if $p \in P$, and any such isomorphism would need to map $d_i \mapsto p \cdot d_i$.

Define $\overline{c}^p$ by $c_i^p = p \cdot d_i + e_i$. Note that an isomorphism sending $b \mapsto p \cdot b$ and fixing $\overline{a}$ and $\overline{e}$ would necessarily map $\overline{c} \mapsto \overline{c}^p$. Thus, if there is such an isomorphism, then $G \models \theta(\overline{a}, \overline{c}^p, p \cdot b)$. Conversely, if $G \models \theta(\overline{a}, \overline{c}^p, p \cdot b)$ then $G \models \exists \overline{x}\, \theta(\overline{a}, \overline{x}, p \cdot b)$, and, by the definition of a Scott family, there must be an isomorphism fixing $\overline{a}$ and mapping $b \mapsto p \cdot b$. Thus,

$$p \in P \Leftrightarrow G \models \theta(\overline{a}, \overline{c}^p, p \cdot b).$$

Since θ is a computable Π_1 formula, and $\overline{c}^p$ can be obtained effectively from p, it follows that P is co-c.e. □

Since there exist co-c.e. sets that are semi-low and noncomputable, we obtain the following result.

Corollary 1. *There is a computable, homogenous, completely decomposable, abelian group, which is Δ_2^0-categorical but not relatively Δ_2^0-categorical.*

5. Degrees of categoricity of abelian p-groups and Boolean algebras

While Δ_n^0-categoricity provides an upper bound on the complexity of isomorphisms between computable copies of a structure, the degrees of categoricity, when they exist, give the exact level of categoricity. Examples of Fokina, Kalimullin, and R. Miller [18], as well as of Csima, Franklin, and Shore [13] showed that for every computable successor ordinal α, every d-c.e. degree in and above $\mathbf{0}^{(\alpha)}$ is the degree of categoricity of some structure. However, for many natural classes, the only degrees of categoricity can be the jumps of $\mathbf{0}$. Cenzer, Harizanov, and Remmel established in [10] that the degrees of categoricity of computable injections structures can only be $\mathbf{0}$, $\mathbf{0}'$ and $\mathbf{0}''$. If follows from results by Calvert, Cenzer, Harizanov, and Morozov [9] that every relatively Δ_2^0-categorical equivalence structure has a degree of categoricity that can only be either $\mathbf{0}$ or $\mathbf{0}'$. Csima and Ng recently announced that the degree of categoricity of an arbitrary computable equivalence structure can only be $\mathbf{0}$, $\mathbf{0}'$, or $\mathbf{0}''$. Frolov [20] showed that the degrees of categoricity of relatively Δ_2^0-categorical linear orders can only be $\mathbf{0}$ and $\mathbf{0}'$. Bazhenov [8] established that the degrees of categoricity of relatively Δ_2^0-categorical (equivalently, Δ_2^0-categorical) Boolean algebras can only be $\mathbf{0}$ and $\mathbf{0}'$. In this section, we prove an analogous result for relatively Δ_2^0-categorical abelian p-groups. We also extend Bazhenov's to relatively Δ_3^0-categorical Boolean algebras.

We will now focus on relatively Δ_2^0-categorical abelian p-groups, where p is a prime number. A group G is called a *p-group* if for all $g \in G$, the order of g is a power of p. By $\mathbb{Z}(p^n)$ we denote the cyclic group of order p^n. By $\mathbb{Z}(p^\infty)$ we denote the quasicyclic (Prüfer) abelian p-group, the direct limit of the sequence $\mathbb{Z}(p^n)$, and also the set of rationals in $[0, 1)$ of the form $\frac{i}{p^n}$ with addition modulo 1. The *length* of an abelian p-group G, $\lambda(G)$, is the least ordinal α such that $p^{\alpha+1}G = p^\alpha G$. Here, $p^0 G = G$, $p^{\alpha+1}G = p(p^\alpha G)$, and $p^\lambda G = \bigcap_{\alpha<\lambda} p^\alpha G$ for limit λ. The divisible part of G, $Div(G)$, is $p^{\lambda(G)}G$ and is a direct summand of G. The group G is said to be *reduced* if $Div(G) = \{0\}$. For an element $g \in G$, the *height* of g, $ht(g)$, is ∞ if $g \in Div(G)$, and is otherwise the least α such that $g \notin p^{\alpha+1}G$. For a computable group G, $ht(g)$ can be an arbitrary computable ordinal. The height of G is the supremum of $\{ht(g) : g \in G\}$. Let $o_G(g)$ be the order of g in G. The *period* of G is $max\{o(g) : g \in G\}$ if this number is finite, and is ∞ otherwise.

Goncharov [21] and Smith [43] independently characterized computably categorical abelian p-groups as those that can be written in one of the following two forms:

(a) $\bigoplus_l \mathbb{Z}(p^\infty) \oplus F$, where $l \leq \omega$ and F is a finite group; or

(b) $\bigoplus_n \mathbb{Z}(p^\infty) \oplus H \oplus \bigoplus_\omega \mathbb{Z}(p^k)$, where $n, k \in \omega$ and H is a finite group.

For these groups, computable categoricity and relative computable categoricity coincide (for a proof see also [7]).

In [7], Calvert, Cenzer, Harizanov, and Morozov established that a computable abelian p-group G is relatively Δ^0_2-categorical if and only if:

(i) G is isomorphic to $\bigoplus_l \mathbb{Z}(p^\infty) \oplus H$, where $l \leq \omega$ and H has finite period; or

(ii) All elements in G are of finite height (equivalently, G is reduced with $\lambda(G) \leq \omega$).

Using these two characterizations, it is easy to see the following.

Proposition 1. *A computable abelian p-group G is relatively Δ^0_2-categorical but not computably categorical if and only if it has one of the following two forms:*

(1) $\bigoplus_\omega \mathbb{Z}(p^k) \oplus \bigoplus_\omega \mathbb{Z}(p^m) \oplus \bigoplus_l \mathbb{Z}(p^\infty) \oplus H$, *where* $0 < k < m \leq \omega$, $0 \leq l < \omega$, *and H is of finite period; or*

(2) Every element of G has finite height, but G contains elements of arbitrarily large finite heights.

Theorem 6. *The categoricity degrees of computable relatively Δ^0_2-categorical abelian p-groups can only be* $\mathbf{0}$ *and* $\mathbf{0}'$.

Proof. Obviously, the degree of categoricity of computably categorical abelian p-groups is $\mathbf{0}$. Now, suppose that G is a computable abelian p-group, which is relatively Δ^0_2-categorical but not computably categorical. We will show that G has the degree of categoricity $\mathbf{0}'$.

We use the characterization of relatively Δ^0_2-categorical but not computably categorical abelian p-groups from Proposition 1. We will handle the two cases separately. In what follows, G_s is a finite part of G at stage s obtained in a computable way. We also fix the modulus function μ of $\emptyset'$ for the rest of the proof.

First Case. Consider elements $x \in G$ with $x \neq 0$, $p \cdot x = 0$ and $ht(x) = k-1$. Note that $\mathbb{Z}(p^k)$ contains such an element (indeed, $p-1$ such elements). By the observation that $G \cong \bigoplus_{\omega} \mathbb{Z}(p^k) \oplus G$, we may assume that we have an effective enumeration $(a_n)_{n \in \omega}$ of infinitely many elements of this sort.

We will build a second computable copy A such that the first $\mu(n)$ elements of A contain at most n elements of the desired sort. Then, given any isomorphism $f : G \cong A$, the function $n \mapsto f(a_n)$ would necessarily dominate μ. Thus, any isomorphism from G to A would compute $\emptyset'$.

The construction is now straightforward. By $dom(h)$ we denote the domain and by $ran(h)$ the range of a function h. We will build a Δ^0_2 homomorphism $h : G \cong A$ and arrange that $A = ran(h) \oplus \bigoplus_{\omega} \mathbb{Z}(p^m)$. We begin with $h_0 = \emptyset$.

At stage $s+1$, for every $n \leq s$, we consider every $x \in G$ with $n \leq x \leq s$, $x \neq 0$, $p \cdot x = 0$ and $[ht(x)]^{G_s} < k$. For each such element, if $h_s(x) \leq \mu_s(n)$, we define $h_{s+1}(x)$ as some new large element. This requires that we also define $h_{s+1}(y)$ for every y dividing such an x, to be some new large element. We let $h_{s+1}(x) = h_s(x)$ for every other x. We then extend the domain of h_{s+1} to the next element of G. We let h_{s+1} induce the group operation on its range *via* pull-back.

Let $D_{s+1} = ran(h_s) - ran(h_{s+1})$. Note that every element of D_s has height less than k. We add new elements to extend D_{s+1} to a copy of $\bigoplus_{l} \mathbb{Z}(p^m)$ for some $l < \omega$. Also, for every $a \in A_{s+1} - ran(h_{s+1})$ and every $b \in ran(h_{s+1})$, if A does not yet have an element corresponding to $a+b$, we add an appropriate element now. This completes the stage $s+1$.

Now we argue by induction that h is a total Δ^0_2 function. To start, fix $x \in G$ with $x \neq 0$ and $p \cdot x = 0$. If $h_{s+1}(x) \neq h_s(x)$, then either our construction was deliberately redefining $h(x)$, or it was required to redefine $h(x)$ because it deliberately redefined $h(z)$ for some z that x divides. The only such z's are of the form $q \cdot x$ for $1 \leq q < p$. Let s_0 be such that $\mu_{s_0}(q \cdot x) = \mu(q \cdot x)$ for $1 \leq q < p$. Then at any stage $s > s_0$ with $h_{s+1}(x) \neq h_s(x)$, we have $h_{s+1}(q \cdot x) > \mu_s(q \cdot x) = \mu(q \cdot x)$ since $h_{s+1}(q \cdot x)$ is chosen to be large. Then at any stage $t > s$, $h_t(q \cdot x) > \mu(q \cdot x) = \mu_t(q \cdot x)$, and so we will have $h_{t+1}(x) = h_t(x)$, and hence $h(x)$ will reach a limit.

Now, consider $y \in G$ with $p^{i+1} \cdot y = 0$ for $1 \leq i < \omega$. Then $p \cdot (p^i \cdot y) = 0$, and $h_{s+1}(y) \neq h_s(y)$ only when $h_{s+1}(p^i \cdot y) \neq h_s(p^i \cdot y)$. Since we have just argued that $h(p^i \cdot y)$ reaches a limit, it follows that $h(y)$ reaches a limit.

Note that $A = ran(h) \oplus \bigoplus_{\omega} \mathbb{Z}(p^m)$ by construction. It follows that $A \cong G$. It also follows that every $x \in A - ran(h)$ with $p \cdot x = 0$ has height at least $m - 1 \geq k$. Finally, our construction ensures that there are at most n elements $x \in G$ with $p \cdot x = 0$, $ht(x) < k$ and $h(x) < \mu(n)$. Thus, there are at most n elements $x \in A$ with $p \cdot x = 0$, $ht(x) < k$ and $x < \mu(n)$, as desired.

Second Case. By a result of Khisamiev [31] and independently of Ash, Knight and Oates [5], we know that

$$G \cong \mathbb{Z}(p^{k_0}) \oplus \mathbb{Z}(p^{k_1}) \oplus \cdots,$$

where the sequence $(k_i)_{i \in \omega}$ is uniformly computable from below. That is, there is a computable function $g : \omega \times \omega \to \omega$ such that for all i and s, $g(i,s) \leq g(i,s+1)$, and for all i, $k_i = \lim_s g(i,s)$. Fix such a function g. By our assumptions about G, we know that the k_i's are unbounded.

We will construct a computable function h and a Δ^0_2 function ι such that:

(1) For all i and s, $h(i,s) \leq h(i,s+1)$;
(2) $\iota : \omega \to \omega$ is a bijection;
(3) For all i, $\lim_s h(i,s) = \lim_s g(\iota(i),s)$; and
(4) For all n and all $x \in G$ with $x < \mu(n)$ and $x \neq 0$, we have $ht(x)+1 < \lim_s h(2n,s)$ (recall that μ is the modulus function for $\emptyset'$).

We will then let $A = \mathbb{Z}(p^{\lim_s h(0,s)}) \oplus \mathbb{Z}(p^{\lim_s h(1,s)}) \oplus \cdots$ By the first property above, this is a computable structure. By the second and the third properties, $A \cong G$. By the fourth property, given an isomorphism $f : A \cong G$, for any element x of the $(2n)$th summand of A with $x \neq 0$ and $p \cdot x = 0$, it must be that $f(x) \geq \mu(n)$. Thus, f computes $\emptyset'$.

It remains to construct h and ι. We begin with $\iota_0 = \emptyset$ and $h(i,0) = 0$ for all i.

At stage $s+1$, if there is an n with $2n \in dom(\iota_s)$ and an $x \in G$ with $x < \mu(n)$, $x \neq 0$ and $[ht(x)]^{G_s} \geq h(2n,s)$, we search for a large pair (j,t) with $g(j,t) > h(2n,s)$, and define $\iota_{s+1}(2n) = j$ and $h(2n,s+1) = g(j,t)$. We then choose a large m and define $\iota_{s+1}(2m+1) = \iota_s(2n)$. We let $\iota_{s+1}(k) = \iota_s(k)$ for every other k.

We then choose the least $a \notin dom(\iota_{s+1})$ and the least $b \notin ran(\iota_{s+1})$, and define $\iota_{s+1}(a) = b$. Then, for every $i \in dom(\iota_{s+1})$ with $h(i,s+1)$ not yet defined, we define $h(i,s+1) = \max\{g(\iota_{s+1}(i),s+1), h(i,s)\}$. For every $i \notin dom(\iota_{s+1})$, we define $h(i,s+1) = 0$. This completes the stage $s+1$.

First, note that, by construction, $h(i,s) \leq h(i,s+1)$ for every i and s.

Next, we argue that ι is a total Δ^0_2 function. Note that, by construction, for every i, there is eventually a stage s_0 such that $\iota_s(i)$ is defined for all $s \geq s_0$. If i is odd, then $\iota_s(i) = \iota_{s_0}(i)$ for all $s \geq s_0$. If, instead, $i = 2n$, then at every stage s with $\iota_s(i) \neq \iota_s(i+1)$, we have $h(i,s+1) \geq h(i,s)+1$. Let $u = \max\{ht(x) : x \in G \wedge x < \mu(n)\}$. Thus, for sufficiently large s_1, $h(i,s_1) > u$, and then $h(i,s) = h(i,s_1)$ for all $s \geq s_1$.

Next, we argue that ι is surjective. If $b = \iota_{s_0}(a)$, then either $b = \iota_s(a)$ for all $s > s_0$, or there is a stage $s_1 > s_0$ with $b = \iota_{s_1}(c)$ for some odd c. By construction, ι never changes on odd inputs, so $b = \iota_s(c)$ for all $s \geq s_1$. By construction, every element is eventually added to the range of some ι_s, so every element is in $ran(\iota)$.

By induction on s, $h(i,s) \leq \lim_s g(\iota_s(i),s)$ for all i and s, and so, in particular, $\lim_s h(i,s)$ exists and equals at most $\lim_s g(\iota(i),s)$. On the other hand, $h(i,s) \geq g(\iota_s(i),s)$ for all i and s, by construction, and so $\lim_s h(i,s) = \lim_s g(\iota(i),s)$, as desired.

Finally, for all n and all $x \in G$ with $x < \mu(n)$ and $x \neq 0$, we have $ht(x)+1 < \lim_s h(2n,s)$, since we deliberately increase $h(2n,s)$ whenever this appears to be false. This completes the proof. □

We now turn to relatively Δ^0_3-categorical Boolean algebras. A Boolean algebra $\mathcal{B}$ is *atomic* if for every $a \in \mathcal{B}$, there is an atom $b \leq a$. An equivalence relation $\sim$ on a Boolean algebra $\mathcal{A}$ is defined by:

$$a \sim b \text{ iff each of } a \cap \overline{b} \text{ and } b \cap \overline{a} \text{ is } \emptyset \text{ or a union of finitely many atoms of } \mathcal{A}.$$

A Boolean algebra $\mathcal{A}$ is a 1-*atom* if $\mathcal{A}/\sim$ is a two-element algebra. A Boolean algebra $\mathcal{A}$ is *rank* 1 if $\mathcal{A}/\sim$ is a nontrivial atomless Boolean algebra. McCoy [36] proved that a countable rank 1 atomic Boolean algebra is isomorphic to the interval algebra $I(2 \cdot \eta)$.

Goncharov and Dzgoev [24], and independently Remmel [41] and LaRoche [33] established that a computable Boolean algebra is computably categorical (also, relatively computably categorical) if and only if it has finitely many atoms. In [35], McCoy established that a Boolean algebra is relatively Δ^0_2-categorical if and only if it is a finite direct sum of algebras that are atoms, atomless, or 1-atoms. Furthermore, in [36], McCoy characterized relatively Δ^0_3-categorical Boolean algebras as those computable Boolean algebras that can be expressed as finite direct sums of algebras that are atoms, atomless, 1-atoms, rank 1 atomic, or isomorphic to the interval algebra $I(\omega + \eta)$. In our next theorem, we will use this characterization.

Theorem 7. *The degrees of categoricity of relatively Δ^0_3-categorical Boolean algebras can only be $\mathbf{0}$, $\mathbf{0}'$ and $\mathbf{0}''$.*

Proof. Fix a relatively Δ^0_3-categorical Boolean algebra $\mathcal{B}$. If $\mathcal{B}$ is a finite join of atoms, 1-atoms, and atomless Boolean algebras, then $\mathcal{B}$ is relatively Δ^0_2-categorical, and so its degree of categoricity is either $\mathbf{0}$ or $\mathbf{0}'$ by Bazhenov's result in [8]. Otherwise, $\mathcal{B}$ has a summand that is either rank 1 atomic or isomorphic to the interval algebra $I(\omega + \eta)$.

All of the potential summands in the characterization of relatively Δ^0_3-categorical Boolean algebras have computable isomorphic copies in which the set of finite elements (that is, the elements a with $a \sim 0$) is computable. We will show that both the rank 1 atomic algebra and $I(\omega+\eta)$ have computable isomorphic copies in which the set of finite elements is Σ^0_2-complete. It will follow that $\mathcal{B}$ has a computable isomorphic copy in which the set of finite elements is computable, and another computable isomorphic copy in which it is Σ^0_2-complete, and so any isomorphism between these two copies will compute $\emptyset''$.

We begin with the rank 1 atomic algebra. Let $\mathcal{C}$ be a computable copy of this algebra in which the set of atoms is computable. Let $\{a_i : i \in \omega\}$ be the set of all atoms of $\mathcal{C}$. We will create an algebra $\mathcal{A}$ by extending $\mathcal{C}$. Let $\phi(i,x)$ be a computable formula such that

$$i \in \emptyset'' \Leftrightarrow \exists^{<\infty} x\ \phi(i,x).$$

At every step s, we will consider whether $\phi(i,s)$ holds. The first time $\phi(i,s)$ holds, we choose three large elements b^0_i, b^1_i and b^2_i and use them to partition a_i into three pieces. That is,

$$b^0_i \wedge b^1_i = b^1_i \wedge b^2_i = b^2_i \wedge b^0_i = 0$$

and

$$b^0_i \vee b^1_i \vee b^2_i = a_i.$$

At the second step at which we see $\phi(i,s)$ hold, we repeat the process on b^0_i and b^2_i. See the following diagrams.

We then let $\mathcal{A}$ be the Boolean algebra generated by $\mathcal{C}$ along with these newly added elements. Note that every element of $\mathcal{A}$ is the join of an element from $\mathcal{C}$ and some of these new elements (among b^σ_i's). That is, for all $d \in \mathcal{A}$, $d = c \vee b^{\sigma_0}_{i_0} \vee b^{\sigma_1}_{i_1} \vee \cdots \vee b^{\sigma_k}_{i_k}$ for some $c \in \mathcal{C}$ and some $b^{\sigma_0}_{i_0}, \ldots, b^{\sigma_k}_{i_k}$.

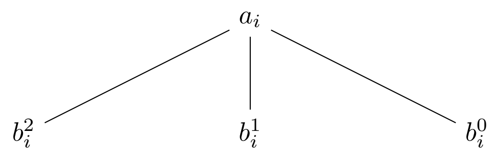

FIGURE 2. Working with rank 1 atomic, the first time we see $\phi(i,s)$ hold.

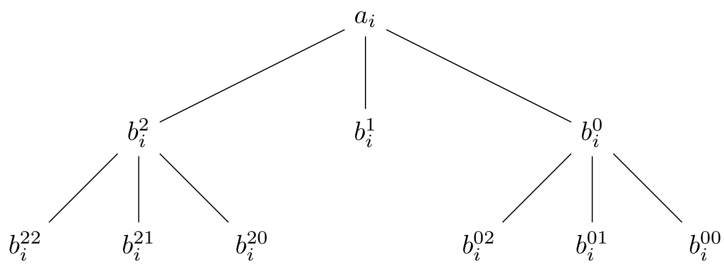

FIGURE 3. Working with rank 1 atomic, the second time we see $\phi(i,s)$ hold.

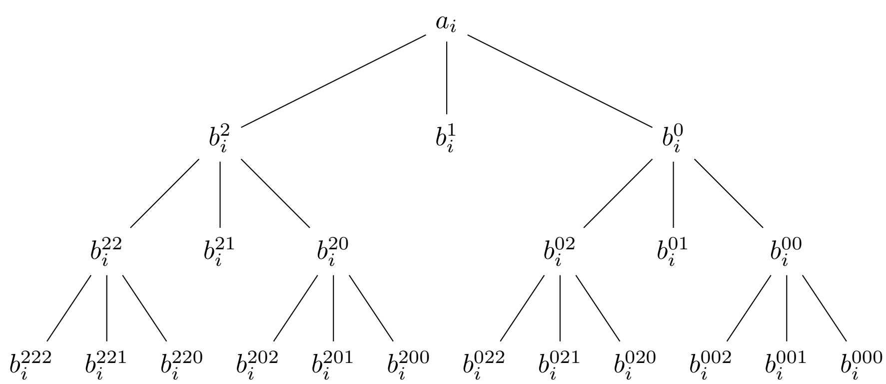

FIGURE 4. Working with rank 1 atomic, the third time we see $\phi(i,s)$ hold.

Observe that a_i is infinite in $\mathcal{A}$ if and only if $\phi(i,x)$ holds for infinitely many x, which is if and only if $i \notin \emptyset''$. Also, a_i necessarily bounds an atom in $\mathcal{A}$, e.g., b_i^1. Finally, if a_i is infinite, then it can be partitioned into two infinite elements, e.g., b_i^0 and $b_i^1 \vee b_i^2$. Since every element of $\mathcal{C}$ bounds an atom, and every infinite element of $\mathcal{C}$ can be partitioned into two infinite elements, it follows that the same holds for every element of $\mathcal{A}$. This characterizes the rank 1 atomic algebra. Thus $\mathcal{A} \cong \mathcal{C}$, and $\mathcal{A}$ is as desired.

Next, consider $I(\omega+\eta)$. Again, let $\mathcal{C}$ be a computable copy of $I(\omega+\eta)$ in which the set of atoms is computable. Let $\{a_i : i \in \omega\}$ be the set of all atoms of $\mathcal{C}$. We again create $\mathcal{A}$ extending $\mathcal{C}$. Let $\phi(i,x)$ be as before. At every step s, if $\phi(i,s)$ holds, we add new elements below a_i. The first time $\phi(i,s)$ holds, we partition $a_i = b_i^0 \vee b_i^1$. The second time it holds, we partition b_i^0 and b_i^1. See the diagrams.

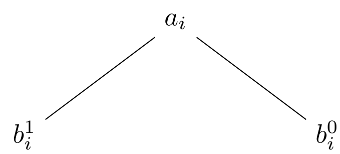

FIGURE 5. Working with $I(\omega+\eta)$, the first time we see $\phi(i,s)$ hold.

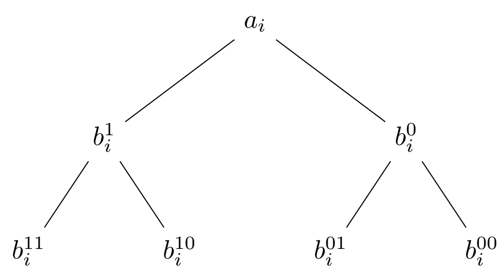

FIGURE 6. Working with $I(\omega+\eta)$, the second time we see $\phi(i,s)$ hold.

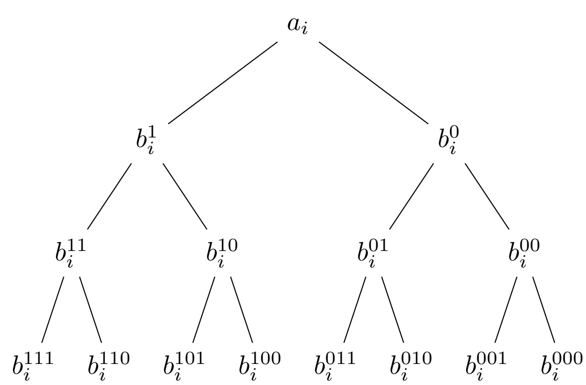

FIGURE 7. Working with $I(\omega+\eta)$, the third time we see $\phi(i,s)$ hold.

We again let $\mathcal{A}$ be the Boolean algebra generated by $\mathcal{C}$ along with these new elements. The isomorphism type of $I(\omega+\eta)$ is characterized by three properties: there are infinitely many atoms; any element that bounds infinitely many atoms also bounds an atomless element; and no two disjoint elements both bound infinitely many atoms. Since every atom of $\mathcal{A}$ is bounded by an atom of $\mathcal{C}$, every atomless element of $\mathcal{C}$ is still atomless in $\mathcal{A}$, and every atom of $\mathcal{C}$ is either atomless or finite in $\mathcal{A}$, the second and the third properties are inherited from $\mathcal{C}$ to $\mathcal{A}$. Meanwhile, the first property is ensured by the fact that $\emptyset''$ is infinite. Thus $\mathcal{A} \cong \mathcal{C}$.

This completes the proof. $\square$

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