GROUPS WITH ORDERINGS OF ARBITRARY ALGORITHMIC COMPLEXITY

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We give general sufficient conditions that a computable group admit bi-orderings of arbitrary computability-theoretic complexity in a strong sense. We apply this result to show that a large class of computable, finitely presented, residually nilpotent groups admit bi-orderings in every *truth-table degree*, a refinement of the Turing degrees. This class includes a wide variety of important groups such as finitely generated free groups, surface groups and certain nilpotent groups.

1. Introduction

Ordered algebraic structures are ubiquitous in mathematics and have been studied since Dedekind, Hölder, and Hilbert, and it is important to understand their constructive properties. Here, we consider linear orderings of the elements of a given group that respect the algebraic structure and study the algorithmic complexity of the admitted orderings.

A linear ordering of the elements of a group, $\prec$, is a *left ordering* if for any group elements a, b, and c, we have

$$a \prec b \implies ca \prec cb.$$

In other words, the ordering is invariant under the group acting on the left. If $\prec$ is simultaneously left-invariant and right-invariant, it is a *bi-ordering*. In the discussion that follows, statements about *orderings* apply to both left ordering and bi-orderings.

The classical study of collections of orderings of various types admitted by a given algebraic structure has a long history, and although much is understood for certain classes of structures, there are basic questions that remain unanswered. (See, for example, Fuchs [20], Kopytov and Medvedev [30], and Mura and Rhemtulla [36] for introduction and survey of early results.)

The corresponding computability-theoretic properties of the orderings of algebraic structures have also been investigated. In 1979, Metakides and Nerode [35] showed that families of orderings of computable fields are in exact correspondence to the collection of effectively closed subsets of the Cantor space, Π^0_1 subsets of 2^ω. A subset of 2^ω is a Π^0_1 *class* (also called *effectively closed*) if it is the collection of paths through a computable subtree of $2^{<\omega}$. By a *subtree*, we mean a subset of $2^{<\omega}$ that is closed under initial segments. A subtree of $2^{<\omega}$ is computable if its set of nodes is computable.

Theorem 1.1:

(1) The class of orderings of a recursively presented field is a Π^0_1 class.
(2) Let C be a non-empty Π^0_1 class of sets in 2^ω. Then there is a recursively presented formally real field F, and a homeomorphism from C onto the order space of F, which is Turing degree (in fact, many-one degree) preserving.

A large body of research in computability theory exists surrounding the properties of Π^0_1 classes and their members (see, for example, [7]), and since these classes characterize the collections of orderings of computable fields, this problem is well-studied. The situation for groups, however, is more complicated.

Although the collection of left orderings or bi-orderings of a computable group always forms a Π^0_1 class, not every Π^0_1 class can be so realized. Solomon explained in [45] and [46] that no such correspondence between collections of orderings and Π^0_1 classes can exist for the collections of orderings of computable groups as a consequence of the existence of a Π^0_1 class containing pairwise Turing-incomparable elements. He went on to show that the degrees of a computably bounded Π^0_1 class cannot, in general, be realized by the degrees of the orderings of a computable torsion-free abelian group. These negative results motivate continued study of computability-theoretic properties of orderings admitted by a given computable group, particularly a non-abelian group.

Progress has been made for certain classes of groups. For example, computable, torsion-free, abelian groups of finite rank greater than 1 admit orderings in every Turing degree, and those with infinite rank admit orderings in every Turing degree above that of the halting set, $\mathbf{0}'$ (see [45, 46]). In 1986, Downey and Kurtz [12] constructed a computable abelian group of infinite rank admitting no computable ordering. The collection of Turing degrees realized by the orderings of this (or any such) group cannot, however, exactly coincide with the cone of degrees above $\mathbf{0}'$ (this is a consequence of the Jockusch-Soare low basis theorem [27]).

For computable, infinite-rank, torsion-free, abelian groups, it is always possible to find an ordering in every Turing degree capable of computing the dependence algorithm for the computable divisible closure of the group. It is natural to wonder if the Turing degree of the requisite low ordering of the group presented by Downey and Kurtz sits in the cone of degrees above the corresponding dependence algorithm and is thus naturally realized, in a certain sense. The construction in [12] may however be modified (see [10]) to ensure that the degree of the dependence algorithm of the computable divisible closure of the group produced is of complete degree (that of the halting set), and so the low orderings of this group do not arise as a consequence of the existence of a low degree dependence algorithm.

More recently, in [28], Kach, Lange, and Solomon showed that there are computable abelian torsion-free groups with computable orderings but not orderings of every Turing degree, demonstrating that the spectrum of Turing degrees achieved by admitted orderings need not be upwards closed.

There are some results of similar flavor for non-abelian groups. Solomon [45, 46] showed that computable, torsion-free, properly n-step nilpotent groups admit bi-orderings of all Turing degrees at or above $\mathbf{0}^{(n)}$. In [23], it was shown that it is possible to construct a computable copy of the free group on a countable infinity of generators admitting no computable left orderings.

In this article, we consider a large class of residually nilpotent groups that includes finitely generated free groups as well as surface groups, and show that they admit bi-orderings of every truth-table degree (see Definition 2.2). We also show that the members of a certain class of nilpotent groups similarly admit bi-orderings in every truth-table degree. In Section 2, we review basic definitions and facts about orderings on groups, and about strong reducibilities and strong degrees in computability theory. In Section 3, we establish general conditions that suffice for a computable group to admit bi-orderings in every truth-table degree. In Section 4, we apply this result to certain classes of groups of importance in algebra and low-dimensional topology. In an Appendix we provide a worked-out example illustrating the technical result in Lemma 5.2. See [44] and [39] for background in general computability theory and notation.

2. Basic definitions and facts

2.1. *Orderings of groups*

A (partial) left ordering $\prec$ of a group G is a binary, irreflexive, antisymmetric, transitive relation on the elements of G, which is left-invariant with respect to the group operation. That is, for every $a, b \in G$ and all $c \in G$, we have

$$a \prec b \implies c \cdot a \prec c \cdot b,$$

where $\cdot$ denotes the group operation. We will usually omit explicit notation for a group operation and write ab for $a \cdot b$. A (partial) right-ordering is defined similarly, and a (partial) bi-ordering is one that is simultaneously left- and right-invariant. When the word "partial" is not used, we mean a total ordering.

An equivalent definition for a left ordering, a right ordering, and a bi-ordering may be given in terms of the *positive cone* of the ordering. For a left ordering, right ordering, or a bi-ordering $\prec$ on G, the corresponding positive cone $P_\prec$ is the set of all elements of G that are greater than the identity element e_G in this ordering, together with the identity itself. That is,

$$g \in P_\prec \iff (g \succ e_G \vee g = e_G).$$

Note that specifying a positive cone is sufficient to specify an ordering: To determine whether $a \prec b$ holds, we need only check whether $a^{-1}b \in P_\prec$ for a left ordering or a bi-ordering, or whether $ba^{-1} \in P_\prec$ for a right-ordering. It is easy to see that $\prec$ and $P_\prec$ have the same m-degree when G

is a computable group. (The cone $P_\prec$ is 1-reducible to $\prec$, but the reverse reducibility can only be an m-reduction in general.)

A subset P of G is the positive cone of a bi-ordering of G if the following conditions hold of P:

(1) $P \cdot P \subseteq P$, i.e., P is a sub-semigroup of G;
(2) $P \cap P^{-1} = \{e_G\}$, where P^{-1} is the set of inverses of elements of P; such a semigroup is said to be *pure*;
(3) $P \cup P^{-1} = G$; and
(4) for each $g \in G$, $gPg^{-1} \subseteq P$, i.e., P is a *normal* sub-semigroup of G.

We will often identify an ordering with its positive cone.

2.2. *Strong reducibilities*

There are many notions of strong reducibilities (strong in the sense that they imply Turing reducibility). Here we are concerned with truth-table reducibility, which is most conveniently defined in terms of another strong reduction notion: weak truth-table (or bounded-Turing) reducibility. Informally, a set A is weak truth-table reducible to another set B if there is a procedure for computing A using oracle B that is *predictable* in that there is a computable function specifying the number of bits of the oracle B that will be needed in the computation of A from B for a given input. The set A is truth-table reducible to B if there is a procedure for computing A from oracle B that, in addition to being predictable, is robust: If another oracle is used in the execution of the algorithm in place of B, the procedure will still halt, though not necessarily with correct output regarding membership in A. The formal definitions are as follows for sets of natural numbers A and B. We use φ_e to denote the function computed by the eth program (Turing machine) in some systematic enumeration of these, and φ_e^A for the function computed by the eth oracle program using oracle A. For any set A, we use $A \upharpoonright n$ to denote the first n bits of A.

Definition 2.1: A set A is *weak truth-table reducible* to set B if there is a computable function h and an index e so that $A(x) = \varphi_e^{B\upharpoonright h(x)}(x)$. We write $A \leq_{wtt} B$ when this is the case, and $A \equiv_{wtt} B$ when both $A \leq_{wtt} B$ and $B \leq_{wtt} A$.

The stronger notion, tt-reducibility, is a refinement of wtt-reducibility.

Definition 2.2: A set A is *truth-table reducible* to B if $A \leq_{wtt} B$ via a function φ_e^B and computable function h, having the additional property that for any string $\sigma \in 2^{<\omega}$ of length $h(x)$, $\varphi_e^\sigma(x)\downarrow$. We write $A \leq_{tt} B$ when this is the case, and $A \equiv_{tt} B$ when both $A \leq_{tt} B$ and $B \leq_{tt} A$.

The relations $\leq_{wtt}$ and $\leq_{tt}$ are pre-orderings of the power set of the natural numbers, and the induced equivalence classes are called *wtt*-degrees and *tt*-degrees, respectively. We will write $deg(A)$ for the Turing degree of the set A, and $deg_{tt}(A)$ for its *tt*-degree:

$$deg_{tt}(A) = \{B \in 2^\omega \mid A \equiv_{tt} B\}.$$

3. Bi-orderings of arbitrary tt-degree

In [11], the authors investigated when a computable group admits left orderings of every Turing degree in an upper cone, where the base of the cone computes a particular family of finite subsets of the group. In particular, they presented general conditions that suffice to ensure a computable group admits left orderings of every Turing degree.

Theorem 3.1: [11] *Let G be a computable group. Let $\mathcal{P}$ be a computable family of finite subsets of $G - \{e\}$ satisfying the following conditions for every $p \in \mathcal{P}$:*

(1) $e \notin \mathrm{sgr}(p)$, *where* $\mathrm{sgr}(p)$ *is the sub-semigroup generated by* p;
(2) $(\exists r_0, r_1 \in \mathcal{P})(\exists g \in G)[r_0, r_1 \supset p \wedge g \in r_0 \wedge g^{-1} \in r_1]$; *and*
(3) $(\forall g \in G - \{e\})(\exists r \in \mathcal{P})[r \supseteq p \wedge (g \in r \vee g^{-1} \in r)]$.

Then G admits a left ordering of every Turing degree.

In [10], Chubb showed that when this family is computably enumerable (c.e.), the same conditions suffice to ensure that G admits a left ordering in every *tt*-degree. Here, we show that stronger conditions provide for bi-orderings in every *tt*-degree, and in the next section give some applications of our general result to specific natural families of orderable groups.

Notation 3.2: For an arbitrary subset C of elements of a group G, let $S(C)$ be the normal sub-semigroup of G generated by the elements in C.

For an ordering $\prec$ of group G, we denote by $P_\prec^+$ the set of strictly positive elements, i.e., $P_\prec^+ =_{def} P_\prec - \{e_G\}$. We call this the strictly positive cone of $\prec$ in G.

Theorem 3.3: *Let G be a computable group, and $\mathcal{P}$ a c.e. family of finite subsets of $G - \{e\}$ satisfying the following conditions for every $p \in \mathcal{P}$:*

(1) $e \notin S(p)$;
(2) $(\exists r_0, r_1 \in \mathcal{P})(\exists g \in G)[r_0, r_1 \supset p \wedge g \in r_0 \wedge g^{-1} \in r_1]$; *and*
(3) $(\forall g \in G - \{e\})(\exists r \in \mathcal{P})[r \supseteq p \wedge (g \in r \vee g^{-1} \in r)]$.

Then G admits a bi-ordering in every tt-degree.

Proof: We construct a map $\mathcal{T} : 2^{<\omega} \to \mathcal{P}$ so that

$$\sigma \sqsubset \tau \implies \mathcal{T}(\sigma) \subseteq \mathcal{T}(\tau),$$

where $\sigma \sqsubset \tau$ denotes that σ is an initial segment of τ. For any $X \in 2^\omega$ we will have that $P_X^+ = \bigcup_n \mathcal{T}(X \upharpoonright n)$ is a pure, normal, sub-semigroup of G that contains exactly one of g and g^{-1} for each non-identity element $g \in G$. In other words, P_X^+ is the strictly positive cone of a bi-ordering of G. We will see that $P_X^+ \equiv_{tt} X$, and so $P_X = P_X^+ \cup \{e\}$ is a bi-ordering of G of the same (arbitrary) tt-degree as X.

Let $G - \{e\} = \{g_0, g_1, \ldots\}$ and $\mathcal{P} = \{p_0, p_1, \ldots\}$ be computable enumerations of $G - \{e\}$ and $\mathcal{P}$, respectively.

Construction

Stage 0. Set $\mathcal{T}(\langle\ \rangle) = p_0$.

Stage $s+1$. At the beginning of this stage, we have $\mathcal{T}$ defined on $2^{\leq s}$. For each σ of length s, find the first r_0 and r_1 in $\mathcal{P}$, and the first $g \in G$ witnessing the satisfaction of condition (2) for $p = \mathcal{T}(\sigma)$. We are free to arrange that $g \in r_1$ and $g^{-1} \in r_0$. Next, find the first g_{j_0} and g_{j_1} appearing in the enumeration of $G - \{e\}$ so that neither these elements nor their inverses are in r_0 and r_1, respectively. Let r_0' (and r_1', respectively) be the first element of $\mathcal{P}$ extending r_0 (r_1, respectively) containing g_{j_0} or $g_{j_0}^{-1}$ (g_{j_1} or $g_{j_1}^{-1}$, respectively). Such r_i' exist by condition (3).

Set $\mathcal{T}(\sigma^\frown i) = r_i'$ for $i = 0, 1$.

This completes the construction of $\mathcal{T}$.

Now, let $\mathbf{x}$ be an arbitrary tt-degree, and X a set with $deg_{tt}(X) = \mathbf{x}$. Define $P_X^+ = \bigcup_s \mathcal{T}(X \upharpoonright s)$ and $P_X = P_X^+ \cup \{e\}$. Our first task is to show that P_X is the positive cone of a bi-ordering of G by verifying that the four conditions given above hold. Second, we check that P_X has the required computability-theoretic properties.

Observe that for any non-identity $g \in G$, there is a stage s so that either $g \in \mathcal{T}(X \upharpoonright s)$ or $g^{-1} \in \mathcal{T}(X \upharpoonright s)$; in fact, if $g = g_i$, then $s = i + 1$.

So, we have $P_X \cup P_X^{-1} = G$. Next, $P_X \cdot P_X \subseteq P_X$ since if for some non-trivial $a, b \in P_X$ we have $ab \notin P_X$, then necessarily $(ab)^{-1} \in P_X$. All three elements, a, b, and $(ab)^{-1}$, must be contained in $p = \mathcal{T}(X \upharpoonright s)$ for some s, and this element $p \in \mathcal{P}$ fails to satisfy condition (1) of the theorem, contrary to assumption. Continuing, $P_X \cap P_X^{-1} = \{e\}$ since e is in both sets, and a non-trivial element in the intersection results again in a violation of condition (1) by the image of some finite initial segment of X under $\mathcal{T}$. Finally, we verify that P_X is normal in G. Assuming otherwise, there must be $g \in G$ and $a \in P_X$ so that $gag^{-1} \notin P_X$. In this case we would have $(gag^{-1})^{-1} = ga^{-1}g^{-1} \in P_X$. Let s be a stage so that both a and $ga^{-1}g^{-1}$ are elements of $p = \mathcal{T}(X \upharpoonright s)$. Then in $S(p)$ we have both of these elements, as well as the conjugate of $ga^{-1}g^{-1}$ by g^{-1} since $S(p)$ is normal. However, this conjugate is a^{-1}, and we see that this again results in a contradiction to condition (1) of the theorem for $p \in \mathcal{P}$.

To see that $P_X \leq_{tt} X$, we observe the following. To determine if $g_i \in P_X$, we use the fact that for each σ of length $i+1$, either g_i or g_i^{-1} is in $\mathcal{T}(\sigma)$. So we have

$$g_i \in P_X \iff g_i \in \mathcal{T}(X \upharpoonright (i+1)).$$

Note that this is a tt-reduction since the initial segment of length $i+1$ of any set X is in the domain of $\mathcal{T}$.

For the reverse reduction, $X \leq_{tt} P_X$, we check whether $i \in X$ via the following algorithm.

Construct $\mathcal{T}$ until the domain includes all nodes of length $i+1$. If $P_X \supseteq \mathcal{T}(\sigma)$ for some σ of length $i+1$, then $i \in X$ if and only if the last bit of σ is 1. If P_X does not contain the image of any σ of length $i+1$, output 0.

If P_X is, in fact, the partial ordering defined by the construction above, this algorithm yields the correct answer to the membership question on X. If it is not, the algorithm halts, but may not yield a correct answer. Thus, the algorithm gives a valid tt-reduction. □

4. Orderings of $\mathbb{Z}^k$

Let $\mathbb{Z}^2$ be the group of pairs of integers under coordinate-wise addition. In [41], Sikora characterized the upper cones of orderings of $\mathbb{Z}^2$ as half-planes (see Figure 1, and in [46], Solomon showed that there are orderings of this group in every Turing degree.

In fact, an ordering of $\mathbb{Z}^2$ will have the same Turing degree as the real that is the slope of the line through the origin, which separates the positive

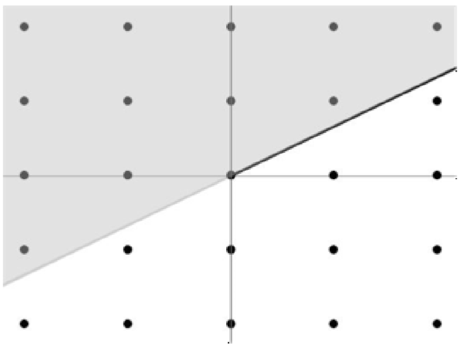

Fig. 1. The lattice points in the shaded area and on the black portion ofthe line separating the half-planes constitute the positive cone of an ordering of $\mathbb{Z}^2$.

and the negative cone (see Lemma 8.6 of [46]). Here, we construct a family of the sort described in Theorem 3.3 to obtain the following result.

Theorem 4.1: *The group $\mathbb{Z}^2$ admits orderings in every tt-degree.*

Proof: Let $\{g_0, g_1, \ldots\}$ be a computable enumeration of $\mathbb{Z}^2 - \{(0,0)\}$ where $g_i = (x_i, y_i)$. We will construct a family of finite subsets of $\mathbb{Z}^2$ having the properties described in Theorem 3.3. We do this by finite approximation.

Let $q > 0$. For the purposes of this construction, we say a point $g_s = (x_s, y_s)$ is *above the line* $y = qx$ if $y_s > qx_s$, or $y_s = qx_s$ and $x_s > 0$. Otherwise, it is *below the line.* In what follows, q will always be a rational number and equal to y_i/x_i for some given x_i and y_i in the positive integers. Note that determining whether a given point of $\mathbb{Z}^2$ is above or below such a line is computable.

Construction

Stage 0. Let $\mathcal{P}_0 = \emptyset$.
Stage $s+1$. Define $\mathcal{P}_{s+1}$ as the elements of $\mathcal{P}_s$ together with the following finite sets:

- All sets $\{g_i\}$ where $0 < i \leq s$ and g_i has $x_i > 0$ and $y_i \geq 0$;
- For each $p \in \mathcal{P}_s$ and each g_t with $t \leq s$, let $q = \min\{y/x \mid (x,y) \in p \wedge x > 0 \wedge y \geq 0\}$. If g_t is above the line $y = qx$, add $p \cup \{g_t\}$, and otherwise add $p \cup \{-g_t\}$.

Let $\mathcal{P} = \bigcup_{i\in\omega} \mathcal{P}_i$. Note that $\mathcal{P}$ is c.e.

End of construction.

Now, we verify the conditions of Theorem 3.3. For any $p \in \mathcal{P}$, all of the elements of p are above the line $y = qx$ where $q = \min\{y/x \mid (x, y) \in p \wedge x > 0 \wedge y \geq 0\}$. Since all the points in p are contained in a half-plane, they are contained in the strictly positive cone of an ordering of $\mathbb{Z}^2$, so $S(p)$ does not contain the identity $(0, 0)$, and hence $\mathcal{P}$ satisfies condition (1).

Next, let $g = g_{s_1}$ be a non-identity element of G and $p \in \mathcal{P}_{s_2}$. At stage $t = \max(s_1, s_2) + 1$, either $p \cup \{g\}$ or $p \cup \{-g\}$ will be added to $\mathcal{P}_t$. So, $\mathcal{P}$ satisfies condition (3).

Finally, we will verify condition (2). Let $p \in \mathcal{P}_s$, and $g_j = (x_j, y_j) \in p$ so that $q_{max} = y_j/x_j = \min\{y/x \mid (x, y) \in p \wedge x, y > 0\}$. If p contains any points in the third quadrant, let $q_{min} = \max\{(y/x \mid (x, y) \in p \wedge x, y < 0\}$, otherwise let $q_{min} = 0$. Note that $q_{min} \lneq q_{max}$. Find the first $g_k = (x_k, y_k)$ in the enumeration so that g_k is in the first quadrant of the plane and $q_{min} \lneq y_k/x_k \lneq q_{max}$, and let $g_{k'} = -g_k$. Let $t = max\{i, j, k, k'\}$, and let n be the cardinality of p. Then by the end of stage $t + 1$, $p \cup \{-g_k\}$ will be added to $\mathcal{P}$, and by the end of stage $t + n + 1$, $p \cup \{g_k\}$ will have be added to $\mathcal{P}$. □

As a corollary, we obtain the following results.

Corollary 4.2: *Every computable group isomorphic to $\mathbb{Z}^k$ for $k > 1$ admits orderings in all tt-degrees. Every computable group isomorphic to $\mathbb{Z}^\omega$, which has a computable basis admits orderings in all tt-degrees.*

Proof: Consider $\mathbb{Z}^k$. Let $\mathbf{x}$ be a *tt*-degree, and $\prec$ an ordering of $\mathbb{Z}^2$ of *tt*-degree $\mathbf{x}$. For any non-identity element $g = (x_1, \ldots, x_k)$ of $\mathbb{Z}^k$, declare g to be greater than the identity in $\mathbb{Z}^k$ if and only if $(0, 0) \prec (x_1, x_2)$ in $\mathbb{Z}^2$, or if $x_1 = x_2 = 0$ and the first non-zero entry of g is positive. Clearly, this ordering is *tt*-equivalent to $\prec$.

Similarly, we obtain the same result for a computable isomorphic copy of $\mathbb{Z}^\omega$ with a computable basis. □

We will use Corollary 4.2 in the proof of Theorem 5.5 at the end of the next section.

5. Orderings of finitely presented residually nilpotent groups

We now apply Theorem 3.3 to a large class of finitely presented, residually nilpotent groups, which are not nilpotent. Recall that for a group G, the *lower central series* of G is the descending sequence of subgroups of G, indexed by ordinals, $\{\gamma_\alpha(G)\}_{\alpha\geq 1}$, defined as follows:

$$\gamma_1(G) = G,$$
$$\gamma_{\alpha+1}(G) = [G, \gamma_\alpha(G)],$$
$$\gamma_\beta(G) = \bigcap_{\alpha<\beta} \gamma_\alpha(G) \text{ when } \beta \text{ is a limit ordinal.}$$

Here, as usual, for subgroups A and B of G, the commutator $[A, B]$ is the subgroup

$$[A, B] = \langle a^{-1}b^{-1}ab \mid a \in A, b \in B\rangle.$$

A group G is *residually nilpotent* if every non-identity element has a non-identity homomorphic image in some nilpotent group. For any group G, the subgroup $\gamma_\omega(G)$ is the smallest subgroup of G for which $G/\gamma_\omega(G)$ is residually nilpotent. Thus, G is residually nilpotent if and only if $\gamma_\omega(G)$ is the trivial group. It is well-known that finitely generated nilpotent groups are supersolvable. Furthermore, as it was shown in [26], supersolvable groups are residually finite. It follows that finitely generated nilpotent groups are finitely presented and have decidable word problem. It then follows that finitely presented residually nilpotent groups have decidable word problem, hence they have computable isomorphic copies.

Let $F_n = \langle x_1, x_2, \ldots, x_n \mid \quad \rangle$ denote the free group with basis $\{x_1, x_2, ..., x_n\}$.

Theorem 5.1: *Let G be a computable, finitely presented, torsion-free group. Let*

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots$$

be the lower central series of G. Assume that $\gamma_\omega(G)$ is trivial, and for every $i \in \{1, 2, \ldots\}$, $\gamma_i(G)/\gamma_{i+1}(G)$ is non-trivial and torsion-free (i.e., G is residually nilpotent but not nilpotent). Then there are bi-orderings on G in all tt-degrees.

Before we start our proof of Theorem 5.1, we note that there are many examples of groups (which we will discuss later) to which Theorem 5.1 applies (for example, all fundamental groups of closed and oriented surfaces of genus $g \geq 2$).

Proof: Let a group

$$G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_k \rangle$$

be computable, torsion-free, and non-nilpotent with lower central series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_\omega(G) = \{e\}.$$

The quotient groups of the lower central series of G are finite-rank (always true for finitely generated groups), and abelian (always true). In addition, assume that the quotient groups of the lower central series are torsion-free.

For $i \geq 1$, let $\gamma_i(G)/\gamma_{i+1}(G)$ have rank k_i, and write $\{h_{i,1}, \ldots, h_{i,k_i}\}$ for the generators of $\gamma_i(G)/\gamma_{i+1}(G)$. Note that if we choose some of the generators to be positive and some to be negative, this choice will induce a lexicographical bi-ordering of $\gamma_i(G)/\gamma_{i+1}(G)$ since this quotient group is abelian. Furthermore, we can make such a choice for each quotient group of the lower central series so that we have a family of orderings, $\{\prec_i\}_{i\geq 1}$. Together these induce a *standard* bi-ordering $\prec$ on G if we declare $g \prec e$ if and only if there is an i so that $g\gamma_{i+1}(G) \prec_i \gamma_{i+1}(G)$ (see [37],[42], [20]).

Our result will be established by an application of Theorem 3.3, so we will construct a family $\mathcal{P}$ of finite subsets of G satisfying the conditions stated in that theorem. The family will consist of two kinds of finite subsets of G: a sort of basic sets, which we will call *seeds* (see definition below), and extensions of these sets, which we will call *sprouts*.

We will describe a scheme for uniformly effectively selecting representatives of the generating elements (cosets) of each of the quotient groups. The seeds will contain representatives of the generating cosets (or inverses of these) from one of the quotient groups, selected in a systematic way. Sprouts will be finite extensions of seeds, each extendible to a standard bi-ordering of G.

The lower central series provides a complete filtration of $G - \{e\}$ by uniformly computably enumerable sets. As a consequence of the fact that the groups constituting the lower central series of a free group F_n are uniformly computable as sets, we have that the elements of the lower central series of G are uniformly c.e. They are each the range of a computable function – the canonical homomorphism from F_n to G – with computable domain. The homomorphism is computable since the group G is computable (see [11]).

For $i \geq 1$, let $\{a_{i,1}, \ldots, a_{i,k_i}\}$ be the set of representatives of the elements of a generating basis for $\gamma_i(G)/\gamma_{i+1}(G)$. That is, for every i, j, we have $a_{i,j} \in h_{i,j}$. We show in Corollary 5.3 below that a choice of representatives of all of these sets, $\{a_{i,j} : i \geq 1 \wedge 1 \leq j \leq k_i\}$, can be made uniformly computably in i, and that the rank of each quotient group can be computed. Each $g \in \gamma_i(G) - \gamma_{i+1}(G)$ is a member of a unique non-trivial coset $C_g = g\gamma_{i+1}(G)$. Since the quotient group is abelian, note that

$$C_g = \left(\prod_{j=1}^{k_i} (a_{i,j}^{\varepsilon_{i,j}})^{c_j} \right) \gamma_{i+1}(G)$$

for some choice of $\varepsilon_{i,j} \in \{-1, 1\}$ and $c_j \in \mathbb{Z}$. So, since $\gamma_{i+1}(G)$ is c.e., we can also computably enumerate C_g.

Construction

We finitely approximate a c.e. family of finite subsets, $\mathcal{P}$, satisfying the conditions in Theorem 3.3 as follows.

Stage 0. $\mathcal{P} = \emptyset$

Stage $s + 1$.

First, add new seeds to $\mathcal{P}$ by adding all sets of the form $\{a_{s,1}^{\varepsilon_{s,1}}, \ldots, a_{s,k_s}^{\varepsilon_{s,k_s}}\}$ where $\varepsilon_{s,j} \in \{-1, 1\}$. (There are only finitely many of these sets, in fact, 2^{k_s}.)

Next, add sprouts to $\mathcal{P}$.

(1) For each seed σ in $\mathcal{P}$, add the set obtained by forming words of length $\leq s$ from the members of σ that represent positive elements in the induced lexicographic ordering of the corresponding quotient group. More precisely, if $\sigma = \{a_{s,1}^{\varepsilon_{s,1}}, \ldots, a_{s,k_s}^{\varepsilon_{s,k_s}}\}$, add all words of length $\leq s$ of the form $\prod_{j=1}^{k_s} (a_{i,j}^{\varepsilon_{i,j}})^{c_j}$, where the first non-zero c_i is positive.

(2) Add all consistent unions of sets already contained in $\mathcal{P}$. (A union would be *inconsistent* if it contained elements g_1 and g_2 with $g_1 g_2 = e$. Again, there are only finitely many sets of this type.)

(3) For each p already in $\mathcal{P}$, add p' constructed as follows: Let p' be the subset generated by adding to p all the elements of $gC_{g,s}$ for $g \in p$, where $C_{g,s}$ is the set of the first s elements enumerated from the unique non-trivial element of the quotient group of the lower central series to which g belongs. (The appropriate C_g can be identified by considering the history of the construction of p.)

This completes the construction of $\mathcal{P}$. □

We first recall the definition of Fox derivative and discuss its basic properties. Let $\mathbb{Z}[F_n]$ denote its integral group ring of the free group F_n. As shown in [18], there is a unique map $\frac{\partial}{\partial x_j} : F_n \to \mathbb{Z}[F_n]$, $j = 1, 2, \ldots, n$, called *derivation*, which satisfies the following conditions:

$$\frac{\partial}{\partial x_j}(x_i) = \begin{cases} 1 \text{ if } j = i \\ 0 \text{ if } j \neq i \end{cases} \text{ for } i = 1, 2, \ldots, n;$$
$$\frac{\partial}{\partial x_j}(uv) = \frac{\partial}{\partial x_j}(u) + u\frac{\partial}{\partial x_j}(v) \text{ for } u, v \in F_n.$$

The map $\frac{\partial}{\partial x_j}$ can be extended linearly to a map $\frac{\partial}{\partial x_j} : \mathbb{Z}[F_n] \to \mathbb{Z}[F_n]$, $j = 1, 2, \ldots, n$, called *Fox derivative.* This allows us to define the mth order derivative for $m > 1$:

$$\frac{\partial^m}{\partial x_{i_m} \partial x_{i_{m-1}} \cdots \partial x_{i_1}}(u) = \frac{\partial}{\partial x_{i_m}}\left(\frac{\partial^{m-1}}{\partial x_{i_{m-1}} \cdots \partial x_{i_1}}(u)\right) \text{ for } u \in F_n.$$

Assume that $X = \{x_1, x_2, \ldots, x_n\}$ is an ordered alphabet. Let X^* be the set of all words over X, and let X_k^* be the set of all words of length k. Define the set of Lyndon words of length k (where $k \geq 1$) to be the set W_k of words $w \in X_k^*$ for which w is lexicographically smaller (denoted by $\prec$) than any of its cyclic rearrangements[a]. Let

$$W = \bigcup_{k \geq 1} W_k$$

denote the set of all Lyndon words over alphabet X. As shown in [9], the size of the set W_k is given by

$$|W_k| = \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right) n^d,$$

where μ is the Möbius function on $\mathbb{Z}_+$ defined by:

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1, \\ (-1)^i & \text{if } m = p_1 p_2 \cdots p_i \text{ where } p_1, \ldots, p_i \text{ are distinct primes,} \\ 0 & \text{if } m \text{ is divisible by a square.} \end{cases}$$

For example, let $X = \{x_1, x_2\}$ and $x_1 \prec x_2$. Then $w = x_1x_1x_2x_1x_2 \in W_4$ since $x_1x_1x_2x_1x_2 \prec x_1x_2x_1x_2x_1$, $x_1x_1x_2x_1x_2 \prec x_2x_1x_2x_1x_1$,

[a]It is worth mentioning that the set W_k can also be defined as the set of words $w \in X_k^*$ such that w is strictly lexicographically smaller than any of its proper right factors.

$x_1x_1x_2x_1x_2 \prec x_1x_2x_1x_1x_2$ and $x_1x_1x_2x_1x_2 \prec x_2x_1x_1x_2x_1$. In this example we also have

$$W_1 = \{x_1, x_2\}, W_2 = \{x_1x_2\}, W_3 = \{x_1x_1x_2, x_1x_2x_2\}, \text{ and } W_4 = \{x_1x_1x_1x_2, x_1x_1x_2x_2, x_1x_2x_2x_2\}.$$

In general, there is an efficient algorithm developed by Duval in [17] for finding elements of W_k for $k \geq 2$.

Define $\psi : \mathbb{Z}[F_n] \to \mathbb{Z}$ by $\psi\left(\sum_{u\in F_n} a_u u\right) = \sum_{u\in F_n} a_u$. Define $D_u^0 : F_n \to \mathbb{Z}$ as follows for $u = x_{i_1}x_{i_2}\cdots x_{i_m} \in X^*$ and $v \in F_n$:

$$D_u^0(v) = \psi\left(\frac{\partial^m}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_m}}(v)\right).$$

For a Lyndon word $w \in W$, a factorization $w = uv$ is called the *standard factorization* of w iff v is a Lyndon word of the maximal length $|v| \geq 1$ (see, for instance, [9]). With each $w \in W$ we associate the *standard commutator* defined recursively as follows:

$$[w] = w \text{ if } w \in X, \text{ and}$$
$$[w] = [[u],[v]] \text{ if } w = uv \text{ is the standard factorization of } w.$$

Let $C_k = \{[w] \in F_n \mid w \in W_k\}$ be the standard commutators of weight $k \geq 1$. For instance, in the example above, we have

$$C_1 = \{x_1, x_2\}, C_2 = \{[x_1, x_2]\}, C_3 = \{[x_1, [x_1, x_2]], [[x_1, x_2], x_2]\}, \text{ and}$$
$$C_4 = \{[x_1, [x_1, [x_1, x_2]]], [[x_1, [x_1, x_2]], x_2], [[[x_1, x_2], x_2], x_2]\}.$$

Let

$$L_m(G) = \gamma_m(G)/\gamma_{m+1}(G).$$

It was shown in [9] that W_k has the following important properties.

(1) If $w \in W_k$ then: (a) For every $e \in X_k^*$, if $e \prec w$, then $D_e^0([w]) = 0$; (b) $D_w^0([w]) = 1$ (see Lemma 3.4 in [9]).
(2) For all $u \in F_n$, $u \in \gamma_k(F_n)$ iff $D_w^0(u) = 0$ for all $w \in \bigcup_{i=1}^{k-1} W_i$ (see Corollary 3.6 in [9]).
(3) The set $\{c\gamma_k(F_n) \mid c \in C_k\}$ is a basis of the free abelian group $L_k(F_n)$, $k \geq 1$ (see Theorem 3.5 in [9]).
(4) The maps $D_w^0 : F_n \to \mathbb{Z}$ for $w \in W_k$ form a basis for the additive group $\operatorname{Hom}(L_k(F_n), \mathbb{Z})$ of homomorphisms of the multiplicative group $L_k(F_n)$ into the additive group $\mathbb{Z}$ (see Theorem 3.5 in [9]).

The following lemma is a direct consequence of an algorithm given in [9] for computing the presentation of $L_m(G)$. We modify this algorithm to make it uniform in $m \geq 1$.

Lemma 5.2: *Let $G = \langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_k \rangle$ be a computable finitely presented group. A presentation of the quotient groups of the lower central series of G, $\{L_m(G)\}_{m<\omega}$, can be computed uniformly in m.*

Proof: Let $R = \{r_1, r_2, \ldots, r_k\}$. By $\langle\langle R \rangle\rangle$ we denote the normal closure of R in F_n. For $m \geq 1$, let $W_m = \{w_1, w_2, ..., w_s\}$ be the set of all Lyndon words of length m over the alphabet $X = \{x_1, x_2, \ldots, x_n\}$, and $C_m = \{b_1, b_2, \ldots, b_s\}$ be the set of the corresponding standard commutators. If $S_m = \{r_{m_1}, r_{m_2}, \ldots, r_{m_l}\} \subseteq F_n$ satisfy the condition

$$\langle\langle R \rangle\rangle \cap \gamma_m(F_n) = \langle\langle S_m \rangle\rangle \leq F_n,$$

then, according to [9], the group $L_m(G)$ has the following presentation

$$L_m(G) = \left\{ \bar{b}_1, \bar{b}_2, \ldots, \bar{b}_s \mid \bar{r}_{m_1}, \bar{r}_{m_2}, \ldots, \bar{r}_{m_l} \right\},$$

where $\bar{b}_j = b_j \gamma_{m+1}(F_n)$, $1 \leq j \leq s$, and

$$\bar{r}_{m_j} = \prod_{k=1}^{s} b_k^{D^0_{w_k}(r_{m_j})} \gamma_{m+1}(F_n),\ 1 \leq j \leq l.$$

Hence, the presentation matrix for $L_m(G)$ is given by

$$M_m(G) = \begin{pmatrix} D^0_{w_1}(r_{m_1}) & D^0_{w_2}(r_{m_1}) & \cdots & D^0_{w_s}(r_{m_1}) \\ D^0_{w_1}(r_{m_2}) & D^0_{w_2}(r_{m_2}) & \cdots & D^0_{w_s}(r_{m_2}) \\ \vdots & \vdots & \cdots & \vdots \\ D^0_{w_1}(r_{m_l}) & D^0_{w_2}(r_{m_l}) & \cdots & D^0_{w_s}(r_{m_l}) \end{pmatrix},$$

The set S_m, $m \geq 1$, defined above is constructed inductively on m as follows. For $m = 1$, one naturally sets $S_1 = \{r_1, r_2, \ldots, r_k\}$. Assume by induction that for $m \geq 1$, the set $S_m = \{r_{m_1}, r_{m_2}, \ldots, r_{m_l}\}$ has been determined. Let $v_1, v_2, \ldots, v_l$ be the rows of $M_m(G)$. The row space of the matrix $M_m(G)$ is generated over $\mathbb{Z}$ by linearly independent vectors $v_{i_1}, v_{i_2}, \ldots, v_{i_{p_m}}$, where $1 \leq i_1 < i_2 < \cdots < i_{p_m} \leq l$, and, to make our procedure uniform in m, we require that $v_{i_1}, v_{i_2}, \ldots, v_{i_{p_m}}$ be the first linearly independent rows chosen from $\{v_1, v_2, \ldots, v_l\}$, while $u_{k_1}, u_{k_2}, \ldots, u_{k_{q_m}}$, $1 \leq k_1 < k_2 < \cdots < k_{q_m} \leq l$, are the remaining (linearly dependent) vectors from $\{v_1, v_2, \ldots, v_l\}$. Clearly, we have $p_m + q_m = l$, and define the corresponding relations $s_j = r_{m_{i_j}}$, $1 \leq j \leq p_m$, and $t_j = r_{m_{k_j}}$, $1 \leq j \leq q_m$.

For every $i \in \{1, 2, \ldots, q_m\}$, there are unique integers $\beta_{i,k} \in \mathbb{Z}$ such that $z_i = t_i \left(\prod_{k=1}^{p} s_k^{\beta_{i,k}} \right)^{-1} \in \gamma_{m+1}(F_n)$. These coefficients can be found by taking $(\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,p_m}) \in \mathbb{Z}^{p_m}$ and then checking whether

$$D_w^0(z_i) = 0 \text{ for all } w \in \bigcup_{j=1}^{m} W_j \text{ (see Corollary 3.6 in [9]).}$$

Let

$$S_{m+1} = \{z_1, z_2, \ldots, z_{q_m}\} \cup \{[s_k, x_j] \mid k \in \{1, 2, ..., p_m\} \wedge j \in \{1, 2, \ldots, n\}.$$

It follows from [8] (see Lemma A5) that

$$\langle\langle R \rangle\rangle \cap \gamma_{m+1}(F_n) = \langle\langle S_{m+1} \rangle\rangle \leq F_n.$$

It follows that the presentation matrix for $L_m(G)$ is given by

$$M'_m(G) = \begin{pmatrix} D_{w_1}^0(s_1) & D_{w_2}^0(s_1) & \cdots & D_{w_s}^0(s_1) \\ D_{w_1}^0(s_2) & D_{w_2}^0(s_2) & \cdots & D_{w_s}^0(s_2) \\ \vdots & \vdots & \cdots & \vdots \\ D_{w_1}^0(s_{p_m}) & D_{w_2}^0(s_{p_m}) & \cdots & D_{w_s}^0(s_{p_m}) \end{pmatrix}.$$

This completes the proof. □

For a finitely generated abelian group A there are nonnegative integers k, r, and if $k > 0$ integers $d_1, d_2, ..., d_k$ such that $1 \leq d_1 \mid d_2 \mid \cdots \mid d_k$ and

$$A \simeq \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_k} \oplus \mathbb{Z}^r.$$

The integers $d_1, d_2, \ldots, d_k$ are called the *elementary divisors* of A, and r is called the *rank* of A – these numbers are unique for A. For our purpose, we assume that $m \leq n$, and let $A = \{x_1, x_2, \ldots, x_n \mid R_1, R_2, \ldots, R_m\}$, where $R_j = \sum_{i=1}^{n} m_{i,j} x_i$ for $m_{ij} \in \mathbb{Z}$ and $1 \leq j \leq m$, be a presentation of A. Here, we outline a construction of an isomorphism $\varphi : A \to B$, where $B = \{z_1, z_2, \ldots, z_k, y_1, y_2, \ldots, y_r \mid d_1 z_1, \ldots, d_k z_k\}$, and $1 \leq d_1 \mid d_2 \mid \cdots \mid d_k$, given the presentation matrix $M = (m_{i,j})_{1 \leq i \leq m,\ 1 \leq j \leq n}$ of A.

Recall that an integer $m \times n$ matrix $D = (d_{i,j})$ with rank $r \leq m$ is said to be in the *Smith normal form* if:

(i) D is diagonal,
(ii) $d_{ii} \in \mathbb{Z}_+$, where $1 \leq i \leq r$,
(iii) $d_{i,i} \mid d_{i+1,i+1}$, where $1 \leq i \leq r - 1$, and
(iv) $d_{i,i} = 0$, where $r + 1 \leq i \leq m$.

As shown in [43], for M, there are invertible integer matrices $U \in \mathrm{GL}(m, \mathbb{Z})$ and $V \in \mathrm{GL}(n, \mathbb{Z})$ such that $D = UMV$ is in the Smith normal form[b]. Let $A_n = \{x_1, x_2, ..., x_n \mid \quad\}$ denote the free abelian group of rank n and let $N = \mathrm{Span}_{\mathbb{Z}}(\{R_1, R_2, ..., R_m\})$. Clearly, $A = A_n/N$. Let $V = (v_{i,j})_{1\le i,j\le n}$ and $V^{-1} = (\overline{v_{i,j}})_{1\le i,j\le n}$. Define

$$z_i = \sum_{j=1}^{n} \overline{v_{i,j}} x_i \text{ for } i = 1, 2, \ldots, r, \text{ and}$$
$$y_{i-r} = \sum_{j=1}^{n} \overline{v_{i,j}} x_i \text{ for } i = r+1, r+2, \ldots, n.$$

Consider a free abelian group $B_n = \{z_1, z_2, \ldots, z_r, y_1, y_2, \ldots, y_{n-r} \mid \quad\}$ (with the basis $\{z_1, z_2, \ldots, z_r, y_1, y_2, \ldots, y_{n-r}\}$). Since $V \in \mathrm{GL}(n, \mathbb{Z})$, the map

$$\varphi : A_n \to B_n \text{ given by}$$
$$\varphi(x_i) = \sum_{j=1}^{r} v_{i,j} z_j + \sum_{j=r+1}^{n} v_{i,j} y_{j-r} \text{ for } i = 1, 2, ..., n,$$

is an isomorphism between free abelian groups of rank n. Let $\overline{R}_j = \varphi(R_j)\, forj = 1, 2, \ldots, m$. Then

$$N' = \mathrm{Span}_{\mathbb{Z}}\left(\{\overline{R}_j \mid 1 \le j \le m\}\right) = \mathrm{Span}_{\mathbb{Z}}(\{d_{i,i} z_i \mid 1 \le i \le r\}),$$

and $\varphi(N) = N'$. Therefore, φ induces an isomorphism of the quotient groups $A = A_n/N$ and $B = B_n/N' = \{z_1, z_2, \ldots, z_r, y_1, y_2, \ldots, y_{n-r} \mid d_1 z_1, \ldots, d_k z_r\}$, where $1 \le d_1 \mid d_2 \mid \cdots \mid d_r$.

The following corollary is an easy consequence of Lemma 5.2 and above considerations.

Corollary 5.3: *Let G be as in the statement of Theorem 5.1. Then a basis of the quotient groups of the lower central series of G, $\{L_m(G)\}_{1\le m<\omega}$, can be found uniformly computably in m.*

Lemma 5.4: *The family $\mathcal{P}$ of finite subsets satisfies the conditions of Theorem 3.3.*

[b]There are many well-known algorithms for computing matrices D, P, and Q (see, for instance, [29, 24]).

Proof: Let $p \in \mathcal{P}$. To see that condition (1) holds, note that p can clearly be extended to the upper cone $P_<$ of some standard bi-ordering of G (all the quotient groups are orderable, and each of these orderings extends to some standard ordering of G). Since $P_< - \{e\}$ is a semigroup containing p, we must have that $e \notin S(p)$.

To see that condition (2) holds, let i be the largest natural number so that there is some $a_{i,j}^{\varepsilon_{i,j}} \in p$. At stage $i+1$, the seeds $\sigma_0 = \{a_{i+1,1}, \dots, a_{i+1,k_{i+1}}\}$ and $\sigma_1 = \{a_{i+1,1}^{-1}, \dots, a_{i+1,k_{i+1}}\}$ will be added to $\mathcal{P}$, and at stage $i+2$, both $r_0 = p \cup \sigma_0$ and $r_1 = p \cup \sigma_1$ will be added to $\mathcal{P}$. These both extend p, one contains $a_{i+1,1}$ and the other contains its inverse, $a_{i+1,1}^{-1}$.

Finally, as in condition (3), let g be a non-identity element of G, and assume that $g \in \gamma_i(G) - \gamma_{i+1}(G)$, and that (a finite set) p has been added to $\mathcal{P}$ at stage s. Then either p contains $\{a_{i,1}^{\varepsilon_{i,1}}, \dots, a_{i,k_i}^{\varepsilon_{i,k_i}}\}$ for some choice of $\varepsilon_{i,j}$'s, if $s \geq i+2$, or an extension $p' \supset p$ contains such a set at stage $i+2$. Since these elements are representatives of generating sets of $\gamma_i(G)/\gamma_{i+1}(G)$, we can write

$$g = \left(\prod_{j=1}^{k_i} (a_{i,j}^{\varepsilon_{i,j}})^{c_j} \right) w$$

for some $w \in \gamma_{i+1}$ and integers c_j. If the first non-zero c_j is positive, then g will be enumerated into some p'' extending p' at some later stage as w appears in the enumeration of $\gamma_{i+1}(G)$. If the first non-zero c_j is negative, then g^{-1} is an element of

$$\left(\prod_{j=1}^{k_i} (a_{i,j}^{\varepsilon_{i,j}})^{-c_j} \right) \gamma_{i+1}(G)$$

(the quotient group is abelian), and so will be enumerated into some p'' extending p' at a later stage.

In conclusion, the conditions of Theorem 3.3 are satisfied, so it follows that G admits a bi-ordering in every tt-degree. □

Solomon [45, 46] showed that n-step nilpotent groups admit left orderings in every Turing degree above $\mathbf{0^{(n)}}$. We now consider finitely presented nilpotent groups satisfying an additional condition and show that they admit bi-orderings of every tt-degree. First, note that finitely presented nilpotent groups have a decidable word problem, so have computable isomorphic copies.

Theorem 5.5: *Let G be a computable, finitely presented, nilpotent group with lower central series*

$$\gamma_1(G) \geq \gamma_2(G) \geq \cdots \geq \gamma_n(G) = \{e\}$$

having the property that for each $i \in \{1, \ldots, n-1\}$, $\gamma_i(G)/\gamma_{i+1}(G)$ is torsion-free, and furthermore, for some $j < n$, $\gamma_j(G)/\gamma_{j+1}(G)$ is isomorphic to $\mathbb{Z}^k$ for $k \geq 2$. Then G admits bi-orderings of every tt-degree.

Proof: We can again make use of a set of canonical representatives of the cosets in the quotient groups of the lower central series of G, obtained from a basis as described above.

Let j be chosen so that the group $\gamma_j(G)/\gamma_{j+1}(G)$ (which is necessarily torsion-free and abelian of finite rank) has rank greater than 1. Then, by Corollary 4.2, this quotient group admits orderings of every *tt*-degree. Choose a (finite) family of orderings, $\{<_i\}_{i<n}$, on the quotient groups so that for all $i \neq j$, $<_i$ is computable, and $<_j$ has arbitrary *tt*-degree $\mathbf{x}$. Define the ordering $\prec$ on G by declaring

$$e_G \prec g \iff g \in \gamma_i(G) - \gamma_{i+1}(G) \text{ and } \gamma_{i+1} <_i g\gamma_{i+1}.$$

Then $\prec$ is a bi-ordering of G of *tt*-degree $\mathbf{x}$. To determine whether a non-identity element $g \in G$ is in the positive cone $P_\prec$, we need to find i so that $g \in \gamma_i(G) - \gamma_{i+1}(G)$ (this can be done computably), then find the canonical representative of its coset (also computable), and finally check to see if the representative is positive or negative according to $<_i$. In the case when $i \neq j$, this is a computable procedure. When $i = j$, we need only query the oracle $<_j$ about the representative of g in its coset. Note that this is a *tt*-reduction, since for any g we can compute the appropriate representative.

On the other hand, to *tt*-compute $<_j$ from $\prec$, given an element of the (computable) set of representatives of elements of $\gamma_j(G)/\gamma_{j+1}(G)$, we need only check whether it is positive according to $\prec$. □

Theorem 5.1 applies to many important classes of groups. One of them is the collection of fundamental groups $S_{g,0}$ of closed and oriented surfaces of genus $g \geq 2$. Such groups have presentations of the form

$$S_{g,0} = \langle x_1, y_1, \ldots,\ x_g, y_g \mid [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] \rangle.$$

They are known to have a decidable word problem, are torsion-free and residually nilpotent (see [3, 19]). The quotient groups $\gamma_i(S_{g,0})/\gamma_{i+1}(S_{g,0})$, $i \geq 1$, are free abelian groups of finite rank[c]. In the Appendix, we give a detailed calculations using the method described in our proof of Lemma 5.2.

Other important examples of groups for which Theorem 5.1 applies can be found in the class of finitely generated one-relator parafree groups (see families of groups 5.1, 5.2, and 5.3 below). Recall that a group G is *parafree* if G is residually nilpotent and for some $k \geq 1$ and all n:

$$G/\gamma_n(G) \simeq F_k/\gamma_n(F_k).$$

It is well-known that a finitely presented parafree group G has a decidable word problem and, clearly, the quotients $\gamma_i(G)/\gamma_{i+1}(G)$ are free abelian groups of finite rank for all $i \geq 1$. In [4], Baumslag introduced an infinite family of finitely generated one-relator parafree but not free groups. These groups are given by the following presentations

$$G_{i,j} = \left\langle a, b, c \mid a = [c^i, a][c^j, b] \right\rangle, \text{ where } ij \neq 0. \tag{5.1}$$

Furthermore, let $\mathbb{F}_{n+2} = \langle s, t, x_1, \ldots, x_n \mid \quad \rangle$ be a free group of rank $n+2$, let $\mathbb{F}_{n+1} = \langle t, x_1, \ldots, x_n \mid \quad \rangle \leq \mathbb{F}_{n+2}$, and $w \in \mathbb{F}_{n+1}$. Assume that w involves x_1 and $w \in \mathbb{F}_{n+2}^{(k)}$, where $\mathbb{F}_{n+2}^{(k)}$ denotes kth term of the derived series of $\mathbb{F}_{n+2}$. Consider the family of groups given by the presentations

$$H_w = \langle s, t, x_1, \ldots, x_n \mid x_1 = w[s, t] \rangle.$$

The family of groups H_w was introduced in [5] by Baumslag. He proved that every H_w is parafree but not free. In particular, groups given by the following presentations

$$H_{i,j} = \left\langle a, s, t \mid a = [a^i, t^j][s, t] \right\rangle, \text{ where } i, j \geq 1, \tag{5.2}$$

are parafree but not free. Finally, in [6], Baumslag and Cleary introduced the family of parafree groups given by the following presentations

$$K_w = \langle t, x_1, \ldots, x_k \mid x_1^m w = t^n \rangle,$$

where $m, n \in \mathbb{Z}_+$, $\gcd(m, n) = 1$, $x_1^m w$ is not a proper power in $\mathbb{F}_{k+1} = \langle t, x_1, \ldots, x_k \mid \quad \rangle$, and $w \in \mathbb{F}_{k+1}^{(1)}$. They showed that K_w is a free group iff

[c]The quotients $\gamma_i(S_g)/\gamma_{i+1}(S_g)$, $i \geq 1$, are free abelian groups since the associated Lie ring of S_g is a free Lie ring by a result in [31].

$n = 1$ or $x_1^m w$ is a primitive element in F_{k+1}. In particular, groups given by the presentations

$$K_{i,j} = \left\langle a, s, t \mid a^i[s,a] = t^j \right\rangle, \text{ where } i \geq 1,\ j \geq 2 \text{ and } \gcd(i,\ j) = 1, \tag{5.3}$$

are all parafree but not free. All of the above groups are one-relator groups for which $G/\gamma_n(G) \simeq F_2/\gamma_n(F_2)$, where G is $G_{i,j}, H_w, H_{i,j}, K_{i,j}$, respectively, and F_2 denotes the usual free group of rank 2.

Our last example of groups that satisfy assumptions of Theorem 5.1 are right-angled Artin groups (*RAAGs*). They are constructed as follows. Let Γ be a finite graph with vertices $V(\Gamma) = \{1, 2, \ldots, n\}$, where $n \geq 2$, and edges $E(\Gamma)$. Define a *right-angled Artin group* A_Γ by the following presentation

$$A_\Gamma = \langle x_1, \ldots, x_n \mid [x_i, x_j],\ (i,j) \in E(\Gamma) \rangle.$$

If Γ is not a complete graph on n vertices and $E(\Gamma) \neq \emptyset$, then A_Γ is neither free abelian of rank n nor a free group of rank n (if $E(\Gamma) = \emptyset$, then A_Γ is a free group, and if Γ is a complete graph on n vertices, then A_Γ is a free abelian group of rank n). As it was shown in [25, 47], all *RAAGs* are biautomatic (consequently have a decidable word problem), bi-orderable (consequently are torsion-free [15]), and are residually nilpotent [13]. We also note here that *RAAGs* play a very important role in geometric group theory. In particular, as it was shown in [14], A_Γ is the fundamental group of a 3-manifold M^3 iff every component of Γ is a tree[d] or a triangle. Furthermore, *RAAGs* also appear in the study of the fundamental groups of the configuration spaces of n distinct points on the graph formed by robots tracks [1], [21].

We now consider the class of torsion-free nilpotent groups, which are very well understood. For example, it is well known that such groups are finitely presented and have decidable word, conjugacy, and isomorphism problems (a result due to Grunewald and Segal [40]). We start with the most elementary example – a *free nilpotent group with n generators of class m*, which we denote by $F(n,m)$. Let W_{m+1} be the set of all Lyndon words of length $m \geq 1$ over the alphabet $X = \{x_1, x_2, \ldots, x_n\}$, and let $C_{m+1} = \{[w] \in F_n \mid w \in W_{m+1}\}$ be the corresponding set of all standard commutators. Consider the presentation of $F(n,m)$:

$$F(n,m) = F_n/\gamma_{m+1}(F_n) = \langle x_1, x_2, \ldots, x_n \mid C_{m+1} \rangle,$$

[d]In the case when components of A_Γ are trees, $A_\Gamma \simeq \pi_1(M^3)$, where M^3 is a graph manifold [14].

Clearly, for $i = 1, 2, \ldots, m$, we have $\gamma_i(F(n,m))/\gamma_{i+1}(F(n,m)) \simeq \mathbb{Z}^{k_i}$, where $k_i = \frac{1}{i}\sum_{d|i} \mu\left(\frac{i}{d}\right) n^d$ and μ is the *Möbius function*. Since parafree groups are torsion-free and residually nilpotent, we see that the quotients of groups (5.1), (5.2), and (5.3) by the corresponding $(k+1)$st term of their lower central series is nilpotent of class k and torsion-free. In fact,

$$G_{i,j}/\gamma_{k+1}(G_{i,j}) \simeq H_{i,j}/\gamma_{k+1}(H_{i,j}) \simeq K_{i,j}/\gamma_{k+1}(K_{i,j}) \simeq F_2/\gamma_{k+1}(F_2).$$

Therefore, Theorem 5.5 applies to all groups mentioned above. Furthermore, as we mentioned before, for a surface groups $S_{g,0}$ of genus $g \geq 2$, we have $\bigcap_{j\geq 0} \gamma_j(S_{g,0}) = \{e\}$, and, for all $j \geq 1$, the quotient groups $\gamma_j(S_{g,0})/\gamma_{j+1}(S_{g,0})$ are free abelian of rank r_n, where

$$r_n = \frac{1}{n}\sum_{d|n} \mu(n/d) \left[\sum_{0\leq i\leq [d/2]} \frac{(-1)^i d}{d-i} \binom{d-i}{i} (2g)^{d-2i} \right].$$

Thus, $S_{g,0}/\gamma_{k+1}(S_{g,0})$ is torsion-free nilpotent of class $k \geq 1$. Hence we can apply Theorem 5.5 to all groups $S_{g,0}/\gamma_{k+1}(S_{g,0})$.

Finally, as we also mentioned before, right-angled Artin groups A_Γ are torsion-free and residually nilpotent so, in particular, we have that for all $k \geq 1$, the groups $A_\Gamma/\gamma_{k+1}(A_\Gamma)$ are torsion-free nilpotent and Theorem 5.5 can be applied.

Appendix

In this appendix we present an example with detailed computation for the purpose of illustrating Lemma 5.2. Let us consider the fundamental group of a surface of genus $g \geq 2$ with no boundary components

$$S_{g,0} = \langle x_1, y_1, \ldots, x_g, y_g \mid r_g \rangle, \text{ where } r_g = [x_1, y_1] \cdots [x_g, y_g].$$

As it was shown independently by Baumslag [3] and Frederick [19], $S_{g,0}$ is residually nilpotent. Moreover, as a one-relator group, $S_{g,0}$ has a decidable word problem, and since r_g is cyclically reduced and not a proper power in $F_{2g} = \langle x_1, y_1, \ldots, x_g, y_g \mid \quad \rangle$, it follows that $S_{g,0}$ is also torsion-free. Furthermore, we see that $r_g \in \gamma_2(S_{g,0})$ and $r_g \notin \gamma_3(S_{g,0})$. It follows from [31] that the group $L_n(S_{g,0}) = \gamma_n(S_{g,0})/\gamma_{n+1}(S_{g,0})$, $n = 1, 2, \ldots$, is a torsion-free abelian group of rank m_n:

$$m_n = \frac{1}{n}\sum_{d|n} \mu(n/d) \left[\sum_{0\leq i\leq [d/2]} (-1)^i \frac{d}{d-i} \binom{d-i}{i} (2g)^{d-2i} \right],$$

where μ is the Möbius function. We now consider the simplest case when $g = 2$, i.e., $S_{2,0} = \langle x_1, y_1, x_2, y_2 \mid r\rangle$, where $r = [x_1, y_1]\,[x_2, y_2]$. Thus, we have

$$m_n = \frac{1}{n}\sum_{d|n}\mu\,(n/d)\left[\sum_{0\le j\le[d/2]}(-1)^j\,\frac{d}{d-j}\binom{d-j}{j}(4)^{d-2j}\right] \text{ for } n = 1, 2, 3, \ldots, \tag{5.4}$$

while for the free group $F_4 = \langle x_1, y_1, x_2, y_2 \mid \quad\rangle$ of rank 4 we have

$$k_n = \frac{1}{n}\sum_{d|n}\mu(\frac{n}{d})4^d. \tag{5.5}$$

From the identities (5.4) and (5.5) we obtain

k_1	k_2	k_3	k_4	m_1	m_2	m_3	m_4
4	6	20	60	4	5	16	52

Since $k_4 = 60$ and $m_4 = 52$ we show the computations only for $L_n\,(S_{2,0})$, $n = 1, 2, 3$. We will follow the notation used in Lemma 5.2.

For $L_1\,(S_{2,0})$, we have that $W_1 = \{x_1, y_1, x_2, y_2\}$ is the set of Lyndon words of length 1, where $x_1 \prec y_1 \prec x_2 \prec y_2$. We have the following basis $C_1 = \{a_1, a_2, a_3, a_4\}$, where $a_1 = x_1\gamma_2(F_4)$, $a_2 = y_1\gamma_2(F_4)$, $a_3 = x_2\gamma_2(F_4)$, and $a_4 = y_2\gamma_2(F_4)$, for $L_1\,(F_4) = \gamma_1(F_4)/\gamma_2(F_4) \simeq \mathbb{Z}^4$. Let $S_1 = \{r_{1,1}\}$, where $r_{1,1} = r$, and since

$$\begin{aligned}
D^0_{x_1}\,(r_{1,1}) &= \psi\left(-x_1^{-1} + x_1^{-1}y_1^{-1}\right) = 0,\\
D^0_{y_1}\,(r_{1,1}) &= \psi\left(-x_1^{-1}y_1^{-1} + x_1^{-1}y_1^{-1}x_1\right) = 0,\\
D^0_{x_2}\,(r_{1,1}) &= \psi\left(-\,[x_1, y_1]\,x_2^{-1} + [x_1, y_1]\,x_2^{-1}y_2^{-1}\right) = 0,\\
D^0_{y_2}\,(r_{1,1}) &= \psi\left(-\,[x_1, y_1]\,x_2^{-1}y_2^{-1} + [x_1, y_1]\,x_2^{-1}y_2^{-1}x_2\right) = 0,
\end{aligned}$$

we have

$\diagdown$	$D^0_{x_1}$	$D^0_{y_1}$	$D^0_{x_2}$	$D^0_{y_2}$
$r_{1,1}$	0	0	0	0

From Lemma 5.2 we obtain the following presentation matrix for $L_1\,(S_{2,0})$:

$$M_1\,(S_{2,0}) = \begin{pmatrix} 0\;0\;0\;0 \end{pmatrix}.$$

Therefore, if $r_{2,1} = r_{1,1}$, we have $S_2 = \{r_{2,1}\}$,

$$L_1\,(S_{2,0}) = \{a_1, a_2, a_3, a_4 \mid \qquad\} \simeq \mathbb{Z}^4$$

and it has the following basis $B_1 = \{a_1, a_2, a_3, a_4\}$.

For $L_2(S_{2,0})$, we have that $W_2 = \{x_1y_1, x_1x_2, x_1y_2, y_1x_2, y_1y_2, x_2y_2\}$ is the set of Lyndon words of length 2, where $x_1y_1 \prec x_1x_2 \prec x_1y_2 \prec y_1x_2 \prec y_1y_2 \prec x_2y_2$. We have the following basis $C_2 = \{a_1, a_2, ..., a_6\}$, where

$$a_1 = [x_1, y_1]\,\gamma_3(F_4),\ a_2 = [x_1, x_2]\,\gamma_3(F_4),\ a_3 = [x_1, y_2]\,\gamma_3(F_4),$$
$$a_4 = [y_1, x_2]\,\gamma_3(F_4),\ a_5 = [y_1, y_2]\,\gamma_3(F_4),\ a_6 = [x_2, y_2]\,\gamma_3(F_4),$$

for $L_2(F_4) = \gamma_2(F_4)/\gamma_3(F_4) \simeq \mathbb{Z}^6$. Since $S_2 = \{r_2\}$, and

$$\begin{aligned}
D^0_{x_1y_1}(r_{2,1}) &= \psi\left(D_{x_1}\left(D_{y_1}(r_{2,1})\right)\right)\\
&= \psi\left(D_{x_1}\left(-x_1^{-1}y_1^{-1} + x_1^{-1}y_1^{-1}x_1\right)\right)\\
&= \psi\left(x_1^{-1} - x_1^{-1} + x_1^{-1}y_1^{-1}\right) = 1,\\
D^0_{x_1x_2}(r_{2,1}) &= \psi\left(D_{x_1}\left(D_{x_2}(r_{2,1})\right)\right)\\
&= \psi\left(x_1^{-1} - x_1^{-1}y_1^{-1} - x_1^{-1} + x_1^{-1}y_1^{-1}\right) = 0,\\
D^0_{x_1y_2}(r_{2,1}) &= \psi\left(D_{x_1}\left(D_{y_2}(r_{2,1})\right)\right)\\
&= \psi\left(x_1^{-1} - x_1^{-1}y_1^{-1} - x_1^{-1} + x_1^{-1}y_1^{-1}\right) = 0,\\
D^0_{y_1x_2}(r_{2,1}) &= \psi\left(D_{y_1}\left(D_{x_2}(r_{2,1})\right)\right)\\
&= \psi\left(x_1^{-1}y_1^{-1} - x_1^{-1}y_1^{-1}x_1 - x_1^{-1}y_1^{-1} - x_1^{-1}y_1^{-1}x_1\right) = 0,\\
D^0_{y_1y_2}(r_{2,1}) &= \psi\left(D_{y_1}\left(D_{y_2}(r_{2,1})\right)\right)\\
&= \psi\left(x_1^{-1}y_1^{-1} - x_1^{-1}y_1^{-1}x_1 - x_1^{-1}y_1^{-1} - x_1^{-1}y_1^{-1}x_1\right) = 0,\\
D^0_{x_2y_2}(r_{2,1}) &= \psi\left(D_{x_2}\left(D_{y_2}(r_{2,1})\right)\right) = \psi\left([x_1, y_1]\,x_2^{-1}y_2^{-1}\right) = 1,
\end{aligned}$$

we have

$\diagdown$	$D^0_{x_1y_1}$	$D^0_{x_1x_2}$	$D^0_{x_1y_2}$	$D^0_{y_1x_2}$	$D^0_{y_1y_2}$	$D^0_{x_2y_2}$
$r_{3,1}$	1	0	0	0	0	1

so, from Lemma 5.2, we obtain the following presentation matrix for $L_2(S_{2,0})$:

$$M_2(S_{2,0}) = \begin{pmatrix}1\ 0\ 0\ 0\ 0\ 1\end{pmatrix}.$$

This matrix has the following Smith normal form

$$SM_2(S_{2,0}) = \begin{pmatrix}1\ 0\ 0\ 0\ 0\ 0\end{pmatrix}.$$

By taking $z_1 = a_1 + a_6$, $z_2 = a_2$, $z_3 = a_3$, $z_4 = a_4$, $z_5 = a_5$, $z_6 = a_6$, we see that

$$\begin{aligned}
L_2(S_{2,0}) &= \{a_1, a_2, \ldots, a_6 \mid a_1 + a_6 = 0\}\\
&= \{z_2, z_3, \ldots, z_6 \mid \quad\} \simeq \mathbb{Z}^5,
\end{aligned}$$

so $B_2 = \{z_2, z_3, \ldots, z_6\}$ is a basis for $L_2(S_{2,0})$, and if $r_{3,1} = [r_{2,1}, x_1]$, $r_{3,2} = [r_{2,1}, y_1]$, $r_{3,3} = [r_{2,1}, x_2]$, $r_{3,4} = [r_{2,1}, y_2]$, we have

$$S_3 = \{r_{3,1}, r_{3,2}, r_{3,3}, r_{3,4}\}.$$

For $L_3(S_{2,0})$, we have that $W_3 = \{x_1x_1y_1,\ x_1x_1x_2,\ x_1x_1y_2,\ x_1y_1y_1,\ x_1y_1x_2,\ x_1x_2y_1,\ x_1y_1y_2,\ x_1y_2y_1,\ x_1x_2x_2,\ x_1x_2y_2,\ x_1y_2x_2,\ x_1y_2y_2,\ y_1y_1x_2,\ y_1y_1y_2,\ y_1x_2x_2,\ y_1x_2y_2,\ y_1y_2x_2,\ y_1y_2y_2,\ x_2x_2y_2,\ x_2y_2y_2\}$ is the set of Lyndon words of length 3 ordered lexicographically. In order to compute all Fox derivatives, we need an additional property of the derivative, Property (3.3) on p. 90 in [18]. That is, if $X = \{x_1, x_2, \ldots, x_n\}$ is an ordered alphabet, X^* is the set of all words over the alphabet X, and X^*_m is the set of all words of length m, then for $a = a_1a_2\cdots a_m \in X^*_m$, $u \in \gamma_k(F_n)$, $v \in \gamma_l(F_n)$, where $k + l = m$, we have

$$D^0_a([u,v]) = D^0_{a_1a_2\ldots a_k}(u)\, D^0_{a_{k+1}a_{k+2}\ldots a_m}(v) - D^0_{a_1a_2\ldots a_l}(v)\, D^0_{a_{l+1}a_{l+2}\ldots a_m}(u).$$

Therefore, since $S_3 = \{r_{3,1}, r_{3,2}, r_{3,3}, r_{3,4}\}$ and $r_{2,1} \in \gamma_2(F_4)$, we have, for example,

$$\begin{aligned}
D^0_{x_1x_1y_1}(r_{3,1}) &= D^0_{x_1x_1y_1}([r_{2,1}, x_1]) \\
&= D^0_{x_1x_1}(r_{2,1})\, D^0_{y_1}(x_1) \\
&\quad - D^0_{x_1}(x_1)\, D^0_{x_1y_1}(r_{2,1}) \\
&= -D^0_{x_1}(x_1)\, D^0_{x_1y_1}(r_{2,1}) = -1, \\
D^0_{x_1x_1x_2}(r_{3,1}) &= D^0_{x_1x_1x_2}([r_{2,1}, x_1]) \\
&= D^0_{x_1x_1}(r_{2,1})\, D^0_{x_2}(x_1) - D^0_{x_1}(x_1)\, D^0_{x_1x_2}(r_{2,1}) \\
&= -D^0_{x_1}(x_1)\, D^0_{x_1x_2}(r_{2,1}) = 0, \\
D^0_{x_1x_1y_2}(r_{3,1}) & \\
&= D^0_{x_1x_1y_2}([r_{2,1}, x_1]) \\
&= D^0_{x_1x_1}(r_{2,1})\, D^0_{y_2}(x_1) - D^0_{x_1}(x_1)\, D^0_{x_1y_2}(r_{2,1}) \\
&= -D^0_{x_1}(x_1)\, D^0_{x_1y_2}(r_{2,1}) = 0, \\
D^0_{x_1y_1y_1}(r_{3,1}) &= D^0_{x_1y_1y_1}([r_{2,1}, x_1]) \\
&= D^0_{x_1y_1}(r_{2,1})\, D^0_{y_1}(x_1) - D^0_{x_1}(x_1)\, D^0_{y_1y_1}(r_{2,1}) \\
&= -D^0_{x_1}(x_1)\, D^0_{y_1y_1}(r_{2,1}) = 0, \\
D^0_{x_1y_1x_2}(r_{3,1}) &= D^0_{x_1y_1x_2}([r_{2,1}, x_1]) \\
&= D^0_{x_1y_1}(r_{2,1})\, D^0_{x_2}(x_1) - D^0_{x_1}(x_1)\, D^0_{y_1x_2}(r_{2,1}) \\
&= -D^0_{x_1}(x_1)\, D^0_{y_1x_2}(r_{2,1}) = 0,
\end{aligned}$$

and, similarly, we find the remaining Fox derivatives. We obtain the following values for Fox derivatives:

$\diagdown$	$r_{3,1}$	$r_{3,2}$	$r_{3,3}$	$r_{3,4}$
$D^0_{x_1x_1y_1}$	-1	0	0	0
$D^0_{x_1x_1x_2}$	0	0	0	0
$D^0_{x_1x_1y_2}$	0	0	0	0
$D^0_{x_1y_1y_1}$	0	1	0	0
$D^0_{x_1y_1x_2}$	0	0	1	0
$D^0_{x_1x_2y_1}$	0	0	0	0
$D^0_{x_1y_1y_2}$	0	0	0	1
$D^0_{x_1y_2y_1}$	0	0	0	0
$D^0_{x_1x_2x_2}$	0	0	0	0
$D^0_{x_1x_2y_2}$	-1	0	0	0

$\diagdown$	$r_{3,1}$	$r_{3,2}$	$r_{3,3}$	$r_{3,4}$
$D^0_{x_1y_2x_2}$	1	0	0	0
$D^0_{x_1y_2y_2}$	0	0	0	0
$D^0_{y_1y_1x_2}$	0	0	0	0
$D^0_{y_1y_1y_2}$	0	0	0	0
$D^0_{y_1x_2x_2}$	0	0	0	0
$D^0_{y_1x_2y_2}$	0	-1	0	0
$D^0_{y_1y_2x_2}$	0	1	0	0
$D^0_{y_1y_2y_2}$	0	0	0	0
$D^0_{x_2x_2y_2}$	0	0	-1	0
$D^0_{x_2y_2y_2}$	0	0	0	1

It follows that the matrix of a presentation for the group $L_3(S_{2,0})$ is given by

$$M_3(S_{2,0}) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let

a_1	$=$	$[x_1,\ [x_1,\ y_1]]\,\gamma_4(F_4)$	a_6	$=$	$[[x_1,\ x_2],\ y_1]\,\gamma_4(F_4)$
a_2	$=$	$[x_1,\ [x_1,\ x_2]]\,\gamma_4(F_4)$	a_7	$=$	$[x_1,\ [y_1,\ y_2]]\,\gamma_4(F_4)$
a_3	$=$	$[x_1,\ [x_1,\ y_2]]\,\gamma_4(F_4)$	a_8	$=$	$[[x_1,\ y_2],\ y_1]\,\gamma_4(F_4)$
a_4	$=$	$[[x_1,\ y_1],\ y_1]\,\gamma_4(F_4)$	a_9	$=$	$[[x_1,\ x_2],\ x_2]\,\gamma_4(F_4)$
a_5	$=$	$[x_1,\ [y_1,\ x_2]]\,\gamma_4(F_4)$	a_{10}	$=$	$[x_1,\ [x_2,\ y_2]]\,\gamma_4(F_4)$

.

a_{11}	$=$	$[[x_1,\ y_2],\ x_2]\,\gamma_4(F_4)$	a_{16}	$=$	$[y_1,\ [x_2,\ y_2]]\,\gamma_4(F_4)$
a_{12}	$=$	$[[x_1,\ y_2],\ y_2]\,\gamma_4(F_4)$	a_{17}	$=$	$[[y_1,\ y_2],\ x_2]\,\gamma_4(F_4)$
a_{13}	$=$	$[y_1,\ [y_1,\ x_2]\,\gamma_4(F_4)]$	a_{18}	$=$	$[[y_1,\ y_2],\ y_2]\,\gamma_4(F_4)$
a_{14}	$=$	$[y_1,\ [y_1,\ y_2]]\,\gamma_4(F_4)$	a_{19}	$=$	$[x_2,\ [x_2,\ y_2]]\,\gamma_4(F_4)$
a_{15}	$=$	$[[y_1,\ x_2],\ x_2]\,\gamma_4(F_4)$	a_{20}	$=$	$[[x_2,\ y_2],\ y_2]\,\gamma_4(F_4)$

We have the following basis $C_3 = \{a_i \mid i = 1, 2, \dots, 20\}$ for $L_3(F_4) = \gamma_3(F_4)/\gamma_4(F_4) \simeq \mathbb{Z}^{20}$. From the presentation matrix we obtain the following presentation for the group $L_3(S_{2,0})$:

$$L_3(S_{2,0}) = \{a_1, \dots, a_{20} \mid -a_1 - a_{10} + a_{11} = 0,\ a_4 - a_{11} + a_{12} = 0,\\ a_5 - a_{19} = 0,\ a_7 + a_{20} = 0\}.$$

Smith normal form for the matrix $M_3\left(S_{2,0}\right)$ is

$$SM_3\left(S_{2,0}\right)=\begin{pmatrix}1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\end{pmatrix}=M_3\left(S_{2,0}\right)V,$$

where $V^{-1}=\left(v_{i,j}\right)_{1\leq i,j\leq 20}$ and V^{-1} has the following nonzero entries $v_{1,1}=-1$, $v_{1,10}=-1$, $v_{1,11}=1$, $v_{2,4}=1$, $v_{2,11}=-1$, $v_{2,12}=1$, $v_{3,5}=1$, $v_{3,19}=-1$, $v_{4,7}=1$, $v_{4,20}=1$, $v_{5,3}=1$, $v_{6,6}=1$, $v_{7,2}=1$, $v_{8,8}=...=v_{20,20}=1$. We let $z_1=-a_1-a_{10}+a_{11}$, $z_2=a_4-a_{11}+a_{12}$, $z_3=a_5-a_{19}$, $z_4=a_7+a_{20}$, $z_5=a_3$, $z_6=a_6$, $z_7=a_2$, $z_8=a_8$, $\ldots$, $z_{20}=a_{20}$. In the new generators, the presentation of $L_3\left(S_{2,0}\right)$ is

$$\begin{aligned}L_3\left(S_{2,0}\right)&=\left\{z_1,z_2,\ldots,z_{20}\mid z_1=0,\ z_2=0,\ z_3=0,\ z_4=0\right\}\\&=\left\{z_5,z_6,...,z_{20}\mid \qquad\right\}\simeq\mathbb{Z}^{16},\end{aligned}$$

so $B_3=\left\{z_5,z_6,\ldots,z_{20}\right\}$ is a basis for $L_3\left(S_{2,0}\right)$. If

$$\begin{aligned}r_{4,1}&=\left[r_{3,1},x_1\right],\ r_{4,2}=\left[r_{3,1},y_1\right],\ r_{4,3}=\left[r_{3,1},x_2\right],r_{4,4}=\left[r_{3,1},y_2\right],\\r_{4,5}&=\left[r_{3,2},x_1\right],\ r_{4,6}=\left[r_{3,2},y_1\right],\ r_{4,7}=\left[r_{3,2},x_2\right],r_{4,8}=\left[r_{3,2},y_2\right],\\r_{4,9}&=\left[r_{3,3},x_1\right],\ r_{4,10}=\left[r_{3,3},y_1\right],\ r_{4,11}=\left[r_{3,3},x_2\right],r_{4,12}=\left[r_{3,3},y_2\right],\\r_{4,13}&=\left[r_{3,4},x_1\right],\ r_{4,14}=\left[r_{3,4},y_1\right],\ r_{4,15}=\left[r_{3,4},x_2\right],r_{4,16}=\left[r_{3,4},y_2\right],\end{aligned}$$

we have

$$S_4=\left\{r_{4,1},r_{4,2},\ldots,r_{4,16}\right\}.$$

We will end our computations at this place since the presentation matrix for $L_4\left(S_{2,0}\right)$ is a 16×60 matrix.

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