6.4 Combinations.

Def. Let S be a set of n elements. A r-combination of S is a subset of r elements of S. Then symbol $\binom{n}{r}$ (n choose r) denotes the # of r-combinations of a set of n elements.

Eg. $S = \{A, B, C, D\}$. (i) Find all 3-combinations of S. (ii) What is $\binom{4}{3}$.

(i) $\{B, C, D\}_{(\text{no } A)}$, $\{A, C, D\}_{(\text{no } B)}$, $\{A, B, D\}_{(\text{no } C)}$, $\{A, B, C\}_{(\text{no } D)}$.

(ii) $\binom{4}{3} = 4$.

Eg. Relation of r-permutations and r-combinations.

Let S be a set of n elements. To choose a r-permutation, we use 2 steps.

(i) choose a r-combination. ($\binom{n}{r}$ ways)

(ii) give an order to these r elements. ($r!$ ways).

$\Rightarrow P(n, r) = \binom{n}{r} \cdot r!$

$\Rightarrow \binom{n}{r} = P(n,r)/r! = \dfrac{n!}{(n-r)!\, r!}$.

Eg. $\binom{8}{5} = \dfrac{8!}{5!\,3!} = \dfrac{8 \cdot 7 \cdot 6}{3!} = 8 \cdot 7 = 56$

Eg. How many ways are there to choose 5 person from a group of 12 to work as a team?

$\binom{12}{5} = \dfrac{12!}{5!\,7!} = \dfrac{12 \cdot 11 \cdot 10 \cdot 9^3 \cdot 8^2}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 12 \times 11 \times 6 = 792$.

Eg. Suppose 2 of the 12 insist to work together. So we have to choose both or neither of them. How many ways are there now to choose a 5-member team

Sol. Both $\rightarrow \binom{12-2}{5-2} = \binom{10}{3} = \frac{10\cdot 9\cdot 8}{2\cdot 3} = 120$

Neither $\rightarrow \binom{10-2}{5} = \binom{10}{5} = \frac{10\times 9\times 8\times 7\times 6}{1\times 2\times 3\times 4\times 5} = 36\times 7 = 252$

$\Rightarrow$ # of ways $= 120 + 252 = 372$.

Eg. If two person refuse to work on the same team, how many ways are there now to form a 5-member team?

Sol. Say a, b won't work together.

choose a $\rightarrow \binom{12-2}{4} = \binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} = 210$.

choose b $\rightarrow \binom{12-2}{4} = \binom{10}{4} = \frac{10\cdot 9\cdot 8\cdot 7}{1\cdot 2\cdot 3\cdot 4} = 210$

choose neither $\rightarrow \binom{10}{5} = \binom{10}{5} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5} = 252$.

$\Rightarrow$ # of ways $= 210 + 210 + 252 = 672$.

Eg. If there are 7 women and 5 men to choose from.

(i) How many different 5-member teams containing 3 men and 2 women are there?

(ii) # of teams containing at least 1 man?

(ii) # of teams containing at most 1 man?

Sol. (i) $\# = \binom{5}{3}\cdot\binom{7}{2} = \frac{5\times 4\times 3}{1\times 2\times 3}\cdot\frac{7\times 6}{1\times 2} = 10\cdot 21 = 210$

(ii) $\sum_{k=1}^{5}\binom{5}{k}\binom{7}{5-k} = \binom{5}{1}\binom{7}{4} + \binom{5}{2}\binom{7}{3} + \binom{5}{3}\binom{7}{2} + \binom{5}{4}\binom{7}{1} + \binom{5}{5}\binom{7}{0}$

$= 5\times\frac{7\times 6\times 5\times 4}{1\times 2\times 3\times 4} + \frac{5\times 4}{2}\,\frac{7\times 6\times 5}{1\times 2\times 3} + \frac{5\times 4\times 3}{1\times 2\times 3}\,\frac{7\times 6}{2} + \frac{5\times 4\times 3\times 2}{1\times 2\times 3\times 4}\cdot 7 + \frac{5\times 4\times 3\times 2\times 1}{1\times 2\times 3\times 4\times 5}\cdot 1$

$= 5\times 35 + 10\times 35 + 5\times 42 + 5\times 7 + 1$

$= 771$.

(iii) $\sum_{k=0}^{1} \binom{5}{k}\binom{7}{5-k} = \binom{5}{0}\binom{7}{5} + \binom{5}{1}\binom{7}{4}$

$= 1 \cdot \frac{7\times6\times5\times4\times3}{1\times2\times3\times4\times5} + 5\,\frac{7\times6\times5\times4}{1\times2\times3\times4}$

$= 21 + 5\cdot 35 = 196.$

Eg. In a poker game, if the cards are dealt randomly, i.e., each hand is equally likely, what is the probability of getting two pairs (not full house).

Sol. S = set of all possible hands

$N(S) = \binom{52}{5}$

A = set of hands with two pairs.

To determine A, we use 3 steps.

1. Choose the denomination of the two pairs $\binom{13}{2}$
2. choose the actual cards of the two pairs $\binom{4}{2}\binom{4}{2}$
3. choose the single card $\binom{44}{1}$

$P(A) = \binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{44}{1} / \binom{52}{5} = 123,552 / 2,598,960 \approx 4.75\%.$

Eg. How many 8-bit strings have exactly 3 1's?

Sol. Choose 3 positions from 8 possible ones for "1".

$\binom{8}{3} = \frac{8\times7\times6}{1\times2\times3} = 56.$

Eg How many different ways are there to order the letters in MISSISSIPPI.

Sol. Step 1. Choose 4 positions for "S"s $\binom{11}{4}$

Step 2. Choose 4 positions for "I"s $\binom{11-4}{4} = \binom{7}{4}$

Step 3 Choose 2 positions for "P"s $\binom{11-8}{2} = \binom{3}{2}$

Step 4 Put in "M" $\binom{11-10}{1} = \binom{1}{1}$

$$\# = \binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1} = \frac{11!}{4!\,\cancel{7!}}\,\frac{\cancel{7!}}{4!\,\cancel{3!}}\,\frac{\cancel{3!}}{2!\,1!}\,\frac{1!}{1!\,0!}$$

$$= 11!/4!\,4!\,2!\,1! = 34{,}650.$$

THM. Suppose we have n objects consisting of

n_1 objects of type 1 (indistinguishable from each other)

n_2 — — — 2 — —

$\vdots$

n_k — — — k — — — .

Then $n = n_1 + \cdots + n_k$, and the # of distinct permutation of these n objects are

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= n!/n_1!\,n_2!\cdots n_k!$$

Eg. Double Counting. To choose a 5-member team with at least 1 man from 5 man and 7 women.

Step 1. Choose a man $\binom{5}{1}$

Step 2. Choose other members $\binom{12-1}{4} = \binom{11}{4}$

$\#$ of teams $= \binom{5}{1}\binom{11}{4} = 1{,}650$. $\boxed{\text{WRONG !}}$

Say five men are 1, 2, 3, 4, 5,

seven women are A, B, C, D, E, F, G.

If we choose "1" in step 1, and 2, A, B, C in step 2, we get 1, 2, A, B, C.

If we choose "2" in step 1, and 1, A, B, C in step 2, we get 2, 1, A, B, C.

These are the same team. So there are double counting.

6.5 r-combination with repetitions allowed.

Def. Let X be a set of n elements. An r-combination with repetition allowed (or multiset of size r) is an unordered selection of r elements of X with repetition allowed.

If $X = \{x_1, \dots, x_n\}$, then an r-combination of X is of the form $[x_{i_1}, \dots, x_{i_r}]$, where $i_1, \dots, i_r \in \{1, \dots, n\}$, and these index may equal each other.

Eg. $X = \{1, 2, 3, 4\}$. Then $[1,1,2] = [1,2,1] = [2,1,1]$
$[1,2,2] \neq [1,1,2]$ as 3-combinations of X with repetition allowed

THM. The number of r-combination with repetition allowed of a set of n elements is $\binom{n+r-1}{r}$.

Proof. Let $X = \{x_1, \dots, x_n\}$. Any ordered n-tuple of non-negative integers $(m_1, \dots, m_n)$ satisfying $m_1 + \dots + m_n = r$ gives a unique r-combination of X with repetition allowed that contains m_i x_i's for $i = 1, \dots, n$. Any r-combination of X with repetition allowed corresponds to a unique such tuple. So
\# of r-combo of X with rep allowed
$= \#$ of non-negative integer solutions of $m_1 + \dots + m_n = r$. $(*)$
Given a solution of $(*)$, we can form a sequence of r X's and $n-1$ $|$'s by $\underbrace{x \cdots x}_{m_1 \text{ many}} | \underbrace{x \cdots x}_{m_2 \text{ many}} | \cdots | \underbrace{x \cdots x}_{m_n \text{ many}}$. This correspondence is 1-1. So \# of non-negative integer solutions of $m_1 + \dots + m_n = r$
$= \#$ of sequences with r X's and $n-1$ $|$'s $= \binom{n+r-1}{r}$.

If we have n categories of objects, s.t., objects in the same category are identical, then there are $\binom{n+r-1}{r}$ (ordered) ways to select r objects from these categories (with repetition allowed).

Eg. (a) How many ways are there to select 15 cans of soft drink from five different types?

(b) If root beer is one of the type, and the selection is random so that each selection is equally likely, what's the probability of getting at least 6 cans of root beer? Exactly 6 cans of root beers?

(a) $\# = \binom{15+5-1}{15} = \frac{19\cdot 18\cdot 17\cdot 16\cdot \cancel{15!}}{\cancel{15!}\ 1\times 2\times 3\times 4} = 19\times 6\times 17\times 2 = 3876.$

(b) # of getting at least 6 root beer $= \binom{9+5-1}{9} = 715.$

Probability $= 715/3876 \approx 18.45\%$

Eg. How many triples of integers (i, j, k) with $1 \le i \le j \le k \le n$ are there.

$\# = \binom{n+3-1}{3} = \frac{(n+2)(n+1)n}{3!} = \frac{n(n+1)(n+2)}{6}.$

Eg.

```
for k := 1 to n
    for j := 1 to k
        for i := 1 to j
            [ inner loop ] (no branching, no lead outside)
        next i
    next j
next k.
```

The inner loop is ran $\binom{n+3-1}{3} = \frac{n(n+1)(n+2)}{6}$ times.

Eg. $x_1+x_2+x_3+x_4=10$

(a) How many non-negative integer solutions are there?

(b) How many positive integer solutions are there?

(a) $\binom{10+4-1}{10} = \binom{13}{10} = \frac{13\times 12\times 11}{1\times 2\times 3} = 13\times 11\times 2 = 286.$

(b) $y_i = x_i - 1$, $y_1+y_2+y_3+y_4=6$

$\binom{6+4-1}{6} = \binom{9}{6} = \frac{9\times 8\times 7}{1\times 2\times 3} = 3\times 4\times 7 = 84.$

Another way to look at (b).

How many positive integer solutions are there of $x_1+\cdots+x_n = r$ (*)?

A solution of gives a sequence of r x's and $n-1$ 1's. If all the x_i's are positive, then no two 1's can be adjacent. Consider $\underbrace{x\,x\,\cdots\,x}_{r\text{ many}}$. We are just trying to insert the 1's in to the $r-1$ gaps between the x's so that no 2 1's are in the same gap. So there are $\binom{r-1}{n-1}$ ways to do that. That is, (*) has $\binom{r-1}{n-1}$ different solutions of positive integers.

	ordered	Unordered
repetition ok	n^k	$\binom{n+k-1}{k}$
no repetitions	$P(n,k)$	$\binom{n}{k}$

(Selecting k elements from a set of n elements).

6.6 Algebra of Combinations.

Eg. $\binom{n}{n}=1$, $\binom{n}{n-1}=n$, $\binom{n}{n-2}=\frac{n(n-1)}{2}$, $\binom{n}{r}=0$ if $r>n$.

Eg. $\binom{n}{r}=\binom{n}{n-r}$

Proof. Algebraic. $\binom{n}{r}=\frac{n!}{r!(n-r)!}=\frac{n!}{(n-(n-r))!\,(n-r)!}=\binom{n}{n-r}$

Combinatoric. $A=\{x_1,\dots,x_n\}$, choosing a subset B of A of r elements $\leftrightarrow$ choosing a subset $A-B$ of A of $n-r$ element. So $\binom{n}{n-r}=\binom{n}{r}$.

Eg. Substitution. From $\binom{n}{n-2}=\frac{n(n-1)}{2}$ we have

$\binom{m+1}{m-1}=\frac{(m+1)m}{2}$, $\binom{s-1}{s-3}=\frac{(s-1)(s-2)}{2}$,

$\binom{n+2}{n}=\frac{(n+2)(n+1)}{2}$.

Pascal's Formula

THM. $\binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}$

Proof. Algebraic. $\binom{n}{r-1}+\binom{n}{r}=\frac{n!}{(r-1)!\,(n-r+1)!}+\frac{n!}{r!\,(n-r)!}$

$=\frac{n!\cdot r}{r!\,(n-r+1)!}+\frac{n!\,(n-r+1)}{r!\,(n-r+1)!}$

$=\frac{n!}{r!(n-r+1)!}(r+n-r+1)=\frac{n!\,(n+1)}{r!\,(n+1-r)!}=\binom{n+1}{r}$.

Combinatoric. $A=\{x_1,\dots,x_{n+1}\}$.

\# of r-combination of $A=\binom{n+1}{r}$

\# - - - - - containing $x_{n+1}=\binom{n}{r-1}$

\# - - - - - - - - - not containg $x_{n+1}=\binom{n}{r}$

$\Rightarrow \binom{n+1}{r}=\binom{n}{r-1}+\binom{n}{r}$

Pascal's Triangle

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}$$

Pascal's formula says that any number in the Pascal's Triangle is equal to the sum of the two numbers on its two shoulders. So if we know all the numbers in a row, we can compute the numbers in the next row.

Since it's much faster to compute summation than multiplication and quotient by computer, this is a faster way to compute $\binom{n}{r}$ for large n and r.

Eg.

$$\begin{array}{cccccc} \binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} \\ \| & \| & \| & \| & \| & \| \\ 1 & 5 & 10 & 10 & 5 & 1 \end{array}$$

$$\begin{array}{ccccccc} \binom{6}{0} & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & \binom{6}{6} \\ \| & \| & \| & \| & \| & \| & \| \\ 1 & 6 & 15 & 20 & 15 & 6 & 0 \end{array}$$

Eg. From $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$,

$$\binom{n+2}{r} = \binom{n+1}{r-1} + \binom{n+1}{r} = \binom{n}{r-2} + \binom{n}{r-1} + \binom{n}{r-1} + \binom{n}{r}$$

$$= \binom{n}{r-2} + 2\binom{n}{r-1} + \binom{n}{r}.$$

for any $2 \le r \le n+2$.

General Inclusion / Exclusion formula.

THM. $A_1, \cdots, A_n$ are finite sets. Then

$$N(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \underbrace{\sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} N(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})}_{\text{sum of } \binom{n}{k} \text{ terms.}}.$$

Eg. $N(A \cup B \cup C \cup D) = N(A) + N(B) + N(C) + N(D) - N(A \cap B)$

$- N(A \cap C) - N(A \cap D) - N(B \cap C) - N(B \cap D) - N(C \cap D)$

$+ N(A \cap B \cap C) + N(A \cap B \cap D) + N(A \cap C \cap D) + N(B \cap C \cap D)$

$- N(A \cap B \cap C \cap D)$.

Eg. There are n different letters $L_1, \dots, L_n$ intended for n different recipients. How many ways are there to send these n letters so that every one gets a wrong letter? $(n \geq 1)$

Let A_i be the set of permutations that R_i gets L_i, and the universal set U be the set of all permutations.

Then $A_1 \cup A_2 \cup \cdots \cup A_n = \{$at least one R_i gets $L_i\}$

and $(A_1 \cup A_2 \cup \cdots \cup A_n)^c = \{$no one gets his letter$\}$.

For any $1 \leq k \leq n$, and $1 \leq i_1 < i_2 < \cdots < i_k \leq n$,

$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k} = \{R_{i_j}$ get L_{i_j}, $j = 1, 2, \dots, k\}$

$N(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = (n-k)!$

$$\Rightarrow N(A_1 \cup \cdots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \leq i_1 < \cdots < i_k \leq n} N(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k})$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \cdot (n-k)! = \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!\,\cancel{(n-k)!}} \cancel{(n-k)!}$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!}$$

$$\Rightarrow N((A_1 \cup \cdots \cup A_n)^c) = N(U) - N(A_1 \cup \cdots \cup A_n)$$

$$= n! - \sum_{k=1}^{n} (-1)^{k+1} \frac{n!}{k!} = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$$

$$= n! \left(\sum_{k=0}^{n} (-1)^k \frac{1}{k!} \right).$$

Note that $e^{-1} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!}$

So
$$\frac{n!}{e} - n! \left(\sum_{k=0}^{n} (-1)^k \frac{1}{k!} \right)$$

$$= n! \sum_{k=n+1}^{\infty} (-1)^k \frac{1}{k!}$$

$$= (-1)^{n+1} \sum_{j=1}^{\infty} n! \left(\frac{1}{(n+2j-1)!} - \frac{1}{(n+2j)!} \right)$$

$$= (-1)^{n+1} \sum_{j=1}^{\infty} \frac{n+2j-1}{(n+2j)!} \cdot n!$$

$$= (-1)^{n+1} \sum_{j=1}^{\infty} \frac{1}{\left(\prod_{k=1}^{2j-2} (n+k)\right) \cdot (n+2j)}$$

But $0 < \sum_{j=1}^{\infty} \frac{1}{\prod_{k=1}^{2j-2}(n+k)\,(n+2j)} < \sum_{j=1}^{\infty} \frac{1}{(n+1)^{2j-2}\,(n+2)} \leq \frac{1}{3} \sum_{j=1}^{\infty} \frac{1}{2^{2j-2}} = 4/9$.

So $\left| \frac{n!}{e} - n! \left(\sum_{k=0}^{n} (-1)^k \frac{1}{k!} \right) \right| < \frac{4}{9} < \frac{1}{2}$, i.e.,

$n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!}$ is the integer closest to $\frac{n!}{e}$. So

$$n! \sum_{k=0}^{n} (-1)^k \frac{1}{k!} = \left[\frac{n!}{e} + \frac{1}{2} \right].$$