

Process-Driven Inference of Biological Network Structure: Feasibility, Minimality, and Multiplicity

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Abstract

A common problem in molecular biology is to use experimental data, such as microarray data, to infer knowledge about the structure of interactions between important molecules in subsystems of the cell. By approximating the state of each molecule as “on” or “off”, it becomes possible to simplify the problem, and exploit the tools of Boolean analysis for such inference. Amongst Boolean techniques, the *process-driven* approach has shown promise in being able to identify putative network structures, as well as stability and modularity properties. This paper examines the process-driven approach more formally, and makes four contributions about the computational complexity of the inference problem, under the “dominant inhibition” assumption of molecular interactions. The first is a proof that the feasibility problem (does there exist a network that explains the data?) can be solved in polynomial-time. Second, the minimality problem (what is the smallest network that explains the data?) is shown to be NP-hard, and therefore unlikely to result in a polynomial-time algorithm. Third, a simple polynomial-time heuristic is shown to produce near-minimal solutions, as demonstrated by simulation. Fourth, the theoretical framework explains how multiplicity (the number of network solutions to realize a given biological process), which can take exponential-time to compute, can instead be accurately estimated by a fast, polynomial-time heuristic.

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Introduction

A central theme in molecular biology is to understand the complex network of interactions between biomolecules and how those interactions contribute to higher-level biological function [1–6]. This problem belongs to the broader field of network analysis: graph theoretic representation of objects in physics, biology, sociology, etc; and the categorization and analysis of graph properties [7,8]. Many important network classes have been well studied, such as random networks [9], small-world networks [10], and scale-free networks [11]. Many important measures and concepts have been defined to characterize network structures, such as degree distribution [12], clustering coefficient [7], the Estrada index [8], entropy-based molecular descriptors [13–16], and network motif [17]. These structural approaches have been used in studying biological networks. For example, topological properties such as scale-free, power-law degree distribution [18] have been found for biomolecular networks. The implications of scale-freeness on the robustness and evolvability of genetic regulatory networks have been studied in [19].

Against the backdrop of the success of structural network analysis is the disappointing reality that for many subsystems in the cell, very little is known about the network structure. Instead, what is observable from data is whether or not the molecules are active at a given moment when a “snapshot” (such as a microarray sample) is taken. Given a collection of such snapshots that

represent the dynamics of the system, one major goal is to address the *network inference problem*: what possible network structures could have resulted in the given dynamics? The network inference problem is made challenging because of the variety of molecule-to-molecule interactions and by the sheer number of molecules in even small subsystems of the larger cell. As reviewed in [20], many important inference approaches have recently emerged, which include clustering-based (e.g., [21]), Boolean-based (e.g., [22]), Bayesian-based (e.g., [23]), *in silico*-based (e.g., [20]), and information-theory-based methods. For example, the algorithm C3NET, based on the estimation of mutual information, is a successful example of information-theory-based methods [24]. Boolean-based analysis is relatively simple and might be suitable to handle large-scale data [25–27]. In [28], the authors investigated the effects of discretisation methods, biological constraints, and stringency of Boolean function assignment on the performance of Boolean network, by using performance indices such as accuracy, precision, specificity and sensitivity. These performance indices indicate the correctness of the inferred network based upon the matches between the inference and reference networks. For example, sensitivity was defined as $TP/(TP+FN)$, where TP (true positive) is the number of network connections that were correctly predicted and FN (false negative) is the number of network connections that were deemed as disconnected. They found that biological constraints have pivotal influence on the network performance over the other factors.

In [29], the Boolean network inference problem has been made computationally tractable by some simplifying assumptions. The goal of this paper is to examine this computational complexity more formally. In particular, we show that the existence problem (is there a network that explains the dynamics?) can be solved quickly, in polynomial-time whereas the minimality problem (what is the smallest network that explains the dynamics?) is NP-hard. Fortunately, a simple heuristic, which while not guaranteed to find the minimal network, does appear to find near-minimal networks very efficiently. Furthermore, our formulation also sheds light on the exponential counting problem of *multiplicity* (also called *designability* by some authors), the number of networks that explain a given dynamics.

There are two simplifying assumptions we make in this paper. One of these is the Boolean molecule-state assumption adopted by most prior work in the literature on Boolean-network models [1,30–32]: at any given moment, a given biomolecule is either “on” (active or highly-expressed) or “off” (inactive or inhibited), and molecules stimulate or inhibit other molecules. The successive application of stimulatory or inhibitory interaction results in the next state for each molecule, and in this manner, the system evolves from state to state. The second assumption we make is the *dominant inhibition* assumption [30,33,34], in which any competition between inhibition and stimulation results in inhibition. These two assumptions allow an analysis that is entirely Boolean in nature and let us, after some algebraic reduction, exploit fast algorithms for the class of Boolean expressions called Horn formulas. The satisfiability of a Horn formula is known to be determinable in polynomial-time [35,36].

The complexity results in this paper imply that large systems can be solved efficiently, in polynomial-time, when the goal is to obtain solution networks or to obtain a characterization of the class of network solutions. On the other hand, when the goal is to find the smallest network, the problem remains hard; however, a simple heuristic approach appears to work well in practice although no theoretical guarantees can be made about the minimality of the heuristic solutions. In either case, both the existence problem and the minimization problem reveal useful characteristics of the network. These include the network “backbone,” a list of edges that must be present in all network solutions, as well as additional groups of edges called network modules or network motifs that are thought to be an important organizing principle for biological networks [37].

The results of the paper are organized as follows. In Section 1, we introduce the Boolean network model and process-driven analysis. In Section 2, we study the feasibility problem and its applications. In Section 3, we discuss the computational complexity of finding a minimal network and develop an algorithm to find an approximated minimal network. In Section 4, we develop a polynomial-time algorithm to estimate multiplicity. Technical details and illustrative examples are further provided in File S1.

Results

1. Boolean Network Model

The starting point for our model is a collection of N interacting molecules, each of which at any given time is modeled as either “on” or “off.” Let $s_i(t) \in \{0,1\}$ denote the state of molecule i and $S(t) = (s_1(t), \dots, s_N(t))$ the state of the system at time t . Here, the change of state is assumed to occur over a large enough time interval for biochemical reactions to complete, and thus time is assumed to be discrete: $t = 0, 1, 2, \dots, T-1$. A sequence of such system states, $\mathbf{S}^* = (S(0), S(1), \dots, S(T-1))$ is what we term a *Boolean process*. Intuitively, in biological terms, a Boolean process

corresponds to discretized time-course data. Thus, a sequence of microarray snapshots taken for a system of molecules over a time course can be converted into this Boolean form by noting which molecules are active and which are not.

In [3], Li *et al.* introduced a specific type of Boolean network model to determine the next state of a particular node i from the current state:

$$s_i(t+1) = \begin{cases} 1 & \sum_{j \neq i} a_{ji} s_j(t) > 0 \\ 0 & \sum_{j \neq i} a_{ji} s_j(t) < 0 \\ 0 & \sum_{j \neq i} a_{ji} s_j(t) = 0 \ \& \ a_{ii} = -1 \\ s_i(t) & \sum_{j \neq i} a_{ji} s_j(t) = 0 \ \& \ a_{ii} = 0, \end{cases} \quad (1)$$

where j ranges over $\{1, 2, \dots, N\}$. Each non-diagonal entry, a_{ji} ($j \neq i$), takes the value $-\gamma$, 1, or 0, depending on whether node j inhibits, stimulates, or does not interact with, node i . The diagonal entries, a_{ii} , take the value -1 (degradation) or 0 (no degradation). The inclusion of degradation allows the model to determine whether degradation plays an important role in the dynamics. Note: we have not explicitly included the case that node i is self-activating ($a_{ii} = 1$), because a self-activating node can be replaced by two different nodes with one activating the other.

The parameter γ models the relative dominance of inhibition over stimulation. Since we assume inhibition dominates stimulation for most biomolecular interactions, $\gamma \geq 1$ in our model. Simulation results reported in [29] show that for the budding yeast network, the cases $\gamma = 3, 4, 5, \dots, \infty$ produce exactly the same dynamics and are only slightly different from the cases $\gamma = 1, 2$; for the fission yeast network, the cases $\gamma = 2, 3, 4, \dots, \infty$ produce exactly the same dynamics and are only slightly different from the case $\gamma = 1$. We therefore follow the “dominant inhibition” assumption [30,33,34] by setting $\gamma = \infty$. This assumption renders a simpler, purely logical representation of Eq. (1), namely:

$$s_i(t+1) = \left(\sum_{j \neq i} (s_j(t) g_{ji}) + s_i(t) \bar{r}_{ii} \right) \prod_{j \neq i} (\bar{s}_j(t) \bar{r}_{ji}), \quad (2)$$

where addition represents the Boolean operator OR, multiplication represents AND, and the bar on a variable represents NOT. The Boolean variable r_{ji} represents a putative inhibitory edge (conventionally drawn in red color) from node j to node i . The Boolean variable g_{ji} represents a putative stimulatory edge (conventionally drawn in green color) from node j to node i . Note that each r or g is modulated by an s -variable because edge ji is active only when $s_j(t) = 1$. For Eq. (2), one additional constraint, $r_{ji} g_{ji} = 0$, is imposed to exclude solutions where an edge interaction is both inhibitory and stimulatory, a possibility absent in Eq. (1). However, under “dominant inhibition” condition, such a solution with an edge being both inhibitory and stimulatory produces exactly the same dynamics as if the edge were only inhibitory because the inhibition dominates over the stimulation. Thus this constraint will only come into play in Section 4 on solution multiplicity, and the Boolean variables r_{ji} and g_{ji} will be treated as independent for the rest of the paper.

To understand Eq. (2), consider just one of the four possible transitions for node i at some time step t , for example, $1 \rightarrow 1$. This transition requires either at least one active stimulatory edge, or no active self degradation if no active stimulatory edge is present, and no active inhibitory edge overall. These three conditions

correspond precisely to the three terms $\sum_{j \neq i} (s_j(t)g_{ji})$, $s_i(t)\bar{r}_{ii}$, and $\prod_{j \neq i} (\overline{s_j(t)r_{ji}})$ in Eq. (2) respectively. To prove Eq. (2), one similarly completes the analysis for the remaining three possible transitions of $1 \rightarrow 0$, $0 \rightarrow 1$, and $0 \rightarrow 0$.

2. Feasibility: Does a Solution Exist?

2.1. Simplification. Equation (2), while compact, is somewhat inconvenient for determining the feasibility of a Boolean process. By a simplification procedure (see File S1 A), Eq. (2) is instead converted into four equations each corresponding to a type of state transition of node i :

$$0 \rightarrow 1 : \sum_{j \in \theta_t(i)} g_{ji} = 1, \quad (3)$$

$$1 \rightarrow 1 : \sum_{j \in \theta_t(i)} g_{ji} + \bar{r}_{ii} = 1, \quad (4)$$

$$0 \rightarrow 0 : \prod_{j \in \theta_t(i)} \bar{g}_{ji} + \sum_{j \in \theta_t(i) \ominus} r_{ji} = 1, \quad (5)$$

$$1 \rightarrow 0 : r_{ii} \prod_{j \in \theta_t(i)} \bar{g}_{ji} + \sum_{j \in \theta_t(i) \ominus} r_{ji} = 1, \quad (6)$$

where $\theta_t(i) = \{j | s_j(t) = 1; j \neq i\}$ represents the set of nodes other than i that are active at time t , $\ominus = \{j | r_{ji} = 0; j \neq i\}$ represents the set of nodes that are found not inhibiting node i (see File S1 A as to how these nodes are identified), and $\theta_t(i) \ominus$ represents the set of nodes in $\theta_t(i)$ but with the ones in \ominus excluded.

2.2. Conjunctive normal form. Equations (3)–(6) represent a Boolean satisfiability (SAT) problem – the problem of deciding whether a setting of variables $\{r_{ji}, g_{ji}\}$ can satisfy a Boolean statement. The general satisfiability problem is usually given in the so-called Conjunctive Normal Form (CNF) [38]. A CNF formula or statement consists of a number of clauses in conjunction, and where each clause features variables or their complements with the OR operator. While a solution to the satisfiability problem of a general CNF statement is NP-complete [39], there are some types of CNF that lend themselves to a polynomial solution: a CNF in Horn-clause form can be solved in polynomial time [35,36].

First, note that the first two equations above are already in the form of CNF clauses. The other two are not because they feature products that are not allowed inside clauses. However, they can be multiplied out to create CNF clauses as follows. Treating the entire sum of r_{ji} 's with the symbol x , each equation is of the form.

$$y_1 y_2 \cdots y_n + x = (y_1 + x)(y_2 + x) \cdots (y_n + x), \quad (7)$$

where $n = |\theta_t(i)|$. Then, with $y_k = \bar{g}_{j(k),i}$ (for $k = 1, 2, \dots, n$) and $x = \sum_{j \in \theta_t(i) \ominus} r_{ji}$, one has

$$\prod_{j \in \theta_t(i)} \bar{g}_{ji} + \sum_{k \in \theta_t(i) \ominus} r_{ki} = \prod_{j \in \theta_t(i)} \left(\bar{g}_{ji} + \sum_{k \in \theta_t(i) \ominus} r_{ki} \right). \quad (8)$$

Note that the conversion takes $n < N$ steps, which is clearly polynomial. In summary, we have the following new forms of

equations:

$$0 \rightarrow 1 : \sum_{j \in \theta_t(i)} g_{ji} = 1, \quad (9)$$

$$1 \rightarrow 1 : \sum_{j \in \theta_t(i)} g_{ji} + \bar{r}_{ii} = 1, \quad (10)$$

$$s \rightarrow 0 : \bar{g}_{ji} + \sum_{j \in \theta_t(i) \ominus} r_{ji} = 1, \quad (11)$$

$$1 \rightarrow 0 : r_{ii} + \sum_{j \in \theta_t(i) \ominus} r_{ji} = 1, \quad (12)$$

where s can be either 0 or 1.

2.3. Horn formula. With the equations transformed into the conjunctive normal form, one can apply a SAT algorithm to determine whether the given Boolean process is feasible or not. Although a general SAT problem is NP-complete [40], some special class of SAT problem can be solved in polynomial time. For example, the 2-SAT problem (the number of literals in a clause is limited to 2), is a polynomial time problem [41]. Another important example, relevant to our case, is the HORNSAT problem.

In mathematical logic, a Horn clause is a clause with at most one positive literal. The positive literal is called the head and the negative literals form the body of the clause. The Horn clause form is named after the logician Alfred Horn, who first pointed out the significance of such clauses in [42]. Indeed, Horn clause forms have played a basic role in logic programming and are important for constructive logic [43–45]. A Horn formula is a propositional formula formed by the conjunction of Horn clauses. In formal logic, Horn-satisfiability, or HORNSAT, is the problem of deciding whether a given Horn formula is satisfiable.

Because HORNSAT can be determined in polynomial-time [35,36,45], this means that the network existence problem (for a given dynamics) can be solved in polynomial-time, as discussed below.

We now show how to convert the particular CNF formula for our Boolean network into a Horn formula. First, recall that we have formed a set of nodes $\ominus = \{j | r_{ji} = 0; j \neq i\}$. In the following, we let lowercase letters a, b, c, \dots denote nodes belonging to \ominus ; leave node i as it is; and let uppercase letters A, B, C, \dots denote the other nodes. With these new notations, the left hand side of Eqs (9)–(12) can be represented by the following clause forms:

$$g_{ai} + g_{bi} + g_{ci} + \cdots, \quad (13)$$

$$\bar{r}_{ii} + g_{ai} + g_{bi} + g_{ci} + \cdots, \quad (14)$$

$$\bar{g}_{di} + r_{Ai} + r_{Bi} + r_{Ci} + \cdots, \quad (15)$$

$$\bar{g}_{Di} + r_{Ai} + r_{Bi} + r_{Ci} + \cdots, \quad (16)$$

$$r_{ii} + r_{Ai} + r_{Bi} + r_{Ci} + \dots, \tag{17}$$

where expressions (13), (14), and (17) are from Eqs. (9), (10), and (12), respectively; expressions (15) and (16) are from Eq. (11). Note that the used subscripts do not correspond to the actual nodes; they only indicate to which set (Θ or not Θ) the nodes belong. For example, the subscript a in Eq. (13) does not necessarily equal the subscript a in Eq. (14), and similarly the subscript D in Eq. (16) is allowed to equal the subscript A in the same equation.

For the nodes $j = a, b, c, \dots$, we define new variables.

$$G_{ji} = \bar{g}_{ji}$$

to replace g_{ji} . That is, we shall use the four variables r_{ji} , \bar{r}_{ji} , G_{ji} , and \bar{G}_{ji} for $j = a, b, c, \dots$. For the nodes $j = A, B, C, \dots$, we define new variables

$$R_{ji} = \bar{r}_{ji}$$

to replace r_{ji} . That is, we shall use the four variables R_{ji} , \bar{R}_{ji} , g_{ji} , and \bar{g}_{ji} for $j = A, B, C, \dots$. The clauses (13)–(17) now turn into

$$\bar{G}_{ai} + \bar{G}_{bi} + \bar{G}_{ci} + \dots, \tag{18}$$

$$\bar{r}_{ii} + \bar{G}_{ai} + \bar{G}_{bi} + \bar{G}_{ci} + \dots, \tag{19}$$

$$G_{di} + \bar{R}_{Ai} + \bar{R}_{Bi} + \bar{R}_{Ci} + \dots, \tag{20}$$

$$\bar{g}_{Di} + \bar{R}_{Ai} + \bar{R}_{Bi} + \bar{R}_{Ci} + \dots, \tag{21}$$

$$r_{ii} + \bar{R}_{Ai} + \bar{R}_{Bi} + \bar{R}_{Ci} + \dots, \tag{22}$$

which are in Horn-clause form.

To summarize, the simplification (to CNF) and conversion steps result in a Horn formula that is solvable in polynomial time.

Theorem: *Given a Boolean process, the feasibility problem (the problem of determining whether or not the process can be realized by a network based on the model Eq. (2)), can be solved in polynomial time.*

File S1 B and C provide two examples for CNF and Horn formula conversions. A different proof of the above theorem using Eq. (2) directly is also given in File S1 D.

3. Finding a Minimal Network

Minimality is an important concept in studying genetic networks [46,47], biochemical networks [48], and cell signaling networks [49,50], for example. In the context of this paper, it refers to making the smallest number of positive assignments ($r_{ji} = 1$ or $g_{ji} = 1$) necessary for the Boolean equations to be satisfied. Because each such an assignment corresponds to an interaction edge in the biomolecular network, a minimal number of positive assignments corresponds to a minimal network — a network that can realize the same biological function but has the smallest number of edges [51]. For example, the budding yeast cell cycle process in [29] can be produced by 3.7×10^{30} networks, according to the present

model Eq. (2). Among these network solutions, there are 40,300 networks that only have 23 edges. The remaining networks all have more than 23 edges. The 40,300 networks are therefore minimal networks because they have the smallest number of edges. The minimality of the budding yeast cell cycle process is therefore 23.

It is important to identify these minimal networks. First, the smallest network that “explains” a biological process helps identify the core relationships between molecules that *must be present* for the biological process to function. Second, in a real biological network, the remaining edges beyond the minimal network often confer some functionality orthogonal to the biological process. For example, for the cell-cycle process, the non-minimal edges were found to confer stability properties [29]. Therefore, identifying minimal networks may well be the key to understanding the important elements of a biological process.

In general, a minimal network solution is difficult to find by chance, because the minimal solutions occupy only a tiny fraction of the whole solution space (e.g., 40,320 out of 3.7×10^{30} for the case of budding yeast cell cycle process). Perhaps for this reason, a large number of algorithms have been developed in the area of learning sparse Boolean functions [48,52–55]. These algorithms usually take minimality as a constraint in learning.

From the computational complexity viewpoint of this paper, one would like to know whether it is even computationally feasible to compute a minimal solution in reasonable time. Some early studies [25,27,56] have considered and established the connection between the minimal Boolean network inference problem and the minimal set cover (MSC) problem. For the present Boolean model, we show that the minimal network problem is of equal complexity to the MSC problem, which is well-known to be NP-complete. The comparison also enables the development of a fast heuristic that, when evaluated with randomly-generated processes, produces near-minimal solutions.

3.1. Minimal set covering (the MSC problem). The Minimal Set Cover (MSC) problem is a classical question in computer science and complexity theory [57,58]. It is a problem “whose study has led to the development of fundamental techniques for the entire field” of approximation algorithms [59], and it was one of Karp’s 21 core NP-complete problems shown to be NP-complete in his landmark 1972 paper [60]. It can be informally described as follows. Given a collection of M sets S_1, \dots, S_M , each is a subset of the integers $\{0, \dots, K-1\}$, what is the minimum number of sets whose union contains all the integers $0, \dots, K-1$? For example, if the collection of sets ($M=8, K=4$) is: $\{S_1 = \{1\}, S_2 = \{0,1\}, S_3 = \{1,3\}, S_4 = \{0\}, S_5 = \{0,2\}, S_6 = \{2,3\}, S_7 = \{2\}, S_8 = \{3\}\}$. Note that sets S_1, S_3, S_5, S_7 together cover the elements, that is, $S_1 \cup S_3 \cup S_5 \cup S_7 = \{0,1,2,3\}$. However, it is a cover of size 4. On the other hand, $S_2 = \{0,1\}$ and $S_6 = \{2,3\}$ also form a cover: $S_2 \cup S_6 = \{0,1,2,3\}$. No single set contains all the elements, and thus the minimal set cover is of size 2.

Next, we explain how an MSC problem can be expressed in Boolean form, using the example above. Consider the Boolean equations.

$$\text{Element 0 : } x_2 + x_4 + x_5 = 1,$$

$$\text{Element 1 : } x_1 + x_2 + x_3 = 1,$$

$$\text{Element 2 : } x_5 + x_6 + x_7 = 1,$$

Element 3 : $x_3 + x_6 + x_8 = 1.$ (23)

Here, the variable $x_j = 1$ if set S_j is part of a set cover. Thus, the first line above says that at least one of the sets S_2, S_4, S_5 needs to be in the cover in order for element 0 to be included. Since all elements need to be included, a solution to the CNF expression.

$$(x_2 + x_4 + x_5)(x_1 + x_2 + x_3)(x_5 + x_6 + x_7)(x_3 + x_6 + x_8) = 1$$

results in a cover. For a minimal cover, we want as few variables $x_j = 1$ as possible.

3.2. The minimal-network problem is at least as hard as the MSC problem. In the standard approach [58], to demonstrate that problem A is hard, one shows that a known hard problem B can be transformed into A . In this manner, a putative fast (polynomial-time) algorithm for A must also be able to solve B quickly and since B is known to be hard, therefore A must be hard. The transformation from B to A needs to occur in polynomial-time so that the entire process takes polynomial-time.

We now show how an arbitrary instance of an MSC problem can be transformed into an equivalent minimal-network problem in polynomial time so that a putative fast algorithm for the minimal network problem would have to solve the MSC problem in polynomial-time. The transformation is best explained via an example – we use the example MSC problem above in Eq. (23).

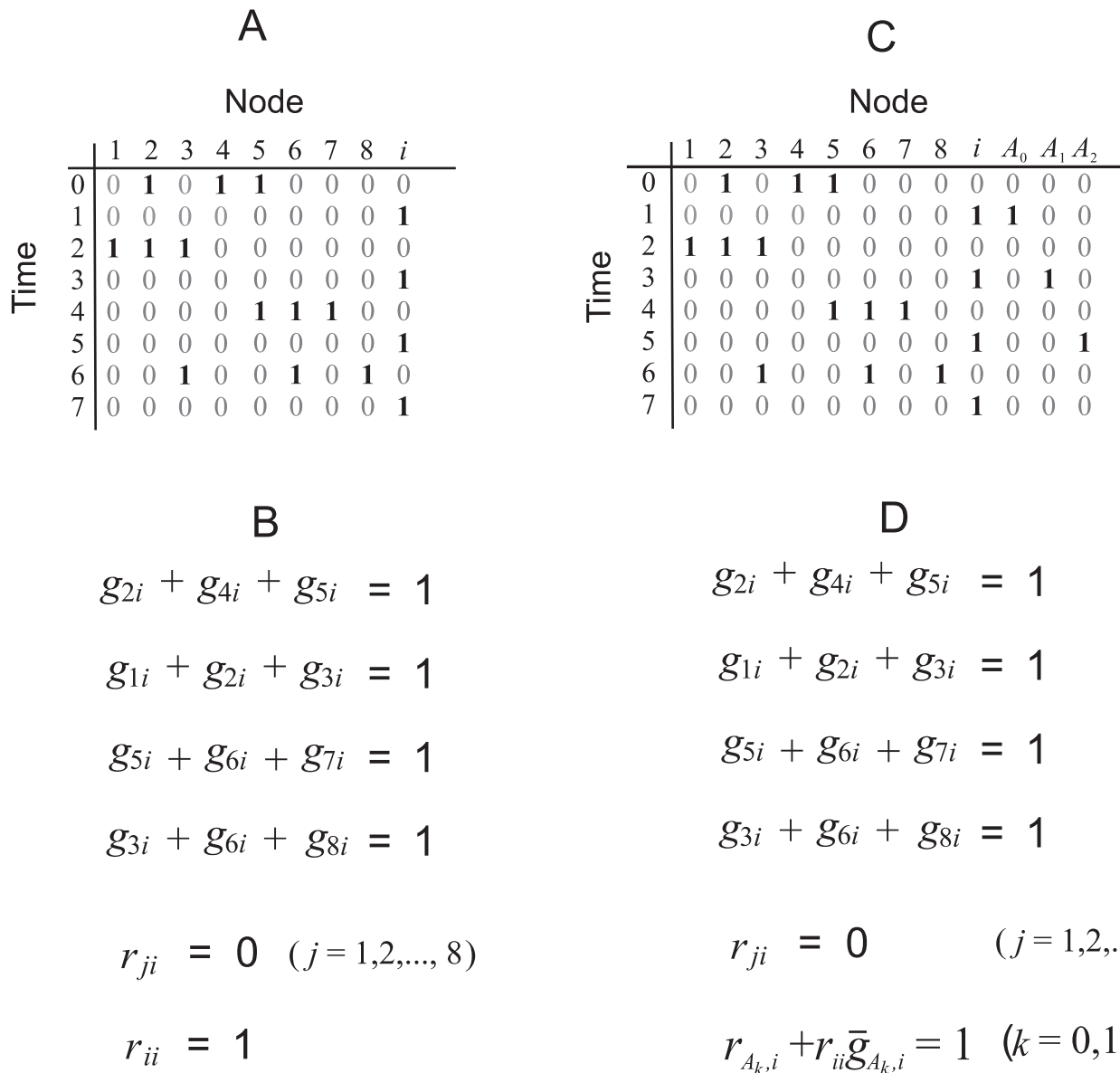


Figure 1. The transformation of a minimal set covering problem into the problem of finding a minimal biological network for a given Boolean process. (A) A Boolean process which does not always have a network solution. (B) The Boolean equations for network edges incoming to node i , according to the process in (A). (C) A Boolean process that is guaranteed to have a network solution. (D) The Boolean equations for network edges incoming to node i , according to the process in (C).
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We need to find a feasible Boolean biological process whose equations match Eq. (23).

Figure 1(A) shows a process with $M+1=9$ nodes and $2K=8$ time steps $0, 1, \dots, 2K-1$. The $(M+1)$ -th column is for a molecule called i , whose transitions will define the desired Boolean process. At time $t=2k$ ($k=0, 1, \dots, K-1$), we set $s_j=1$ for those j appearing in the equation for element k . For example, the equation for set element 3 is $x_3+x_6+x_8=1$; we therefore set $s_3=s_6=s_8=1$ in the row 6 of the process. As for node i , we set $s_i=1$ at rows $t=2k+1$ ($k=0, 1, \dots, K-1$).

Next, consider the Boolean equations for node i , shown in Fig. 1(B). The first four equations are exactly Eq. (23) after replacing g_{ji} with x_j . Notice that in the column for i , whenever node i makes a $0 \rightarrow 1$ transition, the other nodes in the row that are “on” cannot have an inhibitory edge. Thus, $r_{ji}=0$ for these nodes. Similarly, the only way in which the $1 \rightarrow 0$ transition can occur is due to self-degradation, i.e., $r_{ii}=1$.

At first, it appears that this transformation is sufficient. However, notice that nodes $1, \dots, 8$ need to make a $0 \rightarrow 1$ transition in alternate rows. Unfortunately, the transitions rely solely on the activations from node i , which would result in conflicts. Figure 1(A) indicates that node 3 makes a $0 \rightarrow 1$ transition at $t=5$, and therefore there must be a stimulatory edge from node i to node 3 ($g_{i3}=1$). Therefore, this stimulatory edge must have activated node 3 at time $t=3$ (i.e., $s_3(4)=1$), which contradicts the Boolean process and which shows $s_3(4)=0$.

Based on the process in Fig. 1(A), we construct a new process in Fig. 1(C) whose feasibility is guaranteed. The process has $K-1$ additional nodes, denoted by A_k (for $k=0, 1, \dots, K-2$). The node A_k is responsible solely for all the activations at time $t=2k+1$. File S1 E explains why the resulting process is always feasible.

If the minimal network problem for the Boolean process in Fig. 1(C) were solved in polynomial-time, then the equations in Fig. 1(D) and consequently Eq. (23) could be solved in polynomial-time. This would imply that the MSC problem can be solved in polynomial-time and that the NP = P problem is solved in the affirmative. Therefore, it is unlikely that there is a polynomial-time algorithm to find a minimal network.

Theorem: *Given a Boolean process, the problem of finding a minimal network (a network with the smallest number of edges) to realize it, based on the model Eq. (2), is NP-complete.*

3.3. A heuristic algorithm for finding an approximated minimal network. While the association with the cover problem proved that the problem is hard, it also suggests a heuristic. It is known that a simple “greedy” algorithm for the set-cover problem is effective in practice and in fact can be shown to result in covers within $\log(N)$ of optimal [61]. The greedy algorithm for set-cover works as follows: first select the set that covers the most elements. Then, mark the covered elements as “covered.” Then, select the set that covers the most uncovered elements, and repeat in this fashion. The greedy algorithm was used in various biological fields including the inference of sparse Boolean functions [56,62–64].

For our minimal network problem, we first apply the greedy algorithm to only the equations containing stimulatory edges, which approximately identifies a set of minimal stimulatory edges. The results simplify the remaining equations, which contain only inhibitory edges. By applying the greedy algorithm again, an approximated minimal network is found. We consider separately the two cases of $r_{ii}=1$ and $r_{ii}=0$. The algorithm is described as follows.

1. // Start with the first node
 $i \leftarrow 1$
2. // The case $r_{ii} = 1$
 $r_{ii} \leftarrow 1$
Use the greedy algorithm to find a set of stimulatory edges
Use the greedy algorithm to find a set of inhibitory edges
3. // The case $r_{ii} = 0$
 $r_{ii} \leftarrow 0$
Use the greedy algorithm to find a set of stimulatory edges
Use the greedy algorithm to find a set of inhibitory edges
4. Choose the smaller edge set from steps 2 and 3 above for node i
5. // Move onto the next node
 $i \leftarrow i + 1$
if $i > N$ goto 6
else goto 2
endif
6. Stop

3.4. Validation of the algorithm for finding an approximated minimal network.

To test the efficacy of the greedy heuristic, we constructed 90,635 feasible processes, each with $N=11$ and $T=12$. For each of these 90,635 processes, we used the algorithm to seek an approximated minimal network. As a comparison, we also compute the actual minimal networks, by time-consuming brute force enumerations, whereby we can assess how far off from optimal the heuristic’s solutions are. The results are shown in Fig. 2(A). One sees that the approximated minimal network for each of the 68,058 processes (75.1% of total) is a genuine minimal network. The approximated minimal network for each of 18,862 processes (20.8% of total) has only one more edge than the genuine minimal network. Only 4% processes have two or more edges than minimal.

We also constructed 21,000 feasible processes each with $N=15$ and $T=16$. The results for these are shown in Fig. 2(B). One sees that the approximated minimal network for each of the 9030 processes (43% of total) is a genuine minimal network. The approximated minimal network for each of the 7329 processes (34.9% of total) has one more edge than the genuine minimal network. Only 22% processes give deviations more than one edge. We also studied many $N=9$ and $T=10$ processes, with the results presenting in Fig. 2(C), together with the other results.

The above numerical results indicate that the heuristic algorithm is very accurate, with high confidence for at most one additional edge than optimal.

File S1 F and G further provide examples of finding minimal networks.

4. Designability: How Many Solutions?

We consider one more computational aspect of Boolean networks: computing the *designability* of a biological process. The designability of a process is the number of network solutions, that is, the total number of networks that can produce a given process. A process with high designability is likely to be more robust and therefore survive natural selection. If a biological process has only one network solution, for example, then it is unlikely that random mutation would find that network. In other words, processes with high designability are more likely to occur (or be discovered by evolution) than low-designability ones [5,65].

Unfortunately, computing the designability of a process appears to take exponential time, a result that is not unexpected because the problem is similar to many such combinatorial enumeration

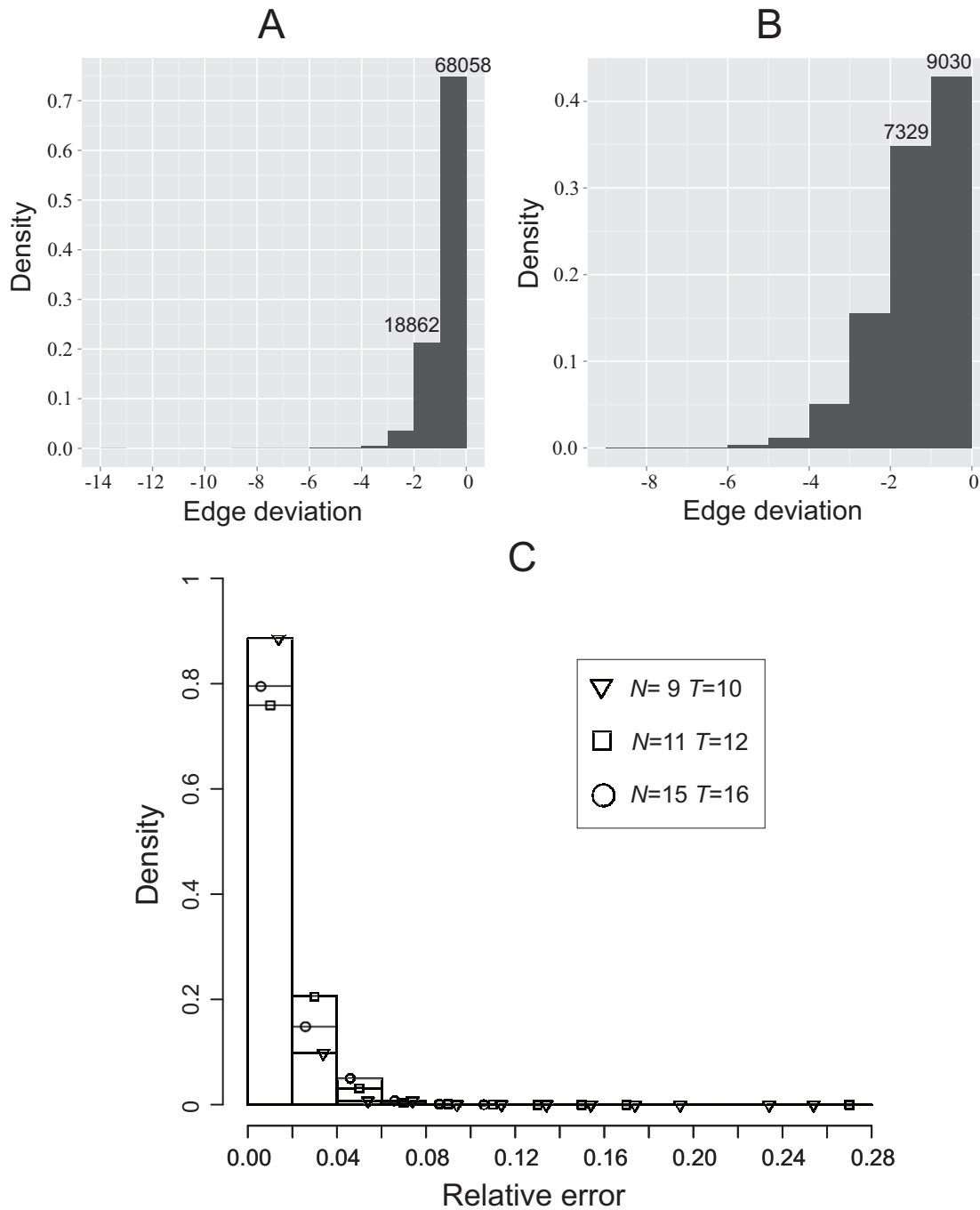


Figure 2. Validation of a heuristic algorithm for finding an approximated minimal network. A large number of feasible Boolean processes were generated. For each process, an approximated minimal network was determined by the heuristic algorithm; the actual minimal network is also computed by time-consuming enumeration. The histogram shows how the number of processes spread over the deviation of estimation (actual minimality minus estimated minimality). (A) 90,635 processes with $N = 11$ and $T = 12$. (B) 21,000 processes with $N = 15$ and $T = 16$. (C) Processes with $N = 9$ and $T = 10$ as compared with other processes. doi:10.1371/journal.pone.0040330.g002

problems [66]. Instead, we seek a fast approximate algorithm. Our approach is to use a type of logistic regression on certain features of a problem. These features are obtained during the simplification procedure described earlier in the feasibility solution.

During simplification, it is easy to obtain the following: x_r , the number of inhibitory edges ($r_{ji} = 1$); x_g , the number of stimulatory edges ($g_{ji} = 1$); $x_{\bar{r}}$, the number of edges that cannot be inhibitory

($\bar{r}_{ji} = 1$); $x_{\bar{g}}$, the number of edges that cannot be stimulatory ($\bar{g}_{ji} = 1$); x_0 , the number of nodes that cannot connect to node i ($\bar{r}_{ji}\bar{g}_{ji} = 1$); x_1 , the number of nodes that can be associated with node i in arbitrary way (inhibitory, stimulatory, or “no connection”).

We now assume that the approximated log-designability (denoted by D) is a linear function of x_k :

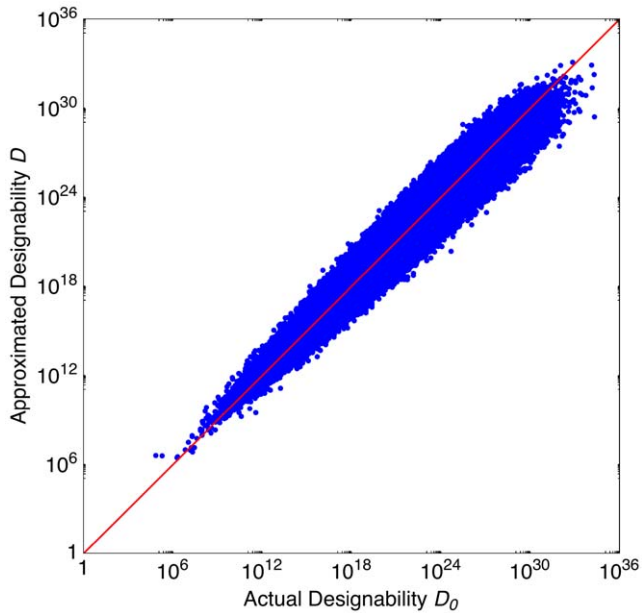


Figure 3. Designability approximation. The 800,000 dots in the picture each represent a random Boolean process. For each dot, the actual designability D_0 and the approximated designability D are represented by the x axis value and the y axis value, respectively. The diagonal corresponds to the ideal fitting.
doi:10.1371/journal.pone.0040330.g003

$$\ln D = b + \sum_k a_k x_k, \quad (24)$$

where $k \in \{r, g, \bar{r}, \bar{g}, 0, 1\}$, b and a_k are unknown coefficients. The coefficients can be estimated by minimizing the difference between the approximated designability D and the exact designability D_0 for some test cases. To this end, we generate 800,000 random Boolean process, each of which has $N = 11$ and $T = 12$. For each process, we calculate the exact designability D_0 and the values of x_k , for $k \in \{r, g, \bar{r}, \bar{g}, 0, 1\}$. We then perform a least-square data fitting to obtain the values of the coefficients. The results are: $b = 64.0796$, $a_r = -0.458234$, $a_g = -0.302389$, $a_{\bar{r}} = -0.0477189$, $a_{\bar{g}} = -0.0368327$, $a_0 = -0.572591$, and $a_1 = 0.398622$.

In Fig. (3), each dot represents one Boolean process; the x and y axes represent D_0 and D , respectively. One sees that the fitting is successful. The dots locate along the diagonal with only small deviations. Therefore, we now have an empirical formula to estimate designability with sensible accuracy. Although it is not very accurate to assume N - and T - independent regression coefficients and use them to estimate designability, the estimation can still be very useful because it becomes very difficult to compute the exact designability for large N and T .

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Discussion

Biological systems are generally large, involving many complicated molecular interactions among numerous working components. In this case, a “coarse grained” description such as a Boolean network model is often a useful step towards understanding a system. Furthermore, a study of general Boolean networks may help elucidate network design principles in biological systems. The Boolean framework in this paper results in a single analytical equation (Eq. 2) to integrate the information about network structure (edge variables) and biological process (state variables). This has rendered the structure-function relationship tractable. When structural information is known, one can use Eq. (2) to study the dynamics and learn the biological function. When the process is known, one can use Eq. (2) to characterize the network space constrained by the process. When both information are partially known, Eq. (2) can still help enumerate the structure-function combinations.

In this paper, we have answered some key questions about the computational complexity of the network inference problem in Boolean networks that feature dominant inhibition. The first is the feasibility problem: is the solution space null? The second is the minimal network problem: what are the fundamental building blocks of the space, namely those networks with the least number of edges? The third is the designability problem: how big is the solution space if it is not null?

Fast algorithms provide the benefit of being able to study many types of processes and to explore the statistics of processes. For example, one is interested in what general features of a process make a process a biological one suited to evolution. Is it the case, for example, that a small minimal network acts as the core with additional edges accumulated during evolution, and does a large multiplicity help maintain stability against mutation?

Beyond a rigorous classification of problem complexity, the present study also offers accurate heuristic algorithms that run in polynomial time. This will be crucial for handling both large-scale datasets and the inevitable statistical noise. The former requires a well-controlled scaling as observed in these polynomial heuristics and the latter requires iterative applications of these algorithms for noise sampling on a trial-and-error basis.

Supporting Information

File S1 A collection of technical details.
(PDF)

Author Contributions

Conceived and designed the experiments: GW CZ RS YR. Performed the experiments: GW. Analyzed the data: GW HC CD CP. Contributed reagents/materials/analysis tools: GW. Wrote the paper: GW RS CZ.

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Supporting Information

A. Substantiation of Eq. (2) according to the four type of state transitions

Equation (2) is in a compact form, which might be inconvenient for various applications. By defining $\theta_t(i) = \{ j \mid s_j(t) = 1; j \neq i \}$, the set of active nodes other than i at time t , Eq. (2) is re-written into four equations each correspond to a type of state transition of node i :

$$0 \rightarrow 1 : \quad \sum_{j \in \theta_t(i)} g_{ji} \cdot \prod_{j \in \theta_t(i)} \bar{r}_{ji} = 1, \quad (\text{S.1})$$

$$1 \rightarrow 1 : \quad \left(\sum_{j \in \theta_t(i)} g_{ji} + \bar{r}_{ii} \right) \cdot \prod_{j \in \theta_t(i)} \bar{r}_{ji} = 1, \quad (\text{S.2})$$

$$0 \rightarrow 0 : \quad \sum_{j \in \theta_t(i)} g_{ji} \cdot \prod_{j \in \theta_t(i)} \bar{r}_{ji} = 0, \quad (\text{S.3})$$

$$1 \rightarrow 0 : \quad \left(\sum_{j \in \theta_t(i)} g_{ji} + \bar{r}_{ii} \right) \cdot \prod_{j \in \theta_t(i)} \bar{r}_{ji} = 0. \quad (\text{S.4})$$

As a simplification, we perform complementation on Eqs. (S.3) and (S.4) so that the right hand sides of all the equations are 1.

$$0 \rightarrow 1 : \quad \sum_{j \in \theta_t(i)} g_{ji} \cdot \prod_{j \in \theta_t(i)} \bar{r}_{ji} = 1, \quad (\text{S.5})$$

$$1 \rightarrow 1 : \quad \left(\sum_{j \in \theta_t(i)} g_{ji} + \bar{r}_{ii} \right) \cdot \prod_{j \in \theta_t(i)} \bar{r}_{ji} = 1, \quad (\text{S.6})$$

$$0 \rightarrow 0 : \quad \prod_{j \in \theta_t(i)} \bar{g}_{ji} + \sum_{j \in \theta_t(i)} r_{ji} = 1, \quad (\text{S.7})$$

$$1 \rightarrow 0 : \quad r_{ii} \prod_{j \in \theta_t(i)} \bar{g}_{ji} + \sum_{j \in \theta_t(i)} r_{ji} = 1. \quad (\text{S.8})$$

From Eqs. (S.5) and (S.6) one immediately finds $r_{ji} = 0$ for some nodes j . By sweeping all the equations ($t = 0, 1, \dots, T - 1$), a set $\Theta = \{ j \mid r_{ji} = 0; j \neq i \}$ is formed to collect all such nodes. Equations (S.7) and (S.8) can be further simplified by substituting $r_{ji} = 0$ (for

$j \in \Theta$) into them:

$$0 \rightarrow 1 : \quad \sum_{j \in \theta_t(i)} g_{ji} = 1, \quad (\text{S.9})$$

$$1 \rightarrow 1 : \quad \sum_{j \in \theta_t(i)} g_{ji} + \bar{r}_{ii} = 1, \quad (\text{S.10})$$

$$0 \rightarrow 0 : \quad \prod_{j \in \theta_t(i)} \bar{g}_{ji} + \sum_{j \in \theta_t(i) \setminus \Theta} r_{ji} = 1, \quad (\text{S.11})$$

$$1 \rightarrow 0 : \quad r_{ii} \prod_{j \in \theta_t(i)} \bar{g}_{ji} + \sum_{j \in \theta_t(i) \setminus \Theta} r_{ji} = 1, \quad (\text{S.12})$$

where the set $\theta_t(i) \setminus \Theta$ contains the nodes in $\theta_t(i)$ but with the ones in Θ excluded.

B. Horn formula representation of Boolean equations for a small process

Table S1: A small Boolean process with $N = 4$ nodes and $T = 4$ time steps.

Time	s_1	s_2	s_3	s_4
0	1	1	0	0
1	1	0	1	0
2	0	0	0	1
3	0	0	0	1

A Boolean process for $N = 4$ nodes and $T = 4$ time steps is given in Table S1. We focus on node $i = 4$ and outline the simplification procedure and the conversion to Horn-clause form in four steps below.

B.1. Boolean equations

$$\begin{aligned}\bar{g}_{1i}\bar{g}_{2i} + r_{1i} + r_{2i} &= 1 \\ (g_{1i} + g_{3i})\bar{r}_{1i}\bar{r}_{3i} &= 1 \\ \bar{r}_{ii} &= 1\end{aligned}$$

B.2. Simplified equations

$$r_{1i} = r_{3i} = r_{ii} = 0$$

and

$$\begin{aligned}\bar{g}_{1i}\bar{g}_{2i} + r_{2i} &= 1 \\ g_{1i} + g_{3i} &= 1\end{aligned}$$

B.3. CNF for the simplified equations

$$(\bar{g}_{1i} + r_{2i})(\bar{g}_{2i} + r_{2i})(g_{1i} + g_{3i}) = 1 \quad (\text{S.13})$$

B.4. Horn formula

From the above expression one sees that nodes 1 and 3 belong to the set $\Theta = \{a, b, c, \dots\}$, node 2 belongs to the set $\{A, B, C, \dots\}$, thus $\bar{g}_{1i} = G_{1i}$ ($g_{1i} = \bar{G}_{1i}$), $g_{3i} = \bar{G}_{3i}$, and $r_{2i} = \bar{R}_{2i}$ as described in Section 3.3. The Horn formula of the above CNF is

$$(G_{1i} + \bar{R}_{2i})(\bar{g}_{2i} + \bar{R}_{2i})(\bar{G}_{1i} + \bar{G}_{3i}) = 1. \quad (\text{S.14})$$

C. Horn formula representation of Boolean equations for a large process

Table S2: A Boolean process with $N = 22$ nodes and $T = 15$ time steps.

Time	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	s_9	s_{10}	s_{11}	s_{12}	s_{13}	s_{14}	s_{15}	s_{16}	s_{17}	s_{18}	s_{19}	s_{20}	s_{21}	s_{22}
0	0	0	1	1	1	1	0	1	1	0	1	1	1	1	1	1	1	0	0	1	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0
2	0	1	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	1	0	1	1	0
3	0	0	0	0	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
4	0	0	0	1	1	0	0	1	1	1	0	1	1	0	0	0	1	0	0	0	1	1
5	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0	0
6	0	0	0	0	1	0	0	1	0	0	0	1	0	1	0	0	0	0	1	0	1	1
7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	1	0
8	0	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0
9	0	0	0	0	1	0	1	1	0	0	0	1	0	0	0	0	0	0	0	1	0	0
10	0	0	0	0	1	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	0	1
11	1	0	0	0	1	1	0	1	1	0	0	1	1	0	1	0	0	1	0	0	1	1
12	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
13	0	1	0	0	0	0	0	0	1	0	0	0	1	0	1	0	0	1	0	1	1	0
14	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0

A larger Boolean process than the previous example has $N = 22$ nodes and $T = 15$ time steps as shown in Table S2. We focus on node $i = 5$. Similarly, we work out the simplification procedure and the conversion to Horn-clause form in four steps below.

C.1. Boolean equations

$$\begin{aligned}
& r_{ii}\bar{g}_{3i}\bar{g}_{4i}\bar{g}_{6i}\bar{g}_{8i}\bar{g}_{9i}\bar{g}_{11,i}\bar{g}_{12,i}\bar{g}_{13,i}\bar{g}_{14,i}\bar{g}_{15,i}\bar{g}_{16,i}\bar{g}_{17,i}\bar{g}_{20,i} + r_{3i} + r_{4i} + r_{6i} + \\
& r_{8i} + r_{9i} + r_{11,i} + r_{12,i} + r_{13,i} + r_{14,i} + r_{15,i} + r_{16,i} + r_{17,i} + r_{20,i} = 1 \\
& \bar{g}_{18,i}\bar{g}_{21,i} + r_{18,i} + r_{21,i} = 1 \\
& \bar{r}_{2i}\bar{r}_{7i}\bar{r}_{10,i}\bar{r}_{18,i}\bar{r}_{20,i}\bar{r}_{21,i} (g_{2i} + g_{7i} + g_{10,i} + g_{18,i} + g_{20,i} + g_{21,i}) = 1 \\
& \bar{r}_{8i}\bar{r}_{20,i}(\bar{r}_{ii} + g_{8i} + g_{20,i}) = 1 \\
& \bar{r}_{4i}\bar{r}_{8i}\bar{r}_{9i}\bar{r}_{10,i}\bar{r}_{12,i}\bar{r}_{13,i}\bar{r}_{17,i}\bar{r}_{21,i}\bar{r}_{22,i} \\
& (\bar{r}_{ii} + g_{4i} + g_{8i} + g_{9i} + g_{10,i} + g_{12,i} + g_{13,i} + g_{17,i} + g_{21,i} + g_{22,i}) = 1 \\
& \bar{r}_{1i}\bar{r}_{13,i}\bar{r}_{18,i}(\bar{r}_{ii} + g_{1i} + g_{13,i} + g_{18,i}) = 1 \\
& r_{ii}\bar{g}_{8i}\bar{g}_{12,i}\bar{g}_{14,i}\bar{g}_{19,i}\bar{g}_{21,i}\bar{g}_{22,i} + r_{8i} + r_{12,i} + r_{14,i} + r_{19,i} + r_{21,i} + r_{22,i} = 1 \\
& \bar{g}_{4i}\bar{g}_{15,i}\bar{g}_{18,i}\bar{g}_{21,i} + r_{4i} + r_{15,i} + r_{18,i} + r_{21,i} = 1 \\
& \bar{r}_{2i}\bar{r}_{3i}\bar{r}_{7i}\bar{r}_{18,i}\bar{r}_{21,i} (g_{2i} + g_{3i} + g_{7i} + g_{18,i} + g_{21,i}) = 1 \\
& \bar{r}_{7i}\bar{r}_{8i}\bar{r}_{12,i}\bar{r}_{20,i}(\bar{r}_{ii} + g_{7i} + g_{8i} + g_{12,i} + g_{20,i}) = 1 \\
& \bar{r}_{8i}\bar{r}_{12,i}\bar{r}_{13,i}\bar{r}_{22,i}(\bar{r}_{ii} + g_{8i} + g_{12,i} + g_{13,i} + g_{22,i}) = 1 \\
& r_{ii}\bar{g}_{1i}\bar{g}_{6i}\bar{g}_{8i}\bar{g}_{9i}\bar{g}_{12,i}\bar{g}_{13,i}\bar{g}_{15,i}\bar{g}_{18,i}\bar{g}_{21,i}\bar{g}_{22,i} + r_{1i} + r_{6i} + \\
& r_{8i} + r_{9i} + r_{12,i} + r_{13,i} + r_{15,i} + r_{18,i} + r_{21,i} + r_{22,i} = 1 \\
& \bar{g}_{21,i} + r_{21,i} = 1 \\
& \bar{g}_{2i}\bar{g}_{9i}\bar{g}_{13,i}\bar{g}_{15,i}\bar{g}_{18,i}\bar{g}_{20,i}\bar{g}_{21,i} + r_{2i} + r_{9i} + r_{13,i} + r_{15,i} + r_{18,i} + r_{20,i} + r_{21,i} = 1. \quad (\text{S.15})
\end{aligned}$$

C.2. Simplified equations

$$\begin{aligned}
& r_{1i} = r_{2i} = r_{3i} = r_{4i} = r_{7i} = r_{8i} = r_{9i} = r_{10,i} = r_{12,i} = \\
& r_{13,i} = r_{17,i} = r_{18,i} = r_{20,i} = r_{21,i} = r_{22,i} = g_{18,i} = g_{21,i} = 0
\end{aligned}$$

and

$$\begin{aligned}
& r_{ii}\bar{g}_3\bar{g}_4\bar{g}_6\bar{g}_8\bar{g}_9\bar{g}_{11,i}\bar{g}_{12,i}\bar{g}_{13,i}\bar{g}_{14,i}\bar{g}_{15,i}\bar{g}_{16,i}\bar{g}_{17,i}\bar{g}_{20,i} \\
& \quad + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i} = 1 \\
& \quad \quad g_{2i} + g_{7i} + g_{10,i} + g_{20,i} = 1 \\
& \quad \quad \quad \bar{r}_{ii} + g_{8i} + g_{20,i} = 1 \\
& \bar{r}_{ii} + g_{4i} + g_{8i} + g_{9i} + g_{10,i} + g_{12,i} + g_{13,i} + g_{17,i} + g_{22,i} = 1 \\
& \quad \quad \quad \bar{r}_{ii} + g_{1i} + g_{13,i} = 1 \\
& \quad \quad r_{ii}\bar{g}_8\bar{g}_{12,i}\bar{g}_{14,i}\bar{g}_{19,i}\bar{g}_{22,i} + r_{14,i} + r_{19,i} = 1 \\
& \quad \quad \quad \bar{g}_4\bar{g}_{15,i} + r_{15,i} = 1 \\
& \quad \quad \quad g_{2i} + g_{3i} + g_{7i} = 1 \\
& \quad \quad \quad \bar{r}_{ii} + g_{7i} + g_{8i} + g_{12,i} + g_{20,i} = 1 \\
& \quad \quad \quad \bar{r}_{ii} + g_{8i} + g_{12,i} + g_{13,i} + g_{22,i} = 1 \\
& \quad \quad r_{ii}\bar{g}_1\bar{g}_6\bar{g}_8\bar{g}_9\bar{g}_{12,i}\bar{g}_{13,i}\bar{g}_{15,i}\bar{g}_{22,i} + r_{6i} + r_{15,i} = 1 \\
& \quad \quad \quad \bar{g}_2\bar{g}_9\bar{g}_{13,i}\bar{g}_{15,i}\bar{g}_{20,i} + r_{15,i} = 1. \tag{S.16}
\end{aligned}$$

C.3. CNF for the left hand of the simplified equations

$$\begin{aligned}
& (r_{ii} + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i}) (\bar{g}_{3i} + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i}) \\
& (\bar{g}_{4i} + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i}) (\bar{g}_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i}) \\
& (\bar{g}_{8i} + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i}) (\bar{g}_{9i} + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i}) \\
& (\bar{g}_{11,i} + r_{6i} + r_{14,i} + r_{15,i} + r_{16,i}) (\bar{g}_{12,i} + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i}) \\
& (\bar{g}_{13,i} + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i}) (\bar{g}_{14,i} + r_{6i} + r_{11,i} + r_{15,i} + r_{16,i}) \\
& (\bar{g}_{15,i} + r_{6i} + r_{11,i} + r_{14,i} + r_{16,i}) (\bar{g}_{16,i} + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i}) \\
& (\bar{g}_{17,i} + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i}) (\bar{g}_{20,i} + r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i}) \\
& (g_{2i} + g_{7i} + g_{10,i} + g_{20,i}) (\bar{r}_{ii} + g_{8i} + g_{20,i}) (\bar{r}_{ii} + g_{1i} + g_{13,i}) \\
& (\bar{r}_{ii} + g_{4i} + g_{8i} + g_{9i} + g_{10,i} + g_{12,i} + g_{13,i} + g_{17,i} + g_{22,i}) \\
& (r_{ii} + r_{14,i} + r_{19,i}) (\bar{g}_{8i} + r_{14,i} + r_{19,i}) (\bar{g}_{12,i} + r_{14,i} + r_{19,i}) (\bar{g}_{14,i} + r_{19,i}) \\
& (\bar{g}_{19,i} + r_{14,i}) (\bar{g}_{22,i} + r_{14,i} + r_{19,i}) (\bar{g}_{4i} + r_{15,i}) \bar{g}_{15,i} (g_{2i} + g_{3i} + g_{7i}) \\
& (\bar{r}_{ii} + g_{7i} + g_{8i} + g_{12,i} + g_{20,i}) (\bar{r}_{ii} + g_{8i} + g_{12,i} + g_{13,i} + g_{22,i}) \\
& (r_{ii} + r_{6i} + r_{15,i}) (\bar{g}_{1i} + r_{6i} + r_{15,i}) (\bar{g}_{6i} + r_{15,i}) (\bar{g}_{8i} + r_{6i} + r_{15,i}) \\
& (\bar{g}_{9i} + r_{6i} + r_{15,i}) (\bar{g}_{12,i} + r_{6i} + r_{15,i}) (\bar{g}_{13,i} + r_{6i} + r_{15,i}) (\bar{g}_{15,i} + r_{6i}) \\
& (\bar{g}_{22,i} + r_{6i} + r_{15,i}) (\bar{g}_{2i} + r_{15,i}) (\bar{g}_{9i} + r_{15,i}) (\bar{g}_{13,i} + r_{15,i}) \bar{g}_{15,i} (\bar{g}_{20,i} + r_{15,i}). \quad (\text{S.17})
\end{aligned}$$

C.4. Horn formula

From C.2 above, one finds that

$$\Theta = \{1, 2, 3, 4, 7, 8, 9, 10, 12, 13, 17, 18, 20, 21, 22\}$$

For nodes in Θ , we use variables such as $(r_{3i}, \bar{r}_{3i}, G_{3i}, \bar{G}_{3i})$. For node i , we still use r_{ii} . For the other nodes, we use variables such as $(R_{6i}, \bar{R}_{6i}, g_{6i}, \bar{g}_{6i})$. We thus turn the above CNF

into the following Horn formula:

$$\begin{aligned}
& (r_{ii} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) (G_{3i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) \\
& (G_{4i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) (\bar{g}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) \\
& (G_{8i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) (G_{9i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) \\
& (\bar{g}_{11,i} + \bar{R}_{6i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) (G_{12,i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) \\
& (G_{13,i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) (\bar{g}_{14,i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) \\
& (\bar{g}_{15,i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{16,i}) (\bar{g}_{16,i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i}) \\
& (G_{17,i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) (G_{20,i} + \bar{R}_{6i} + \bar{R}_{11,i} + \bar{R}_{14,i} + \bar{R}_{15,i} + \bar{R}_{16,i}) \\
& (\bar{G}_{2i} + \bar{G}_{7i} + \bar{G}_{10,i} + \bar{G}_{20,i}) (\bar{r}_{ii} + \bar{G}_{8i} + \bar{G}_{20,i}) (\bar{r}_{ii} + \bar{G}_{1i} + \bar{G}_{13,i}) \\
& (\bar{r}_{ii} + \bar{G}_{4i} + \bar{G}_{8i} + \bar{G}_{9i} + \bar{G}_{10,i} + \bar{G}_{12,i} + \bar{G}_{13,i} + \bar{G}_{17,i} + \bar{G}_{22,i}) \\
& (r_{ii} + \bar{R}_{14,i} + \bar{R}_{19,i}) (G_{8i} + \bar{R}_{14,i} + \bar{R}_{19,i}) (G_{12,i} + \bar{R}_{14,i} + \bar{R}_{19,i}) (\bar{g}_{14,i} + \bar{R}_{19,i}) \\
& (\bar{g}_{19,i} + \bar{R}_{14,i}) (G_{22,i} + \bar{R}_{14,i} + \bar{R}_{19,i}) (G_{4i} + \bar{R}_{15,i}) \bar{g}_{15,i} (\bar{G}_{2i} + \bar{G}_{3i} + \bar{G}_{7i}) \\
& (\bar{r}_{ii} + \bar{G}_{7i} + \bar{G}_{8i} + \bar{G}_{12,i} + \bar{G}_{20,i}) (\bar{r}_{ii} + \bar{G}_{8i} + \bar{G}_{12,i} + \bar{G}_{13,i} + \bar{G}_{22,i}) \\
& (r_{ii} + \bar{R}_{6i} + \bar{R}_{15,i}) (G_{1i} + \bar{R}_{6i} + \bar{R}_{15,i}) (\bar{g}_{6i} + \bar{R}_{15,i}) (G_{8i} + \bar{R}_{6i} + \bar{R}_{15,i}) \\
& (G_{9i} + \bar{R}_{6i} + \bar{R}_{15,i}) (G_{12,i} + \bar{R}_{6i} + \bar{R}_{15,i}) (G_{13,i} + \bar{R}_{6i} + \bar{R}_{15,i}) (\bar{g}_{15,i} + \bar{R}_{6i}) \\
& (G_{22,i} + \bar{R}_{6i} + \bar{R}_{15,i}) (G_{2i} + \bar{R}_{15,i}) (G_{9i} + \bar{R}_{15,i}) (G_{13,i} + \bar{R}_{15,i}) \bar{g}_{15,i} (G_{20,i} + \bar{R}_{15,i}) \cdot (\text{S.18})
\end{aligned}$$

D. The conversion of Eq. (2) into CNF form to solve feasibility problems

During the conversion one often encounters clauses like $\sum_{j=1}^n x_j y_j$, which can be converted into CNF form by Eq. (7) but requires exponential number of clauses (2^n). It is known, however, that there exists a transformation into CNF that avoids exponential increase in size but only preserves the satisfiability of $\sum_j x_j y_j = 1$ as the following

$$\left(\sum_j z_j \right) \prod_j (\bar{z}_j + x_j) \prod_j (\bar{z}_j + y_j) = 1, \quad (\text{S.19})$$

where z_j (for $j = 1, 2, \dots, n$) are auxiliary variables. Moreover, it is known that the equation $x = y$ is equivalent to

$$(x + \bar{y})(\bar{x} + y) = 1. \quad (\text{S.20})$$

Equations (S.19) and (S.20) allow us to convert Eq. (2) into the following CNF form

$$ABCDEFGHI = 1 \quad (\text{S.21})$$

where

$$A = \sum_j y_j \quad (\text{S.22})$$

$$B = \prod_{j \neq i} (\bar{y}_j + s_j(t)) \quad (\text{S.23})$$

$$C = \prod_{j \neq i} (\bar{y}_j + r_{ji}) \quad (\text{S.24})$$

$$D = \prod_{j \neq i} (\bar{y}_i + \bar{s}_j(t) + \bar{g}_{ji} + s_i(t+1)) \quad (\text{S.25})$$

$$E = \bar{y}_i + \bar{s}_i(t) + r_{ii} + s_i(t+1) \quad (\text{S.26})$$

$$F = \sum_j z_j + \bar{s}_i(t+1) \quad (\text{S.27})$$

$$G = \prod_j (\bar{z}_j + s_j(t) + \bar{s}_i(t+1)) \quad (\text{S.28})$$

$$H = \prod_{j \neq i} (\bar{z}_j + g_{ji} + \bar{s}_i(t+1)) \quad (\text{S.29})$$

$$I = \bar{z}_i + \bar{r}_{ii} + \bar{s}_i(t+1) \quad (\text{S.30})$$

and where y_j and z_j are auxiliary variables.

When the state variables $\{s_j(t)\}$ are unknown, the satisfiability of Eq. (S.21) can be in general very difficult to determine. However, if all the state variables $\{s_j(t)\}$ are known,

the satisfiability of Eq. (S.21) for the unknown edge interaction variables $\{r_{ji}\}$ and $\{g_{ji}\}$ including the auxiliary variables can be solved in polynomial time. This is because, after the substitution of the values of $\{s_j(t)\}$, only terms of the following form are left:

$$\sum_j y_j \quad (\text{S.31})$$

$$\bar{y}_j \ (j \neq i) \quad (\text{S.32})$$

$$\bar{y}_j + r_{ji} \ (j \neq i) \quad (\text{S.33})$$

$$\bar{y}_i + \bar{g}_{ji} \ (j \neq i) \quad (\text{S.34})$$

$$\bar{y}_i + r_{ii} \quad (\text{S.35})$$

$$\sum_j z_j \quad (\text{S.36})$$

$$\bar{z}_j \ (j \neq i) \quad (\text{S.37})$$

$$z_i \quad (\text{S.38})$$

$$\bar{z}_j + g_{ji} \ (j \neq i) \quad (\text{S.39})$$

$$\bar{z}_i + \bar{r}_{ii} \quad (\text{S.40})$$

For $j \neq i$, define new variables $Y_j = \bar{y}_j$, $Z_j = \bar{z}_j$, $R_{ji} = \bar{r}_{ji}$, and $G_{ji} = \bar{g}_{ji}$. The above terms turn into

$$\sum_{j \neq i} \bar{Y}_j + y_i \quad (\text{S.41})$$

$$Y_j \ (j \neq i) \quad (\text{S.42})$$

$$Y_j + \bar{R}_{ji} \ (j \neq i) \quad (\text{S.43})$$

$$\bar{y}_i + G_{ji} \ (j \neq i) \quad (\text{S.44})$$

$$\bar{y}_i + r_{ii} \quad (\text{S.45})$$

$$\sum_{j \neq i} \bar{Z}_j + z_i \quad (\text{S.46})$$

$$Z_j \ (j \neq i) \quad (\text{S.47})$$

$$z_i \quad (\text{S.48})$$

$$Z_j + \bar{G}_{ji} \ (j \neq i) \quad (\text{S.49})$$

$$\bar{z}_i + \bar{r}_{ii} \quad (\text{S.50})$$

which are in Horn-clause form because each term has at most one positive literal.

E. The proof that the Boolean process in Fig. 1(C) is feasible

We prove the feasibility of the Boolean process by actually constructing an explicit network solution (Fig. S1). We first give the necessary edge connections and ascertain that these connections are not contradictory. We then focus on each node to demonstrate that it follows the state transitions as prescribed by the process shown in Fig. 1(C).

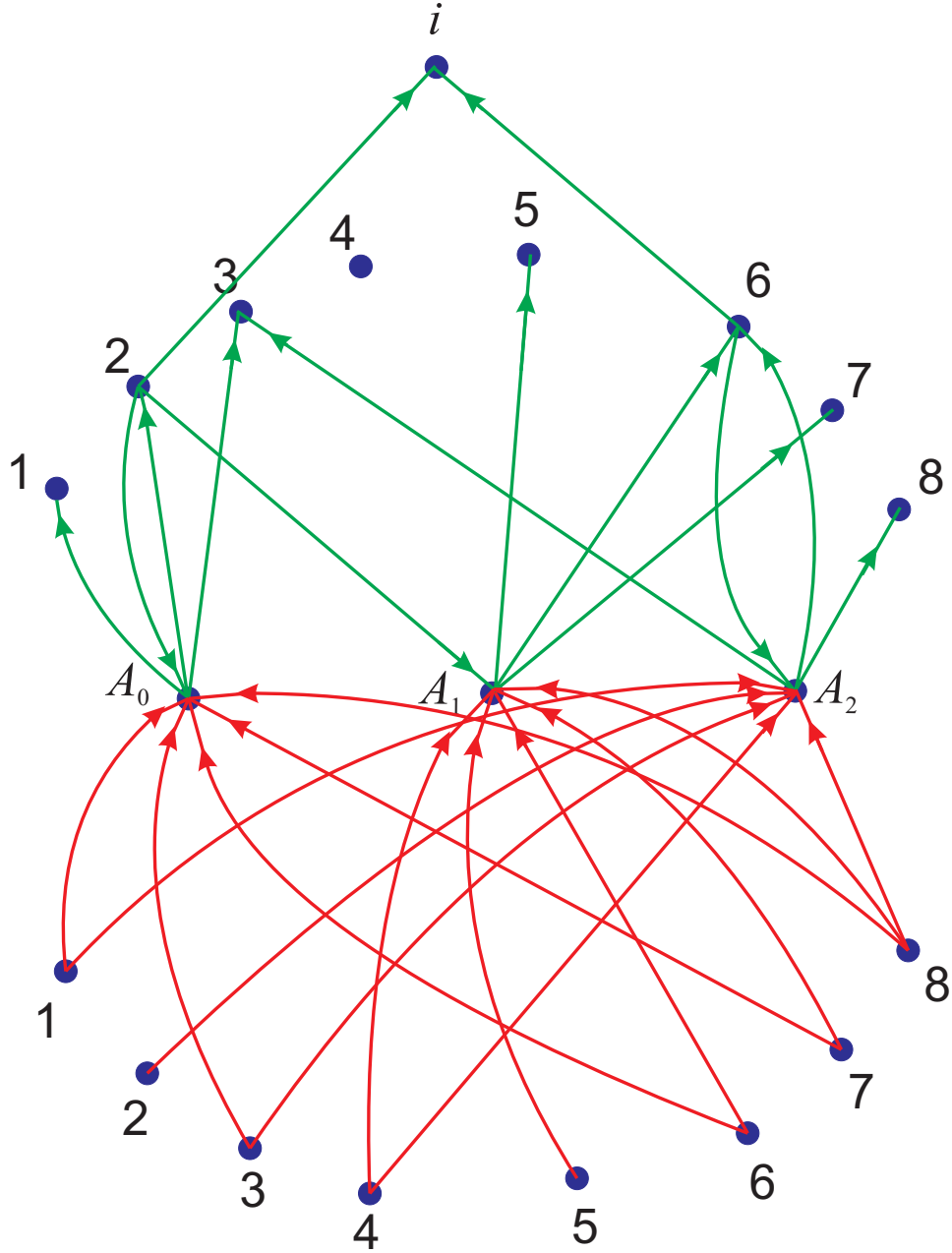


Fig. S1: An example network that can realize the Boolean process in Fig. 1(C). All the nodes are self-degradative. The red (green) arrows represent activation (inhibition). To avoid clutter, nodes $1, 2, 3, \dots, 8$ are duplicated to separate the red arrows from the green ones.

First, we specify all necessary network connections as follows.

1. All the nodes are self-degradative. That is, every node turns off at the next time step if there is no external stimulations.

2. There are no connections among nodes A_0 , A_1 , A_2 , and i .
3. Let θ_τ ($\tau = 0, 1, 2, 3$) denote the set of nodes $j \in \{1, 2, \dots, 8\}$ with $s_j(2\tau) = 1$. That is, $\theta_0 = \{2, 4, 5\}$, $\theta_1 = \{1, 2, 3\}$, $\theta_2 = \{5, 6, 7\}$, and $\theta_3 = \{3, 6, 8\}$. It is required that for each τ , there is at least one node in θ_τ stimulate(s) node i .
4. For a given $\tau \in \{0, 1, 2\}$, node A_τ receives stimulation(s) from at least one node in the set θ_τ . It also receives inhibitions from **all** the nodes $j \in \{1, 2, \dots, 8\}$ excluding θ_τ (denoted by $\bar{\theta}_\tau$).
5. For a given $\tau \in \{0, 1, 2\}$, node A_τ stimulates **all** the nodes in $\theta_{\tau+1}$.

The network in Fig. S1 satisfies the above connection requirements.

Second, we generate the dynamics according to the above network connections, starting from the initial states as given by the $t = 0$ row in Fig. 1(C). At time $t = 1$, nodes i and A_0 are turned on because of the stimulations from θ_0 ; the nodes in θ_0 themselves become 0 because of degradation. At time $t = 2$, the nodes in θ_1 are all turned on by node A_0 ; nodes i and A_0 become 0 because of degradation. Continue in this manner until the entire process is completed.

It is crucial that A_τ is turned on once and only once, for any given τ . Otherwise it would play the same role as node i and would have the same problem experienced by the process in Fig. 1(A). Our network connections guarantee that A_τ is turned on once and only once. Note that A_τ is inhibited by all the $\bar{\theta}_\tau$ nodes. In the following times defined by $t = 2\tau'$ with $\tau' > \tau$, there always exists at least one node $j \in \bar{\theta}_\tau$ that is active, which certainly prevents the re-activation of node A_τ . Otherwise, one has $\bar{\theta}_\tau \subseteq \bar{\theta}_{\tau'}$, namely $\theta_\tau \supseteq \theta_{\tau'}$. This fact implies that the Boolean equation $\sum_{j \in \theta_\tau} x_j = 1$ becomes redundant in terms of $\sum_{j \in \theta_{\tau'}} x_j = 1$. Therefore, $\sum_{j \in \theta_\tau} x_j = 1$ should have been removed in the first place.

Even though we focused on an explicit example, the construction procedure outlined in Section 4.2 and this section works for any minimal set covering problem instance.

F. Use the heuristic algorithm to find a minimal network for the Boolean process in Table S2.

The Boolean process is the same as the one in Supporting Information Section C. We still focus on node $i = 5$.

F.1. The case $r_{ii} = 1$:

In Eq. (S.16), those only containing stimulatory edges are as follows:

$$\begin{aligned}
 g_{2i} + g_{7i} + g_{10,i} + g_{20,i} &= 1, \\
 g_{8i} + g_{20,i} &= 1, \\
 g_{4i} + g_{8i} + g_{9i} + g_{10,i} + g_{12,i} + g_{13,i} + g_{17,i} + g_{22,i} &= 1, \\
 g_{1i} + g_{13,i} &= 1, \\
 g_{2i} + g_{3i} + g_{7i} &= 1, \\
 g_{7i} + g_{8i} + g_{12,i} + g_{20,i} &= 1, \\
 g_{8i} + g_{12,i} + g_{13,i} + g_{22,i} &= 1.
 \end{aligned}$$

By applying the greedy algorithm, we obtain minimum solutions $(g_{1i} \ g_{2i} \ g_{8i})$ and $(g_{2i} \ g_{8i} \ g_{13,i})$. Note that $(g_{1i} \ g_{2i} \ g_{8i})$ not only represents $g_{1i} = g_{2i} = g_{8i} = 1$, but also implies $g_{ji} = 0$ for $j \neq \{1, 2, 8\}$. Since $(g_{1i} \ g_{2i} \ g_{8i})$ renders the remaining equations simpler than $(g_{2i} \ g_{8i} \ g_{13,i})$ does, we choose $(g_{1i} \ g_{2i} \ g_{8i})$. After simplification, the remaining equations for inhibitory edges are:

$$\begin{aligned}
 r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i} &= 1, \\
 r_{14,i} + r_{19,i} &= 1, \\
 r_{6i} + r_{15,i} &= 1, \\
 r_{15,i} &= 1.
 \end{aligned}$$

By applying the greedy algorithm again, we obtain a minimal solution $(r_{14,i} \ r_{15,i})$. Therefore, the minimal solution for the case $r_{ii} = 1$ is $(g_{1i} \ g_{2i} \ g_{8i} \ r_{14,i} \ r_{15,i})$.

F.2. The case $r_{ii} = 0$:

In Eq. (S.16), those only containing stimulatory edges are as follows:

$$g_{2i} + g_{7i} + g_{10,i} + g_{20,i} = 1$$

$$g_{2i} + g_{3i} + g_{7i} = 1$$

By applying the greedy algorithm, we obtain minimum solutions (g_{2i}) and (g_{7i}) . We choose (g_{7i}) since it renders a shorter form for the remaining equations:

$$r_{6i} + r_{11,i} + r_{14,i} + r_{15,i} + r_{16,i} = 1,$$

$$r_{14,i} + r_{19,i} = 1,$$

$$r_{6i} + r_{15,i} = 1.$$

By applying the greedy algorithm again, we obtain a minimal solution $(r_{6i} \ r_{14,i})$. Therefore, the minimal solution for the case $r_{ii} = 0$ is $(g_{7i} \ r_{6i} \ r_{14,i})$.

Comparing the two cases, we obtain the final minimal solution $(g_{7i} \ r_{6i} \ r_{14,i})$ for node i .

complete the other aspects of analysis, we wrote the Boolean equations and simplified them. By using Eq. (24) and the information obtained during simplification, the approximately obtained designability was 5.27×10^{27} , compared with the exact value 3.7×10^{30} .

Since the original network is known, the construction of an approximated minimal network should use edges only existed in the original network. Let r_{ji} be an edge belonging to the original network but r_{ki} not. Even though the assignment $r_{ki} = 1$ satisfies more equations than the assignment $r_{ji} = 1$, one should still select r_{ji} . As a result, this modified greedy algorithm will always yield a subnetwork of the original network, and, according to Section 3, the subnetwork has a high probability to be a genuine minimal network. If the original network contains one and only one minimal network, then the modified greedy algorithm has a high probability to obtain the genuine minimal network, although it was designed to seek an approximated one.

We applied this modified greedy algorithm many times and found that it always yields the *same* subnetwork (Fig. S2(C)). Therefore, the subnetwork may well be a genuine minimal network and it is highly likely that the original network contains no other minimal networks. Indeed, by enumerating the Boolean equations, we found that the biological process in Fig. S2(A) has in total 40,320 minimal networks, each of which has 23 edges. Among the minimal networks there is one and only one contained in the budding yeast network (Fig. S2(B)), which is exactly what was found by the modified greedy algorithm, Fig. S2(C).

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