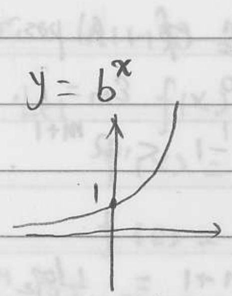
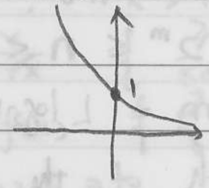
9.4. Exponential and Logarithmic Functions.

$y = b^x$ is defined on $\mathbb{R}$ with range $\mathbb{R}_+$

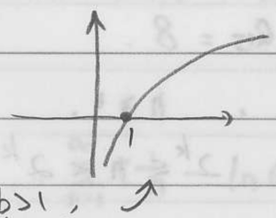


$b>1$, ↗
$\lim_{x\to -\infty} b^x = 0$
$\lim_{x\to \infty} b^x = \infty$

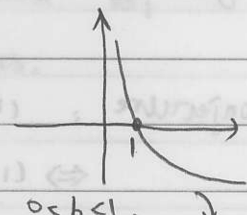


$0<b<1$, ↘
$\lim_{x\to \infty} b^x = 0$
$\lim_{x\to -\infty} b^x = \infty$

$y = \log_b x$ is the inverse of b^x. Domain $= \mathbb{R}^+$, range $= \mathbb{R}$



$b>1$, ↗
$\lim_{x\to 0} \log_b x = -\infty$
$\lim_{x\to\infty} \log_b x = \infty$



$0<b<1$, ↘
$\lim_{x\to 0} \log_b x = \infty$
$\lim_{x\to\infty} \log_b x = -\infty$.

Orders: For any $r>0$, $e^x = \Omega(x^r)$, $\neq O(x^r)$

$\ln x = O(x^r)$, $\neq \Omega(x^r)$

For any $b>1$, $r>0$, $b^x = e^{(\ln b)x} = \Omega((\ln b)^r x^r) = \Omega(x^r)$

$\neq O((\ln b)^r x^r) = O(x^r)$.

$\log_b x = \frac{1}{\ln b} \ln x = O(x^r) \neq \Omega(x^r)$.

Eg. If $2^k \le x < 2^{k+1}$, then $k \le \log_2 x < k+1 \Rightarrow \lfloor \log_2 x \rfloor = k$.

Thus, if $n \in \mathbb{Z}^+$ is not an integer power of 2, then

$\lfloor \log_2 n \rfloor = \lfloor \log_2 (n-1) \rfloor$. Specially, if $n>1$ is an odd integer, then $\lfloor \log_2 n \rfloor = \lfloor \log_2 (n-1) \rfloor$.

Eg. Length of the binary representation of a positive integer.

$$n = \sum_{k=0}^{m} \varepsilon_k 2^k, \quad \varepsilon_k = 1, 0, \quad \varepsilon_m = 1.$$

$$\Rightarrow \quad 2^m \le n \le \sum_{k=0}^{m} 2^k = 2^{m+1} - 1 < 2^{m+1}.$$

$$\Rightarrow \quad m = \lfloor \log_2 n \rfloor.$$

$\Rightarrow$ Length of the binary rep $= m+1 = \lfloor \log_2 n \rfloor + 1.$

Eg. Solve $\begin{cases} a_k = 2a_{\lfloor \frac{k}{2} \rfloor}, & k \ge 2 \\ a_1 = 1 & \end{cases}$.

Eesy to see that $a_1 = 1$, $a_2 = a_3 = 2$, $a_4 = a_5 = a_6 = a_7 = 4$,

$a_8 = a_9 = \cdots = a_{15} = 8$.

Conjecture : (1) $a_n = 2^{\lfloor \log_2 n \rfloor}$, $n \ge 1$.

$\Leftrightarrow$ (1)' $a_n = 2^k$ for $2^k \le n < 2^{k+1}$ $\forall k \ge 0$.

Prove (1)' by induction on k.

(i) $k = 0 \Rightarrow 2^0 = 1$, $2^{0+1} = 2$. $1 \le n < 2 \Rightarrow n = 1$.

$a_1 = 1 = 2^0$. So (1)' is true for $k=0$.

(ii) Assume $a_n = 2^k$ for $2^k \le n < 2^{k+1}$. Assume $2^{k+1} \le m < 2^{k+2}$. Then $2^k \le \lfloor \frac{m}{2} \rfloor < 2^{k+1} \Rightarrow a_{\lfloor \frac{m}{2} \rfloor} = 2^k$

$\Rightarrow a_m = 2a_{\lfloor \frac{m}{2} \rfloor} = 2^{k+1}$. Thus, (1)' is true for $k+1$.

(i) & (ii) $\Rightarrow$ (1)' is true for $\forall k$.

$\Rightarrow$ (1) $a_n = 2^{\lfloor \log_2 n \rfloor}$, $n \ge 1$.

Eg. $x + x\log_2 x = \Theta(x \log_2 x)$.

$$\lim_{x \to \infty} \left| \frac{x + x\log_2 x}{x \log_2 x} \right| = \lim_{x \to \infty} \left(\frac{1}{\log_2 x} + 1 \right) = 1 > 0$$

$\Rightarrow x + x\log_2 x = \Theta(x \log_2 x)$.

Eg. $b, c > 1$, $\log_b x = \Theta(\log_c x)$. $\left(\frac{\log_b x}{\log_c x} = \log_c b > 0 \ \forall x > 0 \right)$.

Eg. $\ln(1+x) \le x$ for $x > -1$.

Let $f(x) = x - \ln(1+x)$, $x > -1$.

$$f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} \begin{cases} \ge 0, & x \ge 0 \\ \le 0, & -1 < x \le 0 \end{cases}$$

$$f(0) = 0 - \ln 1 = 0, \quad f(x) \begin{cases} \nearrow & x \ge 0 \\ \searrow & -1 < x \le 0. \end{cases}$$

$\Rightarrow$ If $x > 0$, then $f(x) \ge f(0) = 0$.

If $x \le 0$, then $f(x) \ge f(0) = 0$.

$\Rightarrow f(x) \ge 0 \quad \forall x \in (-1, \infty)$. $\Rightarrow \ln(1+x) \le x$, $\forall x > -1$.

Big M-Test. If $0 \le a_n \le b_n$, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ also converges.

Eg. $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \ln\left(1 + \frac{1}{n}\right)\right)$ converges.

(i) $\ln\left(1+\frac{1}{n}\right) \le \frac{1}{n} \quad \forall n \ge 1. \Rightarrow \frac{1}{n} - \ln\left(1+\frac{1}{n}\right) \ge 0.$

(ii) $\ln\left(1+\frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right) = -\ln\frac{n}{n+1} = -\ln\left(1 - \frac{1}{n+1}\right) \ge -\left(-\frac{1}{n+1}\right) = \frac{1}{n+1}.$

$$\Rightarrow \frac{1}{n} - \ln\left(1+\frac{1}{n}\right) \le \frac{1}{n} - \frac{1}{n+1}.$$

But $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{N\to\infty} \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{N\to\infty}\left(1 - \frac{1}{N+1}\right) = 1.$

$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right) = C, \quad C \in (0, \infty).$$

Note
$$\sum_{n=1}^{N}\left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right) = \sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\ln\left(\frac{n+1}{n}\right)$$
$$= \sum_{n=1}^{N}\frac{1}{n} - \sum_{n=1}^{N}\left(\ln(n+1) - \ln(n)\right) = \sum_{n=1}^{N}\frac{1}{n} - \ln(N+1)$$

$$\Rightarrow C = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right) = \lim_{N\to\infty}\sum_{n=1}^{N}\left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right) = \lim_{n\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n} - \ln(N+1)\right).$$

$$\lim_{N\to\infty}\frac{\sum_{n=1}^{N}\frac{1}{n}}{\ln N} = \lim_{N\to\infty}\frac{\sum_{n=1}^{N}\frac{1}{n} - \ln(N+1) + \ln(N+1)}{\ln N}$$
$$= \lim_{N\to\infty}\frac{\sum_{n=1}^{N}\frac{1}{n} - \ln(N+1)}{\ln N} + \lim_{N\to\infty}\frac{\ln(N+1)}{\ln N}$$
$$= \frac{C}{\infty} + \lim_{N\to\infty}\frac{\frac{1}{N+1}}{\frac{1}{N}} = 1 > 0.$$

$$\Rightarrow \sum_{n=1}^{N}\frac{1}{n} = \Theta(\ln N).$$