4.1-2. Sequences and Mathematical Induction.

Given integers $m \leq n$, a sequence starting at the m-th term, and ending in the n-th term is a map $A: \{m, m+1, \cdots, n\} \to \mathbb{R}$. The value of A at k is denoted by A_k (instead of $A(k)$). We denote this sequence by $\{A_k\}_{k=m}^{n}$, or $A_m, A_{m+1}, \cdots, A_n$. We can also consider infinite sequences, i.e., when $n = \infty$ or $m = -\infty$ or both.

Sum: $\sum_{k=m}^{n} A_k = A_m + A_{m+1} + \cdots + A_n$.

When $\{A_k\}$ is an infinite sequence, this is also called a series, and is define by a limit.

Product. $\prod_{k=m}^{n} A_k = A_m \cdot A_{m+1} \cdots A_n$.

(i) $\sum_{k=m}^{n} A_k + \sum_{k=m}^{n} B_k = \sum_{k=m}^{n} (A_k + B_k)$

(ii) $C \sum_{k=m}^{n} A_k = \sum_{k=m}^{n} C \cdot A_k$

(iii) $\left(\prod_{k=m}^{n} a_k\right) \cdot \left(\prod_{k=m}^{n} b_k\right) = \prod_{k=m}^{n} (a_k \cdot b_k)$.

Change of subindex.

$$\sum_{k=m}^{n} A_k = \sum_{i=m+l}^{n+l} A_{i-l} \qquad (i - l = k)$$

$$\prod_{k=m}^{n} A_k = \prod_{i=m+l}^{n+l} A_{i-l}$$

Mathematical Induction.

Let $P(n)$ be a property defined for integers n, and a be a fixed integer. (I) If (i) $P(a)$ is true (ii) $P(k)$ is true $\Rightarrow P(k+1)$ is true for $\forall k \geq a$, then $P(n)$ is true for $\forall n \geq a$.

Slightly stronger.

(II) If (i) $P(a)$ is true, (ii) $P(a), P(a+1), \cdots, P(k)$ are true $\Rightarrow P(k+1)$ is true for $\forall k \geq a$ then $P(n)$ is true for $\forall n \geq a$.

(Consider $Q(k) = P(a) \wedge P(a+1) \wedge \cdots \wedge P(k)$. Then (II) reduces to (I).)

Eg. For any $n \geq 8$, $3x+5y=n$ has non-negative integer solutions.

Proof. If $n=8$, $x=y=1$ is a solution. Assume that $k \geq 8$ and $3x+5y=k$ has a non-negative integer solution (x_0, y_0). (i) If $y_0 \neq 0$, then (x_0+2, y_0-1) is a solution of $3x+5y=k+1$ $(3(x_0+2)+5(y_0-1) = 3x_0+5y_0+6-5 = k+1)$

(ii) If $y_0 = 0$, then $3x_0 = 8 \Rightarrow x_0 \geq 3$. So $(x_0-3, 2)$ is a solution to $3x+5y=k+1$.

So $3x+5y=k+1$ has a non-negative integer solution.

$\Rightarrow$ $3x+5y=n$ has non-negative integer solutions for $\forall n \geq 8$.

Eg. Prove that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Proof. $n=1 \Rightarrow \sum_{k=1}^{1} k = 1$, $\frac{1(1+1)}{2} = 1$. (true for $n=1$)

Assume that $\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$. Then $\sum_{k=1}^{m+1} k = \sum_{k=1}^{m} k + (m+1)$

$= \frac{m(m+1)}{2} + (m+1) = \frac{(m+1)(m+2)}{2}$ (true for $m+1$)

$\Rightarrow \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for $\forall n \geq 1$.

Eg. $\sum_{k=0}^{n} ar^k = a\frac{r^{n+1}-1}{r-1}$ for any $n \geq 0$, $r \neq 1$.

Proof. $n=0 \Rightarrow \sum_{k=0}^{0} ar^k = a$. $a\frac{r-1}{r-1} = a$. ($n=1$ true)

Assume that $\sum_{k=0}^{m} ar^k = a\frac{r^{m+1}-1}{r-1}$. Then

$\sum_{r=0}^{m+1} ar^k = \left(\sum_{r=0}^{m} ar^k\right) + ar^{m+1} = a\frac{r^{m+1}-1}{r-1} + ar^{m+1}$

$= a\frac{r^{m+1}-1+r^{m+2}-r^{m+1}}{r-1} = a\frac{r^{m+2}-1}{r-1}$ ($m+1$ true)

$\Rightarrow \sum_{k=0}^{n} ar^k = a\frac{r^{n+1}-1}{r-1}$ $\forall n \geq 0$, $r \neq 1$.

Mathematical Induction (III) Fix $l \geq 0$. If (i) $P(a), P(a+1), \cdots, P(a+l)$ are true; (ii) For any $k \geq a$, $P(k), P(k+1), \cdots, P(k+l)$ are true $\Rightarrow P(k+l+1)$ is true, then $P(n)$ is true for $\forall n \geq a$.

Eg. There are n different letters $L_1, \cdots, L_n$ intended for n different recipients $R_1, \cdots, R_n$, respectively. Prove by induction on n that there are $n!\left(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\right)$ ways to send the letters so that every recipient gets a wrong letter.

Proof. Let A_n be the set of ways to send the n letters so that every recipient gets a wrong letter. And $a_n = N(A_n)$.

Assume $n \geq 3$. Let B_k be the subset of A_n consisting of ways that R_1 gets L_k. Of course, $B_1 = \phi$. For $1 < k \leq n$, consider what letter does R_k get. Let C_k be the subset of B_k consisting ways R_k gets L_1, and D_k be the subset of B_k consisting ways R_k does not get L_1. Then $B_k = C_k \cup D_k$, $C_k \cap D_k = \phi$. $N(C_k) = a_{n-2}$, $N(D_k) = a_{n-1}$.

$\Rightarrow N(B_k) = a_{n-2} + a_{n-1}$. But $\{B_2, B_3, \cdots, B_n\}$ is a pertition of A_n. So $a_n = N(A_n) = \sum_{k=2}^{n} B_k = (n-1)(a_{n-2} + a_{n-1})$.

Substitute n by $m+2$, then $a_{m+2} = (m+1)(a_m + a_{m+1})$ for $m = 1, 2, \cdots$.

Use Math Induction (II) $(l=1)$.

(i) $n=1$, $a_n = 0$ and $n!\left(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\right) = 0$.

$n=2$, $a_n = 1$ and $n!\left(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\right) = 1$.

So $a_n = n!\left(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\right)$ for $n = 1, 2$.

(ii) Assume $m \geq 1$, and $a_m = m!\left(\sum_{k=0}^{m} \frac{(-1)^k}{k!}\right)$, $a_{m+1} = (m+1)!\left(\sum_{k=0}^{m+1} \frac{(-1)^k}{k!}\right)$.

Then $a_{m+2} = (m+1)(a_m + a_{m+1}) = (m+1)\left(\sum_{k=0}^{m} \frac{(-1)^k m!}{k!} + \sum_{k=0}^{m+1} \frac{(-1)^k (m+1)!}{k!}\right)$

$= (m+1)\left(\sum_{k=0}^{m} \frac{(-1)^k (m! + (m+1)!)}{k!}\right) + (m+1)(-1)^{m+1}\frac{(m+1)!}{(m+1)!}$

$= \left(\sum_{k=0}^{m} (-1)^k \frac{(m+1)! + (m+1)! \cdot (m+1)}{k!}\right) + (m+1)(-1)^{m+1}$

$= \left(\sum_{k=0}^{m} (-1)^k \frac{(m+2)!}{k!}\right) + (-1)^{m+1}\frac{(m+2)!}{(m+1)!} + (-1)^{m+2}\frac{(m+2)!}{(m+2)!}$

$$= \sum_{k=0}^{m+2} (-1)^k \frac{(m+2)!}{k!} \quad (\text{true for } m+2).$$

$\Rightarrow$ true for $\forall n \geq 1$, i.e.,

$$a_n = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

6.7 The Binominal THM.

$(a+b)^2 = (a+b)(a+b) = aa + ab + ba + bb$

$(a+b)^3 = (a+b)(a+b)(a+b) = aaa + aab + aba + abb$
$\qquad + baa + bab + bba + bbb$

$(a+b)^4 = (a+b)(a+b)(a+b)(a+b)$.

each term in the expasion corresponds to a string of length 4 over $\{a, b\}$. And any such string gives you a unique term.

But multiplication is commutative. So many of these terms are equal. Indeed, as long as the # of a's and # of b's in two string are the same, they correspond to the equal terms. So $a^i b^{4-i}$ appears $\binom{4}{i}$ times in the sum, $i=0,1,\dots,4$.

$\Rightarrow (a+b)^4 = \sum_{i=0}^{4} \binom{4}{i} a^i b^{4-i}$.

In general, a term in the expasion for $(a+b)^n$ correspond to a string of length n over $\{a, b\}$. And any string corresponds to a unique term. So $a^i b^{n-i}$ appears $\binom{n}{i}$ times in the sum. This combinatorically proved

THM. $(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$ for $\forall\, n \geq 0$

Algebraic Proof by Induction.

(i) $n=0$, $l.h.s = (a+b)^0 = 1$, $r.h.s = \sum_{i=0}^{0} \binom{0}{i} a^i b^{0-i} = 1$.

(ii) Assume $(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i}$ for some $k \geq 0$.

Then $(a+b)^{k+1} = (a+b)(a+b)^k = (a+b) \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i}$

$= \sum_{i=0}^{k} \binom{k}{i} a^{i+1} b^{k-i} + \sum_{i=0}^{k} \binom{k}{i} a^i b^{k-i+1}$

$= \sum_{j=1}^{k+1} \binom{k}{j-1} a^j b^{k+1-j} + \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j+1}$

$= \binom{k}{0} a^0 b^{k+1-0} + \sum_{j=1}^{k} \left(\binom{k}{j-1} + \binom{k}{j}\right) a^j b^{k+1-j} + \binom{k}{k+1-1} a^{k+1} b^{k+1-k-1}$

Note $\binom{k+1}{j} = \binom{k}{j} + \binom{k}{j-1}$

$= \binom{k+1}{0} a^0 b^{k+1-0} + \sum_{j=1}^{k} \binom{k+1}{j} a^j b^{k+1-j} + \binom{k+1}{k+1} a^{k+1} b^{k+1-(k+1)}$

$= \sum_{j=0}^{k+1} \binom{k+1}{j} a^j b^{k+1-j}$.

Note, the coefficients of $(a+b)^n$ are the n-th row of the Pascal's Triangle.

$$\begin{matrix} & & & & 1 & & & & & 0 \\ & & & 1 & & 1 & & & & 1 \\ & & 1 & & 2 & & 1 & & & 2 \\ & 1 & & 3 & & 3 & & 1 & \cdots & 3 \\ 1 & & 4 & & 6 & & 4 & & 1 \cdots & 4 \end{matrix}$$

Eg (a) $(a+b)^5 = \sum_{i=0}^{5} \binom{5}{i} a^i b^{5-i} = b^5 + 5ab^4 + 10a^2b^3 + 10a^3b^2 + 5a^4b + a^5$

(b) $(x-4y)^4 = \sum_{i=0}^{4} \binom{4}{i} x^{4-i} (-4y)^i$

$$= x^4 + 4x^3(-4y) + 6x^2(-4y)^2 + 4x^1(-4y)^3 + (-4y)^4$$

$$= x^4 - 16x^3y + 96x^2y^2 - 256xy^3 + 256y^4$$

Eg $\sum_{k=0}^{n} \binom{n}{k} = ?$

(i) Let $a=b=1$ in the Binomial THM.

$\Rightarrow \sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$

(ii) Let X be a set consisting of n elements.

Then $\binom{n}{k}$ = # of subsets of X of k elements, $k = 0,1,2,\ldots,n$.

$\Rightarrow \sum_{k=0}^{n} \binom{n}{k}$ = # of subsets of $X = 2^n$.

Eg. $\sum_{k=0}^{n} \binom{n}{k} 9^k = \sum_{k=0}^{n} \binom{n}{k} 9^k 1^{n-k} = (9+1)^n = 10^n$

Multiplication of polynomials

$f(x) = \sum_{k=0}^{m} a_k x^k \quad g(x) = \sum_{l=0}^{n} b_l x^l$.

$f(x) \cdot g(x) = \sum_{i=0}^{m+n} c_i x^i$, where

$$c_i = \sum_{\substack{k+l=i \\ 0 \le k \le m \\ 0 \le l \le n}} a_k b_l .$$

If we fix that $a_k = 0$ if $k > m$, and $b_l = 0$ if $l > n$.

then $c_i = \sum_{k=0}^{i} a_k b_{i-k}$.

Eg. (1) $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, let $a=x$, $b=1$ in BTHM.

(2) Prove $2^{n-1} = \frac{1}{n}\left(\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n}\right)$ for $n \geq 1$

and $0 = \sum_{k=1}^{n} k\binom{n}{k}(-1)^k$ for $n \geq 2$.

$$n(1+x)^{n-1} = \frac{d}{dx}(1+x)^n = \frac{d}{dx}\sum_{k=0}^{n}\binom{n}{k}x^k = \sum_{k=0}^{n}\binom{n}{k}\frac{d}{dx}(x^k)$$
$$= \sum_{k=1}^{n} k\binom{n}{k}x^k. \quad (k=0 \text{ term vanishes since } \tfrac{d}{dx}(1)=0)$$

When $n \geq 1$, plug in $x=1$, and get

$$n2^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}. \Rightarrow 2^{n-1} = \frac{1}{n}\left(\sum_{k=1}^{n} k\binom{n}{k}\right).$$

When $n \geq 2$, plug in $x=-1$, $\Rightarrow$

$$\sum_{k=1}^{n} k\binom{n}{k}(-1)^k = n(1-1)^{n-1} = 0.$$

(3) Prove $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$.

Algebraic Proof. $\binom{2n}{n}$ = coefficient of x^n in $(1+x)^{2n} = (1+x)^n \cdot (1+x)^n$.

$$\Rightarrow \binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}.$$

Combinatoric Proof. Let A, B be two disjoint sets of n elements. $C = A \cup B$. $N(C) = 2n$.

$\binom{2n}{n}$ = # of subsets of C of n elements.

$\binom{n}{k}\binom{n}{n-k}$ = # of subset of C of n elements with k elements from A, $n-k$ elements from B.

$\Rightarrow \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$ = # of subsets of C of n elements

$\Rightarrow \binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$.

Note $\binom{n}{n-k} = \binom{n}{k}$, we have $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$.

4.3-4 More on mathematical induction

Eg. $\prod_{k=2}^{n}(1-\frac{1}{k}) = \frac{1}{n}$ for $n \geq 2$

(i) $n=2 \Rightarrow \prod_{k=2}^{2}(1-\frac{1}{k}) = 1-\frac{1}{2} = \frac{1}{2}$, i.e., $n=2$ true.

(ii) Assume $\prod_{k=2}^{m}(1-\frac{1}{k}) = \frac{1}{m}$ for some $m \geq 2$.

Then $\prod_{k=2}^{m+1}(1-\frac{1}{k}) = (1-\frac{1}{m+1})\prod_{k=2}^{m}(1-\frac{1}{k}) = \frac{m}{m+1}\cdot\frac{1}{m} = \frac{1}{m+1}$,

i.e., $n=m+1$ true.

(i) & (ii) $\Rightarrow \prod_{k=2}^{n}(1-\frac{1}{k}) = \frac{1}{n}$ for $\forall\, n \geq 2$.

(Note $1-\frac{1}{k} = \frac{k-1}{k}$. So $\prod_{k=2}^{n}(1-\frac{1}{k}) = \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n-2}{n-1}\cdot\frac{n-1}{n} = \frac{1}{n}$.

Eg. $3 \mid 2^{2n}-1$ $\quad n \geq 1$.

(i) $n=1$, $2^2-1=3$. So $n=1$ true.

(ii) Assume $3 \mid 2^{2m}-1$. $2^{2(m+1)}-1 = 4\cdot 2^{2m}-1 = 3\cdot 2^{2m}+(2^{2m}-1)$.

So $3 \mid 2^{2(m+1)}-1$.

(i) & (ii) $\Rightarrow 3 \mid 2^{2n}-1$, $n \geq 1$.

(Note $2^{2n} = 4^n$, $\sum_{k=0}^{n-1}4^k = \frac{4^n-1}{4-1} = \frac{4^n-1}{3}$. $\Rightarrow 2^{2n}-1 = 3\left(\sum_{k=0}^{n-1}4^k\right)$.)

Eg. $2n+1 < 2^n$, $n \geq 3$.

(i) $n=3 \Rightarrow 2n+1=7$, $2^n=8$. $7<8$.

(ii) Assume $2m+1 < 2^m$ for some $m \geq 3$. Then

$2^{m+1} = 2\cdot 2^m > 2(2m+1) = 2m+1+2m+1 > 2m+1+2 = 2(m+1)+1$.

(i) & (ii) $\Rightarrow 2n+1 < 2^n$ $\forall\, n \geq 3$.

Eg. The sequence $\{a_k\}_{k=0}^{\infty}$ satisfies $a_0 = a$, $a_k = r a_{k-1}$.

Show $a_k = ar^k$, $k \geq 0$.

(i) $a_0 = a = ar^0$. (ii) Assume $a_m = ar^m$, then $a_{m+1} = r a_m = ar^{m+1}$.

(i) & (ii) $\Rightarrow a_k = ar^k$, $k \geq 0$.

Eg. Any integer >1 is divisible by a prime number.

(i) $n=2$ is prime. $\Rightarrow$ true for $n=2$.

(ii) Assume any integer k with $m \geq k > 1$ is divisible by a prime for some $k \geq 2$. Consider $m+1$. If $m+1$ is prime, then $m+1 \mid m+1$. If $m+1$ is not prime, then $\exists\ 1 < k \leq m$, s.t. $k \mid m+1$. But $\exists$ prime number $p \mid k$. So $p \mid m+1$.

$\Rightarrow$ $m+1$ is divisible by a prime number

(i) & (ii) $\Rightarrow$ the proposition.

Eg. The sequence $\{a_k\}_{k=0}^{\infty}$ satisfies $a_1 = 0$, $a_k = 3a_{\lfloor \frac{k}{2} \rfloor} + 2$.

Show that $2 \mid a_k$ $k \geq 1$.

(i) $k=1 \Rightarrow a_k = a_1 = 0$. $2 \mid 0$.

(ii) Assume a_k is even when $1 \leq k \leq m$ for some $m \geq 1$.

$m+1 \geq 2$, so $1 \leq \lfloor \frac{m+1}{2} \rfloor \leq m$. $\Rightarrow$ $a_{\lfloor \frac{m+1}{2} \rfloor}$ is even

$\Rightarrow$ $2 \mid a_{m+1} = 3a_{\lfloor \frac{m+1}{2} \rfloor} + 2$.

(i) & (ii) $\Rightarrow$ $2 \mid a_k$, $\forall k \geq 1$.

Binary Integer Representation

Eg. For any $n \geq 1$, there is a unique $r \geq 0$, and sequence $\{c_j\}_{j=0}^{r}$, s.t., $c_r = 1$, $c_j = 0$ or 1, $0 \leq j \leq r$, $n = \sum_{j=0}^{r} c_j 2^j$.

Proof. Existance.

(i) $n=1$, Choose $r=0$, $c_0 = 1$. $1 = 1 \cdot 2^0 = 1$.

(ii) Assume that, for an $m \geq 1$, any integer k satisfying $1 \leq k \leq m$, has a binary integer rep. Consider $m+1$. (A) if m is even, then $1 \leq \frac{m}{2} \leq m$. $\exists\ r \geq 0$, $\{c_j\}_{j=0}^{r}$, s.t. $c_r = 1$, $c_j = 0$ or 1, and $\frac{m}{2} = \sum_{j=0}^{r} c_j 2^j$. Then $m+1 = \left(\sum_{j=1}^{r+1} c_{j-1} 2^j\right) + 1 \cdot 2^0$. Then $m+1$ has a binary integer rep. (B) if m is odd, then $\frac{m+1}{2}$ is an integer, and $1 \le \frac{m+1}{2} \le m$. So

$\exists\ r \geq 0$, $\{c_j\}_0^r$, s.t., $c_r = 1$, $c_j = 0$ or 1, and $\frac{m+1}{2} = \sum_{j=0}^{r} c_j 2^j$.

So $m+1 = \left(\sum_{j=1}^{r+1} c_{j-1} 2^j\right) + 0 \cdot 2^0$.

(i) & (ii) $\Rightarrow$ any $n \geq 1$ has a binary integer rep.

Uniqueness.

(i) $n=1$. It's clear that we have to have $r=0$, $c_0=1$.

So the binary integer rep of 1 is unique.

(ii) Assume for any $1 \le k \le m$, k has a unique binary rep.

Consider $m+1$. Let $\{c_i\}_0^r$, $\{d_j\}_0^s$ be any two binary reps of $m+1$, i.e., $c_r = d_s = 1$, $c_i, d_j = 0, 1$. $m+1 = \sum_{i=0}^{r} c_i 2^i = \sum_{j=0}^{s} d_j 2^j$.

If $r < s$, then $m+1 = \sum_{i=0}^{r} c_i 2^i \le \sum_{i=0}^{r} 2^i = 2^{r+1} - 1 < 2^{r+1} \le 2^s \le \sum_{j=0}^{s} d_j 2^j = m+1$.

This is a contradiction. If $s < r$, we can again prove $m+1 < m+1$, which is again a contradiction. So $r = s$, and the two binary reps are $\{c_i\}_0^r$ $\{d_j\}_0^r$, $c_r = d_r = 1$, $c_i, d_j = 1$ or 0. $m+1 = \sum_{i=0}^{r} c_i 2^i = \sum_{j=0}^{r} d_j 2^j$.

Consider $m+1-2^r$. If $m+1-2^r = 0$, then $\sum_{i=0}^{r-1} c_i 2^i = \sum_{j=0}^{r-1} d_j 2^j = 0$.

$\Rightarrow c_i = d_j = 0$, $0 \le i, j \le r-1$. $\Rightarrow \{c_i\}_0^r = \{d_j\}_0^r$.

If $m+1-2^r \neq 0$, then $1 \le m+1-2^r \le m$. By induction hypothesis, $\{c_i\}_0^{r-1} = \{d_j\}_0^{r-1}$. $\Rightarrow \{c_i\}_0^r = \{d_j\}_0^r$.

$\Rightarrow m+1$ has a unique binary integer rep.

(i) & (ii) $\Rightarrow$ The binary integer rep of any n is unique.

Well-Ordering Principle for the Integers.

THM. Let $\mathbb{Z}$ be the set of integers, and X a non-empty subset of $\mathbb{Z}$. If $\exists\ k \in \mathbb{Z}$, s.t., all elements of X are $\geq k$, then $\exists$ an element x_0 of X, s.t., $x_0 \le x$ for any $x \in X$.

Quotient - Remainder

Eg. If $n \in \mathbb{Z}$, $d \in \mathbb{Z}$ and $d > 0$, the $\exists$ a unique ordered pair $(q, r) \in \mathbb{Z} \times \mathbb{Z}$, s.t., $n = dq + r$, $0 \le r < d$.

Proof. Let $X = \{x \in \mathbb{Z} \mid x = n - dk$ for some $k \in \mathbb{Z}$, and $x \geq 0\}$.
First, $X \neq \phi$. Indeed, if $n \geq 0$, then $n = n - d \cdot 0 \in X$, and, if $n < 0$, then $x = n - nd = n(1-d) \in X$. But all elements of X are ≥ 0, So $\exists\ \underset{= n - dq}{r} \in X$, s.t., for $\forall x \in X$, $r \le x$.
If $r \geq d$, then $r - d \geq 0$, and $r - d = n - dq - d = n - d(q+1)$.
So $r - d \in X \Rightarrow r - d \geq r$. This is a contradiction.
So $0 \le r < d$. And (q, r) is a pair satisfying the conditions. If there are two such pairs, say, (q_1, r_1) and (q_2, r_2), then $n = q_1 d + r_1 = q_2 d + r_2$, $0 \le r_1, r_2 < d$.
So $-d < r_1 - r_2 < d$, $q_2 - q_1 = \frac{r_1 - r_2}{d}$,
$\Rightarrow -1 < q_2 - q_1 < 1 \Rightarrow q_2 = q_1 \Rightarrow r_1 = r_2$.
So the pair (q, r) is unique.