Transverse Khovanov-Rozansky Homologies

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Transverse Links in the Standard Contact S^3

A contact structure ξ on an oriented 3-manifold M is an oriented tangent plane distribution such that there is a 1-form α on M satisfying $\xi = \ker \alpha$, $d\alpha|_\xi > 0$ and $\alpha \wedge d\alpha > 0$. Such a 1-form is called a contact form for ξ.

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Clearly, any smooth counterclockwise closed braid around the z-axis can be smoothly isotoped into a transverse closed braid around the z-axis without changing the braid word.

The Transverse Markov Theorem

Transverse Markov moves:

- Braid group relations generated by
  - $\sigma_i \sigma_i^{-1} = \sigma_i^{-1} \sigma_i = \emptyset$,
  - $\sigma_i \sigma_j = \sigma_j \sigma_i$, when $|i - j| > 1$,
  - $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.
- Conjugations: $\mu \leftrightsquigarrow \eta^{-1} \mu \eta$.
- Positive stabilizations and destabilizations: $\mu \ (\in \mathbf{B}_m) \leftrightsquigarrow \mu \sigma_m \ (\in \mathbf{B}_{m+1})$.

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Two transverse closed braids are transverse isotopic if and only if the two braid words are related by a finite sequence of transverse Markov moves.

So there is a one-to-one correspondence between transverse isotopy classes of transverse links and closed braids modulo transverse Markov moves.

Contact Framing

ξ_{st} admits a nowhere vanishing basis $\{\partial_x + y\partial_z, \partial_y - x\partial_z\}$. For each transverse link L, this basis induces a contact framing of L. If two transverse links are transverse isotopic, then they are isotopic as framed links.

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An invariant for transverse links is called classical if it depends only on the framed link type of the transverse link. Otherwise, it is called non-classical or effective.

The Khovanov-Rozansky Homology

Khovanov and Rozansky introduced an approach to construct link homologies using matrix factorizations by:

1. Choose a base ring R and a potential polynomial $p(x) \in R[x]$.
2. Define matrix factorizations associated to MOY graphs using this potential $p(x)$.
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This approach has been carried out for the following potential polynomials:

- $x^{N+1} \in \mathbb{Q}[x]$ (the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology);
- $ax \in \mathbb{Q}[a, x]$ (the HOMFLYPT homology);
- $x^{N+1} + \sum_{l=1}^{N} \lambda_l x^l \in \mathbb{Q}[x]$ (deformed $\mathfrak{sl}(N)$ Khovanov-Rozansky homology);
- $x^{N+1} + \sum_{l=1}^{N} a_l x^l \in \mathbb{Q}[a_1, \ldots, a_N, x]$ (the equivariant $\mathfrak{sl}(N)$ Khovanov-Rozansky homology).

Transverse Khovanov-Rozansky Homologies

For $N \geq 1$, applying Khovanov and Rozansky's matrix factorization construction to $ax^{N+1} \in \mathbb{Q}[a, x]$, one gets a chain complex $\mathcal{C}_N$. For each link diagram D, the homology $\mathcal{H}_N(D)$ of $\mathcal{C}_N(D)$ is a $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$-graded $\mathbb{Q}[a]$-module.

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Theorem (W)

Suppose $N \geq 1$. Let B be a closed braid. Every transverse Markov move on B induces an isomorphism of $\mathcal{H}_N(B)$ preserving the $\mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 3}$-graded $\mathbb{Q}[a]$-module structure.

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Question

Is $\mathcal{H}_N$ an effective invariant for transverse links?

Decategorification

$$\mathcal{P}_N(B) := \sum_{(\varepsilon,i,j,k)\in\mathbb{Z}_2\oplus\mathbb{Z}^{\oplus 3}} (-1)^i \tau^{\varepsilon} \alpha^j \xi^k \dim_{\mathbb{Q}} \mathcal{H}_N^{\varepsilon,i,j,k}(B) \in \mathbb{Z}[[\alpha,\xi]][\alpha^{-1},\xi^{-1},\tau]/(\tau^2-1)$$

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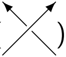
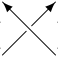
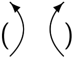
1. *$\mathcal{P}_N$ is invariant under transverse Markov moves.*
2. $\alpha^{-1}\xi^{-N}\mathcal{P}_N($  $) - \alpha\xi^N\mathcal{P}_N($  $) = \tau(\xi^{-1}-\xi)\mathcal{P}_N($  $)$.
3. $\mathcal{P}_N(U^{\sqcup m}) = (\tau\alpha^{-1}[N])^m\left(\frac{1}{1-\alpha^2} + \frac{\left(\frac{\tau\alpha\xi^{-1}+\xi^{-N}}{\xi^{-N}-\xi^N}\right)^m - 1}{\tau\alpha\xi^{-N-1}+1}\right)$*, where* $U^{\sqcup m}$ *is the m-strand closed braid with no crossings and* $[N] := \frac{\xi^{-N}-\xi^N}{\xi^{-1}-\xi}$.
4. *Parts 1–3 above uniquely determine the value of $\mathcal{P}_N$ on every closed braid.*

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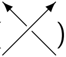
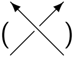
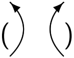
2. $\alpha^{-1}\xi^{-N}\mathcal{P}_N($  $) - \alpha\xi^N\mathcal{P}_N($  $) = \tau(\xi^{-1}-\xi)\mathcal{P}_N($  $).$
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4. *Parts 1–3 above uniquely determine the value of $\mathcal{P}_N$ on every closed braid.*

It is not clear if $\mathcal{P}_N$ is effective. But $\mathcal{P}_N$ does not detect flype moves. ($\mu\sigma_m^k\nu\sigma_m^{\pm1} \leftrightsquigarrow \mu\sigma_m^{\pm1}\nu\sigma_m^k$, where $\mu,\nu\in\mathbf{B}_m$.)

Module Structure

Theorem (W)

Let $H_N(B)$ be the $\mathfrak{sl}(N)$ Khovanov-Rozansky homology of a closed braid B, and $(\varepsilon, i, k) \in \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2}$.

1. $H_N^{\varepsilon,i,k}(B) \cong \mathcal{H}_N^{\varepsilon,i,\star,k}(B)/(a-1)\mathcal{H}_N^{\varepsilon,i,\star,k}(B)$.

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1. $H_N^{\varepsilon,i,k}(B) \cong \mathcal{H}_N^{\varepsilon,i,\star,k}(B)/(a-1)\mathcal{H}_N^{\varepsilon,i,\star,k}(B)$.
2. *As a $\mathbb{Z}$-graded $\mathbb{Q}[a]$-module,*

$$\mathcal{H}_N^{\varepsilon,i,\star,k}(B) \cong (\mathbb{Q}[a]\{sl(B)\}_a)^{\oplus l} \oplus (\mathbb{Q}[a]\{sl(B)+2\}_a)^{\oplus(\dim_{\mathbb{Q}} H_N^{\varepsilon,i,k}(B)-l)} \oplus (\bigoplus_{q=1}^{n} \mathbb{Q}[a]/(a)\{s_q\}),$$

where

- $\{s\}_a$ *means shifting the a-grading by* s,
- l *and* n *are finite non-negative integers determined by* B *and the triple* (ε, i, k),
- $\{s_1, \ldots, s_n\} \subset \mathbb{Z}$ *is a sequence determined up to permutation by* B *and the triple* (ε, i, k),
- $sl(B) \leq s_q \leq c_+ - c_- - 1$ *and* $(N-1)s_q \leq k - 2N + 2c_-$ *for* $1 \leq q \leq n$, *where* $c_\pm$ *is the number of* $\pm$ *crossings in* B.

Negative Stabilization

Theorem (W)

Let L be a transverse closed braid, and L_- a transverse closed braid obtained from L by a single negative stabilization. Then the chain complex $\mathcal{C}_N(L_-)\{2,0\}$ is isomorphic to the mapping cone of the standard quotient map $\pi_0 : \mathcal{C}_N(L) \to \mathcal{C}_N(L)/a\mathcal{C}_N(L)$.

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Thus, if $\mathscr{H}_N(L)$ is the homology of $\mathcal{C}_N(L)/a\mathcal{C}_N(L)$, there is a long exact sequence

$$\cdots \to \mathcal{H}_N^{\varepsilon,i-1}(L)\{-2,0\} \xrightarrow{\pi_0} \mathscr{H}_N^{\varepsilon,i-1}(L)\{-2,0\} \to \mathcal{H}_N^{\varepsilon,i}(L_-) \to \mathcal{H}_N^{\varepsilon,i}(L)\{-2,0\} \xrightarrow{\pi_0} \cdots$$

Negative Stabilization (cont'd)

Theorem (W)

Let B be a closed braid and B_- a stabilization of B. Set $s = sl(B)$. Then for any $(i,k) \in \mathbb{Z}^{\oplus 2}$, there are a long exact sequence of $\mathbb{Z}$-graded $\mathbb{Q}[a]$-modules

$$
\begin{tikzcd}
\cdots \arrow[r] & \mathcal{H}_N^{s-1,i,\star,k}(B_-) \arrow[r] & \mathcal{H}_N^{s,i-1,\star,k+N+1}(B)\{-1\}_a \arrow[dll] \\
H_N^{s,i-1,k+N+1}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{s-1\}_a \arrow[r] & \mathcal{H}_N^{s-1,i+1,\star,k}(B_-) \arrow[r] & \cdots
\end{tikzcd}
$$

and a short exact sequence of $\mathbb{Z}$-graded $\mathbb{Q}[a]$-modules

$$0 \to H_N^{s,i,k}(B) \otimes_{\mathbb{Q}} \mathbb{Q}[a]\{s\}_a \to \mathcal{H}_N^{s,i,\star,k}(B_-) \to \mathcal{H}_N^{s-1,i-1,\star,k+N+1}(B)\{-1\}_a \to 0.$$

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- Denote by U_0 the transverse unknot with self linking -1 and by U_m the transverse unknot obtained from U_0 by m negative stabilizations.
- Then every transverse unknot is transverse isotopic to U_m for some $m \geq 0$.

Transverse Unknots (cont'd)

$$\begin{aligned}
\mathcal{F} &:= \bigoplus_{l=0}^{N-1} \mathbb{Q}[a]\,\langle 1 \rangle\,\{-1, -N+1+2l\},\\
\mathcal{T} &:= \bigoplus_{l=0}^{\infty} \mathbb{Q}[a]/(a)\,\langle 1 \rangle\,\{-1, N+1+2l\},
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$$\mathcal{H}_N(U_0) \cong \mathcal{F} \oplus \mathcal{T},$$
$$\mathcal{H}_N(U_1) \cong \mathcal{F} \oplus \mathcal{T}\,\langle 1\rangle\,\{-1, -N-1\}\|1\|,$$

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$$\mathcal{H}_N(U_1) \cong \mathcal{F} \oplus \mathcal{T} \langle 1 \rangle \{-1, -N-1\} \|1\|,$$

and, for $m \geq 2$,

$$\mathcal{H}_N(U_m) \cong \mathcal{F}\{-2(m-1), 0\} \oplus \mathcal{T} \langle m \rangle \{-m, -m(N+1)\} \|m\| \oplus \bigoplus_{l=1}^{m-1} \mathcal{F}/a\mathcal{F} \langle l \rangle \{-2m+l, -l(N+1)\} \|l+1\|,$$

where "$\|l\|$" means shifting the homological grading by l.