A colored $\mathfrak{sl}(N)$-homology for links in S^3

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- I will introduce an $\mathfrak{sl}(N)$-homology associated to links colored by integers, which generalizes the Khovanov-Rozansky $\mathfrak{sl}(N)$-homology.
- The construction of this colored $\mathfrak{sl}(N)$-homology uses matrix factorizations over rings of symmetric polynomials.
- I conjecture that this colored $\mathfrak{sl}(N)$-homology decategorifies to the quantum $\mathfrak{sl}(N)$-polynomial of links colored by exterior powers of the defining representation.

Rings of Symmetric and Partially Symmetric Polynomials

- An alphabet is a set $\mathbb{X} = \{x_1, \ldots, x_m\}$ of finitely many indeterminants. Denote by $\mathrm{Sym}(\mathbb{X})$ the ring of symmetric polynomials in $\mathbb{X}$ with complex coefficients. The grading on $\mathrm{Sym}(\mathbb{X})$ is given by $\deg x_j = 2$.

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- Let $\mathbb{X}_1, \ldots, \mathbb{X}_l$ be a collection of pairwise disjoint alphabets. Denote by $\mathrm{Sym}(\mathbb{X}_1|\cdots|\mathbb{X}_l)$ the ring of polynomials in $\mathbb{X}_1 \cup \cdots \cup \mathbb{X}_l$ over $\mathbb{C}$ that are symmetric in each $\mathbb{X}_i$, which is naturally a $\mathrm{Sym}(\mathbb{X}_1 \cup \cdots \cup \mathbb{X}_l)$-module. This is a free module whose structure is known.

Simple Symmetric Polynomials

For an alphabet $\mathbb{X} = \{x_1, \dots, x_m\}$,

- $$\begin{aligned} \text{elementary:} \quad & X_k := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq m} x_{i_1} x_{i_1} \cdots x_{i_k}, \\ \text{complete:} \quad & h_k(\mathbb{X}) := \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq m} x_{i_1} x_{i_1} \cdots x_{i_k}, \\ \text{power sum:} \quad & p_k(\mathbb{X}) := \sum_{i=1}^{m} x_i^k. \end{aligned}$$

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- $$\begin{aligned} \mathrm{Sym}(\mathbb{X}) &= \mathbb{C}[X_1, \ldots, X_m] = \mathbb{C}[h_1(\mathbb{X}), \ldots, h_m(\mathbb{X})] \\ &= \mathbb{C}[p_1(\mathbb{X}), \ldots, p_m(\mathbb{X})] \end{aligned}$$

Cohomology of Complex Grassmannian

Denote by $G_{m,N}$ the complex (m, N) Grassmannian. Let $\mathbb{X}$ and $\mathbb{Y}$ be alphabets of m and $N - m$ indeterminants.

- Usual cohomology:

$$H^*(G_{m,N}; \mathbb{C}) \cong \mathrm{Sym}(\mathbb{X})/(h_{N+1-m}(\mathbb{X}), h_{N+2-m}(\mathbb{X}), \dots, h_N(\mathbb{X}))$$

as graded $\mathbb{C}$-algebras.

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- $GL(N;\mathbb{C})$-equivariant cohomology:

$$H^*_{GL(N;\mathbb{C})}(G_{m,N};\mathbb{C}) \cong \mathrm{Sym}(\mathbb{X}|\mathbb{Y})$$

  as graded $\mathrm{Sym}(\mathbb{X}\cup\mathbb{Y})$-algebras.

Grading Shifts

Let M be a graded vector space. For $j \in \mathbb{Z}$, define $M\{q^j\}$ to be M with grading shifted by j, i.e. $M\{q^j\} = M$ as ungraded R-modules and, for every homogeneous element $m \in M$, $\deg_{M\{q^j\}} m = j + \deg_M m$. More generally, let $f(q) = \sum_{j=k}^{l} a_j q^j$ be a Laurent polynomial whose coefficients are non-negative integers. Define

$$M\{f(q)\} = \bigoplus_{j=k}^{l} (\underbrace{M\{q^j\} \oplus \cdots \oplus M\{q^j\}}_{a_j-\text{fold}}).$$

Quantum Integers

Quantum integers are a particular class of such Laurent polynomials. We use the following definitions:

$$[j] := \frac{q^j - q^{-j}}{q - q^{-1}},$$

$$[j]! := [1] \cdot [2] \cdots [j],$$

$$\begin{bmatrix} j \\ k \end{bmatrix} := \frac{[j]!}{[k]! \cdot [j-k]!}.$$

It is well known that

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = q^{-mn} \sum_{\lambda=(\lambda_1 \geq \cdots \geq \lambda_m):\ l(\lambda) \leq m,\ \lambda_1 \leq n} q^{2|\lambda|}.$$

Graded Matrix Factorizations

Fix an integer $N > 0$. Let R be a graded commutative unital $\mathbb{C}$-algebra, and w a homogeneous element of R with $\deg w = 2N + 2$.

A graded matrix factorization M over R with potential w is a collection of two graded free R-modules M_0, M_1 and two homogeneous R-module homomorphisms $d_0 : M_0 \to M_1$, $d_1 : M_1 \to M_0$ of degree $N + 1$, called differential maps, s.t.

$$d_1 \circ d_0 = w \cdot \mathrm{id}_{M_0}, \qquad d_0 \circ d_1 = w \cdot \mathrm{id}_{M_1}.$$

We usually write M as

$$M_0 \xrightarrow{d_0} M_1 \xrightarrow{d_1} M_0.$$

Koszul Matrix Factorizations

If $a_0, a_1 \in R$ are homogeneous s.t. $\deg a_0 + \deg a_1 = 2N + 2$, then denote by $(a_0, a_1)_R$ the graded matrix factorization

$$R \xrightarrow{a_0} R\{q^{N+1-\deg a_0}\} \xrightarrow{a_1} R,$$

which has potential $a_0 a_1$.

Koszul Matrix Factorizations (cont'd)

More generally, if $a_{1,0}, a_{1,1}, \dots, a_{k,0}, a_{k,1} \in R$ are homogeneous with $\deg a_{j,0} + \deg a_{j,1} = 2N + 2$, then define

$$\left(\begin{array}{cc} a_{1,0}, & a_{1,1} \\ a_{2,0}, & a_{2,1} \\ \dots & \dots \\ a_{k,0}, & a_{k,1} \end{array} \right)_R$$

to be the tenser product

$$(a_{1,0}, a_{1,1})_R \otimes_R (a_{2,0}, a_{2,1})_R \otimes_R \cdots \otimes_R (a_{k,0}, a_{k,1})_R,$$

which is a graded matrix factorization with potential $\sum_{j=1}^{k} a_{j,0} \cdot a_{j,1}$.

The Categories $\mathrm{HMF}_{R,w}$ and $\mathrm{hmf}_{R,w}$

If M, M' are both graded matrix factorizations over R with potential w, then $\mathrm{Hom}_R(M, M')$ is a graded $\mathbb{Z}_2$-chain complex of R-modules. Its homology, $\mathrm{Hom}_{\mathrm{HMF}}(M, M')$, is the R-module of homotopy classes of morphisms of matrix factorizations from M to M'. Denote by $\mathrm{Hom}_{\mathrm{hmf}}$ the $\mathbb{C}$-subspace of $\mathrm{Hom}_{\mathrm{HMF}}(M, M')$ of homogenous elements of bi-degree $(0, 0)$.

Category	Objects	Morphisms
$\mathrm{HMF}_{R,w}$	all homotopically finite graded matrix factorizations over R of potential w with quantum gradings bounded below	$\mathrm{Hom}_{\mathrm{HMF}}$
$\mathrm{hmf}_{R,w}$	same as above	$\mathrm{Hom}_{\mathrm{hmf}}$

The Krull-Schmidt Property

An additive category $\mathcal{C}$ is called Krull-Schmidt if

- every object of $\mathcal{C}$ is isomorphic to a finite direct sum $A_1 \oplus \cdots \oplus A_n$ of indecomposible objects of $\mathcal{C}$;
- and, if $A_1 \oplus \cdots \oplus A_n \cong A'_1 \oplus \cdots \oplus A'_l$, where $A_1, \ldots A_n, A'_1, \ldots, A'_l$ are indecomposible objects of $\mathcal{C}$, then $n = l$ and there is a permutation σ of $\{1, \ldots, n\}$ such that $A_i \cong A'_{\sigma(i)}$ for $i = 1, \ldots, n$.

Theorem (Khovanov-Rozansky)

If R is a polynomial ring with homogeneous indeterminants of positive gradings and w is a homogeneous element of R with $\deg w = 2N + 2$, then $\mathrm{hmf}_{R,w}$ and $\mathrm{hCh}^b(\mathrm{hmf}_{R,w})$ are both Krull-Schmidt, where $\mathrm{hCh}^b(\mathrm{hmf}_{R,w})$ is the homotopy category of bounded chain complexes over $\mathrm{hmf}_{R,w}$.

MOY Graphs

An (embedded) MOY graph is an oriented plane graph with each edge colored by a non-negative integer such that

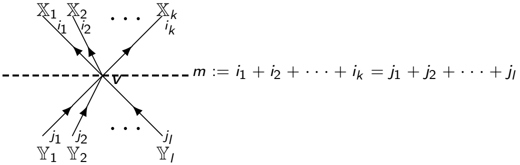
- for every vertex v with valence at least 2, the sum of integers coloring the edges entering v is equal to the sum of integers coloring the edges leaving v,
- through each such vertex v of Γ, there is a straight line L_v so that all the edges entering v enter through one side of L_v and all edges leaving v leave through the other side of L_v.

Markings

A marking of an MOY graph Γ consists the following:

1. A finite collection of marked points on Γ such that
   - every edge of Γ has at least one marked point;
   - all the end points (vertices of valence 1) are marked;
   - none of the interior vertices (vertices of valence at least 2) is marked.
2. An assignment of pairwise disjoint alphabets to the marked points such that the alphabet associated to a marked point on an edge of color m has m indeterminants.

The Matrix Factorization Associated to a Vertex

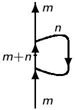


Let $\mathbb{X} = \mathbb{X}_1 \cup \cdots \cup \mathbb{X}_k$ and $\mathbb{Y} = \mathbb{Y}_1 \cup \cdots \cup \mathbb{Y}_l$. Denote by X_j and Y_j the j-th elementary symmetric polynomials in $\mathbb{X}$ and $\mathbb{Y}$.

$$C(v) := \left(\begin{array}{cc} U_1 & X_1 - Y_1 \\ U_2 & X_2 - Y_2 \\ \dots & \dots \\ U_m & X_m - Y_m \end{array} \right)_{\mathrm{Sym}(\mathbb{X}_1|\dots|\mathbb{X}_k|\mathbb{Y}_1|\dots|\mathbb{Y}_l)} \{q^{-\sum_{1\leq s<t\leq k} i_s i_t}\},$$

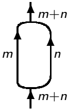
where U_j is homogeneous of degree $2N + 2 - 2j$ and $\sum_{j=1}^{m} (X_j - Y_j) U_j = p_{N+1}(\mathbb{X}) - p_{N+1}(\mathbb{Y})$.

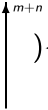
Decompositions (I & II)

(I) $C($ 

$) \simeq C($ 

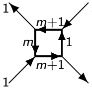
$)\{\begin{bmatrix} N-m \\ n \end{bmatrix}\}\langle n \rangle.$

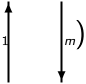
(II) $C($ 

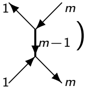
$) \simeq C($ 

$)\{\begin{bmatrix} m+n \\ n \end{bmatrix}\}.$

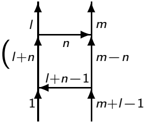
Decompositions (III & IV)

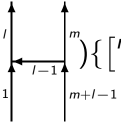
(III) $C($ 

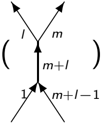
$) \simeq C($ 

$) \oplus C($ 

$)\{[N-m-1]\}\langle 1\rangle.$

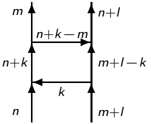
(IV) $C($ 

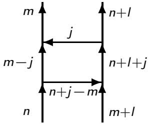
$) \simeq C($ 

$)\{\begin{bmatrix} m-1 \\ n \end{bmatrix}\} \oplus C($ 

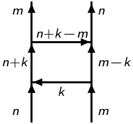
$)\{\begin{bmatrix} m-1 \\ n-1 \end{bmatrix}\}.$

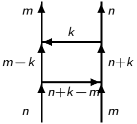
Decompositions (V)

(V) $C($ 

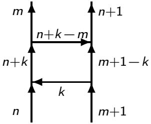
$) \simeq \bigoplus_{j=\max\{m-n,0\}}^{m} C($ 

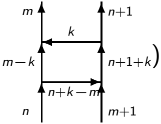
$)\{\begin{bmatrix} l \\ k-j \end{bmatrix}\}.$

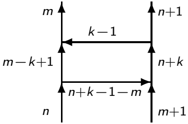
$l = 0 \Rightarrow C($ 

$) \simeq C($ 

$).$

$l = 1 \Rightarrow C($ 

$) \simeq C($ 

$) \oplus C($ 

$).$

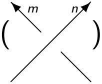
Marking of Colored Link Diagrams

Recall that N is a fixed positive integer. (It is the "N" in "$\mathfrak{sl}(N)$".) Given a diagram D of a link whose components are colored by integers $\in \{1, \ldots, N\}$. A marking of D consists the following:

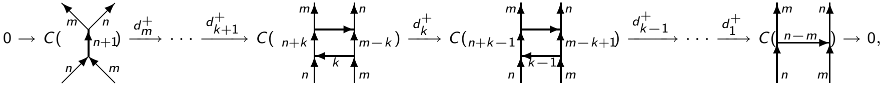
1. A finite collection of marked points on D such that
   - every arc between two crossings has at least one marked point;
   - none of the crossings is marked.
2. An assignment of pairwise disjoint alphabets to the marked points such that the alphabet associated to a marked point on an arc of color m has m indeterminants.

The Chain Complex of a Colored Crossing

Assume $n \geq m$ and temporarily forget the quantum grading shifts.

$C($ 

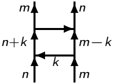
$)$ should be an object of $\mathrm{hCh}^{\mathrm{b}}(\mathrm{HMF})$ of the form

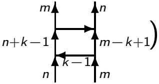


where d_k^+ is homogeneous of quantum degree 1.
If we assume d_k^+ is not homotopic to 0, then there is a unique chain complex of this form. (If $m = n = 1$, then this chain complex is isomorphic to that defined by Khovanov and Rozansky.)

The Chain Complex of a Colored Crossing (cont'd)

- The lowest quantum grading of

$\mathrm{Hom}_{\mathrm{HMF}}(C($ 

$), C($ 

$))$

is 1 and the space of homogeneous elements of quantum degree 1 is 1-dimensional.

The Chain Complex of a Colored Crossing (cont'd)

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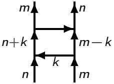
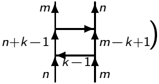
$\mathrm{Hom}_{\mathrm{HMF}}(C($ 

$), C($ 

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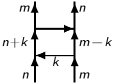
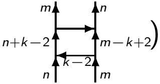
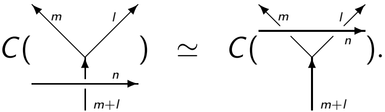
$\mathrm{Hom}_{\mathrm{HMF}}(C($ 

$), C($ 

$))$ is 4.

Fork Sliding

Lemma

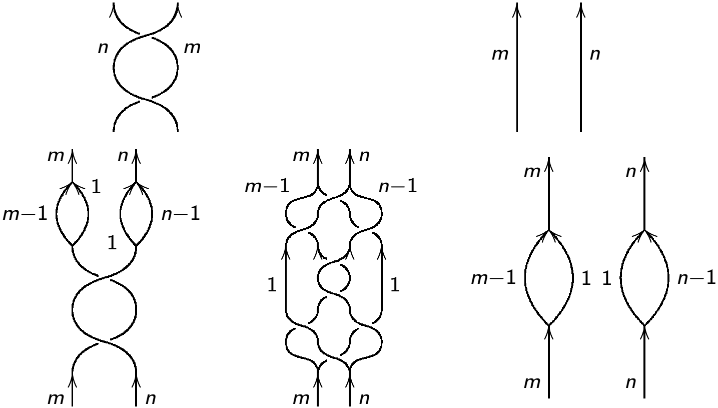


Reidemeister Moves – Main Theorem

Theorem

The $\mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z}$-graded colored $\mathfrak{sl}(N)$-homology is invariant under Reidemeister moves.

Proof



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- Is the $\mathbb{Z}_2$-grading concentrated?
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- Functorality? (Khovanov and Rozansky's proof should carry over. But the algebra looks much harder.)

Generalizations

- Lee-Gornik deformation. (Definition and invariance look easy. The Lee-Gornik basis is hard to construct. We can probably still get colored $\mathfrak{sl}(N)$-Rasmussen invariants and genus bounds.)

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- Categorification of the $\mathfrak{sl}(N)$-invariant for 3-manifolds. (Holy Grail?)

Related Research

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- Webster's **categorify'em all** approach.