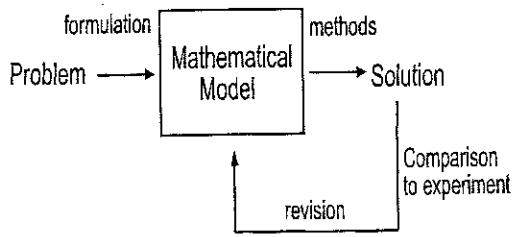

Figure 1.1 Schematic of the modeling process.

1.1 Dimensional Analysis

1.1.1 The Program of Applied Mathematics

Applied mathematics is a broad subject area in the mathematical sciences dealing with those topics, problems, and techniques that have been useful in analyzing real-world phenomena. In a very limited sense it is a set of methods that are used to solve the equations that come out of science, engineering, and other areas. Traditionally, these methods were techniques used to examine and solve ordinary and partial differential equations, and integral equations. At the other end of the spectrum, applied mathematics is applied analysis, or the theory that underlies the methods. But, in a broader sense, applied mathematics is about mathematical modeling and an entire process that intertwines with the physical reality that underpins its origins.

By a **mathematical model** we mean an equation, or set of equations, that describes some physical problem or phenomenon that has its origin in science, engineering, economics, or some other area. By **mathematical modeling** we mean the process by which we formulate and analyze the model. This process includes introducing the important and relevant quantities or variables involved in the model, making model specific assumptions about those quantities, solving the model equations by some method, and then comparing the solutions to real data and interpreting the results. Often the solution method involves computer simulation or approximation. This comparison may lead to revision and refinement until we are satisfied that the model accurately describes the physical situation and is predictive of other similar observations. This process is depicted schematically in Fig. 1.1. Therefore the subject of mathematical modeling involves physical intuition, formulation of equations, solution methods, and analysis. A good mathematical model is simple, applies to many situations, and in predictive.

Overall, in mathematical modeling the overarching objective is to make

sense of the world as we observe it, often by inventing caricatures of reality. Scientific exactness is sometimes sacrificed for mathematical tractability. Model predictions depend strongly on the assumptions, and changing the assumptions changes the model. If some assumptions are less critical than others, we say the model is robust to those assumptions. They help us clarify verbal descriptions of nature and the mechanisms that make up natural law, and they help us determine which parameters and processes are important, and which are unimportant.

Another issue is the level of complexity of a model. With modern computer technology it is tempting to build complicated models that include every possible effect we can think of, with large numbers of parameters and variables. Simulation models like these have their place, but computer runs do not always allow us to discern which are the important processes, and which are not. Of course, the complexity of the model depends upon the data and the purpose, but it is usually a good idea to err on the side of simplicity, and then build in complexity as it is needed or desired.

Finally, authors have tried to classify models in several ways—stochastic vs. deterministic, continuous vs. discrete, static vs. dynamic, quantitative vs. qualitative, descriptive vs. explanatory, and so on. In our work we are interested in modeling the underlying reasons for the phenomena we observe (explanatory) rather than fitting the data with formulas (descriptive) as is often done in statistics. For example, fitting measurements of the size of an animal over its lifetime by a regression curve is descriptive, and it gives some information. But describing the dynamics of growth by a differential equation relating growth rates, food assimilation rates, and energy maintenance requirements tells more about the underlying processes involved.

The reader is already familiar with many models. In an elementary science or calculus course we learn that Newton's second law, force equals mass time acceleration, governs mechanical systems like falling bodies; Newton's inverse-square law of gravitation describes the motion of the planets; Ohm's law in circuit theory dictates the voltage drop across a resistor in terms of the current; the law of mass action in chemistry describes how fast chemical reactions occur; or the logistics equation models growth and competition in population.

The first step in modeling is to select the relevant variables (independent and dependent) and parameters that describe the problem. Physical quantities have *dimensions* like time, distance, degrees, and so on, or corresponding *units* like seconds, meters, and degrees Celsius. The equations we write down as models must be dimensionally correct. Apples cannot equal oranges. Verifying that each term in our model has the same dimensions is the first task in obtaining a correct equation. Also, checking dimensions can often give us insight into what a term in the model might be. We always should be aware

of the dimensions of the quantities, both variables and parameters, in a model, and we should always try to identify the physical meaning of the terms in the equations we obtain. A general rule is to always let the physical problem drive the mathematics, and not vice-versa.

It would be a limited view to believe that applied mathematics consists only of developing techniques and algorithms to solve problems that arise in a physical or applied context. Applied mathematics deals with all the stages of the modeling process, not merely the formal solution. It is true that an important aspect of applied mathematics is studying, investigating, and developing procedures that are useful in solving mathematical problems: these include analytic and approximation techniques, numerical analysis, and methods for solving differential and integral equations. It is more the case, however, that applied mathematics deals with all phases of the problem. Formulating the model and understanding its origin in empirics are crucial steps. Because there is a constant interplay between the various stages, the scientist, engineer, or mathematician must understand each phase. For example, the solution stage sometimes involves making approximations that lead to a simplification; the approximations often come from a careful examination of the physical reality, which in turn suggests what terms may be neglected, what quantities (if any) are small, and so on. The origins and analysis are equally important. Indeed, physical insight forces us toward the right questions and at times leads us to the theorems and their proofs. In fact, mathematical modeling has been one of the main driving forces for mathematics itself.

In the first part of this chapter our aim is to focus on the first phase of the modeling process. Our strategy is to formulate models for various physical systems while emphasizing the interdependence of mathematics and the physical world. Through study of the modeling process we gain insight into the equations themselves. In addition to presenting some concrete examples of modeling, we also discuss two techniques that are useful in developing and interpreting the model equations. One technique is *dimensional analysis*, and the other is *scaling*. The former permits us to understand the dimensional (meaning length, time, mass, etc.) relationships of the quantities in the equations and the resulting implications of dimensional homogeneity. Scaling is a technique that helps us understand the magnitude of the terms that appear in the model equations by comparing the quantities to intrinsic reference quantities that appear naturally in the physical situation.

1.1.2 Dimensional Methods

One of the basic techniques that is useful in the initial, modeling stage of a problem is the analysis of the relevant quantities and how they must relate to each other in a dimensional way. Simply put, apples cannot equal oranges; equations must have a consistency to them that precludes every possible relationship among the variables. Stated differently, equations must be dimensionally homogeneous. These simple observations form the basis of the subject known as **dimensional analysis**. The methods of dimensional analysis have led to important results in determining the nature of physical phenomena, even when the governing equations were not known. This has been especially true in continuum mechanics, out of which the general methods of dimensional analysis evolved.

The cornerstone result in dimensional analysis is known as the **Pi theorem**. The Pi theorem states that if there is a physical law that gives a relation among a certain number of dimensioned physical quantities, then there is an equivalent law that can be expressed as a relation among certain dimensionless quantities (often noted by $\pi_1, \pi_2, \ldots$, and hence the name). In the early 1900s, E. Buckingham gave a proof of the Pi theorem for special cases, and now the theorem often carries his name. Birkhoff (1950) can be consulted for a bibliography and history.

To communicate the flavor and power of this classic result, we consider a calculation made by the British applied mathematician G. I. Taylor in the late 1940s to compute the yield of the first atomic explosion after viewing photographs of the spread of the fireball. In such an explosion a large amount of energy E is released in a short time (essentially instantaneously) in a region small enough to be considered a point. From the center of the explosion a strong shock wave spreads outward; the pressure behind it is on the order of hundreds of thousands of atmospheres, far greater than the ambient air pressure whose magnitude can be accordingly neglected in the early stages of the explosion. It is plausible that there is a relation between the radius of the blast wave front r, time t, the initial air density ρ, and the energy released E. Hence, we *assume* there is a physical law

$$g(t, r, \rho, E) = 0, \tag{1.1}$$

which provides a relationship among these quantities. The Pi theorem states that there is an equivalent physical law between the independent dimensionless quantities that can be formed from t, r, E, and ρ. We note that t has dimensions of time, r has dimensions of length, E has dimensions of mass $\cdot$ length$^2 \cdot$ time^{-2}, and ρ has dimensions of mass $\cdot$ length^{-3}. Hence, the quantity $r^5\rho/t^2E$ is dimensionless, because all of the dimensions cancel out of this quantity (this is easy to check). It is not difficult to observe that no other independent dimensionless

quantities can be formed from t, r, e, and ρ. The Pi theorem then guarantees that the physical law (1.1) is equivalent to a physical law involving only the dimensionless quantities; in this case

$$f\left(\frac{r^5\rho}{t^2E}\right) = 0, \tag{1.2}$$

because there is only one such quantity, where f is some function of a single variable. From (1.2) it follows that the physical law must take the form (a root of (1.2))

$$\frac{r^5\rho}{t^2E} = C,$$

or

$$r = C\left(\frac{Et^2}{\rho}\right)^{1/5}, \tag{1.3}$$

where C is a constant. Therefore, just from dimensional reasoning it has been shown that the radius of the wave front depends on the two-fifths power of time. Experiments and photographs of explosions confirm this dependence. The constant C depends on the dimensionless ratio of the specific heat at constant pressure to the specific heat at constant volume. By fitting the curve (1.3) to experimental data of r versus t, the initial energy yield E can be computed, since C and ρ are known quantities. (See Exercise 3.) Although this calculation is only a simple version of the original argument given by Taylor, we infer that dimensional reasoning can give crucial insights into the nature of a physical process and is an invaluable tool for the applied mathematician, scientist, or engineer.

As an aside, there is another aspect of dimensional analysis that is important in engineering, namely the design of small-scale models (say of an airplane or ship) that behave like their real counterparts. A discussions of this important topic is not treated here, but can be found in many engineering texts, especially books on fluid mechanics.

Example 1.1

Those who bike have certainly noticed that the force F (mass times length, per time-squared) of air resistance appears to be positively related to their speed v (length per time) and cross-sectional area A (length-squared). But force involves mass, so it cannot depend only upon v and A. Therefore, let us add fluid, or air, density ρ (mass per length-cubed), and assume a relation of the form

$$F = f(\rho, A, v),$$

for some function f to be determined. What are the possible forms of f? To be dimensionally correct, the right side must be a force. What powers of ρ, A, and v would combine to give a force, that is,

$$F = k\rho^x A^y v^z,$$

where k is some constant without any units (i.e., dimensionless). If we denote mass by M, length by L, and time by T, then the last equation requires

$$MLT^{-2} = (ML^{-3})^x(L^2)^y(LT^{-1})^z.$$

Equating exponents of M, L, and T gives

$$x = 1, \quad -3x + 2y + z = 1, \quad -z = -2.$$

Therefore $x = 1$, $y = 1$, and $z = 2$. So the only relation must be

$$F = k\rho A v^2.$$

Consequently, the force must depend upon the square of the velocity, a fact often used in problems involving fluid resistance, and it must be proportional to density and area. Again, substantial information can be obtained from dimensional arguments alone.

EXERCISES

1. A pendulum executing small vibrations has period P and length l, and m is the mass of the bob. Can P depend only on l and m? If we assume P depends on l and the acceleration g due to gravity, then show that $T =$ constant $\cdot\sqrt{L/g}$.

2. A detonation wave moves through a high explosive with velocity D, thereby initiating an explosion which releases an energy per unit mass, e. If there is a relationship between e and D, what can be concluded?

3. In the blast wave problem take $C = 1$ (a value used by Taylor) in (1.3), and use $\rho = 1.25$ kg/m^3. Some of the radius (m) vs. time (milliseconds) data for the Trinity explosion is given in the following table:

t	0.10	0.52	1.08	1.5	1.93	4.07	15.0	34.0
r	11.1	28.2	38.9	44.4	48.7	64.3	106.5	145

Using these data, estimate the yield of the Trinity explosion in kilotons (1 kiloton equals $4.186(10)^{12}$ joules). Compare your answer to the actual yield of approximately 21 kilotons.

4. In the blast wave problem assume, instead of (1.1), a physical law of the form
$$g(t, r, \rho, e, P) = 0,$$
where P is the ambient pressure. By inspection, find two independent dimensionless parameters formed from t, r, ρ, e, and P. Naming the two dimensionless parameters π_1 and π_2 and assuming the law is equivalent to
$$f(\pi_1, \pi_2) = 0,$$
does it still follow that r varies like the two-fifths power of t?

5. The law governing the distance x an object falls in time t in a field of constant gravitational acceleration g with no air resistance is $x = \frac{1}{2}gt^2$. How many independent dimensionless quantities can be formed from t, x, and g? Rewrite the physical law in terms of dimensionless quantities. Can the distance a body falls depend on the mass m as well? That is, can there be a physical law of the form $f(t, x, g, m) = 0$?

6. A ball tossed upward with initial velocity v reaches a height at time t given by $x = -\frac{1}{2}gt^2 + vt$. Show this law is dimensionally consistent. Find two independent dimensionless variables (call them y and s) and rewrite the law only in terms of the dimensionless quantities. Plot the dimensionless law in the sy plane, and argue that the single, dimensionless plot contains all the information in the graphs of all the plots of $x = -\frac{1}{2}gt^2 + vt$ (x vs t), for all possible parameter values g and v.

1.1.3 The Pi Theorem

It is generally true that a physical law
$$f(q_1, q_2, q_3, \ldots, q_m) = 0 \tag{1.4}$$
relating m quantities $q_1, q_2, \ldots, q_m$ is equivalent to a physical law that relates the dimensionless quantities that can be formed from $q_1, q_2, \ldots, q_m$. This is the content of the Pi theorem. Before making a formal statement, however, we introduce some basic terminology.

First, the m quantities $q_1, q_2, \ldots, q_m$, which are like the quantities t, r, ρ, and e in the blast wave example, are dimensioned quantities. This means that they can be expressed in a natural way in terms of certain selected fundamental dimensions $L_1, L_2, \ldots, L_n$ ($n < m$), appropriate to the problem being studied. In the blast wave problem, time T, length L, and mass M can be taken to be the fundamental dimensions, since each quantity t, r, ρ, and e can be expressed

in terms of T, L, and M. For example, the dimensions of the energy e are ML^2T^{-2}. In general, the dimensions of q_i, denoted by $[q_i]$, can be written in terms of the fundamental dimensions as
$$[q_i] = L_1^{a_{1i}} L_2^{a_{2i}} \cdots L_n^{a_{ni}} \tag{1.5}$$
for some choice of exponents $a_{1i}, \ldots, a_{ni}$. If $[q_i] = 1$, then q_i is said to be **dimensionless**. The $n \times m$ matrix
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ a_{21} & \cdots & a_{2m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$
containing the exponents is called the **dimension matrix**. The elements in the ith column give the exponents for q_i in terms of the powers of $L_1, \ldots, L_n$.

The fundamental assumption regarding the physical law (1.4) goes back to the simple statement that apples cannot equal oranges. We assume that (1.4) is **unit free** in the sense that it is independent of the particular set of units chosen to express the quantities $q_1, q_2, \ldots, q_m$. We are distinguishing the word **unit** from the word **dimension**. By units we mean specific physical units like seconds, hours, days, and years; all of these units have dimensions of time. Similarly, grams, kilograms, slugs, and so on are units of the dimension mass. Any fundamental dimension L_i has the property that its units can be changed upon multiplication by the appropriate conversion factor $\lambda_i > 0$ to obtain $\overline{L}_i$ in a new system of units. We write
$$\overline{L}_i = \lambda_i L_i, \quad i = 1, \ldots, n.$$
The units of derived quantities q can be changed in a similar fashion. If
$$[q] = L_1^{b_1} L_2^{b_2} \cdots L_n^{b_n},$$
then
$$\overline{q} = \lambda_1^{b_1} \lambda_2^{b_2} \cdots \lambda_n^{b_n} q$$
gives q in the new system of units. The physical law (1.4) is said to be independent of the units chosen to express the dimensional quantities $q_1, q_2, \ldots, q_m$, or *unit-free*, if $f(q_1, \ldots, q_m) = 0$ and $f(\overline{q}_1, \ldots, \overline{q}_m) = 0$ are equivalent physical laws. More formally:

Definition 1.2

The physical law (1.4) is unit-free if for all choices of real numbers $\lambda_1, \ldots, \lambda_n$, with $\lambda_i > 0$, $i = 1, \ldots, n$, we have $f(\overline{q}_1, \ldots, \overline{q}_m) = 0$, if, and only if, $f(q_1, \ldots, q_m) = 0$.

Example 1.3

The physical law

$$f(x,t,g) \equiv x - \tfrac{1}{2}gt^2 = 0 \tag{1.6}$$

relates the distance x a body falls in a constant gravitational field g to the time t. In the cgs system of units, x is given in centimeters (cm), t in seconds, and g in cm/ sec^2. If we change units for the fundamental quantities x and t to inches and minutes, then in the new system of units

$$\overline{x} = \lambda_1 x, \quad \overline{t} = \lambda_2 t,$$

where $\lambda_1 = \frac{1}{2.54}$ (in/cm) and $\lambda_2 = \frac{1}{60}$ (min/sec). Because $[g] = LT^{-2}$, we have

$$\overline{g} = \lambda_1 \lambda_2^{-2} g.$$

Then

$$f(\overline{x},\overline{t},\overline{g}) = \overline{x} - \tfrac{1}{2}\overline{g}\overline{t}^2 = \lambda_1 x - \tfrac{1}{2}(\lambda_1\lambda_2^{-2}g)(\lambda_2 t)^2 = \lambda_1\left(x - \tfrac{1}{2}gt^2\right).$$

Therefore (1.6) is unit-free.

Theorem 1.4

(Pi Theorem) Let

$$f(q_1, q_2, \ldots, q_m) = 0 \tag{1.7}$$

be a unit free physical law that relates the dimensional quantities $q_1, q_2, \ldots, q_m$. Let $L_1, \ldots, L_n (n < m)$ be fundamental dimensions with

$$[q_i] = L_1^{a_{1i}} L_2^{a_{2i}} \cdots L_n^{a_{ni}}, \quad i = 1, \ldots, m,$$

and let $r = \text{rank } A$, where A is the dimension matrix (1.5). Then there exists $m-r$ independent dimensionless quantities $\pi_1, \pi_2, \ldots, \pi_{m-r}$ that can be formed from $q_1, \ldots, q_m$, and the physical law (1.7) is equivalent to an equation

$$F(\pi_1, \pi_2, \ldots, \pi_{m-r}) = 0, \tag{1.8}$$

expressed only in terms of the dimensionless quantities.

The proof of the Pi theorem is presented later. For the present we analyze an example. Before continuing, however, we note that the existence of a physical law (1.7) is an assumption. In practice one must conjecture which are the relevant variables in a problem and then apply the machinery of the theorem. The resulting dimensionless physical law (1.8) must be checked by experiment, or whatever, in an effort to establish the validity of the original assumptions.

Example 1.5

(Heat transfer)At time $t = 0$ an amount of heat energy e, concentrated at a point in space, is allowed to diffuse outward into a region with temperature zero. If r denotes the radial distance from the source and t is time, the problem is to determine the temperature u as a function of r and t . This problem can be formulated as a boundary value problem for a partial differential equation (the heat equation, as in Chapter 6), but here we see what can be learned from a careful dimensional analysis of the problem. The first step is to make an assumption regarding which quantities affect the temperature u. Clearly t, r, and e are relevant quantities. It also is reasonable that the heat capacity c, with dimensions of energy per degree per volume, of the region plays a role, as will the rate at which heat diffuses outward. The latter is characterized by the thermal diffusivity k of dimensions length-squared per time. Therefore, we conjecture a physical law of the form

$$f(t, r, u, e, k, c) = 0,$$

which relates the six quantities, t, r, u, e, k, c. The next step is to determine a minimal set of fundamental dimensions $L_1, \ldots, L_n$ by which the six-dimensional quantities can be expressed. A suitable selection would be the four quantities T (time), L (length), Θ (temperature), and E (energy). Then

$$[t] = T, \quad [e] = E, \quad [r] = L,$$
$$[k] = L^2T^{-1}, \quad [u] = \Theta, \quad [c] = E\Theta^{-1}L^{-3},$$

and the dimension matrix A is given by

$$\begin{array}{c} \\ T \\ L \\ \Theta \\ E \end{array} \begin{array}{c} \begin{array}{cccccc} t & r & u & e & k & c \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 2 & -3 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

Here $m = 6$, $n = 4$, and the rank of the dimension matrix is $r = 4$. Consequently, there are $m - r = 2$ dimensionless quantities that can be formed from t, r, u, e, k, and c. To find them we proceed as follows: If π is dimensionless, then for some choice of $\alpha_1, \ldots, \alpha_6$

$$\begin{aligned} 1 = [\pi] &= [t^{\alpha_1} r^{\alpha_2} u^{\alpha_3} e^{\alpha_4} k^{\alpha_5} c^{\alpha_6}] \\ &= T^{\alpha_1} L^{\alpha_2} \Theta^{\alpha_3} E^{\alpha_4} (L^2T^{-1})^{\alpha_5} (E\Theta^{-1}L^{-3})^{\alpha_6} \\ &= T^{\alpha_1-\alpha_5} L^{\alpha_2+2\alpha_5-3\alpha_6} \Theta^{\alpha_3-\alpha_6} E^{\alpha_4+\alpha_6}. \end{aligned}$$

Therefore, the exponents must vanish and we obtain four homogeneous linear equations for $\alpha_1, \ldots, \alpha_6$, namely

$$\begin{aligned} \alpha_1 - \alpha_5 &= 0, \\ \alpha_2 + 2\alpha_5 - 3\alpha_6 &= 0, \\ \alpha_3 - \alpha_6 &= 0, \\ \alpha_4 + \alpha_6 &= 0. \end{aligned}$$

The coefficient matrix of this homogeneous linear system is just the dimension matrix A. From elementary matrix theory the number of independent solutions equals the number of unknowns minus the rank of A. Each independent solution will give rise to a dimensionless variable. Now the method unfolds and we can see the origin of the rank condition in the statement of the Pi theorem. By standard methods for solving linear systems we find two linearly independent solutions

$$\alpha_1 = -\tfrac{1}{2}, \quad \alpha_2 = 1, \quad \alpha_3 = \alpha_4 = 0, \quad \alpha_5 = -\tfrac{1}{2}, \quad \alpha_6 = 0,$$

and

$$\alpha_1 = \frac{3}{2}, \quad \alpha_2 = 0, \quad \alpha_3 = 1, \quad \alpha_4 = -1, \quad \alpha_5 = \frac{3}{2}, \quad \alpha_6 = 1.$$

These give two dimensionless quantities

$$\pi_1 = \frac{r}{\sqrt{kt}}$$

and

$$\pi_2 = \frac{uc}{e}(kt)^{3/2}.$$

Therefore the Pi theorem guarantees that the original physical law $f(t, r, u, e, k, c) = 0$ is equivalent to a physical law of the form

$$F(\pi_1, \pi_2) = 0.$$

Solving for π_2 gives

$$\pi_2 = g(\pi_1)$$

for some function g, or

$$u = \frac{e}{c}(kt)^{-3/2} g\left(\frac{r}{\sqrt{kt}}\right). \tag{1.9}$$

Again, without solving an equation governing the diffusion process, we are able via dimensional analysis to argue that the temperature of the region varies according to (1.9). For example, we can conclude that the temperature near the source $r = 0$ falls off like $t^{-3/2}$.

1.1.4 Proof of the Pi Theorem

To prove the Pi theorem we must demonstrate two propositions.

(i) Among the quantities $q_1, \ldots, q_m$ there are $m-r$ independent dimensionless variables that can be formed, where r is the rank of the dimension matrix A.

(ii) If $\pi_1, \ldots, \pi_{m-r}$ are the $m-r$ dimensionless variables, then (1.7) is equivalent to a physical law of the form $F(\pi_1, \ldots, \pi_{m-r}) = 0$.

The proof of (i) is straightforward; the general argument proceeds exactly like the construction of the dimensionless variables in Example 1.5. It makes use of the familiar result in linear algebra that the number of linearly independent solutions of a set of n homogeneous equations in m unknowns is $m-r$, where r is the rank of the coefficient matrix. For, let π be a dimensionless quantity. Then

$$\pi = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m} \tag{1.10}$$

for some $\alpha_1, \alpha_2, \ldots, \alpha_m$. In terms of the fundamental dimensions $L_1, \ldots, L_n$,

$$\begin{aligned} \pi &= (L_1^{a_{11}} L_2^{a_{21}} \cdots L_n^{a_{n1}})^{\alpha_1} (L_1^{a_{12}} L_2^{a_{22}} \cdots L_n^{a_{n2}})^{\alpha_2} \cdots (L_1^{a_{1m}} L_2^{a_{2m}} \cdots L_n^{a_{nm}})^{\alpha_m} \\ &= L^{a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1m}\alpha_m} \cdots L_n^{a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nm}\alpha_m}. \end{aligned}$$

Because $[\pi] = 1$, the exponents vanish, or

$$\begin{aligned} a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1m}\alpha_m &= 0 \\ &\vdots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nm}\alpha_m &= 0 \end{aligned} \tag{1.11}$$

By the aforementioned theorem in linear algebra, the system (1.11) has exactly $m-r$ independent solutions $[\alpha_1, \ldots, \alpha_m]$. Each solution gives rise to a dimensionless variable via (1.10), and this completes the proof of (i). The independence of the dimensionless variables is in the sense of linear algebraic independence.

The proof of (ii) makes strong use of the hypothesis that the law is unit-free. The argument is subtle, but it can be made almost transparent if we examine a particular example.

Example 1.6

Consider the unit-free law

$$f(x, t, g) = x - \tfrac{1}{2}gt^2 = 0 \tag{1.12}$$

for the distance a particle falls in a gravitational field. If length and time are chosen as fundamental dimensions, a straightforward calculation shows there is a single dimensionless variable given by

$$\pi_1 = t^2 \frac{g}{x}.$$

The remaining variable g can be expressed as $g = x\pi_1/t^2$, and we can define a function G by

$$G(x, t, \pi_1) \equiv f(x, t, x\pi_1/t^2).$$

Clearly the law $G(x, t, \pi_1) = 0$, or

$$x - \tfrac{1}{2}\left(\frac{x\pi_1}{t^2}\right) t^2 = 0, \tag{1.13}$$

is equivalent to (1.12) and is unit-free because f is. Then F is defined by

$$F(\pi_1) \equiv G(1, 1, \pi_1) = 1 - \tfrac{1}{2}\pi_1 = 0,$$

which is equivalent to (1.13) and (1.12).

We present the proof of (ii) in the special case when $m = 4$, $n = 2$, and $r = 2$. The notation will be easier, and the general argument for arbitrary m, n, and r proceeds in exactly the same manner. Therefore we consider a unit-free physical law

$$f(q_1, q_2, q_3, q_4) = 0, \tag{1.14}$$

with

$$[q_j] = L_1^{a_{1j}} L_2^{a_{2j}}, \quad j = 1, \ldots, 4,$$

where L_1 and L_2 are fundamental dimensions. The dimension matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

is assumed to have rank $r = 2$. If π is a dimensionless quantity, then

$$\pi = q_1^{\alpha_1} q_2^{\alpha_2} q_3^{\alpha_3} q_4^{\alpha_4},$$

where the exponents α_1, α_2, α_3, α_4 satisfy the homogeneous system (1.11), which in this case is

$$\begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \alpha_1 + \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \alpha_2 + \begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} \alpha_3 + \begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix} \alpha_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{1.15}$$

We wish to determine α_1, α_2, α_3, α_4, and the form of the two dimensionless variables. Without loss of any generality, we can assume the first two columns of A are linearly independent. This is because we can rearrange the indices

on the q_j so that the two independent columns appear as the first two. Then columns three and four can be written as linear combinations of the first two, or

$$\begin{pmatrix} a_{13} \\ a_{23} \end{pmatrix} = c_{31} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + c_{32} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}, \tag{1.16}$$

$$\begin{pmatrix} a_{14} \\ a_{24} \end{pmatrix} = c_{41} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + c_{42} \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} \tag{1.17}$$

for some constants c_{31}, c_{32}, c_{41}, and c_{42}. Substituting into (1.15) gives

$$(\alpha_1 + c_{31}\alpha_3 + c_{41}\alpha_4) \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} + (\alpha_2 + c_{32}\alpha_3 + c_{42}\alpha_4) \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The left side is a combination of linearly independent vectors, and therefore the coefficients must vanish,

$$\alpha_1 + c_{31}\alpha_3 + c_{41}\alpha_4 = 0,$$
$$\alpha_2 + c_{32}\alpha_3 + c_{42}\alpha_4 = 0.$$

Therefore, we can solve for α_1 and α_2 in terms of α_3 and α_4 and write

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \alpha_3 \begin{pmatrix} -c_{31} \\ -c_{32} \\ 1 \\ 0 \end{pmatrix} + \alpha_4 \begin{pmatrix} -c_{41} \\ -c_{42} \\ 0 \\ 1 \end{pmatrix}.$$

The two vectors on the right represent two linearly independent solutions of (1.15); hence, the two dimensionless quantities are

$$\pi_1 = q_1^{-c_{31}} q_2^{-c_{32}} q_3,$$
$$\pi_2 = q_1^{-c_{41}} q_2^{-c_{42}} q_4.$$

Next define a function G by

$$G(q_1, q_2, \pi_1, \pi_2) \equiv f(q_1, q_2, \pi_1 q_1^{c_{31}} q_2^{c_{32}}, \pi_2 q_1^{c_{41}} q_2^{c_{42}}).$$

The physical law

$$G(q_1, q_2, \pi_1, \pi_2) = 0 \tag{1.18}$$

holds if, and only if, (1.14) holds, and therefore (1.18) is an equivalent physical law. Since $f = 0$ is unit free, it easily follows that (1.18) is unit-free (note that $\overline{\pi}_1 = \pi_1$, $\overline{\pi}_2 = \pi_2$ under any change of units; that is, dimensionless variables have the same value in all systems of units). For the final stage of the argument we show that (1.18) is equivalent to the physical law

$$G(1, 1, \pi_1, \pi_2) = 0, \tag{1.19}$$

which will give the result, for (1.19) implies $F(\pi_1, \pi_2) = 0$, where $F(\pi_1, \pi_2) \equiv G(1, 1, \pi_1, \pi_2)$. Because (1.18) is a unit-free, we must have

$$G(\overline{q}_1, \overline{q}_2, \pi_1, \pi_2) = 0, \tag{1.20}$$

where

$$\overline{q}_1 = \lambda_1^{a_{11}} \lambda_2^{a_{21}} q_1, \quad \overline{q}_2 = \lambda_1^{a_{12}} \lambda_2^{a_{22}} q_2$$

for *every* choice of the conversion factors $\lambda_1, \lambda_2 > 0$. We select λ_1 and λ_2 so that $\overline{q}_1 = \overline{q}_2 = 1$. We are able to make this choice because

$$\lambda_1^{a_{11}} \lambda_2^{a_{21}} q_1 = 1, \quad \lambda_1^{a_{12}} \lambda_2^{a_{22}} q_2 = 1 \tag{1.21}$$

implies

$$\begin{aligned} a_{11} \ln \lambda_1 + a_{21} \ln \lambda_2 &= -\ln q_1, \\ a_{12} \ln \lambda_1 + a_{22} \ln \lambda_2 &= -\ln q_2. \end{aligned} \tag{1.22}$$

And, because the coefficient matrix

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

in (1.22) is nonsingular (recall the assumption that the first two columns of the dimension matrix A are linearly independent), the system (1.22) has a unique solution $(\ln \lambda_1, \ln \lambda_2)$, from which λ_1 and λ_2 can be determined to satisfy (1.21). Thus (1.19) is an equivalent physical law and the argument is complete.

The general argument for arbitrary m, n, and r can be found in Birkhoff (1950), from which the preceding proof was adapted. This classic book also provides additional examples and historical comments.[1]

In performing a dimensional analysis on a problem, two judgments are required at the beginning.

1. The selection of the pertinent variables.
2. The choice of the fundamental dimensions.

The first is a matter of experience and may be based on intuition or experiments. There is, of course, no guarantee that the selection will lead to a useful formula after the procedure is applied. Second, the choice of fundamental dimensions may involve tacit assumptions that may not be valid in a given problem. For example, including mass, length, and time but not force in a given problem assumes there is some relation (Newton's second law) that plays

[1] A simple proof of the Pi theorem in a linear algebraic setting is contained in W. D. Curtis, J. D. Logan, & W. A. Parker, 1982. Dimensional analysis and the Pi theorem, *Linear Algebra and Its Applications* **47**, 117–126.

an important role and causes force not to be an independent dimension. As a specific example, a small sphere falling under gravity in a viscous fluid is observed to fall, after a short time, at constant velocity. Since the motion is un-accelerated we need not make use of the proportionality of force to acceleration, and so force can be treated as a separate, independent fundamental dimension. In summary, intuition and experience are important ingredients in applying the dimensional analysis formalism to a specific physical problem.

EXERCISES

1. The speed v of a wave in deep water is determined by its wavelength λ and the acceleration g due to gravity. What does dimensional analysis imply regarding the relationship between v, λ, and g?
2. An ecologist postulated that that there is a relationship among the mass m, density ρ, volume V, and surface area S of certain animals. Discuss this conjecture in terms of dimensionless analysis.
3. A small sphere of radius r and density ρ is falling at constant velocity v under the influence of gravity g in a liquid of density ρ_l and viscosity μ (given in mass per length per time). It is observed experimentally that
$$v = \frac{2}{9} r^2 \rho g \mu^{-1} \left(1 - \frac{\rho_l}{\rho}\right).$$
Show that this law unit-free.
4. A physical phenomenon is described by the quantities P, l, m, t , and ρ, representing pressure, length, mass, time, and density, respectively. If there is a physical law
$$f(P, l, m, t, \rho) = 0$$
relating these quantities, show that there is an equivalent physical law of the form $G(l^3\rho/m, t^6 P^3/m^2\rho) = 0$.
5. A physical system is described by a law $f(E, P, A) = 0$, where E, P, and A are energy, pressure, and area, respectively. Show that $PA^{3/2}/E = \text{const}$.
6. The length L of an organism depends upon time t, its density ρ, its resource assimilation rate a (mass per area per time), and its resource use rate b (mass per volume per time). Show that there is a physical law involving two dimensionless quantities.
7. A piece of shrapnel of density ρ is driven off an explosive device at velocity v. The density of the explosive is ρ_e, and E is its Gurney energy (joules/kilogram), or the specific energy available in the explosive to do work. Determine how the velocity of the shrapnel depends upon E.

8. In an indentation experiment, a slab of metal of thickness h is subjected to a constant pressure P on its upper surface by a cylinder of radius a. The technician then measures the vertical displacement of U of the indentation. The displacement also depends upon two material properties, Poisson's ratio υ, which is dimensionless, and the Lamé constant μ, which has dimensions $\mathrm{M/L^3T^2}$, where M is mass, L is length, and T is time. Determine a set of dimensionless variables and show that the functional form of U is
$$U = aG\left(\frac{P}{\mu a^2}, \frac{h}{a}, \upsilon\right).$$

9. In modeling the digestion process in insects, it is believed that digestion yield rate Y, in mass per time, is related to the concentration C of the limiting nutrient, the residence time T in the gut, the gut volume V, and the rate of nutrient nutrient breakdown r, given in mass per time per volume. Show that for fixed T, r, C, the yield is positively related to the gut volume.

10. A chemical **C** flows continuously into a reactor with concentration C_i and volumetric flow rate q (volume/time). While in the reactor, which has volume V, the substances are continuously stirred and a chemical reaction **C**$\rightarrow$ products, with rate constant k (1/time), consumes the chemical. The mixture exits the reactor at the same flow rate q. The concentration of **C** in the reactor at any time t is $C = C(t)$, and $C(0) = C_0$. Use dimensional analysis to deduce that
$$C = C_i F(tk, a, b),$$
where $a = C_0/C_i$ and $b = Vk/q$ are dimensionless constants and F is some function.

11. The problem is to determine the power P that must be applied to keep a ship of length l moving at a constant speed V. If it is the case, as seems reasonable, that P depends on the density of water ρ, the acceleration due to gravity g, and the viscosity of water ν (in length-squared per time), as well as l and V, then show that
$$\frac{P}{\rho l^2 V^3} = f(\mathrm{Fr}, \mathrm{Re}),$$
where Fr is the Froude number and Re is the Reynolds number defined by
$$\mathrm{Fr} \equiv \frac{V}{\sqrt{lg}}, \quad \mathrm{Re} \equiv \frac{Vl}{\nu}$$

12. A spherical gas bubble with ratio of specific heats γ is surrounded by an infinite sea of liquid of density ρ_l. The bubble oscillates with growth and contraction periodically with small amplitude at a well-defined frequency ω. Assuming a physical law
$$f(P, R, \rho_l, \omega, \gamma) = 0,$$
where P is the mean pressure inside the bubble and R is the mean radius, show that the frequency must vary inversely with the mean radius R. The constant γ is dimensionless.

13. A perfect gas in equilibrium has specific energy E (energy per mass), temperature T, and Boltzmann constant k (specific energy per degree). Derive a functional relationship of the form $E = f(k, T)$.

14. In tests for fuel economy, cars are driven at constant speed V on a level highway. With no acceleration, the force of propulsion F must be in equilibrium with other forces, such as the air resistance, and so on. Assume that the variables affecting F are the velocity V, the rate C that fuel is burned, in volume per time, and the amount of energy K in a gallon of fuel, in mass per length per time-squared. Determine F as a function of V, C, and K.

15. Across a cell membrane of thickness w the mass concentration of a biochemical molecule is C_i on the inside of the cell, and C_o on the outside. The diffusion constant in the membrane is d, measured in length-squared per time. Use dimensional analysis (not guessing) to obtain the most general relationship for the flux φ of the molecules (mass/(area·time) through the membrane in terms of the other quantities.

1.2 Scaling

1.2.1 Characteristic Scales

Another procedure useful in formulating a mathematical model is *scaling*. Roughly, scaling means selecting new, usually dimensionless variables and reformulating the problem in terms of those variables. Not only is the procedure useful, but it often is a necessity, especially when comparisons of the magnitudes of various terms in an equation must be made in order to neglect small terms. This idea is particularly crucial in the application of perturbation methods to identify small and large parameters. Further, scaling usually reduces the

number of parameters in a problem, thereby leading to great simplification, and it identifies what combinations of parameters are important.

For motivation let us suppose that time t is a variable in a given problem, measured in units of seconds. If the problem involved the motion of a glacier, clearly the unit of seconds is too fast because significant changes in the glacier could not be observed on the order of seconds. On the other hand, if the problem involved a nuclear reaction, then the unit of seconds is too slow; all of the important action would be over before the first second ticked. Evidently, every problem has an intrinsic **time scale**, or *characteristic time* t_c, which is appropriate to the given problem. This is the shortest time for discernible changes to be observed in the physical quantities. For example, the characteristic time for glacier motion would be of the order of years, whereas the characteristic time for a nuclear reaction would be of the order of microseconds. Some problems have multiple time scales. A chemical reaction, for example, may begin slowly and the concentration change little over a long time; then, the reaction may suddenly go to completion with a large change in concentration over a short time. There are two time scales involved in such a process. Other examples are in the life sciences, where multi-scale processes are the norm. Spatial scales vary over as much as 10^{15} orders of magnitude as we progress from processes involving genes, proteins, cells, organs, organisms, communities, and ecosystems; time scales vary from times that it takes for protein to fold to times for evolution to occur. Several scales can occur in the same problem. Yet another example occurs in fluid flow, where the processes of heat diffusion, advection, and possible chemical reaction all have different scales.

Once a characteristic time has been identified, at least for a part of a process, then a new dimensionless variable $\bar{t}$ can be defined by

$$\bar{t} = \frac{t}{t_c}$$

If t_c is chosen correctly, then the *dimensionless time* $\bar{t}$ is neither too large nor too small, but rather of order unity. The question remains to determine the time scale t_c for a particular problem, and the same question applies to other variables in the problem (e.g., length, concentration, and so on). The general rule is that the characteristic quantities are formed by taking combinations of the various dimensional constants in the problem and should be roughly the same order of magnitude of the quantity itself.

After characteristic scales, which are built up from the parameters in the model, are chosen for the independent and dependent variables, the model can then reformulated in terms of the new dimensionless variables. The result will be a model in dimensionless form, where all the variables and parameters in the problem are dimensionless. This process is called **non-dimensionalization**, or

scaling a problem. By the Pi theorem, it is guaranteed that we can always non-dimensionlize a problem. The payoff is simpler model that is independent of units and dimensions and that has fewer parameters, which is an economy of savings.

Example 1.7

(Population growth) Let $p = p(t)$ denote the population of an animal species located in a fixed region at time t. The simplest model of population growth is the classic **Malthus model**,[2] which states that the per capita growth rate $\frac{1}{p}\frac{dp}{dt}$ is constant. This means that the growth rate $\frac{dp}{dt}$ is proportional to the population p, or $\frac{dp}{dt} = rp$, where r is the *growth rate*, given in dimensions of inverse-time. Easily, the Malthus model predicts that the population will grow exponentially for all time, that is, $p = p_0 e^{rt}$, where p_0 is the initial population. Many books on an impending world population explosion have been written with the Malthusian model as a premise. Clearly, however, as a population grows, intraspecific competition for food, living space, and natural resources limit the growth. We can modify the Malthus model to include a competition term. The simplest approach is to notice that if there are p individuals in the system, then the number of encounters, which is a measure of competition, is approximately p^2; therefore we subtract a term proportional to p^2 from the growth rate to obtain the model

$$\frac{dp}{dt} = rp\left(1 - \frac{p}{K}\right), \quad p(0) = p_0. \tag{1.23}$$

This is saying that the per capita growth rate is not constant, but decreases linearly with population. The parameter K is the *carrying capacity*, which is the number of individuals that the ecosystem can sustain. When $p = K$ the growth rate is zero. The model (1.23) is called the **logistics model**[3]; as the population grows, the negative p^2 term will kick in and limit the growth. To reduce (1.23) to dimensionless form we select new dimensionless independent and dependent variables. The time scale and population scale are formed from the constants in the problem, r, K, and p_0. Of these, only r contains the dimensions of time, and therefore we scale time by $1/r$ giving a new, dimensionless time τ defined by

$$\tau = rt.$$

There are two choices for the population scale, K or p_0. Either will work, and

[2] Thomas Malthus (1766–1834) was an English essayist who was one of the first individuals to address demographics and food supply.
[3] The logistics model was introduced by P. Verhulst (1804–1849).

so we select K to obtain a dimensionless population P given by

$$P = \frac{p}{K}$$

Thus, we measure population in the problem relative to the carrying capacity. Using these dimensionless variables, it is straightforward to obtain

$$\frac{dP}{d\tau} = P(1-P), \quad P(0) = \alpha, \tag{1.24}$$

where $\alpha \equiv p_0/K$ is a dimensionless constant. The scaled model (1.24) has only one constant (α), a significant simplification over (1.23) where there are three constants (r, K, p_0). The constant α represents a scaled, initial population. There is a single combination of the parameters in the original that is relevant to the dynamics. The initial value problem (1.24) can be solved by separating variables to obtain

$$P(\tau) = \frac{\alpha}{\alpha + (1-\alpha)e^{-\tau}}. \tag{1.25}$$

Clearly

$$\lim_{\tau \to \infty} P(\tau) = 1.$$

It follows that, confirming our earlier statement, the limiting population p is equal to the carrying capacity K. We observe that there are two *equilibrium populations*, or constant solutions of (1.23), $p = K$ and $p = 0$. The population $p = K$ is an *attractor*; that is, regardless of the initial population, the population $p(t)$ tends to the value K as time gets large.

1.2.2 A Chemical Reactor Problem

A basic problem in chemical engineering is to understand how the concentration of chemical species vary when undergoing a reaction in a chemical reactor. To illustrate the concept of non-dimensionalization and scaling we analyze a simple model of an isothermal, continuously stirred, tank reactor (see Fig. 1.2). The reactor has a fixed volume V, and a chemical **C** of fixed concentration c_i, given in mass per volume, enters the reactor through the feed at a constant flow rate q, given in volume per time; initially the concentration of the chemical in the reactor is c_0. When the chemical enters the reactor, the mixture is perfectly stirred while undergoing a chemical reaction, and then the mixture exits the reactor at the same flow rate q. At any time t, we denote the concentration of the chemical **C** in the reactor by $c = c(t)$. The reactant chemical is assumed to disappear, that is, it is consumed by reaction, with a rate $r = r(c)$, given in mass per unit volume, per unit time. We are thinking of a simple reaction of the form **C** $\rightarrow$ products. Usually reaction rates depend on temperature, but

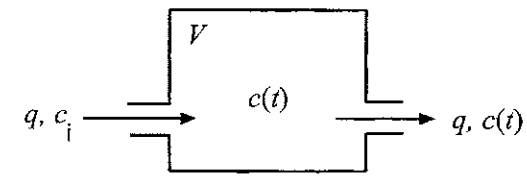

Figure 1.2 Continuously stirred tank reactor.

here we assume that r depends only on the concentration c; this is what makes our problem isothermal. The *perfectly stirred* assumption is an idealization and implies that there are no concentration gradients in the reactor; otherwise, c would also depend on a spatial variable.

To obtain a mathematical model of this problem we look for a physical law. A common principle that is fundamental to all flow problems is mass balance. That is, the time rate of change of the mass of the chemical inside the reactor must equal the rate mass flows in (qc_i), minus the rate that mass flows out (qc), plus the rate that mass is consumed by the reaction (Vr). At any given time the mass of the chemical in the reactor is Vc. In symbols, the mass balance equation is

$$\frac{d}{dt}(Vc(t)) = qc_i - qc(t) - Vr(c(t)).$$

Observe that the factor V is required on the reaction term to obtain consistent dimensions. To fix the idea, let us take the reaction rate r to be proportional to c, that is $r = kc$, where k is the rate constant having dimensions of inverse time. Recall that this is the law of mass action in elementary chemistry, which states that the rate of a reaction is proportional to the product of the concentrations of the reactants. Then, the mathematical model is given by the initial value problem

$$\frac{dc}{dt} = \frac{q}{V}(c_i - c) - kc, \quad t > 0, \tag{1.26}$$

$$c(0) = c_0. \tag{1.27}$$

To non-dimensionalize the problem we choose dimensionless independent and dependent variables. This means we must select a characteristic time and a characteristic concentration by which to measure the real time and concentration. These characteristic values are formed from the constants in the problem: c_i, c_0, V, q, k. Generally, we measure the dependent variable relative to some maximum value in the problem, or any other value that represents the order of magnitude of that quantity. There are two constant concentrations, c_i and c_0, and either one of them is a suitable concentration scale. Therefore, we define a

dimensionless concentration C by

$$C = \frac{c}{c_i}.$$

Therefore, all concentrations in the problem are measured relative to c_i, the concentration of the feed. To select a time scale we observe that there are two quantities with dimensions of time that can be formed from the constants in the problem, V/q and k^{-1}. The former is based on the flow rate, and the latter is based on the reaction rate. So the choice of a time scale is not unique. Either choice leads to a correct dimensionless problem. Let us hold up on making a selection and define a dimensionless time τ by

$$\tau = \frac{t}{T},$$

where T is either V/q or k^{-1}. We recast the model in dimensionless form. By the chain rule

$$\frac{dc}{dt} = \frac{c_i}{T}\frac{dC}{d\tau},$$

and therefore the model becomes

$$\frac{dC}{d\tau} = \frac{qT}{V}(1-C) - kTC, \quad \tau > 0,$$
$$C(0) = \gamma,$$

where γ is a dimensionless constant given by the ratio

$$\gamma = \frac{c_0}{c_i}.$$

If we choose $T = V/q$, then we obtain the dimensionless model

$$\frac{dC}{d\tau} = 1 - C - \beta C, \quad \tau > 0, \tag{1.28}$$
$$C(0) = \gamma, \tag{1.29}$$

where

$$\beta = \frac{kV}{q}$$

is a dimensionless constant representing a ratio of the two time scales. Therefore, the problem has been reduced to dimensionless form, and any results that are obtained are free of any specific set of units that we select. Moreover, the number of parameters has been decreased from five to two, and the problem is simpler. If we choose the other time scale, $T = k^{-1}$, then the problem reduces to the dimensionless form

$$\frac{dC}{d\tau} = \frac{1}{\beta}(1-C) - C, \quad \tau > 0, \tag{1.30}$$
$$C(0) = \gamma. \tag{1.31}$$

Is there any advantage of one over the other? Yes. In some problems, where the terms differ in order of magnitude, it is important to choose the *correct* scale, so that each term reflects the correct magnitude. For example, suppose reaction occurs on a slow time scale compared to the flow through the reactor. Then k is small compared to the flow rate q/V. We write $k \ll q/V$. Therefore the dimensionless parameter β is small, or $\beta \ll 1$. Of the two dimensionless models (1.28)–(1.29) or (1.30)–(1.31), which best reflects this assumption? We expect the reaction term to be small compared to the flow rate term, so the best choice is (1.28)–(1.29), which means we should scale time on the more dominant flow rate. If reaction is fast compared to the flow rate, then β is large ($\beta \gg 1$), and we should choose a reaction time scale, giving (1.30)–(1.31); this choice puts the small term $1/\beta$ correctly on the flow rate term.

Making the correct choice when terms have different orders of magnitude is especially essential when we want to make an approximation by deleting small terms in the equation. This is a very common strategy in equations that we cannot solve, for example, most nonlinear equations. For example, if $\beta \ll 1$ and we scale by by flow rate, the model is (1.28)–(1.29). Ignoring the small term gives the approximation

$$\frac{dC}{d\tau} = 1 - C, \quad C(0) = \gamma,$$

which gives approximation $C(\tau) = 1 + (\gamma - 1)e^{-\tau}$. This approximation is believable; the reaction is slow and the concentration approaches the concentration of the feed. On the other hand, the model (1.30)–(1.31) gives $\beta\frac{dC}{d\tau} = (1-C) - \beta C$, which, upon neglecting the small terms, provides the approximation $C = 1$. This approximation does not even satisfy the initial condition.

In summary, if approximations are to be made by deleting small terms, it is important how we non-dimensionalize the problem. The common scaling strategy is to non-dimensionalize the problem using a generic time scale T that is chosen later to make the coefficients of the terms in the equation reflect their size or reflect what terms balance in the process. Ultimately, proper scaling is learned through experience and careful analysis.

1.2.3 The Projectile Problem

The projectile problem, as first pointed out by Lin and Segel (1974), is a good illustration of the importance of choosing correct scales, particularly when it is desired to make a simplification by neglecting small quantities. Terms in an equation that appear small are not always as they seem, and proper scaling is essential in determining the orders of magnitude of the terms.

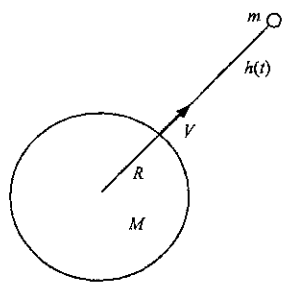

Figure 1.3 The projectile problem.

We analyze the motion of a projectile thrust vertically upward from the surface of the earth. At time $t = 0$ on the surface of the earth, with radius R and mass M, an object of mass m is given a vertical upward velocity of magnitude V. To be determined is the height h above the earth's surface that the mass reaches at time t (see Fig. 1.3). Generally, forces on the object are the gravitational force and the force due to air resistance. But we assume the force due to air resistance is negligible in the particular problem we are considering. In general, as a first approximation it is common to neglect what are believed to be small effects, since in that case the equations are more tractable for analysis. Should the analytic results compare unfavorably with experiment or should a more detailed description be required, then additional effects can be included.

The governing equation, or mathematical model, comes from a physical law. Newton's universal gravitational law, which states that the force between the two objects is proportional to the product of the masses and inversely proportional to the square of the distance between them, where the mass of each object can be regarded as concentrated at its center. By Newton's second law the force equals the mass times acceleration, or

$$m\frac{d^2h}{dt^2} = -G\frac{Mm}{(h+R)^2},$$

where G is the proportionality constant in the universal gravitational law. When $h = 0$, that is, at the earth's surface, the gravitational force equals $-mg$, and therefore

$$\frac{GM}{R^2} = g.$$

Thus

$$\frac{d^2h}{dt^2} = -\frac{R^2g}{(h+R)^2}, \tag{1.32}$$

with initial conditions

$$h(0) = 0, \quad \frac{dh}{dt}(0) = V. \tag{1.33}$$

The initial value problem (1.32) and (1.33) is the mathematical model for the problem.

At this point we undertake a dimensional analysis of the problem and gain considerable insight without actually attempting a solution. From our model the relevant dimensional quantities are t, h, R, V, and g having dimensions

$$[t] = \text{time } (T), \qquad [V] = \text{velocity } (LT^{-1}), \tag{1.34}$$

$$[h] = \text{length } (L), \qquad [g] = \text{acceleration } (LT^{-2}), \tag{1.35}$$

$$[R] = \text{length } (L). \tag{1.36}$$

We can use T (time) and L (length) as fundamental dimensions. Following the procedure described earlier, if π is a dimensionless combination of t, h, R, V, and g, then

$$[\pi] = [t^{\alpha_1}h^{\alpha_2}R^{\alpha_3}V^{\alpha_4}g^{\alpha_5}] = T^{\alpha_1-\alpha_4-2\alpha_5}L^{\alpha_2+\alpha_3+\alpha_4+\alpha_5} = 1.$$

Therefore,

$$\begin{aligned} \alpha_1 - \alpha_4 - 2\alpha_5 &= 0, \\ \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 &= 0. \end{aligned} \tag{1.37}$$

This system has rank two and so there are three independent dimensionless variables. Either by inspection or solving (1.37), we find dimensionless quantities

$$\pi_1 = \frac{h}{R}, \quad \pi_2 = \frac{t}{R/V}, \quad \pi_3 = V/\sqrt{gR}. \tag{1.38}$$

By the Pi theorem, if there is a physical law relating t, h, R, V, and g (and we assume there must be, since in theory we could solve (1.32) and (1.33) to obtain that law), then there is an equivalent law that can be expressed as

$$\frac{h}{R} = f\left(\frac{t}{R/V}, \frac{V}{\sqrt{gR}}\right) \tag{1.39}$$

for some function $f(\pi_2, \pi_3)$.

Actually there is considerable information in (1.39). For example, suppose we are interested in finding the time $t_{\max}$ that is required for the object to reach its maximum height for a given velocity V. Then differentiating (1.39) with respect to t and setting $h'(t)$ equal to zero gives

$$\frac{\partial f}{\partial \pi_2}\left(\frac{t_{\max}}{R/V}, \frac{V}{\sqrt{gR}}\right) = 0,$$

or

$$\frac{t_{\max}}{R/V} = F\left(\frac{V}{\sqrt{gR}}\right) \tag{1.40}$$

for some function F. Remarkably, with little analysis beyond dimensional reasoning, we have found that the time to maximum height depends on the dimensionless combination $V/\sqrt{gR}$. The value in knowing this kind of information lies in the efficiency of (1.40); a *single* graph of $t_{\max}/(R/V)$ vs. $V/\sqrt{Rg}$ contains all of the data of the graphs of $t_{\max}$ versus V for *all* choices of g and R. For example, an experimenter making measurements on different planets of $t_{\max}$ versus V would not need a separate plot of data for each planet. An entire atlas can be replaced by a single map when we plot dimensionless quantities.

Now we select characteristic time and length scales and recast the problem represented by (1.32) and (1.33) into dimensionless form. The problem is more subtle than it originally appears. The general method requires us to choose a new dimensionless dependent variable $\overline{h}$ and independent variable $\overline{t}$ by

$$\overline{t} = \frac{t}{t_c}, \quad \overline{h} = \frac{h}{h_c}, \tag{1.41}$$

where t_c is an intrinsic time scale and h_c is an intrinsic length scale; the values of t_c and h_c should be chosen by taking combinations of the constants in the problem, which in this case are R, V, and g. This problem presents several choices. For a length scale h_c we could take either R or V^2/g. Possible time scales are R/V, $\sqrt{R/g}$, and V/g. Which choice is the most appropriate? Actually, equations (1.41) represent a legitimate transformation of variables for any choice of t_c and h_c; after the change of variables an equivalent problem results. From a scaling viewpoint, however, one particular choice is advantageous. The three choices

$$\overline{t} = \frac{t}{R/V}, \quad \overline{h} = \frac{h}{R}, \tag{1.42}$$

$$\overline{t} = \frac{t}{\sqrt{R/g}}, \quad \overline{h} = \frac{h}{R}, \tag{1.43}$$

and

$$\overline{t} = \frac{t}{Vg^{-1}}, \quad \overline{h} = \frac{h}{V^2g^{-1}} \tag{1.44}$$

lead to the following three dimensionless problems, which are equivalent to (1.32) and (1.33):

$$\epsilon\frac{d^2\overline{h}}{d\overline{t}^2} = -\frac{1}{(1+\overline{h})^2}, \qquad \overline{h}(0) = 0, \quad \frac{d\overline{h}}{d\overline{t}}(0) = 1, \tag{1.45}$$

$$\frac{d^2\overline{h}}{d\overline{t}^2} = -\frac{1}{(1+\overline{h})^2}, \qquad \overline{h}(0) = 0, \quad \frac{d\overline{h}}{d\overline{t}}(0) = \sqrt{\epsilon}, \tag{1.46}$$

and

$$\frac{d^2\overline{h}}{d\overline{t}^2} = -\frac{1}{(1+\epsilon\overline{h})^2}, \qquad \overline{h}(0) = 0, \quad \frac{d\overline{h}}{d\overline{t}}(0) = 1, \tag{1.47}$$

respectively, where ϵ is a dimensionless parameter defined by

$$\epsilon = \frac{V^2}{gR}.$$

To illustrate how difficulties arise in selecting an incorrect scaling, let us modify our original problem by examining the situation when ϵ is known to be a small quantity; that is, V^2 is much smaller than gR. Then one may be tempted, in order to make an approximation, to delete the terms involving ϵ in the scaled problem. Problem (1.45) then becomes

$$(1+\overline{h})^{-2} = 0, \qquad \overline{h}(0) = 0, \quad \frac{d\overline{h}}{d\overline{t}}(0) = 1,$$

which has no solution, and problem (1.46) becomes

$$\frac{d^2\overline{h}}{d\overline{t}^2} = -\frac{1}{(1+\overline{h})^2}, \qquad \overline{h}(0) = 0, \quad \frac{d\overline{h}}{d\overline{t}}(0) = 0,$$

which has no physically valid solution (the graph of $\overline{h}(\overline{t})$ passes though the origin with zero slope and is concave downward, thereby making $\overline{h}$ negative). Therefore it appears that terms involving small parameters cannot be neglected. This is indeed unfortunate because this kind of technique is common practice in making approximations in applied problems. What went wrong was that (1.42) and (1.43) represent *incorrect* scalings; in these cases, terms that appear small may not in fact be small. For example, in the term $\epsilon d^2\overline{h}/d\overline{t}^2$, the parameter ϵ is small but $d^2\overline{h}/d\overline{t}^2$ may be large, and hence the term may not be negligible compared to other terms in the equation.

If, on the other hand, the term $\epsilon\overline{h}$ is neglected in (1.47), then $d^2\overline{h}/d\overline{t}^2 = -1$, or $\overline{h} = \overline{t} - \overline{t}^2/2$, after applying the initial conditions. Therefore

$$h = -\tfrac{1}{2}gt^2 + Vt,$$

and we have obtained an approximate solution that is consistent with our experience with falling bodies close to the earth. In this case we are able to neglect the small term and obtain a valid approximation because the scaling is correct. That (1.44) gives the correct time and length scales can be argued physically. If V is small, then the body will be acted on by a constant gravitational field; hence, launched with speed V, it will uniformly decelerate and reach its maximum height in V/g units of time, which is the characteristic time. It will travel a distance of about (V/g) times its average velocity $\frac{1}{2}(V+0)$, or $V^2/2g$. Hence V^2/g is a good selection for the length scale. Measuring the height relative to the radius of the earth is not a good choice for small initial velocities.

In general, if a correct scaling is chosen, then terms in the equations that appear small are indeed small and may be safely neglected. In fact, one goal

of scaling is to select intrinsic, characteristic reference quantities so that each term in the dimensional equation transforms into a term that the dimensionless coefficient in the transformed term represents the order of magnitude or approximate size of that term. Pictorially,

$$\left[\begin{pmatrix}\text{Dimensional}\\ \text{term}\end{pmatrix}\right] \to \begin{bmatrix}\text{coefficient}\\ \text{representing}\\ \text{the order of}\\ \text{magnitude}\\ \text{of the term}\end{bmatrix} \cdot \left[\begin{pmatrix}\text{dimensionless}\\ \text{factor}\\ \text{of order unity}\end{pmatrix}\right].$$

By *order of unity* we mean a term that is neither extremely large nor small.

EXERCISES

1. Let $u = u(t)$, $0 \le t \le b$, be a given smooth function. If $M = \max|u(t)|$, then u can be scaled by M to obtain the dimensionless dependent variable $U = u/M$. A time scale can be taken as $t_c = M/\max|u'(t)|$, the ratio of the maximum value of the function to the maximum slope. Find M and t_c for the following functions:

   a) $u(t) = A\sin\omega t, \quad t > 0.$

   b) $u(t) = Ae^{-\lambda t}, \quad t > 0.$

   c) $u(t) = Ate^{-\lambda t}, \quad 0 \le t \le 2/\lambda.$

2. Consider a process described by the function $u(t) = 1+e^{-t/\epsilon}$ on the interval $0 \le t \le 1$, where ϵ is a small number. Use Exercise 1 to determine a time scale. Is this time scale appropriate for the entire interval $[0,1]$? (Sketch a graph of $u(t)$ when $\epsilon = 0.05$.) Explain why two time scales might be required for a process described by $u(t)$.

3. The growth rate of an organism is often measured using carbon biomass as the "currency." The von Bertalanffly growth model is

$$m' = ax^2 - bx^3,$$

where m is its biomass, x is some characteristic length of the organism, a is its biomass assimilation rate, and b is its biomass use rate. Thus, it assimilates nutrients proportional to its area, and it uses nutrients proportional to its volume. Assume $m = \rho x^3$ and rewrite the model in terms of the length x. Determine the dimensions of the constants a, b, and ρ. Select time and length scales ρ/b and a/b, respectively, and reduce the problem to dimensionless form. If $x(0) = 0$, find the length x at time t. Does this seem like a reasonable model?

4. A mass hanging on a spring is given a positive initial velocity V from equilibrium. The ensuing displacement $x = x(t)$ is governed by

$$\begin{aligned} mx'' &= -ax|x'| - kx, \\ x(0) &= 0, \quad x'(0) = V, \end{aligned}$$

where x is the displacement, $-ax|x'|$ is a nonlinear damping force, and $-kx$ is a linear restoring force of the spring. What are the possible time scales, and on what physical processes are they based. If the restoring force is small compared to the damping force, choose appropriate time and spatial scales and non-dimensionalize the model.

5. The dynamics of a nonlinear spring–mass system is described by

$$\begin{aligned} mx'' &= -ax' - kx^3, \\ x(0) &= 0, \quad mx'(0) = I, \end{aligned}$$

where x is the displacement, $-ax'$ is a linear damping term, and $-kx^3$ is a nonlinear restoring force. Initially, the displacement is zero and the mass m is given an impulse I that starts the motion.

   a) Determine the dimensions of the constants I, a, and k.

   b) Recast the problem into dimensionless form by selecting dimensionless variables $\tau = t/T$, $u = ax/I$, where the time scale T is yet to be determined.

   c) In the special case that the mass is very small, choose an appropriate time scale T and find the correct dimensionless model. (A small dimensionless parameter should occur on the terms involving the mass in the original model.)

6. In a classic work modeling the outbreak of the spruce bud worm in Canada's balsam fir forests, researchers proposed that the bud worm population $n = n(t)$ was governed by the law

$$\frac{dn}{dt} = rn\left(1 - \frac{n}{K}\right) - P(n),$$

where r and K are the growth rate and carrying capacity, respectively, and $P(n)$ is a bird *predation* term given by

$$P(n) = \frac{bn^2}{a^2 + n^2},$$

where a and b are positive constants.

   a) Determine the dimensions of the constants a and b.

b) Graph the predation rate $P(n)/b$ for $a = 1, 5, 10$ and make a qualitative statement about the effect that the parameter a has on the model.

c) Select dimensionless variables $N = n/a$ and $\tau = t/(a/b)$ and reduce the differential equation to dimensionless form (introduce constants $q \equiv K/a$ and $s \equiv ar/b$).

d) Researchers are often interested in equilibrium populations, or constant solutions of the differential equation. Working with the dimensionless model, and ignoring the zero population, show that there is always at least one equilibrium population, and show that there may be two or three, depending on the values of the parameters. (Hint: Plot the growth rate and the predation rate on the same set of axes.) Find the equilibrium populations when $q = 12$ and $s = 0.25$ and when $q = 35$ and $s = 0.4$.

e) In the case $q = 35$ and $s = 0.4$ use a numerical differential equations solver to graph the population curve when the initial population is given by $N(0) = 40$, 25, 2, and 0.05, respectively.

7. A rocket blasts off from the earth's surface. During the initial phase of flight, fuel is burned at the maximum possible rate α, and the exhaust gas is expelled downward with velocity β relative to the velocity of the rocket. The motion is governed by the following set of equations:

$$m'(t) = -\alpha, \quad m(0) = M,$$
$$v'(t) = \frac{\alpha\beta}{m(t)} - \frac{g}{(1 + x(t)/R)^2}, \quad v(0) = 0,$$
$$x'(t) = v(t), \quad x(0) = 0,$$

where $m(t)$ is the mass of the rocket, $v(t)$ is the upward velocity, $x(t)$ is the height above the earth's surface, M is the initial mass, g is the gravitational constant, and R is the radius of the earth. Reformulate the problem in terms of dimensionless variables using appropriate scales for m, x, v, t. (Hint: Scale m and x by obvious choices; then choose the time scale and velocity scale to ensure that the terms in the v equation and x equation are of the same order. Assume that the acceleration is due primarily to fuel burning and that the gravitational force is relatively small.)

8. In the chemical reactor problem assume that the reaction is $\mathbf{C} + \mathbf{C} \rightarrow$ products, and the chemical reaction rate is $r = kc^2$, where k is the rate constant. What is the dimension of k? Define dimensionless variables and reformulate the problem in dimensionless form. Solve the dimensionless problem to determine the concentration.

9. The temperature $T = T(t)$ of a chemical sample in a furnace at time t is governed by the initial value problem

$$\frac{dT}{dt} = qe^{-A/T} - k(T - T_f), \quad T(0) = T_0,$$

where T_0 is the initial temperature of the sample, T_f is the temperature in the furnace, and q, k, and A are positive constants. The first term on the right side is the heat generation term, and the second is the heat loss term given by Newton's law of cooling.

a) What are the dimensions of the constants q, k, A?

b) Reduce the problem to dimensionless form using T_f as the temperature scale and choosing a time scale to be one appropriate to the case when the heat loss term is large compared to the heat generated by the reaction.

10. A ball of mass m is tossed upward with initial velocity V. Assuming the force caused by air resistance is proportional to the square of the velocity of the ball and the gravitational field is constant, formulate an initial value problem for the height of the ball at any time t. Choose characteristic length and time scales and recast the problem in dimensionless form.

11. A particle of mass m moves in one dimension on the x-axis under the influence of a force

$$F(x, t) = -\frac{k}{x^2}e^{-t/a},$$

where k and a are positive constants. Initially the particle is located at $x = L$ and has zero velocity.

a) Set up an initial value problem for the location $x = x(t)$ of the particle at time t.

b) Identify the length and time scales in the problem, and introduce dimensionless variables in two different ways, reformulating the problem in dimensionless form in both cases.

12. An aging spring is sometimes modeled by a force $F(x, t) = -kxe^{-t/a}$, where k is the stiffness of the spring and a is constant. Set up and nondimensionalize a model for oscillation of a mass m on an aging spring if $x(0) = L$ and $x'(0) = V$.

13. (Fishery management) A fishing industry on the east coast has fished an area to near depletion and has stopped its activity (assume this occurs at time zero with fish population x_0). The management assumes that the fish will recover and grow logistically with growth rate r and carrying

capacity K. The question is: when should fishing resume? Let $x(t)$ be the fish population, $b(t)$ the number of fishing boats, and q be the catchability, given in units of per boat per time, so that the harvesting rate is $h(t) = qb(t)x(t)$.

a) If t_f is the time fishing is resumed, find differential equations that govern the fish population for $t > t_f$ and $t < t_f$.

b) Assume that the fleet is constant, $b(t) = B$, and non-dimensionalize the model in Part (a), choosing the growth rate as the time scale and the carrying capacity as the population scale.

c) Determine the (dimensionless) time τ that fishing begins as a function of the dimensionless fleet size $\beta = qB/r$, subject to the condition that the fish population should remain at constant for times greater than τ. Translate this relation into dimensioned form to find $t_f = t_f(B)$.

d) Under the conditions of Part (c), what is the optimum number of boats and the maximum sustainable catch, i.e., the maximum harvesting rate?

e) Suppose p is the selling price per fish, c_b is the cost of maintenance per boat per time, w is a fisherman's wage per time, and n is the number of fishermen per boat. Argue that the rate money is earned from the fishing is $ph(t) - b(t)(c_b + nw)$. If the value of money is discounted at rate R over time, conclude that the net profit can be written as a functional of the fleet size as

$$J(b(t)) = \int_0^\infty e^{-Rt}[ph(t) - b(t)(c_b + nw)]\, dt.$$

f) Under the condition of a constant fleet size, $b(t) = B$, at what time t_f should the fleet resume fishing to maximize the profit $J(B)$? What is the optimal fleet size?

g) Use parameter values $R = 0.7$, $p = 2$, $q = 0.5$, $K = 50$, $r = 0.55$, $x_0 = 2$, $c_b + nw = 2$, to plot $J(B)$ in Part (f), and find the maximum starting time, the optimal fleet size, and the maximum profit. (This exercise was adapted from Illner *et al* (2005).)

14. A pendulum of length l with a bob of mass m executes a (dimensionless) angular displacement $\theta = \theta(t)$ from its attachment point, with $\theta = 0$ when the pendulum is vertically downward. See Fig. 1.4. Use Newton's law to derive the equation of motion for the pendulum by noting that the acceleration is d^2s/dt^2, where $s = l\theta$ is the length of the circular arc traveled

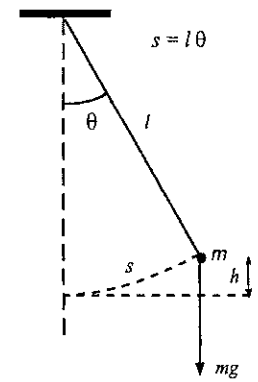

Figure 1.4 Pendulum.

by the bob; the force is $-mg$ vertically downward, and only the component tangential to the arc of oscillation affects the motion. The model is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

If the bob is released from a small angle θ_0 at time $t = 0$, formulate a dimensionless initial value problem describing the motion. (Observe that the radian measurement θ is dimensionless, but one can still scale θ by θ_0.)

15. The initial value problem for the damped pendulum equation is

$$\frac{d^2\theta}{dt^2} + k\frac{d\theta}{dt} + \frac{g}{l}\sin\theta = 0,$$
$$\theta(0) = \theta_0, \quad \theta'(0) = \omega_0.$$

a) Find three time scales and comment upon what process each involves. Which involves the period of undamped oscillations?

b) Non-dimensionalize the model with a time scale appropriate to expecting a small-amplitude, oscillatory behavior.

1.3 Differential Equations

The first subsection below contains a very brief review of elementary solution techniques for first- and second-order differential equations. The Exercises