7.2-3 1-1, onto, inverse, pigeonhole

Def. A function $f: X \to Y$ is called 1-1 (or injective) if, and only if, for $\forall x_1, x_2 \in X$, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$, or. equivalently, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

In an arrow diagram, this means any two arrows point into two different elements of Y.

f is not one-one if and only if $\exists\, x_1, x_2 \in X$, s.t., $x_1 \neq x_2$ but $f(x_1) = f(x_2)$.

Eg. $f_1: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $f_1(x) = x^2$, is 1-1
$f_2: \mathbb{R} \to \mathbb{R}$, $f_2(x) = x^2$ is not 1-1.

Def. A function $f: X \to Y$ is called onto (or surjective), if and only if for $\forall y \in Y$, $\exists\, x \in X$, s.t., $f(x) = y$, i.e., for $\forall y \in Y$, $f^{-1}(y) \neq \phi$.

In an arrow diagram, this means any element of Y has an arrow pointing into it.

f is not onto if and only if $\exists\, y \in Y$, s.t., $f^{-1}(y) = \phi$.

Eg. $g_1: \mathbb{R} \to \mathbb{R}$, $g_1(x) = 4x-1$, is onto
$g_2: \mathbb{Z} \to \mathbb{Z}$, $g_2(x) = 4x-1$, is not onto.

Def. A function $f: X \to Y$ is called bijective if and only if it's both injective and surjective.

Eg. $f: X \to Y$, X & Y are both finite sets.

If f is injective, then $N(X) \le N(Y)$.

If f is surjective, then $N(X) \geq N(Y)$.

If f is bijective, then $N(X) = N(Y)$.

Def. If $f: X \to Y$ is a bijection, then there is a unique function $g: Y \to X$, s.t., $\forall x \in X, y \in Y$, $y = f(x)$ if and only if $x = g(y)$. We usually denote g by $f^{-1}: Y \to X$.

THM. If $f: X \to Y$ is a bijection, then $f^{-1}: Y \to X$ is also a bijection, and $(f^{-1})^{-1} = f$.

Eg. $f_1: \mathbb{R} \to \mathbb{R}$, $f_1(x) = 4x - 1$.

$f_1^{-1}: \mathbb{R} \to \mathbb{R}$, $f_1^{-1}(y) = \frac{y+1}{4}$.

$f_2: \mathbb{R} \to \mathbb{R}_{>0}$, $f_2(x) = e^x$.

$f_2^{-1}: \mathbb{R}_{>0} \to \mathbb{R}$, $f_2^{-1}(y) = \ln y$.

THM. Let I_1, I_2 be two (possibly infinite) intervals in $\mathbb{R}$, and $f: I_1 \to I_2$ a continuous function. f is injective if and only if f is monotone.

Eg. $f: \mathbb{R} \to \mathbb{R}_{>0}$, $f(x) = a^x$, $a > 0$.

$f'(x) = (\ln a)\, a^x$. Then the sign of $f'(x)$ is constant.

$\Rightarrow$ f is monotone. Also, it's easy to see that f is onto, so f is a bijection $\Rightarrow f^{-1}: \mathbb{R}_{>0} \to \mathbb{R}$ is well-defined.

We denote f^{-1} by $\log_a$.

Eg. If $a, b \neq 1$, then $\log_b x = \frac{\log_a x}{\log_a b}$ for $\forall x \in \mathbb{R}_{>0}$.

Just need show $\log_a b \cdot \log_b x = \log_a x$.

$$a^{(\log_a b \cdot \log_b x)} = (a^{\log_a b})^{\log_b x} = b^{\log_b x} = x = a^{\log_a x}$$

$$\Rightarrow \log_a b \cdot \log_b x = \log_a x \Rightarrow \log_b x = \frac{\log_a x}{\log_a b}.$$

$$\log_2 5 = \ln 5 / \ln 2.$$

Eg. T is the set of all finite length bit strings.

$g: T \to T$ is the function given by

$g(s)$ = the string obtained by writing bits in s in the reverse order.

Then g is bijective, and $g^{-1} = g$.

Pigeonhole Principle.

THM. Let X, Y be finite sets.

(i) If $N(X) > N(Y)$, then there are no injective functions from X to Y.

(ii) If $N(X) > kN(Y)$ for some integer $k \geq 0$, then, for any function $f: X \to Y$, there exists a $y \in Y$, s.t., $N(f^{-1}(y)) \geq k+1$. ("$k=1$ in (ii)" $\Rightarrow$ (i))

Eg. A drawer contains 10 white socks and 10 black socks. You pull out some of these socks with looking at them. What's the least number of socks pulled out to ensure you have a pair of the same color?

Sol. 3. (2 is not enough (B, W).)

Eg. $A = \{1,2,3,4,5,6,7,8\}$. Find the least # n, s.t., any subset of A of at least n elements contain a pair of numbers having a sum of 9.

Sol. 4 is not enough. $\{1,2,3,4\}$.

5 is enough. Consider $X = \{\{1,8\}, \{2,7\}, \{3,6\}, \{4,5\}\}$.

For any subset B of A, Define $f: B \to X$ by mapping a number to the pair containing it. If $N(B) \geq 5$, then f is not injective. $\Rightarrow \exists$ a pair in X that is contained in B. $\Rightarrow B$ contains two numbers adding to 9.

Eg. $a, b \in \mathbb{Z}_+$, a/b is a finite decimal or an infinite decimal that is periodic after several digits.

Proof. By the Quotient - Remainder THM, $\exists!$ sequences of integers $\{r_i\}_{i=0}^{\infty}$, $\{q_i\}_{i=0}^{\infty}$, s.t., $0 \leq r_i < b$, $0 \leq q_0$, for $j \geq 1$, $0 \leq q_j < 9$

$$\begin{cases} a = b q_0 + r_0 \\ 10 r_{i-1} = b q_i + r_i, \quad i \geq 1 \end{cases} \quad (*)$$

Then the decimal of a/b is $a/b = q_0 . q_1 q_2 q_3 \cdots$

Case I: If one of the $r_i = 0$, then $r_j = 0$ for any $j \geq i$. So the decimal is finite.

Case II If none of r_i is 0, then $r_i = 1, 2, \cdots, b-1$. Consider $r_1, r_2, \cdots, r_b$. two of these are equal by Pigeonhole Principle. Say $r_i = r_{i+k}$. Then, by $(*)$, it's easy to see that $r_{i+j} = r_{i+k+j}$ $\forall j \geq 0$, and $q_{i+j} = q_{i+k+j}$ $\forall j \geq 1$. So

$$a/b = q_0 . q_1 \cdots q_i \overline{q_{i+1} \cdots q_{i+k}}$$

Note that $i+k \leq b$. So the length of non-periodic part + length of one period $\leq b$.

Contrapositive Form of Pigeonhole Principle.

X, Y are finite sets. $f: X \to Y$ satisfies that $N(f^{-1}(y)) \leq k$ for $\forall\, y \in Y$. Then $N(X) \leq kN(Y)$.

Eg. 42 students are using 12 computers. Each student use one and only one computer. No computer is used by more than 6 stundents. Show that there are at least 5 computers that are used by 3 or more students.

Proof. By contradiction. Assume there are at most 4 computers used by 3+ students. Then at least 8 computers are used by 2- students $\Rightarrow$ There are at most 16 students using these 8 computers. $\Rightarrow$ At least $42-16=26$ are using the other 4. By Pigeonhole, at least 1 computor is used by 7 students. Contradiction!

Eg. Find the least number k, s.t., any subset of $A = \{1, 2, 3, \cdots, 2n\}$ with at least k elements contains two consequitive integers.

Sol. Consider $Y = \{\{1,2\}, \{3,4\}, \cdots, \{2n-1, 2n\}\}$.

Clearly, we can choose $B \subset A$ to be $\{1, 3, 5, \cdots, 2n-1\}$, so that B has no consequitive integers, and $N(B) = n$.

So $k > n$. Consider $n+1$. Any subset C of A of size $n+1$ must contain one of the pairs by Pigeonhole. So $k \leq n+1$. $\Rightarrow$ $k = n+1$.

Eg. Prove that among any 6 people in the world, we can always find 3 of them that every two of the three have shaked hands before, or 3 of them that none of the 3 has shaked hands with any other of the 3.

Proof. relation) type a: two people shaked hands before.

relation) type b: - - - - - - - never shaked - - - - .

Let H be one of the 6 people. He has 5 relations with the other 5 people. 3 of these must be of the type, say these 3 people are A, B, C. If any two of A, B, C have the same type of relation as their relations with H, then we find our 3 people. If no two people of A, B, C have the same type of relation as their relation with H, then A, B, C are the 3 people we are looking for.

THM. X, Y are finite sets, and $N(X) > kN(Y)$ for some $k \in \mathbb{Z}_{\geq 0}$. Let $f: X \to Y$ be a function. Then $\exists\ y \in Y$, s.t., $N(f^{-1}(y)) \geq k+1$.

Proof. Assume otherwise, i.e., $N(f^{-1}(y)) \leq k$ for $\forall\ y \in Y$. Write $n = N(Y)$, and $Y = \{y_1, y_2, \cdots, y_n\}$. Then $\{f^{-1}(y_1), \cdots, f^{-1}(y_n)\}$ is a partition of X, i.e.,

$X = \bigcup_{i=1}^{n} f^{-1}(y_i)$ and $f^{-1}(y_i) \cap f^{-1}(y_j) = \phi$ when $i \neq j$.

$\Rightarrow N(X) = \sum_{i=1}^{n} N(f^{-1}(y_i)) \leq \sum_{i=1}^{n} k = nk = kN(Y) < N(X)$.

Contradiction!

THM. X, Y are finite sets, and $N(X) = N(Y)$.
$f: X \to Y$ is injective if and only if it's surjective.
Proof. If f is injective, but not surjective, let Y' = range of f. Then $Y' \subsetneq Y$. So $N(Y') < N(Y) = N(X)$.
By Pigeon hole, there is no injective function from $X \to Y'$.
So f is not injective. Contradiction.

So injectivity $\Rightarrow$ surjectivity.

If f is surjective, then $\{ f^{-1}(y) \mid y \in Y \}$ is a partition of X, and $N(X) = \sum_{y \in Y} N(f^{-1}(y)) \geq \sum_{y \in Y} 1 = N(Y)$, $(*)$
with equality if and only if $f^{-1}(y) = 1 \quad \forall y \in Y$.
But $N(X) = N(Y)$. So the equality is assumed in $(*)$.
$\Rightarrow N(f^{-1}(y)) = 1 \quad \forall y \in Y. \Rightarrow f$ is injective.