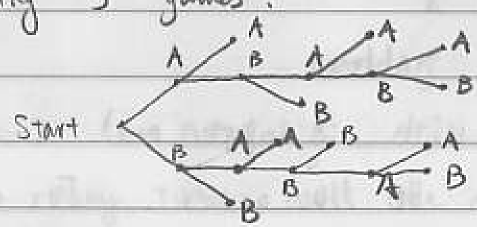
6.2 Possibility Trees and the Multiplication Rule. 1

Eg1. A, B are playing a tournament. Any one wins two games in a row or 3 in total wins the tournament.

(a) How many ways can the tournament be played out.

(b) Assuming all the ways of playing the tournament are equally likely, what is the probability of the tournament lasting 5 games?

(a) 

10 ways, each corresponds to a terminal point.

(b) 4 ways need 5 games. $P(E) = 4/10 = 20\%$

The Multiplication Rule

If an operation has k steps, and, for $1 \le l \le k$, the l-th step can be performed in n_l ways, then the whole operation can be performed in $n_1 n_2 \cdots n_k$ ways.

Eg2. A PIN is a sequence of four symbols chosen from the 26 letters (case insensitive) and the 10 digits, with repetition allowed. How may different PINs are there?

$$\# = (10+26)^4 = 36 \cdot 36 \cdot 36 \cdot 36 = 1,679,616.$$

Eg3. A_1, A_2, A_3, A_4 have n_1, n_2, n_3, n_4 elements, respectively. How many elements does $A_1 \times A_2 \times A_3 \times A_4$ have?

$$N(A_1 \times A_2 \times A_3 \times A_4) = n_1 n_2 n_3 n_4$$

Eg 4 PIN without repetition

(a) How many length-4 PIN without repetitions are there?

(b) If all PINs are equally likely, what is the probability of getting a PIN with no repetitions.

(a) $\# = 36 \cdot 35 \cdot 34 \cdot 33 = 1,413,720$

(b) $P(E) = 1,413,720 / 1,679,616 \approx 84.17\%$

Eg 5. Input / Output tables.

Consider a circuit with 2 input and 1 output signals, where each signal can be 0 or 1. For each such circuit, we can construct a table, e.g.

I_1	I_2	O
1	1	1
1	0	1
0	1	0
0	0	1

How many distinct such input / output tables are there.

$\# = 2 \times 2 \times 2 \times 2 = 16$.

Def. Let n be a positive integer, and S a finite set. A string of length n over S is an ordered n-tuple of elements of S written without parentheses or commas. The elements of S are called the characters. The null string over S is defined to be the string with on characters, and is usually denoted by ε and said to have length 0.

If $S = \{0, 1\}$, then the strings over S are called bit strings.

Eg 6 A PIN is a string of length 4 over
$S = \{x \mid x \text{ is a letter or a digit}\}$.

Eg 7. Counting loops

for $i := 1$ to 4
  for $j := 1$ to 3
    [inner loop] (no branch statement here that lead out of the inner loop)
  next j
next i

How many times will the inner be iterated when the algorith is implement and run?

$\# = 4 \times 3 = 12$.

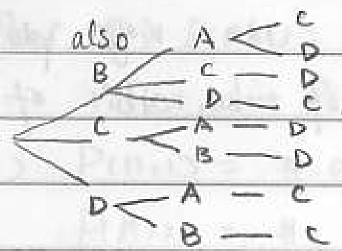
Eg 8. How many strings over $\{A, B, C, D\}$ of length 3 satisfy:

(1) there are no repeated characters in the string.
(2) first character is not A
(3) the last character is C or D.

Sol. Step 1. Choose the last character, (2 ways)
Step 2 Choose the first character, (2 ways)
Step 3 choose the middle charetter, (2 ways)

$\# = 2 \times 2 \times 2 = 8$.

See also



Permutations Given a finite set S, a permutation of S is an ordering of the objects of S in a row.

THM. If $N(S) = n \geq 1$, then there are $n!$ permutations of S.

Proof. By multiplication rule. (n ways to choose the 1st element, $(n-1)$ way for the 2nd, --, 1 way for the last.)

Eg 9. (a) How may ways can the letters in the word computer be arranged in a row?

(b) How may such arrangements has the substring CO?

(c) If all arrangements are equally likely, what's the probability of the arrangement to contain CO?

(a) $\# = 8! = 40,320$

(b) $\# = 7! = 5,040$

(c) $P(E) = 7!/8! = 1/8$.

Eg 10 How may ways are there the arrange 6 objects around a circle? (Only relative position matters).

Sol. Objects are A, B, -, F. Fix the position of A. There are $5! = 120$ ways to arrange the other 5 objects. Any other arrangement can be rotated into one of the above arrangement. So there are 120 ways to arrange them.

(We are counting the # of the orbit of a group action.)

S is a set of n elements. $0 \le r \le n$.

Def. An r-permutation of S is a string of length r over S that has no repeated characters.

Then numbe of r-permutations of a set of n elements is denoted by $P(n,r)$.

THM. $P(n,r) = n(n-1)\cdots(n-r+1)$ (1)

$= n!/(n-r)!$ (2)

Proof. (1) use Multiplication Rule.

(2) Note $n(n-1)\cdots(n-r+1) = \dfrac{n(n-1)\cdots(n-r+1)(n-r)\cdots 1}{(n-r)\cdots 1}$

Eg 11 Compute $P(5,2)$, $P(7,4)$, $P(5,5)$

$P(5,2) = 5!/(5-2)! = 5\cdot 4 = 20$

$P(7,4) = 7!/(7-4)! = 5040/6 = 840$.

$P(5,5) = 5!/(5-5)! = 5!/0! = 120$.

Eg 12. (a) How many 3-permutation of $\{B, Y, T, E, S\}$ are there

(b) How many such 3-permutation start with B.

Sol. (a) $\# = P(5,3) = 5!/(5-3)! = 60$

(b) $\# = P(4,2) = 4!/(4-2)! = 12$.

Eg 13. Prove that $P(n,1) + P(n,2) = n^2$

Proof. (I) $P(n,1) = n!/(n-1)! = n$, $P(n,2) = n!/(n-2)! = n(n-1)$

$\Rightarrow P(n,1) + P(n,2) = n + n(n-1) = n^2$.

(II) $P(n,1) = \#$ of 2-strings over $\{1, \dots, n\}$ with repetition

$P(n,2) = \#$ - - - - - - without.

$\Rightarrow P(n,1) + P(n,2) = \#$ - - - - - - $= n^2$.

6.3 Counting Elements of sets.

Addition Rule: THM If $\{A_1, \ldots, A_n\}$ is a partition of a finite set A, i.e., (i) $A = A_1 \cup \cdots \cup A_n$, (ii) $A_1, \ldots, A_n$ are mutually disjoint, then $N(A) = \sum_{i=1}^{n} N(A_i)$.

Eg 1. A password of a computer consists 1-3 lower case letters from the alphabet with repetition allowed. How many different passwords are possible?

A = set of all passwords. A_i = set of passwords of length i.

Then $\{A_1, A_2, A_3\}$ is a partition of A. So

$N(A) = N(A_1) + N(A_2) + N(A_3) = 26 + 26^2 + 26^3 = 18,278$.

Eg 2. How many integers from 100 to 999 inclusive are divisible by 5?

A = set of 3-digit integers divisible by 5.

A_1 = set of 3-digit integers end in 0.

A_2 = — — — — — 5.

Then $\{A_1, A_2\}$ is a partition of A.

$N(A_1) = 9 \times 10 \times 1 = 90$, $N(A_2) = 9 \times 10 \times 1 = 90$

$\Rightarrow N(A) = 90 + 90 = 180$.

Difference Rule: THM. If B is a subset of a finite set A, then $N(A-B) = N(A) - N(B)$.

Eg 3. A PIN is a string of 4 symbols chosen from the 26 letters and 10-digits. (a) How many PINs contain repetition?

(b) If all PINs are equally likely, what's the probability of a random PIN containing rep.

(a) # of PIN = 36^4 = 1,679,616.

# of PIN w/o rep = $36 \times 35 \times 34 \times 33$ = 1,413,720.

# of PIN w/ rep = 1,679,616 − 1,413,720 = 265,896

(b) P = 265,896/1,679,616 ≈ 15.8%.

Probability of the Complement of an Event.

If S is a sample space, and A is an event in S, then $P(A^c) = 1 - P(A)$.

Eg4. A Python identifier is a string of length ≤ 8 of symbols choosen from upper and lower case letters, "_", and digits, s.t., the first symbol is not a digit. 29 of these strings are reserved as keywords, and hence, unusable as identifiers. How many usable Python identifiers are there.

A = set of Python identifiers

A_i = — — — of length i, $i = 1, \ldots, 8$.

Then $\{A_i \mid i = 1, \ldots, 8\}$ is a partition of A.

$N(A_i) = 53 \cdot 63^{i-1}$, $i = 1, \ldots, 8$

$\Rightarrow N(A) = \sum_{i=1}^{8} 53 \cdot 63^{i-1} = 53 \dfrac{63^8 - 1}{63 - 1}$ = 212,133,167,002,880.

$\Rightarrow$ # of usable Python identifiers = $N(A) - 29$

= 212,133,167,002,851

Eg 5. A class B IP address is a bit string starting with 10 of length 32. Then first 16 digits give the network ID, the other 16 digits give the host ID. A host ID may not consist of either all 0's or all 1's.

(a) How many class B networks can there be?

(b) How many host ID's can there be in a class B network?

Sol. (a) # of class B networks is at most $2^{14} = 16,384$.

(b) # of host ID's in a class B network is at most $2^{16} - 2 = 65,536 - 2 = 65,534$.

Inclusion/Exclusion THM. If A, B, C are finite sets, then

$$N(A \cup B) = N(A) + N(B) - N(A \cap B),$$

$$N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C).$$

Eg 6. (i) How many integers from 1 to 1000 inclusive are multiples of 3 or multiples of 5.

(ii) How many are neither?

Sol. (i) $A = \{x \mid x = 1, \cdots, 1000,\ 3 \mid x\}$

$B = \{x \mid x = 1, \cdots, 1000,\ 5 \mid x\}$

$A \cap B = \{x \mid x = 1, \cdots, 1000,\ 15 \mid x\}$.

$N(A) = N\{3, 3\times 2, \cdots, 3 \times 333\} = 333 - 1 + 1 = 333$

$N(B) = N\{5, 5 \times 2, \cdots, 5 \times 200\} = 200 - 1 + 1 = 200$.

$N(A \cap B) = N\{15, 15 \times 2, \cdots, 15 \times 66\} = 66 - 1 + 1 = 66$.

$N(A \cap B) = 333 + 200 - 66 = 467$.

(ii) $\# = 1000 - 467 = 533$.

Eg 7. In a class of 50 students,

30 know Java,

18 know C++

26 know C#

9 know Java & C++

16 know Java & C#

8 know C++ & C#,

47 know at least one of these 3 languages.

(a) How many of these students knows none of the 3 languages?

(b) — — — — all — — — —

(c) How many know Java and C++, but not C#?

How many know Jave but not C# or C++ ?

Sol

(a) $\# = 50 - 47 = 3$.

(b) $J = \{\text{students know Java}\}$, $P = \{C{+}{+}\}$, $S = \{C\#\}$.

$$47 = N(J \cup P \cup S) = N(J) + N(P) + N(S) - N(J \cap P) - N(J \cap S) - N(P \cap S) + N(J \cap P \cap S)$$

$$= 30 + 18 + 26 - 9 - 16 - 8 + N(J \cap P \cap S)$$

$$\Rightarrow N(J \cap P \cap S) = 6$$

(c) $N((J \cap P) \setminus S) = N((J \cap P) \setminus (J \cap P \cap S))$

$$= N(J \cap P) - N(J \cap P \cap S) = 9 - 6 = 3.$$

$$N(J \setminus (P \cup S)) = N((J \setminus P) \cap (J \setminus S))$$

$$= N(J \setminus P) + N(J \setminus S) - N((J \setminus P) \cup (J \setminus S))$$

$$= N(J \setminus (J \cap P)) + N(J \setminus (J \cap S)) - N(J \setminus (P \cap S))$$

$$= N(J) - N(J \cap P) + N(J) - N(J \cap S) - N(J \setminus (J \cap P \cap S))$$

$$= N(J) - N(J \cap P) + N(J) - N(J \cap S) - N(J) + N(J \cap P \cap S)$$

$$= 30 - 9 + 30 - 16 - 30 + 6 = 11$$