Notes for a Third Edition

of

A COURSE IN FUNCTIONAL ANALYSIS

(Second edition, third printing)

by

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GTM 96

This is a list of additions for my book *A Course in Functional Analysis* (Second Edition, Second Printing). I have a separate list of corrections for the latest printing. If a third edition ever comes into existence (an unlikely event), these additions will likely find their way into it. The following mathematicians have helped me to compile this list: R B Burckel, Pei-Yuan Wu,

I would appreciate any corrections or comments you have.

Page	**Line**	**Comment**
12		The proof that (b) implies (d) is too complicated. Here is an easier one. The definition of continuity implies there is a $\delta > 0$ such that $\lvert L(h)\rvert \le 1$ whenever $\|h\| < \delta$. Thus for any non-zero vector h, $\lvert L(\delta h/\|h\|)\rvert \le 1$. This implies (d) with $c = 1/\delta$.
30		In Exercise 11, A^* is not defined until the next section.
69–70		A simpler proof of Theorem 3.1 is as follows. Let $\{e_1, \ldots, e_d\}$ be a Hamel basis for $\mathcal{X}$ and for each $x = \sum_j \xi_j e_j$ in $\mathcal{X}$, define $\|x\|_\infty = \max\{\lvert\xi_j\rvert : 1 \le j \le d\}$. It is easy to see that $\|.\|_\infty$ is a norm on $\mathcal{X}$. It will be shown that $\|.\|$ and $\|.\|_\infty$ are equivalent. If $f : \mathbb{F}^d \to \mathbb{R}$ is the function $f(\xi_1, \ldots, \xi_d) = \|\sum_j \xi_j e_j\|$, it is easy to show that f is continuous. Since $K \equiv \{\xi = (\xi_1, ..., \xi_d) \in \mathbb{F}^d : \max\{\lvert\xi_j\rvert : 1 \le j \le d\} = 1\}$ is a compact set, f attains its maximum and minimun values on K. Let α and β be points in K with $f(\alpha) \le f(\xi) \le f(\beta)$ for all ξ in K. If $a = f(\alpha)$ and $b = f(\beta)$, then for every $x = \sum_j \xi_j e_j$ in $\mathcal{X}$ with $\|x\|_\infty = 1$, we have that $a \le \|x\| \le b$. So if x is any nonzero vector in $\mathcal{X}$, $a \le \|\, x/\|x\|_\infty \,\| \le b$, or $a\|x\|_\infty \le \|x\| \le b\|x\|_\infty$. Thus the two norms are equivalent. ■
82–83		Another proof that Banach limits exist. Let $\mathcal{M} = \{x \in \ell^\infty : \lim_n n^{-1} \sum_{j=1}^n x(j) \text{ exists }\}$. It follows that $\mathcal{M}$ is a nonempty linear manifold in ℓ^∞. Define $f : \mathcal{M} \to \mathbb{F}$ by $f(x) = \lim_n n^{-1} \sum_{j=1}^n x(j)$. Clearly f is a linear functional and, almost as clearly, $\|f\| = 1$. By Corollary 6.8 there is a linear functional L on ℓ^∞ with $\|L\| = 1$ and $L(x) = f(x)$ for all x in $\mathcal{M}$. It is straightforward to check that L satisfies (a) and (b). The proof of (c) is as in the book. To prove (d), note that for any x in ℓ^∞, $n^{-1} \sum_{j=1}^n [x(j) - x(j+1)] = n^{-1}[x(1) - x(n+1)] \to 0$. Thus $x - x' \in \mathcal{M}$ and so $L(x - x') = f(x - x') = 0$. ■
92		Here is another proof that T^{-1} is not continuous. Let $\{\epsilon_i\}$ be a Hamel basis that contains the orthonormal basis $\{e_n\}$ of $\mathcal{X} = \ell^2$ and put $x_n = e_1 + ... + e_n$. So $\|x_n\|_1 = n$ and $\|x_n\| = n^{1/2}$. Hence $\|T^{-1}\| \ge \|x_n\|_1/\|x_n\| = n^{1/2}$.
96	-1	It suffices to assume that $\mathcal{Y}$ is a normed space.
97	6-8	The argument can be simplified as follows. If $x \in \mathcal{X}$, then $\|A_n x\| \le \|A_n\|\|x\| \le M\|x\|$. Letting $n \to \infty$ shows that $\|Ax\| \le M\|x\|$.
102	10–19	The proof of Proposition 1.11 can be simplified as follows. After defining c, let $V = \operatorname{int} A$. Note that $U \equiv b - \frac{1-t}{t}(V - a)$ is an open set containing b. Since $b \in \operatorname{cl} A$, $U \cap A \neq \emptyset$. Let $d \in U \cap A$ and put $W = td + (1-t)V$. Since A is convex, W is an open subset of A. Moreover the fact that $d \in U$ implies that $td \in tb - (1-t)(V - a) = tb + (1-t)a - (1-t)V = c - (1-t)V$. It follows that $c \in td + (1-t)V = W$. Hence $c \in \operatorname{int} A$.

109	8	We need that $\mathcal{M}$ is closed so it must be shown that f is continuous. Here is a proof. Since $f \leq q < 1$ on H, $f > -1$ on $-H$. Thus $\{x : \lvert f(x)\rvert < 1\}$ contains $H \cap (-H)$, an open neighborhood of 0. The linearity of f now shows that f is continuous at 0, hence everywhere.
171	Ex 8	Condition (b) follows from (a), so that (a) is necessary and sufficient for the boundedness of A.
222	Ex 1	State explicitly as part of the exercise the following. If $\mathcal{A}$ is any Banach algebra with identity and $h : \mathcal{A} \to \mathbb{C}$ is a nonzero homomorphism, then $\lVert h\rVert = 1$.
234	-7	Proposition 1.11(e) can be extended to normal elements as follows. Since $\lVert a^2\rVert^2 = \lVert a^{*^2} a^2\rVert = \lVert (a^*a)^*(a^*a)\rVert = \lVert a^*a\rVert^2 = \lVert a\rVert^4$, we have that $\lVert a^2\rVert = \lVert a\rVert^2$. Now continue as in the book.
288	4–13	This paragraph is reproving something and can be simplified as follows. Put $\phi = \phi_e$. Observe that $A - \phi(N) \in W^*(N)$ and $[A - \phi(N)]e - 0$. Since e is a separating vector for $W^*(N)$, $A - \phi(N) = 0$.