8.1 Recursively defined sequences

Def. A recurrence relation for a sequence $\{a_k\}_{k=0}^{\infty}$ is a formular that expresses a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \dots, a_{k-i}$, where i, k are integers, and $k \geq i \geq 1$. The initial conditions for such a recurrence relation specify values of $a_0, \dots, a_{i-1}$, if i is fixed, or $a_0, a_1, \dots, a_m$ if i depends on k.

Eg. $\{C_k\}_{k=0}^{\infty}$, $\begin{cases} C_k = C_{k-1} + kC_{k-2} + 1, & k \geq 2 \\ C_0 = 1, \; C_1 = 2 \end{cases}$

$C_2 = C_1 + 2C_0 + 1 = 5,$

$C_3 = C_2 + 3C_1 + 1 = 12,$

$C_4 = C_3 + 4C_2 + 1 = 12 + 20 + 1 = 33.$

Eg. $\{S_k\}_{k=0}^{\infty}$ $S_k = 3S_{k-1} - 1$, $k \geq 1 \longleftrightarrow S_{k+1} = 3S_k - 1$, $k \geq 0$

Eg. Sequences satisfying the same recurrence relation can be different, because they may have different initial values.

$\{a_k\}_{k=0}^{\infty}$ $\begin{cases} a_k = 3a_{k-1}, & k \geq 1 \\ a_0 = 1 \end{cases}$, $\{b_k\}_{k=0}^{\infty}$ $\begin{cases} b_k = 3b_{k-1}, & k \geq 1 \\ b_0 = 2 \end{cases}$.

$a_k = 3^k$, $b_k = 2 \cdot 3^k$.

Eg. $\{a_k\}_{k=0}^{\infty}$, $a_n = \frac{(-1)^n}{n!}$, then $\{a_k\}$ satisfies

$a_k = \frac{-a_{k-1}}{k}$, $k \geq 1$, $a_0 = 1$.

Eg. Tower of Hanoi

There are 3 poles on a board, and n disks, numbered $1, 2, \dots, n$, each with a hole in the center. In the initial stage, all disks are stacked on one pole,

and, from top to bottem, the disks are $1, 2, 3, \cdots, n$.
Each step, the player can move one disk on the top of a stack to another pole, so that smaller numbered disks are on top of largered numbered disks. How many moves is needed to move all disk to another pole?

Solution: Let a_n = # of moves needed. Then $a_n = 2a_{n-1} + 1$, $n \geq 2$. (Need at least a_{n-1} moves before moving #n, then move #n, then another a_{n-1} moves to move all other disks onto #n.)

Clearly, $a_1 = 1$.

So $\{a_n\}_{n=1}^{\infty}$ satisfies $\begin{cases} a_n = 2a_{n-1} + 1 & n \geq 2 \\ a_1 = 1 \end{cases}$

$(a_n + 1) = 2(a_{n-1} + 1)$. Let $b_n = a_n + 1$. Then

$b_n = 2b_{n-1}$, $b_1 = 2$. $\Rightarrow b_n = 2^n$.

$\Rightarrow a_n = 2^n - 1$.

Eg. Fibonacci is a type of self-manufacturing robot. After a Fibonacci is assembled, it needs 1 hour to download and install software. Then it assembles a Fibonacci every hour. We finished assemble the first Fibonacci at time $t = 0$ (hour). How many Fibonacci's do we have at time $t = 10$ (hours).

Solution. Let F_n = # of Fibonacci's at $t = n$ (hours).
Then $F_0 = F_1 = 1$. $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$.
(At $t = n$ (hours), we have F_{n-1} Fibonacci's with software installed, and F_{n-2} newly assembled Fibonacci's.)

$F_2 = 1 + 1 = 2$, $F_3 = 2 + 1 = 3$, $F_4 = 3 + 2 = 5$, $F_5 = 5 + 3 = 8$, $F_6 = 8 + 5 = 13$

$F_7 = 13+8 = 21$, $F_8 = 21+13 = 34$, $F_9 = 34+21 = 55$,

$F_{10} = 55+34 = 89$.

Eg. How many bit strings of length n do not contain the substring 11?

Solution. Let S_n = # of such bit strings.

# of such bit string starting with 0 = S_{n-1}.

If such a bit string starts with 1, then the first 2 bits are 10. So # of such bit string starting with 1 = S_{n-2}.

$\Rightarrow$ $S_n = S_{n-1} + S_{n-2}$, $n \geq 2$.

Initial values are $S_0 = 1$ (empty string), $S_1 = 2$.

$$\begin{cases} S_n = S_{n-1} + S_{n-2}, & n \geq 2 \\ S_0 = 1, \quad S_1 = 2. \end{cases}$$

Note that $S_k = F_{k+1}$ for $k = 0, 1, 2, \cdots$.

Eg. Let X be a set of n elements. A partition of X into r subsets is a subset P of $\mathcal{P}(X) \setminus \{\phi\}$ of r elements, s.t. (i) the union of elements of P is X. (ii) any two elements of P are disjoint. How many different partion of X into r subsets are there?

Write $X = \{x_1, \cdots, x_n\}$

$S_{n,r}$ = # of partions of X into r subsets.

# of partions of X into r subsets with one of the subsets is $\{x_n\}$ = $S_{n-1, r-1}$.

# of partions of X into r subsets, none of which is $\{x_n\}$ = $r S_{n-1, r}$.

$\Rightarrow S_{n,r} = S_{n-1,r-1} + r\, S_{n-1,r}$

The initial values are: $S_{1,1} = 1$, $S_{1,r} = 0$ if $r \neq 1$.

Let $S_{n,r} = T_{n,r} \cdot \frac{r^n}{r!}$

$$T_{n,r} \frac{r^n}{r!} = T_{n-1,r-1} \frac{r^{n-1}}{(r-1)!} + r\, T_{n-1,r} \frac{r^{n-1}}{r!}$$

$$\Rightarrow \begin{cases} T_{n,r} = T_{n-1,r-1} + T_{n-1,r} \\ T_{1,1} = 1, \ T_{1,r} = 0 \text{ if } r \neq 1. \end{cases}$$

Using Math Induction, it's easy to show that $T_{n,r} = \binom{n-1}{r-1}$.

(i) $n=1$, $T_{1,1} = 1 = \binom{n-1}{r-1} = \binom{0}{0} = 1$

$T_{1,r} = 0 = \binom{0}{r-1}$ if $r \neq 1$.

(ii) Assume $T_{n-1,r} = \binom{n-2}{r-1}$ for a given $n \geq 2$, and all $r \in \mathbb{Z}$.

Then $T_{n,r} = T_{n-1,r-1} + T_{n-1,r} = \binom{n-2}{r-2} + \binom{n-2}{r-1} = \binom{n-1}{r-1}$.

(i) & (ii) $\Rightarrow T_{n,r} = \binom{n-1}{r-1}$

So $S_{n,r} = T_{n,r} \cdot \frac{r^n}{r!} = \binom{n-1}{r-1} \cdot \frac{r^n}{r!}$.

8.2 Solving Recurrence Relation by Iteration.

$\{a_k\}_{k=0}^{\infty}$ $\quad \begin{cases} a_k = ca_{k-1}+d, & k \geq 1 \\ a_0 = a \end{cases}$

Case (I) $d = 0$.

$$\begin{cases} a_k = c \cdot a_{k-1}, & k \geq 1 \\ a_0 = a \end{cases} \quad (a \neq 0,\ c \neq 0.)$$

$$a_k = c \cdot a_{k-1}$$
$$a_{k-1} = c\, a_{k-2}$$
$$\vdots$$
$$a_1 = c\, a_0$$

$\rightsquigarrow$ $a_k \cdot a_{k-1} \cdots a_1 = c^k a_{k-1} a_{k-2} \cdots a_1 a_0$

$\Rightarrow$ $a_k = ac^k$, $k = 0, 1, 2, \cdots$

So $\{a_k\}_{k=0}^{\infty}$ is a geometric sequence

Case (II) $c = 1$.

$$\begin{cases} a_k = a_{k-1} + d, & k \geq 1 \\ a_0 = a \end{cases}$$

$$a_k = a_{k-1} + d$$
$$a_{k-1} = a_{k-2} + d$$
$$\vdots$$
$$a_1 = a_0 + d$$

$\rightsquigarrow$ $a_k + a_{k-1} + \cdots + a_1 = kd + a_{k-1} + a_{k-2} + \cdots + a_0$

$\Rightarrow$ $a_k = a_0 + kd$, $k \geq 0$. $\rightsquigarrow$ arithmetic sequence.

Case (III) General case, $c \neq 1$, $d \neq 0$.

Method (i) $a_k = c a_{k-1} + d$ $\quad\leadsto\quad$ $a_k = c a_{k-1} + d$

$a_{k-1} = c a_{k-2} + d$ $\qquad\qquad c a_{k-1} = c^2 a_{k-2} + cd$

$\vdots \qquad\qquad \vdots$

$a_1 = c a_0 + d \qquad\qquad c^{k-1} a_1 = c^k a_0 + c^{k-1} d$

$\Rightarrow a_k + c a_{k-1} + \cdots + c^{k-1} a_1 = c a_{k-1} + \cdots + c^{k-1} a_1 + c^k a_0 + \sum_{l=0}^{k-1} c^l d$

$\Rightarrow a_k = c^k a_0 + d \sum_{l=0}^{k-1} c^l = c^k a_0 + d \cdot \frac{c^k - 1}{c-1}$, $k \geq 0$

Method (ii) Solve $x = cx + d \Rightarrow x = \frac{d}{1-c}$.

$\begin{cases} a_k = c a_{k-1} + d \\ x = cx + d \end{cases} \Rightarrow (a_k - x) = c(a_{k-1} - x)$.

$\Rightarrow$ If $b_k = a_k - x = a_k + \frac{d}{c-1}$, then $b_k = c \cdot b_{k-1}$,

$\Rightarrow \{b_k\}_{k=0}^{\infty}$ is a geometric sequence

$\Rightarrow b_k = b_0 c^k = (a + \frac{d}{c-1}) c^k$

$\Rightarrow a_k = b_k + x = (a + \frac{d}{c-1}) c^k - \frac{d}{c-1}$, $k \geq 0$.

General form of linear first order recurrence relation

$\begin{cases} a_k = f(k) a_{k-1} + g(k), & k \geq 1, \\ a_0 = a, \end{cases}$

where f and g are know functions.

$a_k = f(k) a_{k-1} + g(k)$ $\qquad\Rightarrow\qquad$ $a_k = f(k) a_{k-1} + g(k)$

$a_{k-1} = f(k-1) a_{k-2} + g(k-1)$ $\qquad f(k) a_{k-1} = f(k) f(k-1) a_{k-2} + f(k) g(k-1)$

$\vdots$

$a_1 = f(1) a_0 + g(1)$ $\qquad (\prod_{j=0}^{i-1} f(k-j)) a_{k-i} = (\prod_{j=0}^{i} f(k-j)) a_{k-i-1} + (\prod_{j=0}^{i-1} f(k-j)) g(k-i)$

$(\prod_{j=0}^{k-2} f(k-j)) a_1 = (\prod_{j=0}^{k-1} f(k-j)) a_0 + (\prod_{j=0}^{k-2} f(k-j)) g(1)$

$$a_k + \sum_{i=1}^{k-1}\left(\prod_{j=0}^{i-1} f(k-j)\right) a_{k-i}$$
$$= \sum_{i=1}^{k-1}\left(\prod_{j=0}^{i-1} f(k-j)\right) a_{k-i} + \left(\prod_{j=0}^{k-1} f(k-j)\right) a_0 + \sum_{i=0}^{k-1}\left(\prod_{j=0}^{i-1} f(k-j)\right) g(k-i)$$

$$\Rightarrow a_k = \left(\prod_{j=0}^{k-1} f(k-j)\right) a_0 + \sum_{i=0}^{k-1}\left(\prod_{j=0}^{i-1} f(k-j)\right) g(k-i)$$
$$= \left(\prod_{l=1}^{k} f(l)\right) a_0 + \sum_{i=1}^{k}\left(\prod_{j=i+1}^{k} f(j)\right) g(i), \quad k \geq 0.$$

Eg. $\begin{cases} a_k = k\, a_{k-1} + k! , & k \geq 1, \\ a_0 = a \end{cases}$

$\Rightarrow f(k) = k, \quad g(k) = k!$

$\prod_{l=1}^{k} f(l) = \prod_{l=1}^{k} l = k!$

$\left(\prod_{j=i+1}^{k} f(j)\right) g(i) = \left(\prod_{j=i+1}^{k} j\right) i! = k!$

$$\Rightarrow a_k = k! \cdot a + \sum_{i=1}^{k} k! = k!\,(a+k), \quad k \geq 0.$$

8.3 Second-Order Linear Homogeneous Recurrence with Constant Coefficients

Lemma. $f(x_1, x_2, \cdots, x_{i+1})$ is a function of $i+1$ variables.
$\{a_k\}_{k=0}^{\infty}$, $\{b_k\}_{k=0}^{\infty}$ satisfies the recurrence relations
$a_k = f(k, a_{k-1}, \cdots, a_{k-i})$, $b_k = f(k, b_{k-1}, \cdots, b_{k-i})$, $k \geq i$.
If $a_0 = b_0$, $a_1 = b_1$, $\cdots$, $a_{i-1} = b_{i-1}$, then $a_k = b_k$, $\forall k \geq 0$.
Proof. By induction of type III.

Consider a sequence $\{a_k\}_{k=0}^{\infty}$ satisfying
$$\begin{cases} a_k = A a_{k-1} + B a_{k-2}, & k \geq 2 \quad (*) \\ a_0 = \alpha, \ a_1 = \beta, \end{cases}$$
where A, B, α, β are constant numbers.

Case I, $t^2 = At + B$ has two distinct solutions t_1, t_2.
$t_j^2 = A t_j + B$, $j = 1, 2$, and
$t_j^k = A t_j^{k-1} + B t_j^{k-2}$, $j = 1, 2$, $k \geq 2$.
So both $\{t_1^k\}_{k=0}^{\infty}$, $\{t_2^k\}_{k=0}^{\infty}$ satisfy $(*)$.
Consider $b_k = C t_1^k + D t_2^k$, where C, D are constant.
Then $\{b_k\}_{k=0}^{\infty}$ satisfies $(*)$.
Now solve $\begin{cases} C + D = \alpha \\ C t_1 + D t_2 = \beta \end{cases}$
$$\Rightarrow \quad C(t_1 - t_2) = \beta - \alpha t_2 \quad \Rightarrow \quad C = \frac{\beta - \alpha t_2}{t_1 - t_2}$$
$$D(t_1 - t_2) = \alpha t_1 - \beta \quad \Rightarrow \quad D = \frac{\alpha t_1 - \beta}{t_1 - t_2}$$
Plug these C, D back into b_k. We get a sequence $\{b_k\}_{k=0}^{\infty}$,
where $b_k = \frac{\beta - \alpha t_2}{t_1 - t_2} \cdot t_1^k + \frac{\alpha t_1 - \beta}{t_1 - t_2} t_2^k$, $k \geq 0$.
It's easy to see that $b_0 = a_0 = \alpha$, $b_1 = a_1 = \beta$, and
both $\{a_k\}$, $\{b_k\}$ satisfy $(*)$. So, by the lemma,

$$a_k = b_k = \frac{\beta - \alpha t_2}{t_1 - t_2} t_1^k + \frac{\alpha t_1 - \beta}{t_1 - t_2} t_2^k, \quad k \geq 0.$$

Case II. $t^2 = At + B$ has a single solution t_0.

$\Rightarrow \quad t^2 - At - B = (t - t_0)^2 \Rightarrow A = 2t_0, \quad B = -t_0^2$.

Clearly, $\{t_0^k\}_{k=0}^{\infty}$ still satisfies $(*)$.

Consider the sequence $\{c_k\}_{k=0}^{\infty}$, $c_k = k\, t_0^k$, $k \geq 0$.

$$\begin{aligned} A c_{k-1} + B c_{k-2} &= A(k-1) t_0^{k-1} + B(k-2) t_0^{k-2} \\ &= k(A t_0^{k-1} + B t_0^{k-2}) - A t_0^{k-1} - 2B t_0^{k-2} \\ &= k\, t_0^{k-2}(At_0 + B) - \cancel{2t_0 \cdot t_0^{k-1}} - \cancel{2(-t_0^2) t_0^{k-2}} \\ &= k\, t_0^{k-2} \cdot t_0^2 = k\, t_0^k = c_k \end{aligned}$$

$\Rightarrow \{c_k\}$ satisfies $(*)$ too.

Let $\{b_k\}_{k=0}^{\infty}$ be $\quad b_k = E \cdot t_0^k + F c_k = E t_0^k + F \cdot k\, t_0^k$, $k \geq 0$

Then $\{b_k\}$ satisfies $(*)$.

Solve $\begin{cases} E = \alpha \\ E t_0 + F t_0 = \beta \end{cases} \Rightarrow \begin{cases} E = \alpha \\ F = \dfrac{\beta - \alpha t_0}{t_0} \end{cases}$

Plug E & F into $\{b_k\}$, then $b_k = \left(\alpha + \frac{\beta - \alpha t_0}{t_0} k\right) t_0^k$, $k \geq 0$.

This $\{b_k\}_{k=0}^{\infty}$ still satisfies $(*)$, and $b_0 = \alpha$, $b_1 = \beta$.

Thus, by lemma, $a_k = b_k = \left(\alpha + \frac{\beta - \alpha t_0}{t_0} k\right) t_0^k$, $k \geq 0$.

Eg. Fibonacci sequence $\begin{cases} F_k = F_{k-1} + F_{k-2}, & k \geq 2 \\ F_0 = F_1 = 1 \end{cases}$

Solve $t^2 = t + 1 \Rightarrow t = \frac{1 \pm \sqrt{5}}{2}$.

$\Rightarrow \quad F_k = C\left(\frac{1+\sqrt{5}}{2}\right)^k + D\left(\frac{1-\sqrt{5}}{2}\right)^k$, $k \geq 0$.

C, D satisfy $\begin{cases} C + D = 1 \\ C \cdot \frac{1+\sqrt{5}}{2} + D\, \frac{1-\sqrt{5}}{2} = 1 \end{cases} \Rightarrow \begin{cases} C + D = 1 \\ C - D = \frac{1}{\sqrt{5}} \end{cases}$

$\Rightarrow \begin{cases} C = \dfrac{1 + \frac{1}{\sqrt{5}}}{2} = \dfrac{1}{\sqrt{5}} \cdot \dfrac{1+\sqrt{5}}{2} \\ D = \dfrac{1 - \frac{1}{\sqrt{5}}}{2} = -\dfrac{1}{\sqrt{5}} \cdot \dfrac{1-\sqrt{5}}{2} \end{cases}$

$$\Rightarrow F_k = C\left(\frac{1+\sqrt5}{2}\right)^k + D\left(\frac{1-\sqrt5}{2}\right)^k = \frac{1}{\sqrt5}\left(\left(\frac{1+\sqrt5}{2}\right)^{k+1} - \left(\frac{1-\sqrt5}{2}\right)^{k+1}\right), \ k\geq 0.$$

Note that F_k is always an integer, and

$\left|\frac{1}{\sqrt5}\left(\frac{1-\sqrt5}{2}\right)^{k+1}\right| < \frac{1}{2} \quad \forall k \geq 0$.

So F_k is the integer closest to $\frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{k+1}$, i.e.,

$F_k = \left\lfloor \frac{1}{\sqrt5}\left(\frac{1+\sqrt5}{2}\right)^{k+1} + \frac{1}{2} \right\rfloor$.

Eg. A palindrom of a number n is a sum of integers $a_1 + a_2 + \cdots + a_k = n$, s.t., $a_i = a_{k+1-i}$, $i = 1, \cdots, k$, $a_i > 0$.
Find the # of palindroms of number n.

Let P_n be this number.

Consider a palindrom $a_1 + \cdots + a_k$ of n.

(i) If $a_1 = a_k > 1$, then $(a_1 - 1) + a_2 + a_3 + \cdots + a_{k-1} + (a_k - 1)$ is a palindrom of $n-2$. And any palindrom of $n-2$ gives a unique palindrom of n with $a_1 = a_k > 1$ by add 1 to the first and last term.

(ii) If $a_1 = a_k = 1$, then $a_2 + a_3 + \cdots + a_{k-1}$ is a palindrom of $n-2$, and any palindrom of $n-2$ gives a unique palindrom of n by adding the terms "1" and the beginning and ending.

$\Rightarrow P_n = 2P_{n-2}$.

Solve $t^2 = 2 \Rightarrow t = \pm\sqrt2$.

$\Rightarrow P_n = C(\sqrt2)^n + D(-\sqrt2)^n, \ n \geq 1$.

It's clear that $P_1 = 1$, $P_2 = 2$. So

$$\begin{cases} C\sqrt2 - D\sqrt2 = 1 \\ C\cdot 2 + 2\cdot D = 2 \end{cases} \Rightarrow \begin{cases} C - D = 1/\sqrt2 \\ C + D = 1 \end{cases} \Rightarrow \begin{cases} C = \frac{1+\frac{1}{\sqrt2}}{2} \\ D = \frac{1-\frac{1}{\sqrt2}}{2} \end{cases}$$

$$\Rightarrow P_n = \frac{1+\frac{1}{\sqrt2}}{2}(\sqrt2)^n + \frac{1-\frac{1}{\sqrt2}}{2}(-\sqrt2)^n = \frac{1+\sqrt2}{2}(\sqrt2)^{n-1} + \frac{1-\sqrt2}{2}(-\sqrt2)^{n-1}, \ n\geq 1.$$

If $n = 2k-1$, $P_{2k-1} = \frac{1+\sqrt2}{2}2^{k-1} + \frac{1-\sqrt2}{2}2^{k-1} = 2^{k-1}$.

If $n=2k$, $p_{2k}=\frac{1+\sqrt{2}}{2}2^{k-1}\sqrt{2}-\frac{1-\sqrt{2}}{2}2^{k-1}\sqrt{2}=2^k$.

Together, we have $p_n=2^{\lfloor n/2 \rfloor}$, $n\geq 1$.

This is also easy to prove by induction directly.

Eg. Person A has a dollars and Person B has b dollars. They flip a fair coin. If it's a head, then A give B a dollar, if it's a tail, then B give A a dollar. The game stops when one of them has no money left. What is the probability that A gets all the money at the end?

Sol. Let P_k = the probability that A gets all the money at the end if A has k dollars and B has a+b-k dollars to start with. Then $P_0=0$, $P_{a+b}=1$.

For general $k\in\{1,2,\cdots,a+b-1\}$. The probability of {first toss gives head & A wins in the end} $=\frac{1}{2}P_{k-1}$, and the probability of {first toss gives tail & A wins in the end} $=\frac{1}{2}P_{k+1}$.

$\Rightarrow P_k=\frac{1}{2}P_{k-1}+\frac{1}{2}P_{k+1}$ or $P_{k+1}=2P_k-P_{k-1}$, $k\geq 1$,

i.e., $P_k=2P_{k-1}-P_{k-2}$, $k=2,3,\cdots,a+b$.

So $t^2=2t-1 \Rightarrow t=1$.

$$\Rightarrow P_k=(E+Fk)1^k=E+Fk$$

$$\begin{cases} E+F\cdot 0=P_0=0 \\ E+F(a+b)=P_{a+b}=1 \end{cases} \Rightarrow \begin{cases} E=0 \\ F=1/a+b \end{cases}.$$

$\Rightarrow P_k=k/a+b$, $k=0,1,2,\cdots,a+b$.

Specially, $P_a=a/a+b$.