Incidence Relations and Directed Cycles

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Directed graphs and directed cycles

A **directed graph** is a pair $G = (V(G), E(G))$ of finite sets, where

1. $V(G)$ is the set of vertices of G,
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A **directed cycle** in G is a closed directed path, that is, a sequence $v_0, x_0, v_1, x_1, \ldots, x_{n-1}, v_n, x_n, v_{n+1} = v_0$ satisfying

1. $v_0, v_1, \ldots, v_n$ are pairwise distinct vertices of G,
2. each x_i is an edge of G with initial vertex v_i and terminal vertex v_{i+1}.

Two such sequences represent the same directed cycle if one is a circular permutation of the other.

Cycles packing numbers

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- $\alpha(G) :=$ maximal number of pairwise edge-disjoint directed cycles in G,
- $\tilde{\alpha}(G) :=$ maximal number of pairwise disjoint directed cycles in G,

$\alpha(G)$ is known as the **cycle packing number** of G. We call $\tilde{\alpha}(G)$ the **strong cycle packing number** of G.

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$\alpha(G)$ is known as the **cycle packing number** of G. We call $\tilde{\alpha}(G)$ the **strong cycle packing number** of G.

Our goal is to determine $\alpha(G)$ and $\tilde{\alpha}(G)$ using elementary projective algebraic geometry.

Directed trials, paths and circuits

Given a directed graph G, a **directed trail** in G from a vertex u to a different vertex v is a sequence
$u = v_0, x_0, v_1, x_1, \ldots, x_{n-1}, v_n = v$ such that

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A **directed circuit** in G is a closed trial, that is, a sequence
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Two such sequences represent the same directed circuit if one is a circular permutation of the other.

Disassembling a directed graph

Let G be a directed graph, and v a vertex of G. Assume $\deg_{in} v = n$ and $\deg_{out} v = m$. Set $k_v := \max\{m, n\}$ and $l_v := \min\{m, n\}$.

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To disassemble G at v is to split v into k_v vertices such that

1. l_v of these new vertices have in-degree 1 and out degree 1.
2. $k_v - l_v$ of these new vertices have degree 1 such that
   - if $m \geq n$, then each of these degree 1 vertices has in-degree 0 and out-degree 1;
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We call each graph resulted from disassembling G a **disassembly** of G and denote by $\mathrm{Dis}(G)$ the set of all disassemblies of G.

Disassemblies of a directed graph

Lemma

Let G be a directed graph, and D a disassembly of G.

1. *D is a disjoint union of directed paths and directed cycles.*

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2. *$E(D) = E(G)$ and there is a natural graph homomorphism from D to G that maps each edge to itself and each vertex v in D the vertex in G used to create v.*

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2. *$E(D) = E(G)$ and there is a natural graph homomorphism from D to G that maps each edge to itself and each vertex v in D the vertex in G used to create v.*
3. *Under the above natural homomorphism,*
   - *each directed path in D is mapped to a directed trail in G,*
   - *each directed cycle in D is mapped to a directed circuit in G,*
   - *the collection of all directed cycles in D is mapped to a collection of pairwise edge-disjoint circuits in G.*

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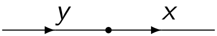
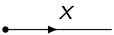
4. *$\alpha(D) \leq \alpha(G)$ and $\alpha(D) = \alpha(G)$ if and only if the collection of all directed cycles in D is mapped to a collection of $\alpha(G)$ pairwise edge-disjoint directed cycles in G.*

Incidence relations, special case

Incidence relations:



$$\Longrightarrow \quad y = x,$$



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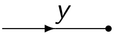

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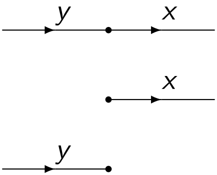

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Let G be a directed graph, and D a disassembly of G. Recall that $E(D) = E(G)$. Define the **incidence set** of D by

$$P(D) = \{p \in \mathbb{CP}^{|E(G)|-1} \mid p \text{ satisfies all incidence relations in } D.\}$$

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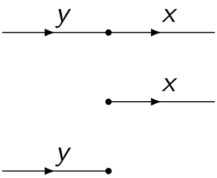

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Clearly, $P(D)$ is a linear subspace of $\mathbb{CP}^{|E(G)|-1}$.

Incidence sets of disassemblies

Lemma

Let G be a directed graph.

1. *For any disassembly D of G, the incidence set $P(D)$ of D is a linear subspace of dimension $\alpha(D) - 1$ of $\mathbb{CP}^{|E(G)|-1}$.*

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1. *For any disassembly D of G, the incidence set $P(D)$ of D is a linear subspace of dimension $\alpha(D) - 1$ of $\mathbb{CP}^{|E(G)|-1}$.*
2. *For any two disassemblies D_1 and D_2 of G, $P(D_1) = P(D_2)$ as linear subspaces of $\mathbb{CP}^{|E(G)|-1}$ if and only if, under the natural homomorphisms from D_1 and D_2 to G, the collections of all directed cycles in D_1 and D_2 are mapped to the same collection of pairwise edge-disjoint circuits in G.*

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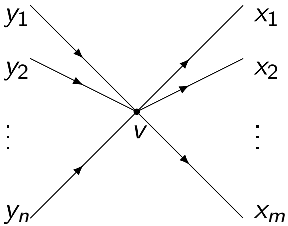
Incidence relations, general case



The set of **incidence relations** at v is

$$\Delta_v := \{e_l(x_1, \ldots, x_m) = e_l(y_1, \ldots, y_n) \mid 1 \leq l \leq \max\{n, m\},\}$$

where e_l is the degree-l elementary symmetric polynomial.

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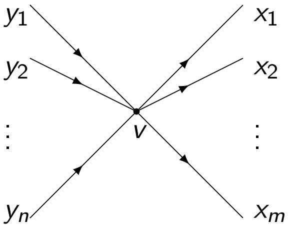

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For a directed graph G, its set of incidence relations is $\Delta(G) := \bigcup_{v \in V(G)} \Delta_v$. The **incidence set** of G is

$$P(G) = \{p \in \mathbb{CP}^{|E(G)|-1} \mid p \text{ satisfies all incidence relations in } G.\}$$

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Proposition

As subsets of $\mathbb{CP}^{|E(G)|-1}$, $P(G) = \bigcup_{D \in \mathrm{Dis}(G)} P(D)$.

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Lemma

Let $x_1, \ldots, x_n$ *and* $y_1, \ldots, y_n$ *be two sequences of complex numbers. Then the following statements are equivalent.*

1. $e_k(x_1, \ldots, x_n) = e_k(y_1, \ldots, y_n)$ *for* $k = 1, \ldots, n$*, where* e_k *is the* k*-th elementary symmetric polynomial.*
2. *There is a bijection* $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ *such that* $x_i = y_{\sigma(i)}$ *for* $i = 1, \ldots, n$.

Irreducible components of the incidence set

Proposition

Let G be a directed graph.

1. *For every maximal*[1] *collection $\mathcal{C}$ of pairwise edge-disjoint directed cycles in G, there is a disassembly $D_{\mathcal{C}}$ of G such that $\mathcal{C}$ is the collection of images of directed cycles in $D_{\mathcal{C}}$ under the natural homomorphism.*

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2. *For any disassembly D of G, $P(D)$ is not a proper subset of $P(D')$ for any $D' \in \mathrm{Dis}(G)$ if and only if the natural homomorphism maps the directed cycles in D to a maximal collection of pairwise edge-disjoint directed cycles in G.*

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3. *The set of irreducible components of $P(G)$ is $\{P(D_{\mathcal{C}}) \mid \mathcal{C}$ is a maximal collection of pairwise edge-disjoint directed cycles in $G.\}$*

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The incidence set determines the cycle packing number

Theorem

Let G be any directed graph. Then:

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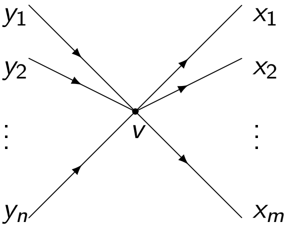
2. $\deg P(G) =$ *the number of distinct collections of* $\alpha(G)$ *edge-disjoint cycles in* G;
3. *There is a bijection between the set of irreducible components of* $P(G)$ *of dimension* $n-1$ *and the set of maximal collections of pairwise edge-disjoint directed cycles in* G *containing exactly* n *directed cycles.*

Collections of pairwise disjoint directed cycles, a stretch



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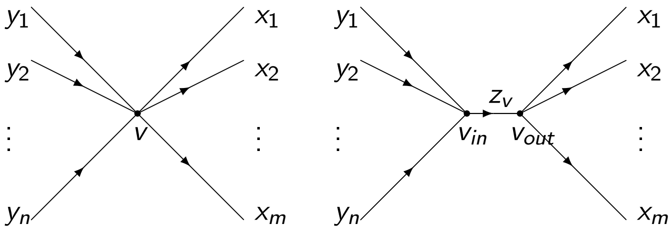

For a directed graph G, denote by B_G obtained by stretching each vertex in G.

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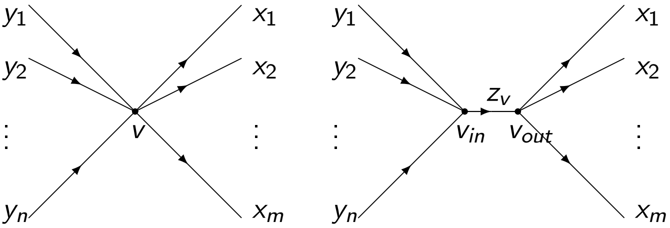

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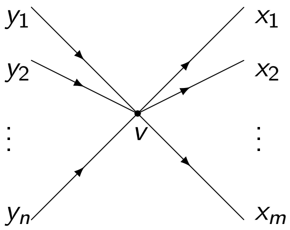

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1. *There is a bijection from the set of directed cycles in G to the set of directed cycles in B_G;*
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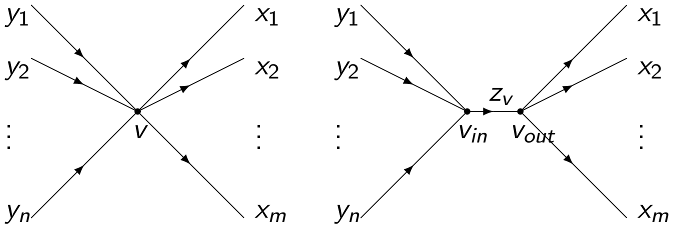

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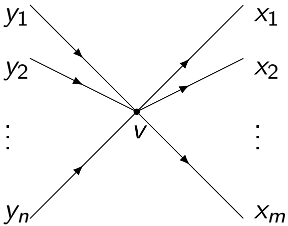
3. $\tilde{\alpha}(G) = \alpha(B_G)$.

The strong incidence set



The set of **strong incidence relations** at v is

$$\begin{aligned}\tilde{\Delta}_v \quad := \quad & \{e_1(x_1, \ldots, x_m) = e_1(y_1, \ldots, y_n)\} \\ & \cup \{e_l(x_1, \ldots, x_m) = 0 \mid 2 \leq l \leq m\} \\ & \cup \{e_l(y_1, \ldots, y_n) = 0 \mid 2 \leq l \leq n\}.\end{aligned}$$

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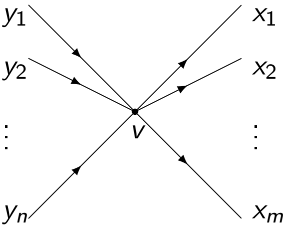

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For a directed graph G, its set of **strong incidence relations** is $\tilde{\Delta}(G) := \bigcup_{v\in V(G)} \tilde{\Delta}_v$. The **strong incidence set** of G is

$$\tilde{P}(G) = \{p \in \mathbb{CP}^{|E(G)|-1} \mid p \text{ satisfies all strong incidence relations in } G.\}$$

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Let G be any directed graph. Then:

1. *$\tilde{P}(G)$ is the union of finitely many linear subspaces of $\mathbb{CP}^{|E(G)|-1}$;*

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Irreducible incidence sets

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Theorem

Let G be a directed graph.

1. *The following statements are equivalent:*
   1.1 $P(G)$ *is irreducible;*
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2. *The following statements are equivalent:*
   2.1 $\tilde{P}(G)$ *is irreducible;*
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See **arXiv:1508.07337** for more related results.