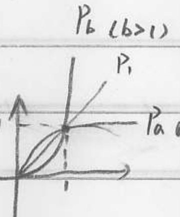
9.1 Real functions and their graphs

Def. If $X, Y \subseteq \mathbb{R}$, then a function $f: X \to Y$ is called a real function. (real valued function of a real variable)

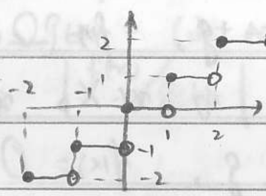
The graph of f is the set $\{(x, f(x)) \mid x \in X\} \subseteq \mathbb{R}^2$.

So $y = f(x)$ iff $(x, y) \in$ graph of f.

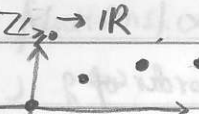


Eg. Power functions : $P_a(x) = x^a$. $P_a: \mathbb{R}^+ \to \mathbb{R}^+$.

Eg. Floor functions : $F: \mathbb{R} \to \mathbb{Z}$, $F(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor \in \mathbb{Z}$, $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$.



Eg. Sequence $A: \mathbb{Z}_{\ge 0} \to \mathbb{R}$. $A(n) = n^{1/2}$



Def. f is a real function, and $M \in \mathbb{R}$.
Then $M \cdot f$ is defined by $(Mf)(x) = M(f(x))$.

If $M \ge 1$, graph of $M \cdot f$ is vertically stretching graph of f by M.
If $0 < M < 1$, -- $M \cdot f$ is vertically compressing graph of f by M.
If $M = 0$, graph of $M \cdot f \subseteq x$-axis.
If $-1 < M < 0$, $M \cdot f$ is flipping graph of f across x-axis then vertically compressing by M.
If $M \le -1$, ——— vertically stretching by M.

Def. f is a real function. $S \subseteq \mathbb{R}$. f is called increasing (resp. decreasing) in S if $\forall x_1, x_2 \in S$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ (resp. $f(x_2) < f(x_1)$)
f is called increasing / decreasing if f is increasing / decreasing on its domain.

Eg. $f(x) = |x|$ is $\begin{cases} \text{increasing} & \text{on } [0,\infty) \\ \text{decreasing} & \text{on } (-\infty, 1] \end{cases}$.

Eg. If f is increasing on S, and $M>0$, then $M \cdot f$ is increasing on S.

9.2 O, Ω, Θ.

Def. f and g are real functions defined on the same set X of nonnegative real numbers.

1. f is of order at least g, $f(x) = \Omega(g(x))$, iff $\exists A, a>0$, s.t., $|f(x)| \geq A|g(x)|$ $\forall x \in X$, $x>a$.
2. f is of order at most g, $f(x) = O(g(x))$, iff $\exists B, b>0$ s.t., $|f(x)| \leq B|g(x)|$ $\forall x \in X$, $x>b$.
3. f is of the same order of g, $f(x) = \Theta(g(x))$, iff $\exists A, B, k>0$, s.t., $A|g(x)| \leq |f(x)| \leq B|g(x)|$ $\forall x \in X$, $x>k$.

THM. All the functions here are defined on the same set X of non-negative real numbers.

(i) f is $O(g)$ and $\Omega(g)$ $\Leftrightarrow$ f is $\Theta(g)$.
(ii) f is $O(g)$ $\Leftrightarrow$ g is $\Omega(f)$
(iii) f is $O(f)$, $\Omega(f)$, $\Theta(f)$
(iv) f is $O(g)$, g is $O(h)$ $\Rightarrow$ f is $O(h)$.
(v) f is $O(g)$, $c \in \mathbb{R}$, $\Rightarrow$ $c \cdot f$ is $O(g)$
(vi) f is $O(h)$, g is $O(k)$ $\Rightarrow$ $f+g$ is $O(G)$, where $G(x) = \max\{|h(x)|, |k(x)|\}$.
(vii) f is $O(h)$, g is $O(k)$ $\Rightarrow$ $f \cdot g$ is $O(h \cdot k)$.

(iii) If $\lim_{x\to\infty} |f(x)/g(x)| = \infty$, then f is not $O(g)$.

$\lim_{x\to\infty} |f(x)/g(x)| = \infty \Rightarrow \forall A,\ \exists M>0$, s.t., $|f(x)| > A|g(x)|$ if $x>M$. $\Rightarrow f$ is not $O(g)$.

☆THM. (i) If $\lim_{x\to\infty} |f(x)/g(x)| = A > 0$, then $f = \Theta(g)$.

(ii) If $\lim_{x\to\infty} |f(x)/g(x)| = 0$, then $f = O(g)$, $g = \Omega(f)$

Proof. (i) Since $A>0$ & $\lim_{x\to\infty} |f(x)/g(x)| = A$, $\exists\, a>0$, s.t., $\forall x > a \Rightarrow \frac{1}{2}A < |f(x)/g(x)| < \frac{3}{2}A$

$\Rightarrow \frac{1}{2}A\,|g(x)| < |f(x)| < \frac{3}{2}A\,|g(x)| \quad \forall x > a$.

(ii) $\lim_{x\to\infty} |f(x)/g(x)| = 0 \Rightarrow \exists\, a>0$, s.t., $\forall x>a \Rightarrow$

$0 \le |f(x)/g(x)| < 1 \Rightarrow |f(x)| < |g(x)| \quad \forall x > a$.

Eg. The inverse of the THM is not true:

$f(x) = x$, $g(x) = (2+\cos x)x$. Then $f = \Theta(g)$,

but $\lim_{x\to\infty} |f(x)/g(x)|$ does not exist.

Orders of power functions.

THM. (i) $x^r = O(x^s) \Leftrightarrow r \le s$. ($x^r$ is not $O(x^s) \Leftrightarrow r > s$)

(ii) $x^r = \Theta(x^s) \Leftrightarrow r = s$.

Proof. (i) $r \le s \Rightarrow |x^r| \le |x^s| \quad \forall x > 1$.

$\Rightarrow x^r = O(x^s)$.

"$\Rightarrow$" $x^r = O(x^s) \Rightarrow \exists A, a>0$, s.t., $x^r \le A x^s$, $\forall x > a$

$\Rightarrow x^{r-s} \le A \quad \forall x > a \Rightarrow r - s \le 0 \Rightarrow r \le s$.

(ii) "$\Leftarrow$" $r = s \Rightarrow x^r = x^s \quad \forall x \Rightarrow x^r = \Theta(x^s)$.

"$\Rightarrow$" $x^r = \Theta(x^s) \Rightarrow \begin{cases} x^r = O(x^s) \Rightarrow r \le s \\ x^s = O(x^r) \Rightarrow r \ge s \end{cases} \Rightarrow r = s$.

Orders of polynomials.

THM. $f(x) = \sum_{i=0}^{m} a_i x^i$, $g(x) = \sum_{j=0}^{n} b_j x^j$, $a_m \ne 0$, $b_n \ne 0$.

(i) $f = O(g) \Leftrightarrow m \le n$. ($f$ is not $O(g) \Leftrightarrow m > n$.)

(ii) $f = \Theta(g) \Leftrightarrow m = n$.

Proof. (i) "⇐" $m \le n \Rightarrow \lim_{x\to\infty} |f(x)/g(x)| = \begin{cases} 0, & m<n \\ |a_m/b_n|, & m=n \end{cases}$.

$\Rightarrow f(x) = O(g(x))$

"⇒" $f(x) = O(g(x))$. But $\lim_{x\to\infty} |f(x)/g(x)| = \begin{cases} \infty, & m>n \\ |a_m/b_n|, & m=n \\ 0, & m<n \end{cases}$.

But (iii) of THM ☆, $m>n$ is not true. $\Rightarrow m \le n$.

(ii) is clear from (i).

Orders of e^x and $\ln x$.

Clearly $e^x, \ln x \to \infty$ as $x \to \infty$.

For any $r>0$, let $n = \lceil r \rceil$, i.e., $n \in \mathbb{Z}$, $n-1 < r \le n$.

Consider $\lim_{x\to\infty} \frac{x^r}{e^x}$, apply L'Hospital's Rule n times, we have $\lim_{x\to\infty} \frac{x^r}{e^x} = \lim_{x\to\infty} r(r-1)(r-2)\cdots(r-n+1) \frac{x^{r-n}}{e^x} = 0$.

since $x^{r-n} \le 1$ when $x>1$, and $e^x \to \infty$ as $x\to\infty$.

So $x^r = O(e^x)$ $\forall r>0$

Also, $\lim_{x\to\infty} \frac{x^r}{e^x} = 0 \Rightarrow \lim_{x\to\infty} \frac{e^x}{x^r} = \infty \Rightarrow e^x$ is not $O(x^r)$ for any $x>0$.

For any $r>0$, $\lim_{x\to\infty} \frac{\ln x}{x^r} = \lim_{x\to\infty} \frac{\frac{1}{x}}{r x^{r-1}} = \lim_{x\to\infty} \frac{1}{r x^r} = 0$.

$\Rightarrow \ln(x) = O(x^r)$ $\forall r>0$, $x^r \neq O(\ln(x))$ $\forall r>0$.

Orders of fractions of combinations of power functions.

THM. $f(x) = \left(\sum_{i=1}^m a_i x^{r_i}\right) / \left(\sum_{j=1}^n b_j x^{s_j}\right)$, where $r_1 < r_2 < \cdots < r_m$, $s_1 < s_2 < \cdots < s_n$, $a_m \neq 0$, $b_n \neq 0$. Then f is $\Theta(x^{r_m - s_n})$.

Proof. $\lim_{x\to\infty} |f(x)/x^{r_m - s_n}| = \lim_{x\to\infty} \left| \left(\sum_{i=1}^m a_i x^{r_i - r_m}\right) / \left(\sum_{j=1}^n b_j x^{s_j - s_n}\right) \right| = \left|\frac{a_m}{b_n}\right| > 0$. $\Rightarrow f(x)$ is $\Theta(x^{r_m - s_n})$.

Orders of partial sums of sequences.

THM. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence, $g: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ a continuous increasing function. Define $b_n = \sum_{i=0}^{n} a_i$.

(i) $a_n = O(g(n)) \Rightarrow b_n = O(\int_0^{n+1} g(x)dx)$.

(ii) $(a_n>0)$ $a_n = \Omega(g(n)) \Rightarrow b_n = \Omega(\int_0^n g(x)dx)$.

Proof. (i) $a_n = O(g(n)) \Rightarrow \forall M, m > 0$, s.t., $|a_n| \leq M|g(n)|, \forall n > m$.

$$\Rightarrow |b_n| \leq \sum_{i=0}^{n} |a_i| = \sum_{i=0}^{m} |a_i| + \sum_{i=m+1}^{n} |a_i|$$
$$\leq \sum_{i=0}^{m} |a_i| + \int_{m+1}^{n+1} M g(x)dx$$
$$\leq \sum_{i=0}^{m} |a_i| + M\int_0^{n+1} g(x)dx \quad , \quad n > m$$
$$\Rightarrow b_n = O(\sum_{i=0}^{m}|a_i| + M\int_0^{n+1} g(x)dx)$$

But $\lim_{n\to\infty} \left|(\sum_{i=0}^{m}|a_i| + M\int_0^{n+1} g(x)dx) / \int_0^{n+1} g(x)dx\right| = M > 0$.

$$\Rightarrow \sum_{i=0}^{m}|a_i| + M\int_0^{n+1} g(x)dx = \Theta(\int_0^{n+1} g(x)dx)$$
$$\Rightarrow b_n = O(\int_0^{n+1} g(x)dx)$$

(ii) $a_n = \Omega(g(n)) \Rightarrow \forall K, k > 0$, s.t., $|a_n| \geq K|g(n)|$ $\forall n > k$ i.e., $a_n \geq K g(n)$, $\forall n > k$.

$$\Rightarrow b_n = \sum_{i=0}^{n} a_i = \sum_{i=0}^{k} a_i + \sum_{i=k+1}^{n} a_i \geq \sum_{i=1}^{k} a_i + \int_k^n K g(x)dx$$
$$= \sum_{i=0}^{k} a_i - K\int_0^k g(x)dx + K\int_0^n g(x)dx$$
$$\Rightarrow b_n = \Omega(K\int_0^n g(x)dx + \sum_{i=0}^{k} a_i - K\int_0^k g(x)dx)$$

But $\lim_{n\to\infty} \left|K\int_0^n g(x)dx + \sum_{i=0}^{k} a_i - K\int_0^k g(x)dx \Big/ \int_0^n g(x)dx\right| = K > 0$.

$$\Rightarrow K\int_0^n g(x)dx + \sum_{i=0}^{k} a_i - K\int_0^k g(x)dx = \Theta(\int_0^n g(x)dx)$$
$$\Rightarrow b_n = \Omega(\int_0^n g(x)dx)$$

Eg. $b_n = \sum_{i=0}^{n} n^k, k>0, \Rightarrow b_n = \Theta(n^{k+1})$.

(i) $\Rightarrow b_n = O(\int_0^{n+1} x^k dx) = O(\frac{1}{k+1} x^{k+1}]_0^{n+1}) = O((n+1)^{k+1})$

(ii) $\Rightarrow b_n = \Omega(\int_0^n x^k dx) = \Omega(\frac{1}{k+1} x^{k+1}]_0^n) = \Omega(n^{k+1})$

But $(n+1)^{k+1} = \Theta(n^{k+1}) \Rightarrow b_n = \Theta(n^{k+1})$.