LEGENDRIAN VERTICAL CIRCLES IN SMALL SEIFERT SPACES

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Abstract. We discuss the relations between the e_0 invariant of a tight contact small Seifert space and the twisting numbers of Legendrian vertical circles in it, and apply these relations to classify up to isotopy tight contact structures on small Seifert spaces with $e_0 \neq 0, -1, -2$.

1. Introduction and Statement of Results

A contact structure ξ on an oriented 3-manifold M is a nowhere integrable tangent plane distribution, i.e., near any point of M, ξ is defined locally by a 1-form α, s.t., $\alpha \wedge d\alpha \neq 0$. Note that the orientation of M given by $\alpha \wedge d\alpha$ depends only on ξ, not on the choice of α. ξ is said to be positive if this orientation agrees with the native orientation of M, and negative if not. A contact structure ξ is said to be co-orientable if ξ is defined globally by a 1-form α. Clearly, an co-orientable contact structure is orientable as a plane distribution, and a choice of α determines an orientation of ξ. Unless otherwise specified, all contact structures in this paper will be co-oriented and positive, i.e., with a prescribed defining form α such that $\alpha \wedge d\alpha > 0$. A curve in M is said to be Legendrian if it is tangent to ξ everywhere. ξ is said to be overtwisted if there is an embedded disk D in M such that ∂D is Legendrian, but D is transversal to ξ along ∂D. A contact structure that is not overtwisted is called tight. Overtwisted contact structures appear to be very "soft". It is proven by Eliashberg in [2] that two overtwisted contact structures are isotopic *iff* they are homotopic as tangent plane distributions. Tight contact structures are more rigid. Classifications of tight contact structures up to isotopy are only know for very limited classes of 3-manifolds. (See, e.g., [3], [4], [5], [6], [7], [12], [13].)

A small Seifert space is a Seifert fibred space with 3 singular fibers over S^2. Any regular fiber f in a small Seifert space $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ admits a canonical framing given by pulling back an arc in the base S^2 containing the projection of f. An embedded circle in $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ is said to be vertical if it is isotopic to a regular fiber. Any vertical circle inherits a canonical framing from the canonical framing of regular fibers. We call this framing Fr.

Definition 1.1. Let ξ be a contact structure on a small Seifert space $M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, and L a Legendrian vertical circle in (M, ξ). The twisting number $t(L)$ of L is the twisting number of $\xi|_L$ along L relative to the canonical framing Fr of L.

In [1], Colin, Giroux and Honda divided the tight contact structures on a small Seifert space into two types: those for which there exists a Legendrian vertical circle with twisting number 0, and those for which no Legendrian vertical circles with twisting number 0 exist. It is proven in [1] that, up to isotopy, the number of tight contact structures of the first type is always finite, and, unless the small Seifert space is also a torus bundle, the number of tight contact structures of the second type is finite too. Their work gives in principle a method to estimate roughly the upper bound of the number of tight contact structures on a small Seifert space. In the present paper, we demonstrate that most small Seifert spaces admit only one of the two types of tight contact structures. To make our claim precise, we need the following invariant. (See [10].)

Definition 1.2. For a small Seifert space $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, define $e_0(M) = \lfloor \frac{q_1}{p_1} \rfloor + \lfloor \frac{q_2}{p_2} \rfloor + \lfloor \frac{q_3}{p_3} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer not greater than x.

Clearly, $e_0(M)$ is an invariant of M, i.e., it does not depend on the choice of the representatives $(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$. Now we can formulate our claim precisely in the following two theorems.

Theorem 1.3. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space. If $e_0(M) \geq 0$, then every tight contact structure on M admits a Legendrian vertical circle with twisting number 0.*

Theorem 1.4. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space. If $e_0(M) \leq -2$, then no tight contact structures on M admit Legendrian vertical circles with twisting number 0.*

In particular, Theorem 1.4 means that, for any small Seifert space $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$, either M or $-M$ does not admit tight contact structures for which there exists a Legendrian vertical circle with twisting number 0, where $-M$ is M with reversed orientation. This is because that $e_0(M) + e_0(-M) = -3$, and, hence, one of $e_0(M)$ and $e_0(-M)$ has to be less than or equal to -2.

It turns out that the case when $e_0(M) = -1$ is the most difficult. Only very weak partial results are known. For example, in [7], Ghiggini and Schönenberger proved that, when $r \leq \frac{1}{5}$, no tight contact structures on the small Seifert space $M(r, \frac{1}{3}, -\frac{1}{2})$ admit Legendrian vertical circles with twisting number 0.

We have following results about the case $e_0(M) = -1$.

Theorem 1.5. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space such that $0 < q_1 < p_1$, $0 < q_2 < p_2$ and $-p_3 < q_3 < 0$.*

(1) If $\frac{q_1}{p_1} + \frac{q_3}{p_3} \geq 0$ or $\frac{q_2}{p_2} + \frac{q_3}{p_3} \geq 0$ or $\frac{q_1}{p_1} + \frac{q_2}{p_2} \geq 1$, then every tight contact structure on M admits a Legendrian vertical circle with twisting number 0.

(2) If $q_3 = -1$, $\frac{q_1}{p_1} < \frac{1}{2p_3 - 1}$ and $\frac{q_2}{p_2} < \frac{1}{2p_3}$, then no tight contact structures on M admit Legendrian vertical circles with twisting number 0.

(3) If $q_1 = q_2 = 1$ and $p_1, p_2 > -2\lfloor \frac{p_3}{q_3} \rfloor$, then no tight contact structures on M admit Legendrian vertical circles with twisting number 0.

As an application of Theorems 1.3 and 1.4, we classify up to isotopy tight contact structures on small Seifert spaces with $e_0 \neq 0, -1, -2$.

For a rational number $r > 0$, there is a unique way to expand $-r$ into a continued fraction

$$-r = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cdots \cfrac{1}{a_{m-1} - \frac{1}{a_m}}}}, \tag{1}$$

where all a_j's are integers, $a_0 = -(\lfloor r \rfloor + 1) \leq -1$, and $a_j \leq -2$ for $j \geq 1$. We denote by $< a_0, a_1, \cdots, a_m >$ the right hand side of equation (1).

Theorem 1.6. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3})$ be a small Seifert space, where p_i, q_i and e_0 are positive integers, s.t., $p_i > q_i$ and g.c.d.$(p_i, q_i) = 1$. Assume that, for $i = 1, 2, 3$, $-\frac{p_i}{q_i} =< b_0^{(i)}, b_1^{(i)}, \cdots, b_{l_i}^{(i)} >$, where all $b_j^{(i)}$'s are integers less than or equal to -2. Then, up to isotopy, there are exactly $|\prod_{i=1}^3 b_0^{(i)} \prod_{j=1}^{l_i}(b_j^{(i)}+1)|$ tight contact structures on M. All these tight contact structures are constructed by Legendrian surgeries of (S^3, ξ_{st}), and are therefore holomorphically fillable contact structures distinguished by their Heegaard Floer invariants.*

Theorem 1.7. *Let $M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$ be a small Seifert space, where p_i and q_i are integers, s.t., $p_i \geq 2$, $q_i \geq 1$ and g.c.d.$(p_i, q_i) = 1$. Assume that, for $i = 1, 2, 3$, $-\frac{q_i}{p_i} =< a_0^{(i)}, a_1^{(i)}, \cdots, a_{m_i}^{(i)} >$, where all $a_j^{(i)}$'s are integers, $a_0^{(i)} = -(\lfloor \frac{q_i}{p_i} \rfloor + 1) \leq -1$, and $a_j^{(i)} \leq -2$ for $j \geq 1$. Then, up to isotopy, there are exactly $|(e_0(M) + 1)\prod_{i=1}^3 \prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ tight contact structures on M. All these tight contact structures are constructed by Legendrian surgeries of (S^3, ξ_{st}), and are therefore holomorphically fillable contact structures distinguished by their Heegaard Floer invariants.*

To understand the proofs in this paper, readers need to be familiar with the techniques developed by Giroux in [8] and Honda in [12]. For those who are not, there is a concise introduction to these techniques in [7].

2. The Twisting Number of Legendrian Vertical Circles

In the rest of this paper, we let Σ be a three hole sphere, and $-\partial\Sigma \times S^1 = T_1 + T_2 + T_3$, where the "$-$" sign means reversing the orientation. We identify T_i to $\mathbb{R}^2/\mathbb{Z}^2$ by identifying the corresponding component of $-\partial\Sigma \times \{\text{pt}\}$ to $(1,0)^T$, and $\{\text{pt}\} \times S^1$ to $(0,1)^T$. Also, for $i = 1, 2, 3$, let $V_i = D^2 \times S^1$, and identify ∂V_i with $\mathbb{R}^2/\mathbb{Z}^2$ by identifying a meridian $\partial D^2 \times \{\text{pt}\}$ with $(1,0)^T$ and a longitude $\{\text{pt}\} \times S^1$ with $(0,1)^T$.

Following Honda, we call a convex torus minimal if it has only two dividing curves.

2.1. The $e_0 \geq 0$ Case.

The $e_0 \geq 0$ case is the simplest case. Theorem 1.3 is a special case of Lemma 2.2, which also implies part (1) of Theorem 1.5.

The following lemma is purely technical.

Lemma 2.1. *Let ξ be a tight contact structure on $\Sigma \times S^1$. Assume that each T_i is minimal convex with dividing curves of slope s_i. Then there exist collar neighborhoods $T_1 \times I$ and $T_2 \times I$ of T_1 and T_2, and a properly embedded vertical convex annulus A in*

$(\Sigma \times S^1) \setminus (T_1 \times I \cup T_2 \times I)$ *connecting* $T_1 \times \{1\}$ *to* $T_2 \times \{1\}$ *with Legendrian boundary satisfying that following:*

(1) $T_1 \times I$ *and* $T_2 \times I$ *are mutually disjoint and disjoint from* T_3*;*
(2) *for* $i = 1, 2$, $T_i \times \{0\} = T_i$ *and* $T_i \times \{1\}$ *is minimal convex with dividing curves of slope* $s'_i \leq s_i$*;*
(3) A *has no* ∂*-parallel dividing curves, and the Legendrian boundary of* A *intersects the dividing sets of* $T_1 \times \{1\}$ *and* $T_2 \times \{1\}$ *efficiently.*

Proof. If both s_1 and s_2 are ∞, then we can isotope T_1 and T_2 slightly to have vertical Legendrian divides. Connect a Legendrian divide of T_1 to a Legendrian divide of T_2 by a properly embedded vertical convex annulus A. Then we are done. If $s_1 = \infty$, but s_2 is finite, then we make T_1 to have vertical Legendrian divides, and T_2 to have vertical Legendrian rulings. Connect a Legendrian divide of T_1 to a Legendrian ruling of T_2 by a properly embedded vertical convex annulus B. Then no dividing curves of B intersects $B \cap T_1$. And we can decrease s_2 to ∞ by isotoping T_2 across the dividing curves of B starting and ending on $B \cap T_2$ through bypass adding. We can keep T_2 disjoint from both T_1 and T_3 through out the isotopy since bypass adding can be done in a small neighborhood of the bypass and the original surface. Then we are back to the case when s_1 and s_2 are both ∞.

Assume $s_i = \frac{q_i}{p_i}$ is finite for $i = 1, 2$, where $p_i > 0$. First, we isotope T_1 and T_2 slightly so that they have vertical Legendrian rulings. Note that the Legendrian rulings always intersect dividing curves efficiently. Then connect a Legendrian ruling of T_1 to a Legendrian ruling of T_2 by a properly embedded vertical convex annulus A in $\Sigma \times S^1$. If A has no ∂-parallel dividing curves, then we are done. If A has a ∂-parallel dividing curve, say on the T_1 side, then, after possibly isotoping A slightly, we can assume there is a bypass of T_1 on A. Adding this bypass to T_1, we get a minimal convex torus T'_1 in $\Sigma \times S^1$ that co-bounds a collar neighborhood of T_1. We can make T'_1 to have vertical Legendrian ruling. By Lemma 3.5 of [12], we have that the slope of the dividing curves of T'_1 is $s'_1 = \frac{q'_1}{p'_1} < s_1$, where $0 \leq p'_1 < p_i$. Now we delete the thickened torus between T_1 and T'_1 from $\Sigma \times S^1$, and repeat the procedure above. This whole process will stop in less than $p_1 + p_2$ steps, i.e, we can either find the collar neighborhoods and the annulus with the required properties, or force one of s_1 and s_2 to decrease to ∞. But the lemma is proved in the latter case. This finishes the proof. $\square$

Lemma 2.2. *Let* $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ *be a small Seifert space such that* $\frac{q_1}{p_1}, \frac{q_2}{p_2} > 0$ *and* $\frac{q_1}{p_1} + \frac{q_3}{p_3} \geq 0$*. Then every tight contact structure on* M *admits a Legendrian vertical circle with twisting number* 0*.*

Proof. Choose $u_i, v_i \in \mathbf{Z}$ such that $0 < u_i < p_i$ and $p_i v_i + q_i u_i = 1$ for $i = 1, 2, 3$. Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix}.$$

Then

$$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Let ξ be a tight contact structure on M. We first isotope ξ to make each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $n_i < 0$, i.e., ∂V_i is minimal convex with dividing curves of slope $\frac{1}{n_i}$ when measured in the coordinates of ∂V_i. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{-n_i q_i + v_i}{n_i p_i + u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(n_i p_i + u_i)} < -\frac{q_i}{p_i}.$$

By Lemma 2.1, we can thicken V_1 and V_2 to V_1' and V_2' such that

(1) V_1', V_2' and V_3 are pairwise disjoint;
(2) for $i = 1, 2$, $T_i' = \varphi_i(\partial V_i')$ is minimal convex with dividing curves of slope $s_i' = -\frac{q_i'}{p} \leq s_i$, where $p, q_i' > 0$;
(3) there exists a properly embedded vertical convex annulus A connecting T_1' to T_2' that has no ∂-parallel dividing curves, and the Legendrian boundary of A intersects the dividing sets of these tori efficiently.

If none of the dividing curves of A is an arc connecting the two components of ∂A, then, by the Legendrian Realization Principle ([8], [12]), we can isotope A to make a vertical circle L on A disjoint from the dividing curves Legendrian. Note that A gives the canonical framing of L, and the twisting number of $\xi|_L$ relative to $TA|_L$ is 0 by Proposition 4.5 of [16]. So $t(L) = 0$.

If there are dividing curves connecting the two components of ∂A. Cut $M \setminus (V_1' \cup V_2' \cup V_3)$ open along A. We get an embedded thickened torus $T_3 \times I$ in M such that $T_3 \times \{0\} = T_3$, and $T_3 \times \{1\}$ is minimal convex with dividing curves of slope $s_3' = \frac{q_1'+q_2'-1}{p}$. Note that

$$s_3' = \frac{q_1' + q_2' - 1}{p} \geq \frac{q_1'}{p} \geq -s_1 > \frac{q_1}{p_1} \geq -\frac{q_3}{p_3} > s_3.$$

According to Theorem 4.16 of [12], there exists a convex torus T in $T_3 \times I$ parallel to T_3 with vertical dividing curves. We can then isotope T to make it in standard form. Then a Legendrian divide of T is a Legendrian vertical circle with twisting number 0. □

Proof of Theorem 1.3 and Theorem 1.5(1). If $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ satisfies that $e_0(M) \geq 0$, then we can assume that $\frac{q_i}{p_i} > 0$ for $i = 1, 2, 3$. It's then clear that $\frac{q_1}{p_1}, \frac{q_2}{p_2} > 0$ and $\frac{q_1}{p_1} + \frac{q_3}{p_3} > 0$. Thus, Lemma 2.2 implies Theorem 1.3.

Now we assume $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ is a small Seifert space such that $0 < q_1 < p_1$, $0 < q_2 < p_2$ and $-p_3 < q_3 < 0$. If $\frac{q_1}{p_1} + \frac{q_3}{p_3} \geq 0$ or $\frac{q_2}{p_2} + \frac{q_3}{p_3} \geq 0$, then Lemma 2.2 applies directly. If $\frac{q_1}{p_1} + \frac{q_2}{p_2} \geq 1$, we apply Lemma 2.2 to $M = M(\frac{q_1}{p_1}, \frac{q_3}{p_3} + 1, \frac{q_2}{p_2} - 1)$. This proves Theorem 1.5(1). □

2.2. The $e_0 \leq -2$ Case.

Definition 2.3. Let ξ be a contact structure on $\Sigma \times S^1$. ξ is said to be inappropriate if ξ is overtwisted, or there exists an embedded $T^2 \times I$ with convex boundary and

I-twisting at least π such that $T^2 \times \{0\}$ is isotopic to one of the T_i's. ξ is called appropriate if it is not inappropriate.

Lemma 2.4. *Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space, and ξ a tight contact structures on M. Suppose that V_1, V_2, V_3 are tubular neighborhoods of the three singular fibers, and $\Sigma \times S^1 = M \setminus (V_1 \cup V_2 \cup V_3)$. Then $\xi|_{\Sigma \times S^1}$ is appropriate.*

Proof. Without loss of generality, we assume ∂V_i is identified with T_i by the diffeomorphism φ_i. $\xi|_{\Sigma\times S^1}$ is clearly tight. If it is inappropriate, then there exists an embedded $T^2 \times I$ with convex boundary and I-twisting at least π such that $T^2 \times \{0\}$ is isotopic to one of the T_i's. Let's say $T^2 \times \{0\}$ is isotopic to T_1. $T^2 \times I$ has I-twisting at least π implies that, for any rational slope s, there is a convex torus T_0 contained in $T^2 \times I$ isotopic to T_1 that has dividing curves of slope s. Specially, we let m be a meridian of ∂V_1, and s the slope of $\varphi_1(m)$. Then the above fact means that we can thicken V_1 so that ∂V_1 has dividing curves isotopic to its meridians, which implies that the thickened V_1 is overtwisted. This contradicts the tightness of ξ. Thus, $\xi|_{\Sigma\times S^1}$ must be appropriate. □

Lemma 2.5 ([5], Lemma 10)**.** *Let ξ be an appropriate contact structure on $\Sigma \times S^1$ such that all three boundary components of $\Sigma \times S^1$ are minimal convex with vertical dividing curves. If Σ_0 is convex with Legendrian boundary that intersects the dividing set of $\partial\Sigma\times S^1$ efficiently, then the dividing set of Σ_0 consists of three properly embedded arcs, each of which connects a different pair of components of $\partial\Sigma_0$.*

The following is a special case of Proposition 6.4 of [1], which also appears in [9] and [13].

Lemma 2.6 ([1], [9], [13])**.** *Isotopy classes of tight contact structures on $\Sigma \times S^1$ such that all three boundary components of $\Sigma\times S^1$ are minimal convex with vertical dividing curves are in 1-1 correspondence with isotopy classes of embedded multi-curves on Σ with 2 fixed end points on each component of $\partial\Sigma$ that have no homotopically trivial components.*

The following lemma from [7] plays a key role in the proof of Theorem 1. For the convenience of readers, we give a detailed proof here.

Lemma 2.7 ([7], Lemma 36)**.** *Let ξ be an appropriate contact structure on $\Sigma \times S^1$. Suppose that T_1, T_2 and T_3 are minimal convex and such that T_1 and T_2 have vertical Legendrian rulings and dividing curves of slope $-\frac{1}{n}$, where $n \in \mathbf{Z}^{>0}$, and T_3 has vertical dividing curves. Let $T_1 \times I$ and $T_2 \times I$ be collar neighborhoods of T_1 and T_2 that are mutually disjoint and disjoint from T_3, and such that, for $i = 1, 2$, $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is minimal convex with vertical dividing curves. If $\xi|_{T_1\times I}$ and $\xi|_{T_2\times I}$ are both isotopic to a given minimal twisting tight contact structure η on $T^2 \times I$ relative to the boundary, then there exists a properly embedded convex vertical annulus A with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap T_1) \cup (A \cap T_2)$ intersects the dividing curves of T_1 and T_2 efficiently.*

Proof. Let $\Sigma' \times S^1 = (\Sigma \times S^1) \setminus [(T_1 \times [0, 1)) \cup (T_2 \times [0, 1))]$, and Σ'_0 a properly embedded convex surface in $\Sigma'\times S^1$ isotopic to $\Sigma'\times\{\mathrm{pt}\}$ that has Legendrian boundary intersecting the dividing set of $\partial\Sigma' \times S^1$ efficiently. Since $\xi|_{\Sigma'\times S^1}$ is appropriate, the dividing set of Σ'_0 consists of three properly embedded arc, each of which connects a different pair of boundary components of Σ'_0. Up to isotopy relative to $\partial\Sigma'_0$, there are infinitely many such multi-arcs on Σ'_0. But, up to isotopy of Σ'_0 which leaves $\partial\Sigma'_0$ invariant, there are only two, each represented by a diagram in Figure 1. Such an isotopy of Σ'_0 extends to an isotopy of $\Sigma' \times S^1$ which, when restricted on a component of $\partial\Sigma' \times S^1$, is a horizontal rotation. Thus, up to isotopy of $\Sigma' \times S^1$, which, when restricted on a component of $\partial\Sigma' \times S^1$, is a horizontal rotation, there are only two appropriate contact structures on $\Sigma' \times S^1$. Now let Φ_t be such an isotopy of $\Sigma' \times S^1$ changing $\xi|_{\Sigma'\times S^1}$ to one of the two standard appropriate contact structures. We extend Φ_t to an isotopy $\widetilde{\Phi}_t$ of $\Sigma \times S^1$, which fixes a neighborhood of $T_1 \cup T_2$, and leaves $T_1 \times I$, $T_2 \times I$ and $\Sigma' \times S^1$ invariant. Note that the relative Euler class of $\xi|_{T_i\times I}$ is $(2k-n, 0)^T$, where k is the number of positive basic slices contained in $(T^2 \times I, \eta)$, and is invariant under $\widetilde{\Phi}_t|_{T_i\times I}$. So $\xi|_{T_i\times I}$ and $\widetilde{\Phi}_{1*}(\xi)|_{T_i\times I}$ have the same relative Euler class, and are both continued fraction blocks satisfying the same boundary condition. According to the classification of tight contact structures on $T^2 \times I$, $\xi|_{T_i\times I}$ and $\widetilde{\Phi}_{1*}(\xi)|_{T_i\times I}$ are isotopic relative to boundary. So $\widetilde{\Phi}_{1*}(\xi)$ satisfies the conditions given in the lemma, and is of one of the two standard form. Thus, up to isotopy fixing T_1, T_2 and leaving T_3 invariant, there are only two appropriate contact structures on $\Sigma \times S^1$ satisfying the given conditions. Rotating the diagram on the left of Figure 1 by $180°$ induces a self-diffeomorphism of $\Sigma \times S^1$ mapping T_1 to T_2 and changing the dividing set of Σ'_0 on the left of Figure 1 to the one on the right. So this self-diffeomorphism is isotopic to a contactomorphism between the two standard appropriate contact structures on $\Sigma \times S^1$. Hence, up to contactomorphism, there is only one such appropriate contact structure on $\Sigma \times S^1$. Thus, we can show the existence of an annulus with the required properties by explicitly constructing such an annulus in a model contact structure on $\Sigma \times S^1$ which satisfies the given conditions.

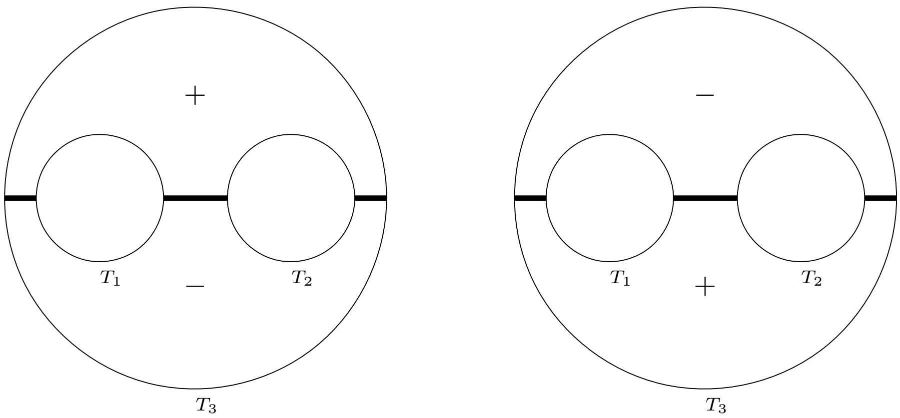

FIGURE 1. The two possible configurations of dividing curves on Σ'_0.

Consider the minimal twisting tight contact structure η on the thickened torus $T^2 \times I$. Note that the vertical Legendrian rulings of $T^2 \times \{0\}$ intersect its dividing curves efficiently. Without loss of generality, we assume that $T^2 \times \{1\}$ has horizontal Legendrian rulings and two vertical Legendrian dividings. We further assume that, for a small $\varepsilon > 0$, $\eta|_{T^2\times[0,\varepsilon]}$ is invariant in the I direction. This is legitimate since $T^2 \times \{0\}$ is convex. So $T^2 \times \{\frac{\varepsilon}{2}\}$ is also a convex torus with vertical Legendrian rulings and dividing curves of slope $-\frac{1}{n}$. Let L be a Legendrian ruling of $T^2 \times \{\frac{\varepsilon}{2}\}$. Since the twisting number of $\eta|_L$ relative to the framing given by $T^2 \times \{\frac{\varepsilon}{2}\}$ is $-n$, we can find a standard neighborhood U of L in $T^2 \times (0, \varepsilon)$ such that ∂U is convex with vertical Legendrian ruling and two dividing curves of slope $-\frac{1}{n}$. Now, we set $\Sigma \times S^1 = (T^2 \times I) \setminus U$, where $T_1 = T^2 \times \{0\}$, $T_2 = \partial U$ and $T_3 = T^2 \times \{1\}$, and let $\xi = \eta|_{\Sigma\times S^1}$. Since η is tight, so is ξ. And there are no embedded thickened tori in $\Sigma \times S^1$ with convex boundary isotopic to T_2 and I-twisting at least π. Otherwise, L would have an overtwisted neighborhood in $T^2\times I$, which contradict the tightness of η. Also, since the I-twisting of η is less than π, there exists no embedded thickened tori in $\Sigma \times S^1$ with convex boundary isotopic to T_1 or T_3 and I-twisting at least π. Thus, ξ is appropriate. Now we choose a vertical convex annulus A_1 in $\Sigma \times S^1$ connecting a Legendrian ruling of T_1 to a Legendrian dividing of T_3, and a vertical convex annulus A_2 in $\Sigma \times S^1$ connecting a Legendrian ruling of T_2 to the other Legendrian dividing of T_3 such that $(T_1\cup A_1)\cap(T_2\cup A_2) = \phi$. The dividing set of A_i consists of n arcs starting and ending on $A_i \cap T_i$. For $i = 1, 2$, we can find a collar neighborhood $T_i \times I$ of T_i, for which $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is convex with dividing set consisting of two circles of slope ∞, by isotoping T_i to engulf all the dividing curves of A_i through bypass adding. Since bypass adding can be done in a small neighborhood of the original surface and the bypass, we can make $T_1 \times I$ and $T_2 \times I$ mutually disjoint and disjoint from T_3. Note that both $T_1 \times I$ and $T_2 \times I$ are minimal twisting. So they are continued fraction blocks satisfying the boundary conditions specified above. Let k_i be the number of positive slices in $T_i \times I$, and $B_i = A_i \cap (T_i \times I)$. Then $2k_i - n = \chi((B_i)_+) - \chi((B_i)_-) = \chi((A_i)_+) - \chi((A_i)_-)$. But $\chi((A_1)_+) - \chi((A_1)_-) = 2k - n$, where k is the number of positive basic slices contained in $(T^2 \times I, \eta)$. So $k_1 = k$. And, since $\eta|_{T^2\times(0,\varepsilon)}$ is I-invariant, we can extend A_2 to a vertical annulus $\widetilde{A}_2$ in $T^2 \times I$ starting at a Legendrian ruling of T_1 and such that $\chi((\widetilde{A}_2)_+) - \chi((\widetilde{A}_2)_-) = \chi((A_2)_+) - \chi((A_2)_-)$. Clearly, $2k - n = \chi((\widetilde{A}_2)_+) - \chi((\widetilde{A}_2)_-)$. So $k_2 = k$. Thus, $k_1 = k_2 = k$. But the isotopy type of a continued fraction block is determined by the number of positive slices in it. Thus, $\xi|_{T_1\times I}$, $\xi|_{T_2\times I}$ and η are isotopic relative to boundary. So our $(\Sigma \times S^1, \xi)$ is indeed a legitimate model. Now we connect a Legendrian ruling of T_1 and a Legendrian ruling of T_2 by a vertical convex annulus A which is contained in $(T^2 \times [0, \varepsilon]) \setminus U$. Then ∂A intersects the dividing sets of T_1 and T_2 efficiently. If A has ∂-parallel diving curves, then $(T^2 \times [0, \varepsilon])$ has non-zero I-twisting, which contradicts our choice of the slice $(T^2 \times [0, \varepsilon])$. Thus, A has no ∂-parallel diving curves. □

Now we are in position to prove Theorem 1.4.

Proof of Theorem 1.4. Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space with $e_0(M) \leq -2$. Without loss of generality, we assume that $p_1, p_2, p_3 > 1$, $0 < q_1 < p_1$, and $q_2, q_3 < 0$. Choose $u_i, v_i \in \mathbf{Z}$ such that $p_iv_i + q_iu_i = 1$ for $i = 1, 2, 3$. Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix}.$$

Then

$$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Assume that ξ is a tight contact structure on M for which there exits a Legendrian vertical circle L in M with twisting number $t(L) = 0$. We first isotope ξ to make $L = \{\mathrm{pt}\} \times S^1 \subset \Sigma \times S^1$, and each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $n_i < 0$, i.e., ∂V_i is minimal convex with dividing curves of slope $\frac{1}{n_i}$ when measured in the coordinates of ∂V_i. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{-n_i q_i + v_i}{n_i p_i + u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(n_i p_i + u_i)}.$$

From our choice of p_i and q_i, one can see that $-1 \le s_1 \le 0$ and $0 \le s_2, s_3 < \infty$. Now, without affecting the properties of L and V_i asserted above, we can further isotope the contact structure ξ to make the Legendrian rulings of T_i to have slope ∞ when measured in the coordinates of T_i.

Pick a Legendrian ruling $\widetilde{L}_i$ on each T_i, and connect L to $\widetilde{L}_i$ by a vertical convex annulus A_i such that $A_i \cap A_j = L$ when $i \neq j$. Let Γ_{A_i} be the dividing set of A_i. Since A_i gives the canonical framing Fr of L, we know that the twisting number of $\xi|_L$ relative to $TA_i|_L$ is 0. This means that $\Gamma_{A_i} \cap L = \phi$. But $\Gamma_{A_i} \cap \widetilde{L}_i \neq \phi$. There are dividing curves of A_i starting and ending on $\widetilde{L}_i$. According to Lemma 3.15 of [12], we can find an embedded minimal twisting slice $T_i \times I$ in $\Sigma \times S^1$, for which $T_i \times \{0\} = T_i$, $T_i \times \{1\}$ is convex with two vertical dividing curves, by isotoping T_i to engulf all the dividing curves of A_i starting and ending on $\widetilde{L}_i$ through bypass adding. Since bypass adding can be done in a small neighborhood of the bypass and the original surface, and the bypasses from different A_i's are mutually disjoint, we can make $T_i \times I$'s pairwise disjoint. By Corollary 4.16 of [12], we can find a minimal convex torus in $T_i \times (0, 1)$ isotopic to T_i with dividing curves of the slope -1. Without loss of generality, we assume that this torus is $T_i \times \{\frac{1}{2}\}$. Moreover, for $i = 2, 3$, we can find another minimal convex torus, say $T_i \times \{\frac{1}{4}\}$, in $T_i \times (0, \frac{1}{2})$ isotopic to T_i with dividing curves of slope 0.

Since the slice $T_i \times I$ is minimal twisting, so is any of its sub-slices. Let's consider the thickened tori $T_i \times [\frac{1}{2}, 1]$. All of these have the same boundary condition, and are minimal twisting. There are only two such tight contact structures up to isotopy relative to boundary. So two of these have to be isotopic relative to boundary. There are 3 cases.

Case 1. $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ are isotopic. We apply Lemma 2.7 to

$$\Sigma' \times S^1 = (\Sigma \times S^1) \setminus (T_1 \times [0, \frac{1}{2}) \cup T_2 \times [0, \frac{1}{2}) \cup T_3 \times [0, 1)).$$

Then there exists a vertical convex annulus A connecting $T_1 \times \{\frac{1}{2}\}$ and $T_2 \times \{\frac{1}{2}\}$ with no ∂-parallel dividing curves that has Legendrian boundary intersecting the dividing

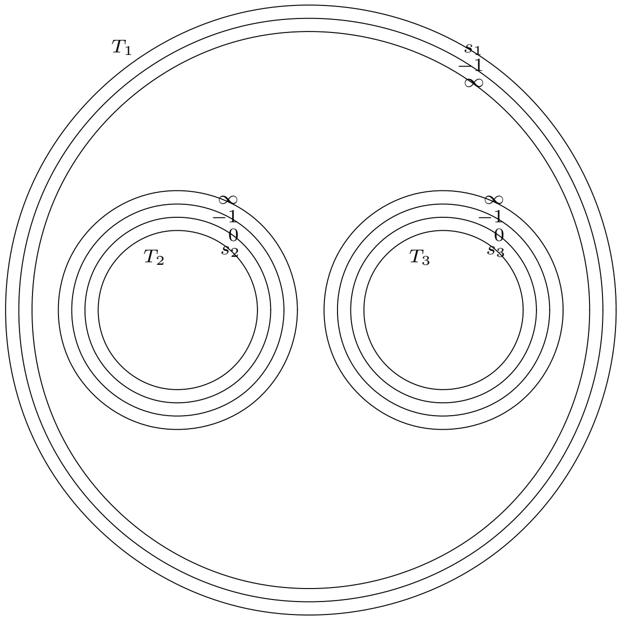

FIGURE 2. Layout of the tori mentioned in the proof of Theorem 1.4 with slopes of dividing curves marked on them.

sets of these tori efficiently. We can extend A across $T_2 \times [\frac{1}{4}, \frac{1}{2}]$ to a convex annulus $\widetilde{A}$ connecting $T_1 \times \{\frac{1}{2}\}$ and $T_2 \times \{\frac{1}{4}\}$ with Legendrian boundary intersecting the dividing sets of these two tori efficiently. Since $T_2 \times [\frac{1}{4}, \frac{1}{2}]$ is minimal twisting, $\widetilde{A} \setminus A$ has no ∂-parallel dividing curves. Thus, $\widetilde{A}$ has no ∂-parallel dividing curves either. Cut $(\Sigma \times S^1) \setminus (T_1 \times [0, \frac{1}{2}) \cup T_2 \times [0, \frac{1}{4}) \cup T_3 \times [0, 1))$ along $\widetilde{A}$, and round the edges. We get a thickened torus $T_3 \times [1, 2]$ embedded in $\Sigma \times S^1$ with minimal convex boundary, where the dividing curves of $T_3 \times \{2\}$ have slope 0. Now we can see that the thickened torus $T_3 \times [0, 2]$ has I-twisting at least π since the dividing curves of $T_3 \times \{\frac{1}{4}\}$ and $T_3 \times \{2\}$ have slope 0 and those of $T_3 \times \{1\}$ have slope ∞. Thus, $\Sigma \times S^1$ is inappropriate. This is a contradiction.

Case 2. $T_1 \times [\frac{1}{2}, 1]$ and $T_3 \times [\frac{1}{2}, 1]$ are isotopic. The proof for this case is identical to that of Case 1 except for interchanging the subindexes 2 and 3.

Case 3. $T_2 \times [\frac{1}{2}, 1]$ and $T_3 \times [\frac{1}{2}, 1]$ are isotopic. Similar to Case 1, we can find a vertical convex annulus B connecting $T_2 \times \{\frac{1}{2}\}$ and $T_3 \times \{\frac{1}{2}\}$ with no ∂-parallel dividing curves that has Legendrian boundary intersecting the dividing sets of these tori efficiently. Extend B across $T_2 \times [\frac{1}{4}, \frac{1}{2}]$ and $T_3 \times [\frac{1}{4}, \frac{1}{2}]$ to a convex annulus $\widetilde{B}$ connecting $T_2 \times \{\frac{1}{4}\}$ and $T_3 \times \{\frac{1}{4}\}$ with Legendrian boundary intersecting the dividing sets of these two tori efficiently. For reasons similar to above, neither component of $\widetilde{B} \setminus B$ has ∂-parallel dividing curves. Thus, $\widetilde{B}$ has no ∂-parallel dividing curves. Cut $(\Sigma \times S^1) \setminus (T_1 \times [0, 1) \cup T_2 \times [0, \frac{1}{4}) \cup T_3 \times [0, \frac{1}{4}))$ along $\widetilde{B}$, and round the edges. We get a thickened torus $T_1 \times [1, 2]$ embedded in $\Sigma \times S^1$ with minimal convex boundary, where the dividing curves of $T_1\times\{2\}$ have slope -1. Now we can see that the thickened torus $T_1\times[0,2]$ has I-twisting at least π since the dividing curves of $T_1\times\{\frac{1}{2}\}$ and $T_1\times\{2\}$ have slope -1 and those of $T_1\times\{1\}$ have slope ∞. Thus, $\Sigma\times S^1$ is inappropriate. This is again a contradiction.

Thus, $M=M(\frac{q_1}{p_1},\frac{q_2}{p_2},\frac{q_3}{p_3})$ admits no tight contact structures for which there exists a Legendrian vertical circle with twisting number 0. □

2.3. **The $e_0=-1$ Case.**

Since part (1) of Theorem 1.5 is already proven, we will concentrate on parts (2) and (3) of Theorem 1.5. We will refine the method used in the $e_0\leq -2$ case to prove these results. Lemmata 2.8 and 2.9 will be the main technical tools used in the proof.

Lemma 2.8. *Let ξ be an appropriate contact structure on $\Sigma\times S^1$. Suppose that T_1, T_2 and T_3 are minimal convex, and such that dividing curves of T_1 and T_2 have slope -1, and T_3 has horizontal dividing curves. Assume that there are pairwise disjoint collar neighborhoods $T_i\times I$ of T_i in $\Sigma\times S^1$ for $i=1,2,3$, such that $T_i\times\{0\}=T_i$ and $T_i\times\{1\}$ is minimal convex with vertical dividing curves. Then $(T_1\times I,\xi|_{T_1\times I})$, $(T_2\times I,\xi|_{T_2\times I})$ and $(T_3\times I,\xi|_{T_3\times I})$ are all basic slices, and the signs of these basic slices can not be all the same, where the sign convention of $(T_i\times I,\xi|_{T_i\times I})$ is given by associating the vector $(0,1)^T$ to $T_i\times\{1\}$.*

Proof. Since ξ is appropriate, each $(T_i\times I,\xi|_{T_i\times I})$ is minimal twisting. From the boundary condition of these slices, we can see these are all basic slices. Assume that all these basic slices have the same sign. Then we have that $(T_1\times I,\xi|_{T_1\times I})$ and $(T_2\times I,\xi|_{T_2\times I})$ are isotopic relative to boundary. We isotope T_1 and T_2 slightly so that they have vertical Legendrian rulings. By Lemma 2.7, we can then find a properly embedded convex vertical annulus A with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A=(A\cap T_1)\cup(A\cap T_2)$ intersects the dividing sets of T_1 and T_2 efficiently. Cut $\Sigma\times S^1$ open along A, we get a thickened torus $T_3\times[0,2]$ such that $T_3\times\{0\}$, $T_3\times\{1\}$ and $T_3\times\{2\}$ are minimal convex, and the slopes of their dividing curves are 0, ∞ and 1, respectively. Note that the slice $(T_3\times[1,2],\xi|_{T_3\times[1,2]})$ has the sign opposite to that of $(T_1\times I,\xi|_{T_1\times I})$, and the slice $(T_3\times[0,1],\xi|_{T_3\times[0,1]})$ has the same sign as that of $(T_1\times I,\xi|_{T_1\times I})$. So $\xi_{T_3\times[0,2]}$ is overtwisted. This is a contradiction. Thus, the signs of the basic slices $(T_1\times I,\xi|_{T_1\times I})$, $(T_2\times I,\xi|_{T_2\times I})$ and $(T_3\times I,\xi|_{T_3\times I})$ can not be all the same. □

The following lemma is a special case of Lemma 37 of [7]. Its proof is quite similar to that of Lemma 2.7 ([7], Lemma 36). We will only give a sketch of it.

Lemma 2.9 ([7], Lemma 37)**.** *Let ξ be an appropriate contact structure on $\Sigma\times S^1$. Suppose that T_1, T_2 and T_3 are minimal convex and such that T_1 has vertical Legendrian rulings and dividing curves of slope $-\frac{1}{n}$, where $n\in\mathbf{Z}^{>0}$, T_2 has vertical Legendrian rulings and dividing curves of slope $\frac{1}{n}$, and T_3 has vertical dividing curves. Let $T_1\times I$ and $T_2\times I$ be collar neighborhoods of T_1 and T_2 that are mutually disjoint and disjoint from T_3, and such that, for $i=1,2$, $T_i\times\{0\}=T_i$ and $T_i\times\{1\}$ is minimal convex with vertical dividing curves. If basic slices of $(T_1\times I,\xi|_{T_1\times I})$ and $(T_2\times I,\xi|_{T_2\times I})$*

are all of the same sign, then there exists a properly embedded convex vertical annulus A with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap T_1) \cup (A \cap T_2)$ intersects the dividing sets of T_1 and T_2 efficiently.

Sketch of proof. Similar to the proof of Lemma 2.7, we can show that, if we prescribe the sign of the basic slices of $(T_1 \times I, \xi|_{T_1 \times I})$ and $(T_2 \times I, \xi|_{T_2 \times I})$, then up to isotopy that fixes T_1, T_2 and leaves T_3 invariant, there are at most two appropriate contact structures on $\Sigma \times S^1$ that satisfies the given conditions each corresponding one of the two diagrams in Figure 1. Since the two layers $T_1 \times I$ and $T_2 \times I$ are not contactomorphic, we can not find a contactomorphism between these two possible appropriate contact structures as before. Instead, we will construct an appropriate contact structure corresponding to each of these two diagrams, and show that each of these admit an annulus with the required properties.

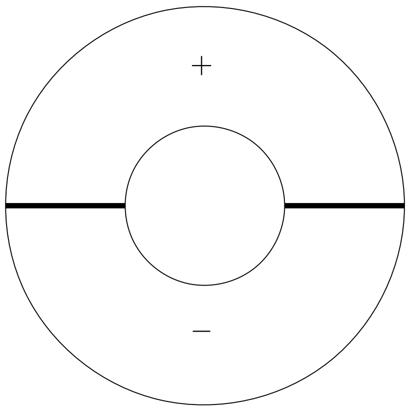

FIGURE 3. Dividing curves on B.

Now consider the tight contact thickened torus $(T_2 \times I, \xi|_{T_2 \times I})$. Like in the proof of Lemma 2.7, we can construct an appropriate contact structure on $\Sigma \times S^1$ satisfying the conditions in the lemma that admits an annulus A with the required properties by "digging out" a vertical Legendrian ruling of a torus in an I-invariant neighborhood of $T_2 \times \{0\}$ parallel to the boundary. Indeed, both of the possible appropriate contact structures can be constructed this way. To see that, we isotope $T_2 \times \{0\}$ and $T_2 \times \{1\}$ lightly to T_2' and T_3' with the same dividing curves and horizontal Legendrian rulings. Then connect a Legendrian ruling of T_2' and a Legendrian ruling of T_3' by a horizontal convex annulus B. The dividing curves of B is given in Figure 3. We can choose the vertical Legendrian ruling to be dug out to intersect one of the two dividing curves of B. These two choices correspond to the two possible configurations of the dividing curves on Σ_0' in Figure 1, and, hence, gives the two possible appropriate contact structures on $\Sigma \times S^1$ satisfying the given conditions. □

Proof of (2) and (3) of Theorem 1.5. Let $M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3})$ be a small Seifert space such that $0 < q_1 < p_1$, $0 < q_2 < p_2$ and $-p_3 < q_3 < 0$. Choose $u_i, v_i \in \mathbf{Z}$ such that $0 < u_i < p_i$ and $p_i v_i + q_i u_i = 1$ for $i = 1, 2, 3$. Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{pmatrix} p_i & u_i \\ -q_i & v_i \end{pmatrix}.$$

Then

$$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Assume that ξ is a tight contact structure on M for which there exits a Legendrian vertical circle L in M with twisting number $t(L) = 0$. We first isotope ξ to make $L = \{\mathrm{pt}\} \times S^1 \subset \Sigma \times S^1$, and each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $n_i < 0$, i.e., ∂V_i is minimal convex with dividing curves of slope $\frac{1}{n_i}$ when measured in the coordinates of ∂V_i. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{-n_i q_i + v_i}{n_i p_i + u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(n_i p_i + u_i)}.$$

Then $-1 \leq s_1, s_2 \leq 0$ and $0 \leq s_3 < 1$. Now, without affecting the properties of L and V_i asserted above, we can further isotope the contact structure ξ to make the Legendrian rulings of T_i to have slope ∞ when measured in the coordinates of T_i. As before, we can find pairwise disjoint collar neighborhoods $T_i \times I$'s in $\Sigma \times S^1$ of T_i's, such that $T_i \times \{0\} = T_i$ and $T_i \times \{1\}$ is minimal convex with vertical dividing curves.

We now prove part (2).

Assume that $q_3 = -1$, $\frac{q_1}{p_1} < \frac{1}{2p_3-1}$ and $\frac{q_2}{p_2} < \frac{1}{2p_3}$. By choosing $n_i \ll -1$, we can make $-\frac{1}{2p_3-1} < s_1 < -\frac{q_1}{p_1}$, $-\frac{1}{2p_3} < s_2 < -\frac{q_2}{p_2}$ and $\frac{1}{p_3+1} < s_3 < \frac{1}{p_3}$. So there is a minimal convex torus in $T_i \times I$ parallel to the boundary, say $T_i' = T_i \times \{\frac{1}{2}\}$, that has dividing curves of slope $-\frac{1}{2p_3-1}$, $-\frac{1}{2p_3}$ and $\frac{1}{p_3+1}$ for $i = 1, 2$ and 3, respectively. Let's consider the layers $T_i \times [\frac{1}{2}, 1]$. $T_1 \times [\frac{1}{2}, 1]$ is a continuous fraction block consisting of $2p_3 - 1$ basic slices. $T_2 \times [\frac{1}{2}, 1]$ is a continuous fraction block consisting of $2p_3$ basic slices. $T_3 \times [\frac{1}{2}, 1]$ consists of 2 continuous fraction blocks, each of which is a basic slice. We can find a minimal convex torus $T_i'' = T_i \times \{\frac{3}{4}\}$ in $T_i \times [\frac{1}{2}, 1]$ parallel to boundary with dividing curves of slope -1 for $i = 1, 2$, and 0 for $i = 3$.

Let the sign of the basic slice $T_3 \times [\frac{3}{4}, 1]$ be $\sigma \in \{+, -\}$. Note that, when $q_3 = -1$, then diffeomorphism $\varphi_3 : \partial V_3 \to T_3$ is given by

$$\varphi_3 = \begin{pmatrix} p_3 & p_3 - 1 \\ 1 & 1 \end{pmatrix}.$$

So the slopes 0 and $\frac{1}{p_3+1}$ of the dividing sets of T_3'' and T_3' correspond to twisting numbers -1 and -2 of Legendrian circles isotopic to the $-\frac{1}{p_3}$-singular fibre. And the basic slice $T_3 \times [\frac{1}{2}, \frac{3}{4}]$ corresponds to a stabilization of a Legendrian circle isotopic to the $-\frac{1}{p_3}$-singular fibre. Since we can freely choose the sign of such a stabilization, we can make the sign of the basic slice $T_3 \times [\frac{1}{2}, \frac{3}{4}]$ to be σ, too.

According to Lemma 2.8, the sign of the basic slices $T_i \times [\frac{3}{4}, 1]$ can not be all the same. Note that we can shuffle the signs of basic slices in a continuous fraction block. So at least one of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ consists only of basic slices of sign $-\sigma$.

Case 1. Assume that all the basic slices in $T_1 \times [\frac{1}{2}, 1]$ are of the sign $-\sigma$. If $T_2 \times [\frac{1}{2}, 1]$ contains p_3 basic slices of the sign $-\sigma$, then we shuffle these signs to the p_3 slices closest to $T_2 \times \{1\}$. Consider the thickened tori $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$ formed by the unions the p_3 basic slices closest to $T_1 \times \{1\}$ and $T_2 \times \{1\}$ in $T_1 \times I$ and $T_2 \times I$, respectively. Remove from M the solid tori bounded by $T_1 \times \{\frac{5}{8}\}$, $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{1\}$. We apply Lemma 2.7 to the resulting $\Sigma \times S^1$ and the thickened tori $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_1 \times \{\frac{5}{8}\})) \cup (A \cap (T_2 \times \{\frac{5}{8}\}))$ intersects the dividing curves of $T_1 \times \{\frac{5}{8}\}$ and $T_2 \times \{\frac{5}{8}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A and round the edges, we get a minimal convex torus $\widetilde{T}_3$ isotopic to T_3 with dividing curves of slope $\frac{1}{p_3}$. This means there exists a thickening $\widetilde{V}_3$ of V_3 with convex boundary $\partial \widetilde{V}_3$ that has two dividing curves isotopic to a meridian. Then $\xi|_{\partial \widetilde{V}_3}$ is overtwisted. This contradicts the tightness of ξ.

If $T_2 \times [\frac{1}{2}, 1]$ contains $p_3 + 1$ basic slices of the sign σ, then we shuffle all these signs to the $p_3 + 1$ slices closest to $T_2 \times \{1\}$. Let $T_2 \times [\frac{5}{8}, 1]$ be the union of these $p_3 + 1$ basic slices. Remove from M the solid tori bounded by $T_1 \times \{1\}$, $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{\frac{1}{2}\}$. Apply Lemma 2.9 to the resulting $\Sigma \times S^1$ and the thickened tori $T_2 \times [\frac{5}{8}, 1]$ and $T_3 \times [\frac{1}{2}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_2 \times \{\frac{5}{8}\})) \cup (A \cap (T_3 \times \{\frac{1}{2}\}))$ intersects the dividing curves of $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{\frac{1}{2}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A, we get a thickened torus $T_1 \times [1, 2]$ embedded in $\Sigma \times S^1$ that has minimal convex boundary such that $T_1 \times \{1\}$ has vertical dividing curves and $T_1 \times \{2\}$ has dividing curves of slope $-\frac{1}{p_3+1}$. Then the thickened torus $T_1 \times [\frac{1}{2}, 2] = (T_1 \times [\frac{1}{2}, 1]) \cup (T_1 \times [1, 2])$ has I-twisting at least π. This again contradicts the tightness of ξ.

But $T_2 \times [\frac{1}{2}, 1]$ is a continuous fraction block consisting of $2p_3$ basic slices. So it either contains p_3 basic slices of the sign $-\sigma$, or contains $p_3 + 1$ basic slices of the sign σ. So, the basic slices in $T_1 \times [\frac{1}{2}, 1]$ can not be all of the sign $-\sigma$.

Case 2. Assume that all the basic slices in $T_2 \times [\frac{1}{2}, 1]$ are of the sign $-\sigma$. If $T_1 \times [\frac{1}{2}, 1]$ contains either p_3 basic slices of the sign $-\sigma$ or $p_3 + 1$ basic slices of the sign σ, then there will be a contradiction just like in Case 1. So the only possible scenario is that $T_1 \times [\frac{1}{2}, 1]$ contains $p_3 - 1$ basic slices of the sign $-\sigma$ or p_3 basic slices of the sign σ. Now we shuffle all the $-\sigma$ signs in $T_1 \times [\frac{1}{2}, 1]$ to the $p_3 - 1$ basic slices closest to $T_1 \times \{1\}$. Let $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$ be the unions the $p_3 - 1$ basic slices closest to $T_1 \times \{1\}$ and $T_2 \times \{1\}$ in $T_1 \times I$ and $T_2 \times I$. Remove from M the solid tori bounded by $T_1 \times \{\frac{5}{8}\}$, $T_2 \times \{\frac{5}{8}\}$ and $T_3 \times \{1\}$, and apply Lemma 2.7 to the resulting $\Sigma \times S^1$ and the thickened tori $T_1 \times [\frac{5}{8}, 1]$ and $T_2 \times [\frac{5}{8}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary

$\partial A = (A \cap (T_1 \times \{\frac{5}{8}\})) \cup (A \cap (T_2 \times \{\frac{5}{8}\}))$ intersects the dividing sets of $T_1 \times \{\frac{5}{8}\}$ and $T_2 \times \{\frac{5}{8}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A and round the edges, we get a minimal convex torus $\widetilde{T}_3$ isotopic to T_3 with dividing curves of slope $\frac{1}{p_3-1}$. This means we can thicken V_3 to a standard neighborhood $\widetilde{V}_3$ of a Legendrian circle isotopic to the $-\frac{1}{p_3}$-singular fibre with twisting number 0. Stabilize this Legendrian circle twice. We get a thickened torus $\widetilde{T}_3 \times [\frac{1}{2}, 2]$ with minimal convex boundary such that $\widetilde{T}_3 \times \{2\} = \widetilde{T}_3$, $\widetilde{T}_3 \times \{\frac{3}{4}\}$ is minimal convex with dividing curves of slope 0, and $\widetilde{T}_3 \times \{\frac{1}{2}\}$ has dividing curves of slope $\frac{1}{p_3+1}$. Since we can choose the signs of these stabilizations freely, we can make both basic slices $\widetilde{T}_3 \times [\frac{1}{2}, \frac{3}{4}]$ and $\widetilde{T}_3 \times [\frac{3}{4}, 2]$ to have the sign $-\sigma$. There exists a minimal convex torus, say $\widetilde{T}_3 \times \{1\}$, in $\widetilde{T}_3 \times [\frac{3}{4}, 2]$ parallel to boundary with vertical dividing curves. Use $\widetilde{T}_3 \times \{1\}$, we can thicken $T_1 \times [\frac{1}{2}, \frac{5}{8}]$ to $\widetilde{T}_1 \times [\frac{1}{2}, 1]$, such that $\widetilde{T}_1 \times [\frac{1}{2}, \frac{5}{8}] = T_1 \times [\frac{1}{2}, \frac{5}{8}]$, and $\widetilde{T}_1 \times \{1\}$ is convex with two vertical dividing curves. Since the basic slice $\widetilde{T}_3 \times [\frac{3}{4}, 2]$ has the sign $-\sigma$, all the basic slices in $\widetilde{T}_1 \times [\frac{5}{8}, 1]$ have the sign σ. Also note that all the basic slices in $\widetilde{T}_1 \times [\frac{1}{2}, \frac{5}{8}] = T_1 \times [\frac{1}{2}, \frac{5}{8}]$ have the sign σ. So we are now in a situation where the basic slices $\widetilde{T}_3 \times [\frac{1}{2}, \frac{3}{4}]$ and $\widetilde{T}_3 \times [\frac{3}{4}, 1]$ both have the sign $-\sigma$, and all the basic slices in $\widetilde{T}_1 \times [\frac{1}{2}, 1]$ have the sign σ. After we thicken $T_2 \times [\frac{1}{2}, \frac{5}{8}]$ to $\widetilde{T}_2 \times [\frac{1}{2}, 1]$, where $\widetilde{T}_2 \times \{1\}$ is minimal convex with vertical dividing curves, we are back to Case 1, which is shown to be impossible. Thus, the basic slices in $T_2 \times [\frac{1}{2}, 1]$ can not be all of the sign $-\sigma$ either.

But, as we mentioned above, one of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ have to consist only of basic slices of sign $-\sigma$. This is a contradiction. Thus, no such ξ exists on M, and, hence, we proved part (2) of Theorem 1.5.

It remains to prove part(3) now.

Assume that $q_1 = q_2 = 1$ and $p_1, p_2 > 2m$, where $m = -\lfloor \frac{p_3}{q_3} \rfloor$. By choosing $n_i \ll -1$, we can make $-\frac{1}{2m} < s_1 < -\frac{1}{p_1}$, $-\frac{1}{2m} < s_2 < -\frac{1}{p_2}$, and $0 < s_3 < -\frac{q_3}{p_3}$. Similar to the proof of part (2), we can find convex a torus $T_i' = T_i \times \{\frac{1}{2}\}$ in $T_i \times I$ parallel to boundary with two dividing curves such that have slope $-\frac{1}{2m}$ for $i - 1, 2$, and 0 for $i = 3$. Then each of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ is a continued fraction block consists of $2m$ basic slices. And $T_3 \times [\frac{1}{2}, 1]$ is a basic slice. Let the sign of the basic slice $T_3 \times [\frac{1}{2}, 1]$ be $\sigma \in \{+, -\}$. For reasons similar to above, one of $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ can not contain basic slices of the sign σ. Without loss of generality, we assume that all basic slices in $T_1 \times [\frac{1}{2}, 1]$ are of the sign $-\sigma$. We now consider the signs of the basic slices in $T_2 \times [\frac{1}{2}, 1]$.

Case 1. Assume that $T_2 \times [\frac{1}{2}, 1]$ contains m basic slices of the sign $-\sigma$. Then we shuffle these signs to the m basic slices in $T_2 \times [\frac{1}{2}, 1]$ closest to $T_2 \times \{1\}$. Denote by $T_1 \times [\frac{3}{4}, 1]$ and $T_2 \times [\frac{3}{4}, 1]$ the unions of the m basic slices in $T_1 \times [\frac{1}{2}, 1]$ and $T_2 \times [\frac{1}{2}, 1]$ closest to $T_1 \times \{1\}$ and $T_2 \times \{1\}$, respectively. Remove from M the solid tori bounded by $T_1 \times \{\frac{3}{4}\}$, $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{1\}$. We apply Lemma 2.7 to the resulting $\Sigma \times S^1$ and the thickened tori $T_1 \times [\frac{3}{4}, 1]$ and $T_2 \times [\frac{3}{4}, 1]$. Then there exists a properly embedded convex vertical annulus A in $\Sigma \times S^1$ with no ∂-parallel dividing curves, whose Legendrian boundary $\partial A = (A \cap (T_1 \times \{\frac{3}{4}\})) \cup (A \cap (T_2 \times \{\frac{3}{4}\}))$ intersects the dividing sets of $T_1 \times \{\frac{3}{4}\}$ and $T_2 \times \{\frac{3}{4}\}$ efficiently. Cutting $\Sigma \times S^1$ open along A and round the edges, we get a thickened torus $T_3 \times [1,2]$ with minimal convex boundary such that $T_3 \times \{1\}$ has dividing curves of slope ∞, and $T_3 \times \{2\}$ has dividing curves of slope $\frac{1}{m}$. Note that $\frac{1}{m} \leq -\frac{q_3}{p_3}$. If $\frac{1}{m} = -\frac{q_3}{p_3}$, then, as above, the existence of $T_3 \times [1,2]$ means that we can thicken V_3 to $\widetilde{V}_3$ such that $\xi|_{\widetilde{V}_3}$ is overtwisted, which contradicts the tightness of ξ. If $\frac{1}{m} < -\frac{q_3}{p_3}$, we can choose s_3 so that $\frac{1}{m} < s_3 < -\frac{q_3}{p_3}$. Then the thickened torus $T_3 \times [0,2] = (T_3 \times I) \cup (T_3 \times [1,2])$ has I-twisting greater than π, which again contradicts the tightness of ξ. So $T_2 \times [\frac{1}{2}, 1]$ can not contain m basic slices of the sign $-\sigma$.

Case 2. Assume that $T_2 \times [\frac{1}{2}, 1]$ contains $m+1$ basic slices of the sign σ. We shuffle one of the σ to the basic slice in $T_2 \times [\frac{1}{2}, 1]$ closest to $T_2 \times \{1\}$. Denote by $T_2 \times [\frac{3}{4}, 1]$ this basic slice. Similar to the proof of Theorem 1.4, we can find a convex vertical annulus A in M satisfying:

(1) A has no ∂-parallel dividing curves;
(2) $\partial A = (A \cap (T_2 \times \{\frac{3}{4}\})) \cup (A \cap (T_3 \times \{\frac{1}{2}\}))$, which is Legendrian and intersects the dividing sets of $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{\frac{1}{2}\}$ efficiently;
(3) A is disjoint from T_1 and the interior of the solid tori in M bounded by $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{\frac{1}{2}\}$.

Note that, since $q_1 = 1$, the diffeomorphism $\varphi_1 : \partial V_1 \to T_1$ is given by

$$\varphi_1 = \begin{pmatrix} p_1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Remove from M the interior of the solid tori in M bounded by $T_2 \times \{\frac{3}{4}\}$ and $T_3 \times \{\frac{1}{2}\}$, and cut it open along A. We get a thickening $\widetilde{V}_1$ of V_1, whose boundary is convex with two dividing curves of slope ∞. Then $\widetilde{V}_1$ is a standard neighborhood of a Legendrian circle isotopic to the $\frac{1}{p_1}$-fibre with twisting number 0. We stabilize this Legendrian circle once. This gives a thickened torus $\widetilde{T}_1 \times [0,2]$ with minimal convex boundary such that $\widetilde{T}_1 \times \{2\} = \partial \widetilde{V}_1$ has dividing curves of slope 0, and $\widetilde{T}_1 \times \{0\}$ has dividing curves of slope $-\frac{1}{p_1 - 1}$, where the slopes are measured in the coordinates of T_1. Since we can choose the sign of the stabilization, we can make the sign of this basic slice σ. Since $-\frac{1}{p_1-1} \geq -\frac{1}{2m}$, we can find minimal convex tori $\widetilde{T}_1 \times \{\frac{1}{2}\}$ and $\widetilde{T}_1 \times \{1\}$ in $\widetilde{T}_1 \times [0,2]$ parallel to the boundary such that $\widetilde{T}_1 \times \{\frac{1}{2}\}$ has dividing curves of slope $-\frac{1}{2m}$, and $\widetilde{T}_1 \times \{1\}$ has dividing curves of slope ∞. Note that $\widetilde{T}_1 \times [\frac{1}{2}, 1]$ is now a continued fraction block consisting of $2m$ basic slices of the sign σ. Now use $\widetilde{T}_1 \times \{1\}$ to thicken $T_2 \times [\frac{1}{2}, \frac{3}{4}]$ to $\widetilde{T}_2 \times [\frac{1}{2}, 1]$ such that $\widetilde{T}_2 \times [\frac{1}{2}, \frac{3}{4}] = T_2 \times [\frac{1}{2}, \frac{3}{4}]$, and $\widetilde{T}_2 \times \{1\}$ is minimal convex with vertical dividing curves. Note that $\widetilde{T}_2 \times [\frac{1}{2}, 1]$ is a continued fraction block that contains at least m basic slices of the sign σ. Now, similar to Case 1, we can find a contradiction. Thus, $T_2 \times [\frac{1}{2}, 1]$ can not contain $m+1$ basic slices of the sign σ either.

But $T_2 \times [\frac{1}{2}, 1]$ contains $2m$ basic slices. So either m of these are of the sign $-\sigma$, or $m+1$ of these are of the sign σ. This is a contradiction. Thus, no such ξ exists on M, and, hence, we proved part (3) of Theorem 1.5. $\square$

3. Counting Tight Contact Structures

3.1. **Continued Fractions.**

In this section, we establish some properties of continued fractions, which will be used to prove Theorems 1.7 and 1.6.

Lemma 3.1. *Let $a_0,\ a_1, \cdots,\ a_m$ be real numbers such that $a_0 \leq -1$, and $a_j \leq -2$ for $1 \leq j \leq m$. Define $\{p_j\}$ and $\{q_j\}$ by*

$$\begin{cases} p_j = -a_j p_{j-1} - p_{j-2},\ j = 0, 1, \cdots, m, \\ p_{-2} = -1,\ p_{-1} = 0, \end{cases}$$

$$\begin{cases} q_j = -a_j q_{j-1} - q_{j-2},\ j = 0, 1, \cdots, m, \\ q_{-2} = 0,\ q_{-1} = 1. \end{cases}$$

Then, for $1 \leq j \leq m$, we have

(1) $-\frac{q_j}{p_j} =< a_0, a_1, \cdots, a_j >$,
(2) $p_j \geq p_{j-1} > 0$, $q_j \geq q_{j-1} > 0$,
(3) $p_j q_{j-1} - p_{j-1} q_j = 1$,
(4) $-\frac{q_j + (a_0+1)p_j}{q_{j-1} + (a_0+1)p_{j-1}} =< a_j, a_{j-1}, \cdots, a_2, a_1 + 1 >$.

Proof. By the definitions of $\{p_j\}$ and $\{q_j\}$, we have $p_0 = 1$, $q_0 = -a_0$, $p_1 = -a_1$, and $q_1 = a_0 a_1 - 1$. Then it's easy to check that the lemma is true for $j = 1$. Assume that the lemma is true for $j - 1 \geq 1$. Then,

$$\begin{aligned} < a_0, a_1, \cdots, a_j > &= < a_0, a_1, \cdots, a_{j-1} - \frac{1}{a_j} > \\ &= -\frac{-(a_{j-1} - \frac{1}{a_j}) q_{j-2} - q_{j-3}}{-(a_{j-1} - \frac{1}{a_j}) p_{j-2} - p_{j-3}} \\ &= -\frac{(a_j a_{j-1} - 1) q_{j-2} + a_j q_{j-3}}{(a_j a_{j-1} - 1) p_{j-2} + a_j p_{j-3}} \\ &= -\frac{a_j(a_{j-1} q_{j-2} + q_{j-3}) - q_{j-2}}{a_j(a_{j-1} p_{j-2} + p_{j-3}) - p_{j-2}} \\ &= -\frac{-a_j q_{j-1} - q_{j-2}}{-a_j q_{j-1} - q_{j-2}} \\ &= -\frac{q_j}{p_j}. \end{aligned}$$

Also, since $q_{j-1} \geq q_{j-2} > 0$ and $-a_j \geq 2$, we have $q_j = -a_j q_{j-1} - q_{j-2} \geq 2q_{j-1} - q_{j-2} \geq q_{j-1} > 0$, and, similarly, $p_j \geq p_{j-1} > 0$.

Furthermore, by definitions of $\{p_j\}$ and $\{q_j\}$,

$$\begin{aligned} p_jq_{j-1} - p_{j-1}q_j &= (-a_jp_{j-1} - p_{j-2})q_{j-1} - p_{j-1}(-a_jq_{j-1} - q_{j-2}) \\ &= p_{j-1}q_{j-2} - p_{j-2}q_{j-1} \\ &= 1. \end{aligned}$$

Finally,

$$\begin{aligned} -\frac{q_j + (a_0+1)p_j}{q_{j-1} + (a_0+1)p_{j-1}} &= -\frac{(-a_jq_{j-1} - q_{j-2}) + (a_0+1)(-a_jp_{j-1} - p_{j-2})}{q_{j-1} + (a_0+1)p_{j-1}} \\ &= \frac{a_j(q_{j-1} + (a_0+1)p_{j-1}) + (q_{j-2} + (a_0+1)p_{j-2})}{q_{j-1} + (a_0+1)p_{j-1}} \\ &= a_j - \frac{1}{< a_{j-1}, \cdots, a_2, a_1+1 >} \\ &= < a_j, a_{j-1}, \cdots, a_2, a_1+1 > . \end{aligned}$$

This shows that the lemma is also true for j. □

Remark 3.2. In the proof of Theorem 1.7 and 1.6, all the a_j's will be integers, and so will the corresponding p_j's and q_j's be. Then, property (3) in Lemma 3.1 implies that $g.c.d.(p_j, q_j) = 1$.

3.2. The $e_0 \geq 1$ Case.

The following lemma is a reformulation of parts (1), (2) and (3) of Lemma 5.1 of [13].

Lemma 3.3. *Let ξ be an appropriate contact structure on $\Sigma \times S^1$ with minimal convex boundary that admits a vertical Legendrian circle with twisting number 0. Assume that dividing curves of T_1, T_2 and T_3 are of slopes -1, -1, $-n$, respectively, where n is an integer greater than 1. Then there is a factorization $\Sigma \times S^1 = L_1 \cup L_2 \cup L_3 \cup (\Sigma' \times S^1)$, where L_i's are embedded thickened tori with minimal twisting and minimal convex boundary $\partial L_i = T_i' - T_i$, s.t., dividing curves of T_i' have slope ∞. The appropriate contact structure ξ is uniquely determined by the signs of the basic slices L_1, L_2 and L_3. The sign convention here is given by associating $(0,1)^T$ to T_i'.*

Proof. We only prove the last sentence. The rest is just part (1) of Lemma 5.1 of [13]. Let Σ_0 be a properly embedded three hole sphere in $\Sigma \times S^1$ isotopic to $\Sigma \times \{\text{pt}\}$, and $\Sigma_0' = \Sigma_0 \cap (\Sigma' \times S^1)$. We isotope Σ_0 so that Σ_0 and Σ_0' are convex with Legendrian boundaries that intersect the dividing curves of $\partial\Sigma \times S^1$ and $\partial\Sigma' \times S^1$ efficiently. Then each component of $\partial\Sigma_0'$ intersects the dividing curves of Σ_0' twice. Since ξ is appropriate, Σ_0' has no ∂-parallel dividing curves. This implies that, up to isotopy relative to boundary and Dehn twists parallel to boundary components, there are only two configurations of dividing curves on Σ_0'. (See Figure 4.) Thus, there are only two tight contact structure on $\Sigma' \times S^1$, up to isotopy relative to boundary and full horizontal rotations of each boundary component.

Let $A_i = \Sigma_0 \cap L_i$. Then the dividing set of each of A_1 and A_2 consists of two arcs connecting the two boundary components. And the dividing set of A_3 consists of two

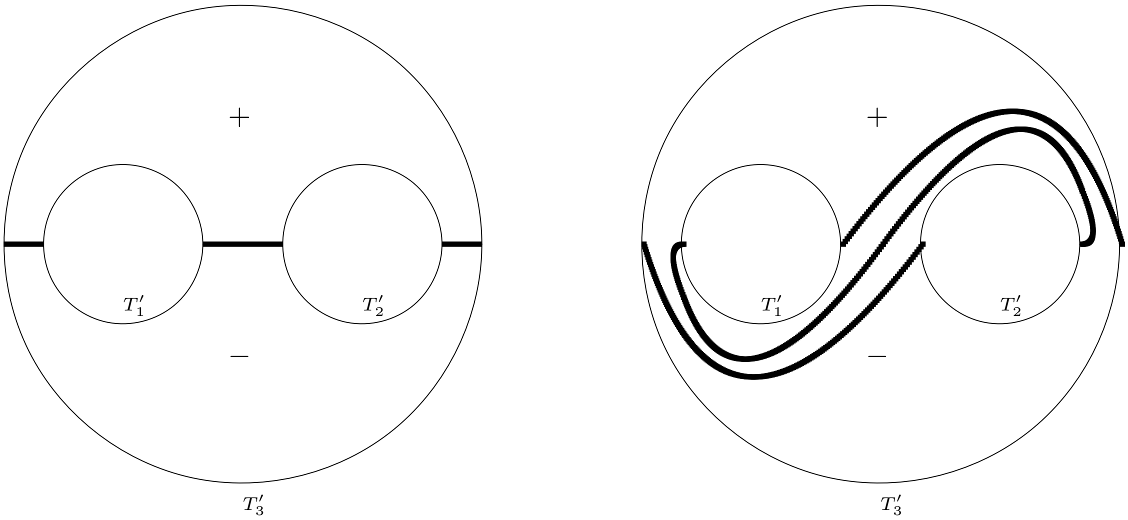

FIGURE 4. Possible configurations of dividing curves on Σ'_0

arcs connecting the two boundary components and $n-1$ ∂-parallel arcs on the T_3 side. From the relative Euler class of $\xi|_{L_3}$, one can see that the half discs bounded by these ∂-parallel arcs must be pairwise disjoint and of the sign opposite to that of L_3. By isotoping Σ_0 relative to $\Sigma'_0 \cup \partial\Sigma_0$, we can freely choose the holonomy of the non-∂-parallel dividing curves of each A_i. This implies that, up to isotopy relative to boundary, there are only two possible configurations of dividing curves on Σ_0 when the signs of L_i's are given. (See Figure 5.)

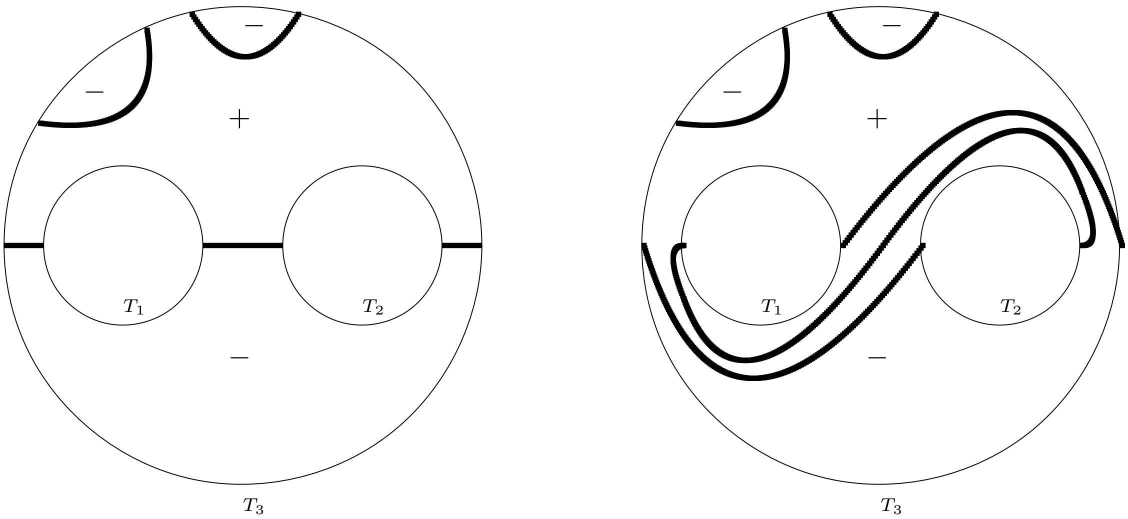

FIGURE 5. Possible configurations of dividing curves on Σ_0. Here, $n=3$, and the layer L_3 is positive

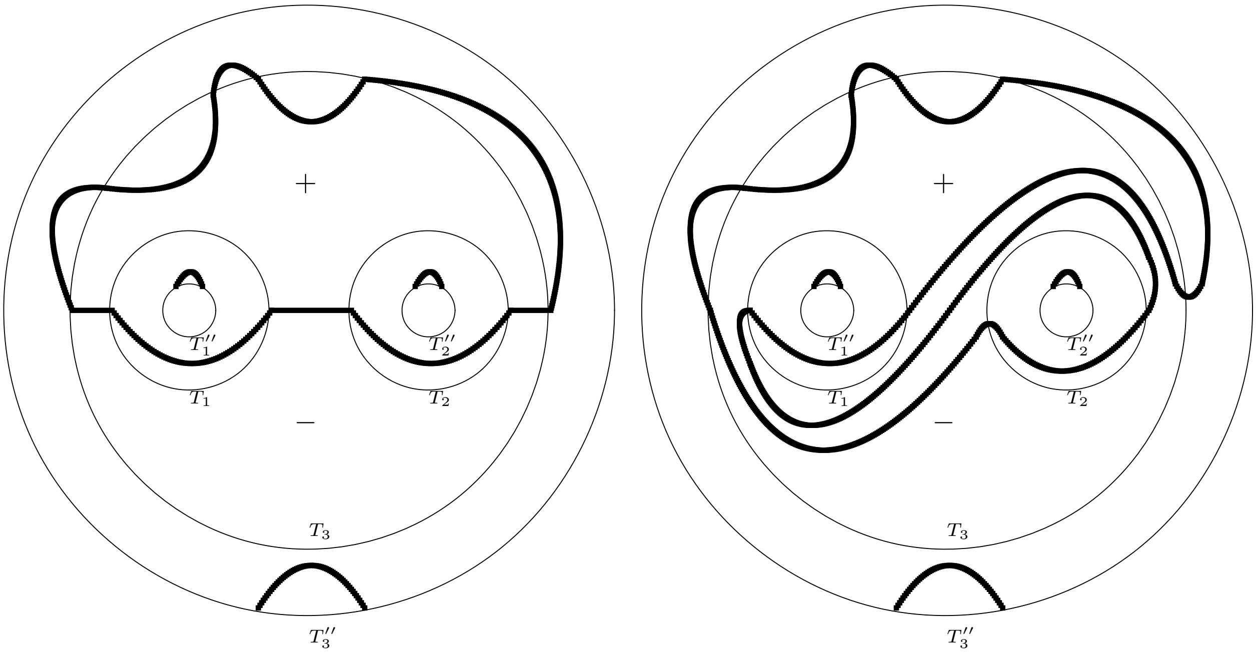

FIGURE 6. After extending to Σ_0'', the two possible configurations become the same. Here, $n = 3$, and the signs of the layers L_1, L_2 and L_3 are $-$, $-$, $+$, respectively.

When the signs of L_i's are mixed, we can extend $(\partial\Sigma \times S^1, \xi)$ to a universally tight contact manifold $(\partial\Sigma'' \times S^1, \xi'')$ by gluing to T_i a basic slice L_i'' of the same sign as L_i for each i, where L_i'' has minimal convex boundary $\partial L_i'' = T_i - T_i''$, and the dividing curves of T_i'' are vertical. Extend Σ_0 across L_i'' to Σ_0'' so that Σ_0'' is convex with Legendrian boundary intersecting the dividing curves of T_i'' efficiently. For $i = 1, 2$, the dividing set of $\Sigma_0'' \cap L_i''$ consists of 1 ∂-parallel arcs on each boundary component. From the relative Euler class of $\xi''|_{L_i''}$, we can see that the half discs on $\Sigma_0'' \cap L_i''$ bounded by these ∂-parallel arcs are of the same sign as the basic slice L_i. The dividing set of $\Sigma_0'' \cap L_3''$ consists of n ∂-parallel arcs on the T_3 side and 1 ∂-parallel arcs on the T_3'' side. From the relative Euler class of $\xi''|_{L_3''}$, we can see that the half discs on $\Sigma_0'' \cap L_3''$ bounded by these ∂-parallel arcs are pairwise disjoint and of the same sign as the basic slice L_3.

Now, one can see that, after the extension, the two possible configurations of dividing curves on Σ_0 become the same minimal configuration of dividing curves on Σ_0''. (See Figure 6.) By Lemma 5.1 of [13], the two configurations correspond to the same universally tight contact structure on $\Sigma \times S^1$. This shows that, when the signs of L_i's are mixed, ξ is uniquely determined by the signs of L_i's. When all the L_i's have the same sign, ξ is virtually overtwisted, and the isotopy type relative to boundary of such a contact structure is determined by the action of the relative Euler class on Σ_0, which is, in turn, determined by the sign of L_3. Thus, when all the L_i's have the same sign, this common sign determines ξ. $\square$

Proof of Theorem 1.6. Define $\{p_j^{(i)}\}$ and $\{q_j^{(i)}\}$ by

$$\begin{cases} p_j^{(i)} = -b_j^{(i)} p_{j-1}^{(i)} - p_{j-2}^{(i)}, \ j = 0, 1, \cdots, l_i, \\ p_{-2}^{(i)} = 0, \ p_{-1}^{(i)} = 1, \end{cases}$$

$$\begin{cases} q_j^{(i)} = -b_j^{(i)} q_{j-1}^{(i)} - q_{j-2}^{(i)}, \ j = 0, 1, \cdots, l_i, \\ q_{-2}^{(i)} = -1, \ q_{-1}^{(i)} = 0. \end{cases}$$

By Lemma 3.1 and Remark 3.2, we have $p_i = p_{l_i}^{(i)}$ and $q_i = q_{l_i}^{(i)}$. Let $u_i = -p_{l_i-1}^{(i)}$ and $v_i = -q_{l_i-1}^{(i)}$. Then $p_i v_i - q_i u_i = 1$.

Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \to T_i$ by

$$\varphi_i = \begin{cases} \begin{pmatrix} p_i & -u_i \\ -q_i & v_i \end{pmatrix}, & i = 1, 2; \\ \begin{pmatrix} p_3 & -u_3 \\ -q_3 - e_0 p_3 & v_3 + e_0 u_3 \end{pmatrix}, & i = 3. \end{cases}$$

Then

$$M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).$$

Let ξ be a tight contact structure on M. By Theorem 1.3, ξ admits a vertical Legendrian circle L with twisting number 0. We first isotope ξ so that there is a vertical Legendrian circle with twisting number 0 in the interior of $\Sigma \times S^1$, and each V_i is a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $t_i < 0$, i.e., ∂V_i is minimal convex with dividing curves of slope $\frac{1}{t_i}$ when measured in the coordinates of ∂V_i. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \begin{cases} \frac{-t_i q_i + v_i}{t_i p_i - u_i} = -\frac{q_i}{p_i} + \frac{1}{p_i(t_i p_i - u_i)}, & i = 1, 2; \\ \frac{-t_3(q_3 + e_0 p_3) + (v_3 + e_0 u_3)}{t_3 p_i - u_3} = -e_0 - \frac{q_i}{p_i} + \frac{1}{p_i(t_i p_i - u_i)}, & i = 3. \end{cases}$$

We choose $t_i << -1$ so that $\frac{1}{b_0^{(i)}+1} < s_i < -\frac{q_i}{p_i}$ for $i = 1, 2$, and $-e_0 + \frac{1}{b_0^{(3)}+1} < s_3 < -e_0 - \frac{q_3}{p_3}$. Using the vertical Legendrian circle L, we can thicken V_i to V_i', s.t., V_i''s are pairwise disjoint, and $T_i' = \partial V_i'$ is a minimal convex torus with vertical dividing curves when measured in coordinates of T_i. By Proposition 4.16 of [12], there exits a minimal convex torus T_i'' in the interior of $V_i' \setminus V_i$ isotopic to T_i that has dividing curves of slope $\frac{1}{b_0^{(i)}+1}$ for $i = 1, 2$, and $-e_0 + \frac{1}{b_0^{(3)}+1}$ for $= 3$. Let V_i'' be the solid torus bounded by T_i'', and $\Sigma'' \times S^1 = M \setminus (V_1'' \cup V_2'' \cup V_3'')$.

Now we count the tight contact structures on $\Sigma'' \times S^1$ and V_i'' that satisfy the given boundary condition. First, we look at V_i''. In the coordinates in ∂V_i, the dividing curves of $T_i'' = \partial V_i''$ have slope $\frac{(b_0^{(i)}+1)q_i + p_i}{(b_0^{(i)}+1)v_i + u_i}$. By part (4) of Lemma 3.1 and the definitions of u_i, v_i, we have that $\frac{(b_0^{(i)}+1)q_i + p_i}{(b_0^{(i)}+1)v_i + u_i} =< b_{l_i}^{(i)}, b_{l_i-1}^{(i)}, \cdots, b_2^{(i)}, b_1^{(i)} + 1 >$. Thus, on each V_i'', there are $|\prod_{j=1}^{l_i}(b_j^{(i)} + 1)|$ tight contact structures that satisfy the given boundary condition. Then we look at $\Sigma'' \times S^1$. The thickened torus L_i bounded by

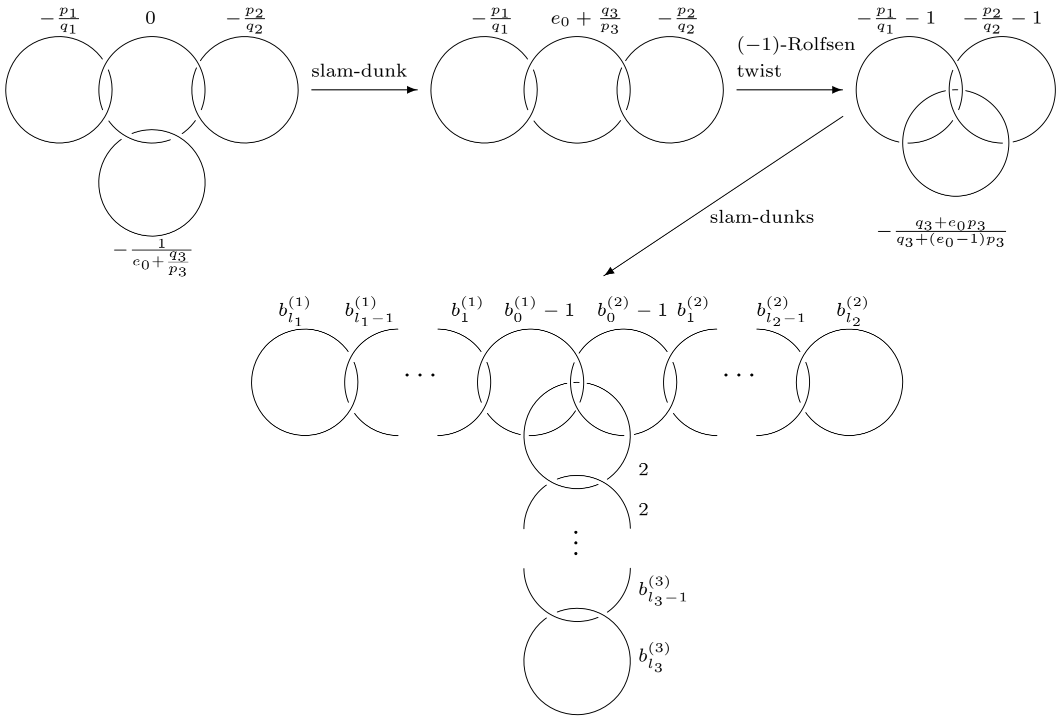

FIGURE 7. Construction of tight contact structures on $M(\frac{q_1}{p_1},\frac{q_2}{p_2},e_0+\frac{q_3}{p_3})$

$T'_i - T''_i$ is a continued fraction block consisting of $|b^{(i)}_0+1|$ basic slices. Let L'_i be the basic slice in L_i closest to T'_i, and $\partial L'_i = T'_i - T'''_i$. Note that T'''_i is a minimal convex torus with dividing curves of slope -1 for $i=1,2$, and $-e_0-1$ for $i=3$. Let $\Sigma'\times S^1 = M\setminus(V'_1\cup V'_2\cup V'_3)$. By Lemma 3.3, the tight contact structure on $(\Sigma'\times S^1)\cup L'_1\cup L'_2\cup L'_2$ is uniquely determined by the signs of the basic slices L'_i. But we can shuffle the signs of the basic slices within a continued fraction block. Let's shuffle all the positive signs in L_i to the basic slices closest to T'_i. Then the sign of L'_i is uniquely determined by the number of positive slices in L_i, and so is the number of positive slices in $L_i\setminus L'_i$. Thus, the tight contact structures on $(\Sigma'\times S^1)\cup L'_1\cup L'_2\cup L'_2$ and $L_i\setminus L'_i$ are uniquely determined by these three numbers. But there are only $|b^{(1)}_0 b^{(2)}_0 b^{(3)}_0|$ ways to choose these three numbers. So there are at most $|b^{(1)}_0 b^{(2)}_0 b^{(3)}_0|$ on $\Sigma''\times S^1$ that satisfy the given boundary condition. Altogether, there are at most $|\prod_{i=1}^3 b^{(i)}_0\prod_{j=1}^{l_i}(b^{(i)}_j+1)|$ tight contact structures on M.

It remains to construct $|\prod_{i=1}^3 b^{(i)}_0\prod_{j=1}^{l_i}(b^{(i)}_j+1)|$ tight contact structures on M by Legendrian surgeries of (S^3,ξ_{st}). We begin with the standard surgery diagram of $M=M(\frac{q_1}{p_1},\frac{q_2}{p_2},e_0+\frac{q_3}{p_3})$. Then, perform a slum-dunk between the 0-component and the $\frac{-1}{e_0+\frac{q_3}{p_3}}$-component, after which the $\frac{-1}{e_0+\frac{q_3}{p_3}}$-component disappears and the original

0-component becomes a $(e_0+\frac{q_3}{p_3})$-component. Next we perform a (-1)-Rolfsen twist on the $(e_0+\frac{q_3}{p_3})$-component, after which the three components remain trivial and have coefficients $-\frac{p_1}{q_1}-1$, $-\frac{p_2}{q_2}-1$ and $-\frac{q_3+e_0p_3}{q_3+(e_0-1)p_3}$. But we have

$$-\frac{p_1}{q_1}-1=<b_0^{(1)}-1,b_1^{(1)},\cdots,b_{l_1}^{(1)}>,$$

$$-\frac{p_2}{q_2}-1=<b_0^{(2)}-1,b_1^{(2)},\cdots,b_{l_2}^{(2)}>$$

and

$$-\frac{q_3+e_0p_3}{q_3+(e_0-1)p_3}=<-2,\cdots,-2,b_0^{(3)}-1,b_1^{(3)},\cdots,b_{l_3}^{(3)}>,$$

where, on the right hand side of the last equation, there are e_0-1 many -2's in front of $b_0^{(3)}-1$. Now, we perform (inverses of) the slam-dunks corresponding to these three continued fractions here, which lead us to the diagram at the bottom of Figure 7. Note that all components in this diagram are trivial. Since the maximal Thurston-Bennequin number of an unknot in (S^3,ξ_{st}) is -1, it's easy to see that there are $|\prod_{i=1}^{3}b_0^{(i)}\prod_{j=1}^{l_i}(b_j^{(i)}+1)|$ ways to realize this diagram by Legendrian surgeries. According to Proposition 2.3 of [10], Theorem 1.2 of [17] and Theorem 2 of [19], these Legendrian surgeries give $|\prod_{i=1}^{3}b_0^{(i)}\prod_{j=1}^{l_i}(b_j^{(i)}+1)|$ pairwise non-isotopic holomorphically fillable contact structures on M distinguished by their Heegaard Floer invariants. □

3.3. The $e_0\leq -3$ Case.

Proof of Theorem 1.7. Define $\{p_j^{(i)}\}$ and $\{q_j^{(i)}\}$ by

$$\begin{cases} p_j^{(i)}=-a_j^{(i)}p_{j-1}^{(i)}-p_{j-2}^{(i)},\ j=0,1,\cdots,m_i,\\ p_{-2}^{(i)}=-1,\ p_{-1}^{(i)}=0,\end{cases}$$

$$\begin{cases} q_j^{(i)}=-a_j^{(i)}q_{j-1}^{(i)}-q_{j-2}^{(i)},\ j=0,1,\cdots,m_i,\\ q_{-2}^{(i)}=0,\ q_{-1}^{(i)}=1.\end{cases}$$

By Lemma 3.1 and Remark 3.2, we have $p_i=p_{m_i}^{(i)}$ and $q_i=q_{m_i}^{(i)}$. Let $u_i=p_{m_i-1}^{(i)}$ and $v_i=q_{m_i-1}^{(i)}$. Then $p_i\geq u_i>0$, $q_i\geq v_i>0$, and $p_iv_i-q_iu_i=1$.

Define an orientation preserving diffeomorphism $\varphi_i:\partial V_i\to T_i$ by

$$\varphi_i=\begin{pmatrix} p_i & u_i\\ q_i & v_i\end{pmatrix}.$$

Then

$$M=M(-\frac{q_1}{p_1},-\frac{q_2}{p_2},-\frac{q_3}{p_3})\cong(\Sigma\times S^1)\cup_{(\varphi_1\cup\varphi_2\cup\varphi_3)}(V_1\cup V_2\cup V_3).$$

Let ξ be a tight contact structure on M. We first isotope ξ to make each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $t_i<-2$, i.e., ∂V_i is minimal convex with dividing curves of slope $\frac{1}{t_i}$ when measured in the coordinates of ∂V_i. Let s_i be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of T_i. Then we have that

$$s_i = \frac{t_i q_i + v_i}{t_i p_i + u_i} = \frac{q_i}{p_i} + \frac{1}{p_i(t_i p_i + u_i)}.$$

The fact $t_i < -2$ implies that $\lfloor \frac{q_i}{p_i} \rfloor < s_i < \frac{q_i}{p_i}$.

After a possible slight isotopy supported in a neighborhood of $T_i = \partial V_i$, we assume that T_i has Legendrian ruling of slope ∞ when measured in the coordinates of T_i. For each i, pick a Legendrian ruling L_i on T_i. Choose a convex vertical annulus $A \subset \Sigma \times S^1$, such that $\partial A = L_1 \cup L_2$, and the interior of A is contained in the interior of $\Sigma \times S^1$. By Theorem 1.4, ξ does not admit Legendrian vertical circles with twisting number 0. So there must be dividing curves of A that connect the two boundary components of A. We isotope T_1 and T_2 by adding to them the bypasses corresponding to the ∂-parallel dividing curves of A. Since bypass adding is done in a small neighborhood of the bypass and the original surface, we can keep V_i's disjoint during this process. Also T_i remains minimal convex after each bypass adding. After we depleted all the ∂-parallel dividing curves of A, each of the remaining dividing curves connects the two boundary components of A. So the slopes of the dividing curves of T_1 and T_2 after the isotopy are $s'_1 = \frac{k_1}{k}$ and $s'_2 = \frac{k_2}{k}$, where $k \geq 1$ and $g.c.d.(k, k_i) = 1$ for $i = 1, 2$. Since $\lfloor \frac{q_i}{p_i} \rfloor < s_i$, We have that, for $i = 1, 2$, $s'_i \geq \lfloor \frac{q_i}{p_i} \rfloor \geq 0$, and, hence $k_i \geq 0$. This is because that, by Lemma 3.15 of [12], $s'_i < \lfloor \frac{q_i}{p_i} \rfloor$ implies $s'_i = \infty$ which contradicts Theorem 1.4. Now, cut M open along $A \cup T_1 \cup T_2$ and round the edges. We get a minimal convex torus isotopic to T_3 with dividing curves of slope $-\frac{k_1+k_2+1}{k}$ when measure in the coordinates of T_3. When measured in the coordinates of ∂V_3, these dividing curves have slope $-\frac{kq_3+(k_1+k_2+1)p_3}{kv_3+(k_1+k_2+1)u_3}$. It's easy to check that $-\frac{kq_3+(k_1+k_2+1)p_3}{kv_3+(k_1+k_2+1)u_3} < -\frac{q_3}{v_3}$. So, by Theorem 4.16 of [12], we can isotope ∂V_3 so that it becomes minimal convex with dividing curves of slope $-\frac{q_3}{v_3}$. Measured in the coordinates of T_3, the slope is 0. This implies that the maximal twisting number of a Legendrian vertical circle is -1.

After an isotopy of ξ, we can find a Legendrian vertical circle L in the interior of $\Sigma \times S^1$ with twisting number -1, and, again, make each V_i a standard neighborhood of a Legendrian circle L_i isotopic to the i-th singular fiber with twisting number $t_i < -2$. As before, we can assume that T_i has Legendrian ruling of slope ∞ when measured in the coordinates of T_i. Let L_i be a Legendrian ruling of T_i. For each i, we choose a convex vertical annulus $A_i \subset \Sigma \times S^1$, s.t., $\partial A_i = L \cup L_i$, the interior of A_i is contained in the interior of $\Sigma \times S^1$, and $A_i \cap A_j = L$ when $i \neq j$. A_i has no ∂-parallel dividing curves on the L side since $t(L)$ is maximal. So the dividing set of A_i consists of two curves connecting L to L_i and possibly some ∂-parallel curves on the L_i side. We now isotope T_i by adding to it the bypasses corresponding to these ∂-parallel dividing curves, and keep V_i's disjoint in this process. After this isotopy, we get a convex decomposition

$$M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3)$$

of M, where each T_i is minimal convex with dividing curves of slope $\lfloor \frac{q_i}{p_i} \rfloor$ when measured in the coordinate of T_i. When measured in coordinates of ∂V_i, the slope of the dividing curves becomes $-\frac{q_i - \lfloor \frac{q_i}{p_i} \rfloor p_i}{v_i - \lfloor \frac{q_i}{p_i} \rfloor u_i} = -\frac{q_i + (a_0^{(i)}+1)p_i}{v_i + (a_0^{(i)}+1)u_i}$.

By part (4) of Lemma 5.1 of [13], there are exactly $2+\lfloor \frac{q_1}{p_1} \rfloor + \lfloor \frac{q_2}{p_2} \rfloor + \lfloor \frac{q_3}{p_3} \rfloor = |e_0(M)+1|$ tight contact structures on $\Sigma \times S^1$ satisfying the boundary condition and admitting no Legendrian vertical circles with twisting number 0. By Theorem 2.3 of [12] and part (4) of Lemma 3.1, there are exactly $|\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ tight contact structures on V_i satisfying the boundary condition. Thus, up to isotopy, there are at most $|(e_0(M)+1)\prod_{i=1}^{3}\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ tight contact structures on M.

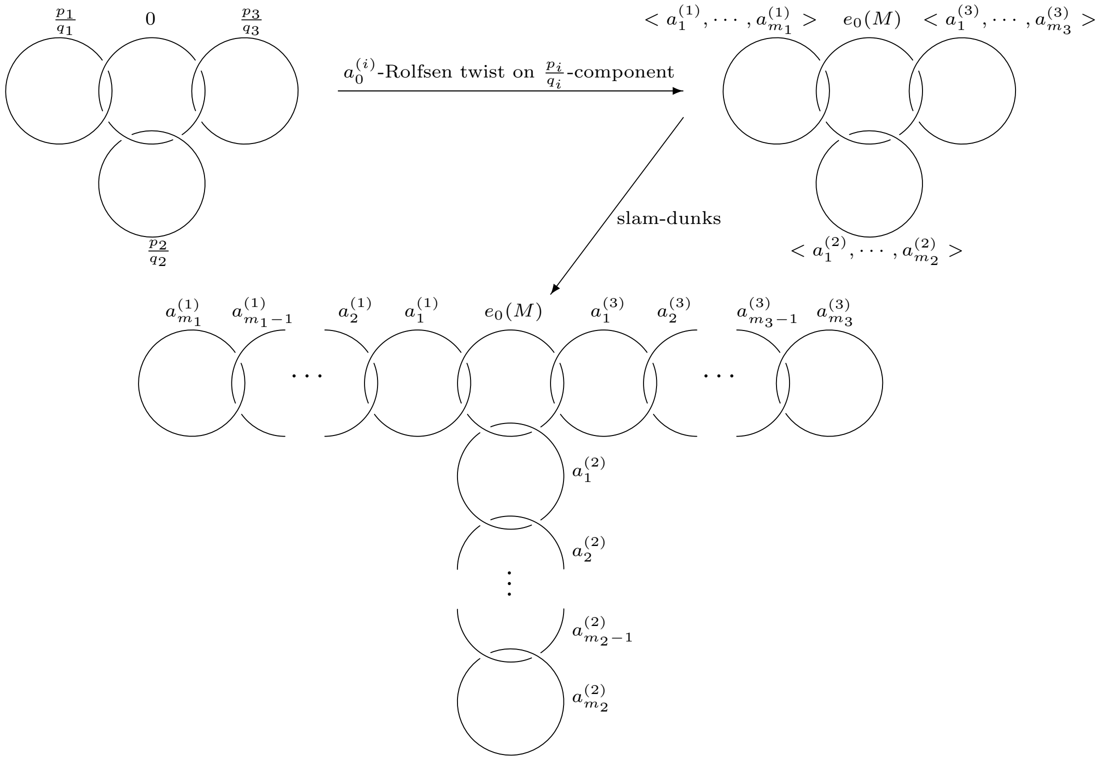

FIGURE 8. Construction of tight contact structures on $M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$

It remains to construct $|(e_0(M)+1)\prod_{i=1}^{3}\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ tight contact structures on M by Legendrian surgeries of (S^3, ξ_{st}). We begin with the standard surgery diagram of $M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$. Then, for each i, perform an $a_0^{(i)}$-Rolfsen twist on the $\frac{p_i}{q_i}$-component. Since $a_0^{(1)} + a_0^{(2)} + a_0^{(3)} = e_0(M)$ and $\frac{p_i}{q_i + a_0^{(i)} p_i} =< a_1^{(i)}, \cdots, a_{m_i}^{(i)} >$, the new surgery coefficients of the four components are $e_0(M)$, $< a_1^{(1)}, \cdots, a_{m_1}^{(1)} >$, $< a_1^{(2)}, \cdots, a_{m_2}^{(2)} >$, and $< a_1^{(3)}, \cdots, a_{m_3}^{(3)} >$. Now, we perform (inverses of) the slam-dunks corresponding to the three continued fractions here, which lead us to the diagram at the bottom of Figure 8. Since the maximal Thurston-Bennequin number of an unknot in (S^3, ξ_{st}) is -1, there are $|(e_0(M)+1)\prod_{i=1}^{3}\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ ways to realize this diagram by Legendrian surgeries. According to Proposition 2.3 of [10], Theorem 1.2 of [17] and Theorem 2 of [19], these Legendrian surgeries give $|(e_0(M)+1)\prod_{i=1}^{3}\prod_{j=1}^{m_i}(a_j^{(i)}+1)|$ pairwise non-isotopic holomorphically fillable contact structures on M distinguished by their Heegaard Floer invariants. $\square$

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