Online Appendix for

“Portfolio Selection with Mental Accounts and Background Risk”

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This Appendix contains proofs of the theoretical results in the paper “Portfolio Selection with Mental Accounts and Background Risk” published in the Journal of Banking and Finance 36, 968–980, April 2012.

The following three results are useful in our proof of Theorem 1.

**Lemma 1.** Fix an account \( m \in \{1, \ldots, M\} \) and a level of expected return \( E \in \mathbb{R} \) for it. The portfolio that minimizes account \( m \)'s variance subject to the restriction that the account has an expected return of \( E \) is given by:

\[
w_E \equiv w_m + \phi_E (w_1 - w_0)
\]

where \( \phi_E = \frac{E - E_m}{BCA} \). Furthermore, we have:

\[
\sigma[r_{w,m}] = \sqrt{\sigma_m^2 + \frac{(E[r_{w,m}] - E_m)^2}{DC}.
\]

**Proof.** Fix an account \( m \in \{1, \ldots, M\} \) and a level of expected return \( E \in \mathbb{R} \) for it. The portfolio that minimizes account \( m \)'s variance subject to the restriction that the account has an expected return of \( E \) solves:

\[
\min_{w \in \mathbb{R}^N} \frac{1}{2} (w' \Sigma w + \Omega_{mm} + 2w' \Psi_m)
\]

\[
s.t. \quad w'1 = 1
\]

\[
w' \mu = E - \nu_m.
\]

A first-order condition for \( w_E \) to solve problem (18) subject to constraints (19) and (20) is:

\[
\Sigma w_E + \Psi_m - \varphi_1 1 - \varphi_2 \mu = 0,
\]
where \( \mathbf{0} \) is the \( N \times 1 \) vector \([0 \cdots 0]'\), and \( \varphi_1 \) and \( \varphi_2 \) are multipliers associated to these constraints.

Using Eq. (21), we have:

\[
w_E = \varphi_1 \Sigma^{-1} \mathbf{1} + \varphi_2 \Sigma^{-1} \mu - \Sigma^{-1} \Psi_m. \tag{22}\]

Premultiplying Eq. (21) by \( \mathbf{1}' \) and using Eq. (19), we obtain:

\[
1 = \varphi_1 C + \varphi_2 A - A_m. \tag{23}\]

Premultiplying Eq. (21) by \( \mu' \) and using Eq. (20), we obtain:

\[
E - \nu_m = \varphi_1 A + \varphi_2 B - B_m, \tag{24}\]

where \( B_m \equiv \mu' \Sigma^{-1} \Psi_m \). Eqs. (23) and (24) imply that:

\[
\varphi_1 = \frac{1 + A_m - \varphi_2 A}{C}, \tag{25}\]

and

\[
\varphi_2 = \frac{E - (1 + A_m) A/C - \nu_m + B_m}{B - A^2/C}. \tag{26}\]

Noting that \( E_m = (1 + A_m) \frac{A}{C} + \nu_m - B_m \), Eq. (16) follows from Eqs. (22), (25), and (26), and the definitions of \( \mathbf{w}_m \), \( \mathbf{w}_0 \), and \( \mathbf{w}_1 \). Using Eq. (22), we have:

\[
\sigma[w'_{w,E,m}] = \sqrt{\frac{2}{\varphi_1^2 C + 2\varphi_1 \varphi_2 A + \varphi_2^2 B + \Omega_{mm} - C_m}}, \tag{27}\]

where \( C_m \equiv \Psi_m' \Sigma^{-1} \Psi_m \). Eqs. (25) and (27) imply that:

\[
\sigma[w'_{w,E,m}] = \sqrt{\left(\frac{1 + A_m - \varphi_2 A}{C}\right)^2 C + 2 \left(\frac{1 + A_m - \varphi_2 A}{C}\right) \varphi_2 A + \varphi_2^2 B + \Omega_{mm} - C_m}. \tag{28}\]

Using Eq. (28) and elementary algebra, we have:

\[
\sigma[w'_{w,E,m}] = \sqrt{\frac{(1 + A_m)^2}{C} + \Omega_{mm} - C_m + \varphi_2^2 \left( B - \frac{A^2}{C} \right)}. \tag{29}\]

Noting that \( \sigma_m^2 = \frac{(1 + A_m)^2}{C} + \Omega_{mm} - C_m \), Eq. (17) follows from Eqs. (26) and (29). \( \Box \)

**Lemma 2.** If \( \alpha_m < \Phi(-\sqrt{D/C}) \), then \( V[1 - \alpha_m, r_{w,m,m}] = -H_{\alpha_m} \).

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Proof. Suppose that $\alpha_m < \Phi(-\sqrt{D/C})$. Using Eq. (4), $w_m$ minimizes account $m$’s variance subject to the restriction that the account has an expected return of $E[r_{w,m,m}]$. Lemma 1 implies that $E[r_{w,m,m}]$ solves:

$$\min_{E \in \mathbb{R}} \ z_m \sqrt{\sigma_m^2 + \frac{(E - E_m)^2}{D/C}} - E.$$  
(30)

A first-order condition for $E[r_{w,m,m}]$ to solve problem (30) is:

$$z_m \frac{(E[r_{w,m,m}] - E_m)/(D/C)}{\sqrt{\sigma_m^2 + (E[r_{w,m,m}] - E_m)^2/(D/C)}} - 1 = 0.$$  
(31)

It follows from Eq. (31) that:

$$E[r_{w,m,m}] = \sqrt{(D/C)^2 \sigma_m^2 + E_m^2}.$$  
(32)

Using Eqs. (17) and (32), we have:

$$\sigma[r_{w,m,m}] = \sqrt{z_m^2 \sigma_m^2 - D/C}.$$  
(33)

Eqs. (4), (32), and (33) imply the desired result. \(\square\)

Lemma 3. Fix any account $m \in \{1, \ldots, M\}$ with $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$. The optimal portfolio within account $m$ is given by $w_m = w_E$ for some $E \in \mathbb{R}$ with $E > E_m$. Furthermore, we have $V[1 - \alpha_m, r_{w,m,m}] = -H_m$.

Proof. Fix any account $m \in \{1, \ldots, M\}$ with $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$. First, we show that $w_m = w_E$ for some $E \in \mathbb{R}$. Suppose by way of a contradiction that $w_m \neq w_E$, where $E = E[r_{w,m,m}]$. It follows from Lemma 1 that $\sigma[r_{w,E,m}] < \sigma[r_{w,m,m}]$. Since $E[r_{w,E,m}] = E[r_{w,m,m}]$ and $\sigma[r_{w,E,m}] < \sigma[r_{w,m,m}]$, Eq. (4) implies that:

$$V[1 - \alpha_m, r_{w,E,m}] < V[1 - \alpha_m, r_{w,m,m}].$$  
(34)

Fix any $E_1 \in \mathbb{R}$ with $E_1 > E[r_{w,m,m}]$. Let $\varepsilon > 0$ be arbitrarily small. Consider portfolio $w_\varepsilon \equiv \varepsilon w_{E_1} + (1 - \varepsilon) w_E$. Note that:

$$E[r_{w,\varepsilon,m}] > E[r_{w,E,m}].$$  
(35)
Since \( \varepsilon \) is arbitrarily small, Eq. (34) implies that:

\[
V[1 - \alpha_m, r_{w_m, m}] < V[1 - \alpha_m, r_{w_m, m}] \leq -H_m,
\]

where the second inequality follows from the definition of \( w_m \). Eqs. (35) and (36) contradict the fact that \( w_m \) is the optimal portfolio within account \( m \). This completes the first part of our proof.

Second, we show that \( E > E_m \). Using Eqs. (4) and (17), we have:

\[
V[1 - \alpha_m, r_{w_E, m}] = z_{\alpha_m} \sqrt{E^2_m + (E[r_{w_E, m}] - E_m)^2 / (D/C) - E[r_{w_E, m}].}
\]

(37)

It follows from Eq. (37) that:

\[
\frac{\partial V[1 - \alpha_m, r_{w_E, m}]}{\partial E[r_{w_E, m}]} = z_{\alpha_m} \frac{(E[r_{w_E, m}] - E_m) / (D/C)}{\sqrt{E^2_m + (E[r_{w_E, m}] - E_m)^2 / (D/C)}} - 1.
\]

(38)

Since \( z_{\alpha_m} > 0 \), Eq. (38) implies that if \( E[r_{w_E, m}] \leq E_m \), then \( \partial V[1 - \alpha_m, r_{w_E, m}] / \partial E[r_{w_E, m}] < 0 \). Hence, we have \( E > E_m \). This completes the second part of our proof.

Third, we show that \( V[1 - \alpha_m, r_{w_m, m}] = -H_m \). Suppose by way of a contradiction that \( V[1 - \alpha_m, r_{w_m, m}] < -H_m \). Fix any \( E_2 \in \mathbb{R} \) with \( E_2 > E[r_{w_m, m}] \). Let \( \xi > 0 \) be arbitrarily small. Consider portfolio \( w_\xi \equiv \xi w_{E_2} + (1 - \xi) w_m \). Note that:

\[
E[r_{w_\xi, m}] > E[r_{w_m, m}]
\]

(39)

and

\[
V[1 - \alpha_m, r_{w_\xi, m}] < -H_m.
\]

(40)

Eqs. (39) and (40) contradict the fact that \( w_m \) is the optimal portfolio within account \( m \). This completes the third part of our proof.\( \square \)

**Proof of Theorem 1.** Fix any account \( m \in \{1, ..., M\} \). First, we show part (i). Suppose that \( \alpha_m \geq \Phi(-\sqrt{D/C}) \). Then:

\[
0 < z_{\alpha_m} \leq \sqrt{D/C}.
\]

(41)
Fix any $E \in \mathbb{R}$. Note that:

$$
\frac{(E[r_{w,E,m}] - E_m) / (D/C)}{\sqrt{\sigma_m^2 + (E[r_{w,E,m}] - E_m)^2 / (D/C)}} < \frac{1}{\sqrt{D/C}}.
$$

(42)

It follows from Eqs. (38), (41), and (42) that:

$$
\frac{\partial V[1 - \alpha_m, r_{w,E,m}]}{\partial E[r_{w,E,m}]} < 0.
$$

(43)

Eq. (43) implies that the optimal portfolio within account $m$ does not exist.

Suppose that $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m > H_{\alpha_m}$. Note that $-H_m < -H_{\alpha_m} = V[1 - \alpha_m, r_{w,m,m}]$. Hence, there exists no portfolio $w$ that meets constraint (5). Therefore, the optimal portfolio within account $m$ does not exist. This completes our proof of part (i).

Second, we show part (ii). Suppose that $\alpha_m < \Phi(-\sqrt{D/C})$ and $H_m \leq H_{\alpha_m}$. Using Lemma 3, we have $E[r_{w,m,m}] > E_m$. Hence, it follows from Lemma 3 and Eq. (17) that:

$$
E[r_{w,m,m}] = E_m + \sqrt{D/C \left( \sigma^2[r_{w,m,m}] - \frac{\alpha_m^2}{2} \right)}.
$$

(44)

Using Eqs. (4), (44) and Lemma 3, we have:

$$
z_{\alpha_m} \sigma[r_{w,m,m}] - E_m - \sqrt{D/C \left( \sigma^2[r_{w,m,m}] - \frac{\alpha_m^2}{2} \right)} = -H_m.
$$

(45)

It follows from Eq. (45) that:

$$
\zeta_1 \sigma^2[r_{w,m,m}] + \zeta_2 \sigma[r_{w,m,m}] + \zeta_3 = 0,
$$

(46)

where $\zeta_1 \equiv z_{\alpha_m}^2 - D/C$, $\zeta_2 \equiv -2z_{\alpha_m} (E_m - H_m)$, and $\zeta_3 \equiv (E_m - H_m)^2 + (D/C) \sigma_m^2$. Using Eq. (46), we have:

$$
\sigma[r_{w,m,m}] = \frac{z_{\alpha_m} (E_m - H_m) \pm \sqrt{(D/C) \left[ (E_m - H_m)^2 - (z_{\alpha_m}^2 - D/C) \sigma_m^2 \right]}}{z_{\alpha_m}^2 - D/C}.
$$

(47)

Eqs. (8)–(11) follow from Lemmas 1 and 3, and Eqs. (44) and (47). This completes our proof of part (ii).
The following result is used in our proof of Corollary 1.

Lemma 4. Consider an investor with a single account who faces account $m$’s background risk and has an objective function given by Eq. (12). The investor’s optimal portfolio is:

$$w_{\gamma m} \equiv w_m + \frac{A}{\gamma_m} (w_1 - w_0).$$

(48)

Proof. Consider an investor with a single account who faces account $m$’s background risk and has an objective function given by Eq. (12). The investor’s optimal portfolio solves:

$$\max_{\mathbf{w} \in \mathbb{R}^N} \ w' \mu + \nu_m - \frac{\gamma_m}{2} \left( w' \Sigma w + \Omega_{mm} + 2w' \Psi_m \right)$$

(49)

$$\text{s.t.} \quad w' \mathbf{1} = 1.$$  

(50)

A first-order condition for $w_{\gamma m}$ to solve problem (49) subject to constraint (50) is:

$$\mu - \gamma_m (\Sigma w_{\gamma m} + \Psi_m) + \lambda_m \mathbf{1} = 0,$$

(51)

where $\lambda_m$ is the multiplier associated with this constraint. Eq. (51) implies that:

$$w_{\gamma m} = \Sigma^{-1} \left( \frac{\mu + \lambda_m \mathbf{1}}{\gamma_m} - \Psi_m \right).$$

(52)

Premultiplying Eq. (52) by $\mathbf{1}'$ and using Eq. (50), we have:

$$1 = \frac{\mathbf{1}'\Sigma^{-1}\mu + \lambda_m \mathbf{1}'\Sigma^{-1} \mathbf{1}}{\gamma_m} - \mathbf{1}'\Sigma^{-1}\Psi_m.$$ 

(53)

Eq. (53) implies that:

$$\lambda_m = \frac{\gamma_m (1 + A_m) - A}{C}.$$ 

(54)

It follows from Eqs. (53) and (54) that:

$$w_{\gamma m} = (1 + A_m) \frac{\Sigma^{-1} \mathbf{1}}{C} - \Sigma^{-1}\Psi_m + \frac{\Sigma^{-1}\mu - A C \Sigma^{-1} \mathbf{1}}{\gamma_m}.$$ 

(55)

The desired result follows from Eq. (55) and the definitions of $w_m, w_0, \text{ and } w_1.$ \(\square\)
Proof of Corollary 1. Fix any account \( m \in \{1, \ldots, M\} \) with \( \alpha_m < \Phi(-\sqrt{D/C}) \) and \( H_m \leq H_{\alpha_m} \).

The desired result follows from Eqs. (8) and (48). □

Proof of Corollary 2. Fix any account \( m \in \{1, \ldots, M\} \) with \( \alpha_m < \Phi(-\sqrt{D/C}) \) and \( H_m \leq H_{\alpha_m} \).

First, we show the ‘if’ part. Suppose that \( \Psi_m = \delta_1 \mathbf{1} + \delta_2 \mu \) for some constants \( \delta_1 \) and \( \delta_2 \). Using the definition of \( \mathbf{w}_m \) and the assumption that \( \Psi_m = \delta_1 \mathbf{1} + \delta_2 \mu \), we have:

\[
\mathbf{w}_m = \left[ 1 + \mathbf{1}'\Sigma^{-1} (\delta_1 \mathbf{1} + \delta_2 \mu) \right] \mathbf{w}_0 - \Sigma^{-1} (\delta_1 \mathbf{1} + \delta_2 \mu).
\]

(56)

It follows from Eq. (56) that:

\[
\mathbf{w}_m = \mathbf{w}_0 - A\delta_2 (\mathbf{w}_1 - \mathbf{w}_0).
\]

(57)

Eqs. (8) and (57) imply that:

\[
\mathbf{w}_m = \mathbf{w}_0 + (\eta_m - A\delta_2) (\mathbf{w}_1 - \mathbf{w}_0).
\]

(58)

Merton (1972) shows that a portfolio \( \mathbf{w} \) is on the mean-variance frontier if and only if:

\[
\mathbf{w} = \theta \mathbf{w}_0 + (1 - \theta) \mathbf{w}_1
\]

(59)

for some \( \theta \in \mathbb{R} \). It follows from Eqs. (58) and (59) that portfolio \( \mathbf{w}_m \) is on the mean-variance frontier. This completes the first part of our proof.

Second, we show the ‘only if’ part. Suppose that \( \mathbf{w}_m \) is on the mean-variance frontier. Using Eqs. (8) and (59), \( \mathbf{w}_m \) is also on this frontier. Hence, Eq. (59) implies that:

\[
\mathbf{w}_m = \theta_m \mathbf{w}_0 + (1 - \theta_m) \mathbf{w}_1
\]

(60)

for some \( \theta_m \in \mathbb{R} \). Using the definition of \( \mathbf{w}_m \) in the left-hand side of Eq. (60), we obtain:

\[
(1 + \mathbf{1}'\Sigma^{-1} \Psi_m) \mathbf{w}_0 - \Sigma^{-1} \Psi_m = \theta_m \mathbf{w}_0 + (1 - \theta_m) \mathbf{w}_1,
\]

(61)

or equivalently:

\[
\Sigma^{-1} \Psi_m = (1 + \mathbf{1}'\Sigma^{-1} \Psi_m - \theta_m) \mathbf{w}_0 - (1 - \theta_m) \mathbf{w}_1.
\]

(62)
Premultiplying Eq. (62) by $\Sigma$, we have:

$$
\Psi_m = \frac{1 + 1'\Sigma^{-1}\Psi_m - \theta_m}{C} = \frac{1 - \theta_m}{A}\mu. \tag{63}
$$

It follows from Eq. (63) that $\Psi_m = \delta_1 1 + \delta_2 \mu$ for some constants $\delta_1$ and $\delta_2$. This completes the second part of our proof.$\square$

The following result is used in our proof of Corollary 3.

**Lemma 5.** Consider an investor with a single account who does not face background risk and has an objective function given by Eq. (13). The investor’s optimal portfolio is:

$$
\omega_\gamma \equiv \omega_0 + \frac{A}{\gamma}(\omega_1 - \omega_0). \tag{64}
$$

**Proof.** Consider an investor with a single account who does not face background risk and has an objective function given by Eq. (13). The investor’s optimal portfolio solves:

$$
\max_{\omega \in \mathbb{R}^N} \omega'\mu - \frac{\gamma}{2}\omega'\Sigma\omega \tag{65}
$$

$$
s.t. \quad \omega'1 = 1. \tag{66}
$$

A first-order condition for $\omega_\gamma$ to solve problem (65) subject to constraint (66) is:

$$
\mu - \gamma\Sigma\omega_\gamma - \lambda 1 = 0, \tag{67}
$$

where $\lambda$ is the multiplier associated with this constraint. Eq. (67) implies that:

$$
\omega_\gamma = \frac{\Sigma^{-1}\mu - \lambda\Sigma^{-1}1}{\gamma}. \tag{68}
$$

Premultiplying Eq. (68) by $1'$ and using Eq. (66), we have:

$$
1 = \frac{1'\Sigma^{-1}\mu - \lambda 1'\Sigma^{-1}1}{\gamma}. \tag{69}
$$
Eq. (69) implies that:
\[ \lambda = \frac{A - \gamma}{C}. \]  
(70)

It follows from Eqs. (68) and (70) that:
\[ w_\gamma = \frac{\Sigma^{-1} 1}{C} + \frac{\Sigma^{-1} \mu - \frac{A}{C} \Sigma^{-1} 1}{\gamma}. \]  
(71)

The desired result follows from Eq. (71). Q.E.D.

**Proof of Corollary 3.** Fix any given account \( m \in \{1, ..., M\} \) with \( \alpha_m < \Phi(-\sqrt{D/C}) \), \( H_m \leq H_{\alpha_m} \), and \( \Psi_m = \delta_1 1 + \delta_2 \mu \) for some constants \( \delta_1 \) and \( \delta_2 \). It follows that Eq. (58) holds. Eqs. (58) and (64) imply the desired result. Q.E.D.

**Proof of Theorem 2.** Suppose that \( \alpha_m < \Phi(-\sqrt{D/C}) \) and \( H_m \leq H_{\alpha_m} \) for any account \( m \in \{1, ..., M\} \). Using Eq. (8), we have:
\[ w_a = \sum_{m=1}^{M} y_m w_m + \sum_{m=1}^{M} y_m \eta_m (w_1 - w_0). \]  
(72)

Noting that \( w_a = \sum_{m=1}^{M} y_m w_m \), the desired result follows from Eq. (72). Q.E.D.

The following result is used in our proof of Corollary 4.

**Lemma 6.** Consider an investor with a single account who faces the aggregate background risk and has an objective function given by Eq. (15). The investor’s optimal portfolio is:
\[ w_{\gamma_a} = w_a + \frac{A}{\gamma_a} (w_1 - w_0). \]  
(73)

**Proof of Lemma 6.** Similar to the proof of Lemma 4 and thus omitted. Q.E.D.

**Proof of Corollary 4.** Suppose that \( \alpha_m < \Phi(-\sqrt{D/C}) \) and \( H_m \leq H_{\alpha_m} \) for any account \( m \in \{1, ..., M\} \). The desired result follows from Eqs. (14) and (73). Q.E.D.
Proof of Corollary 5. Suppose that \( \alpha_m < \Phi(-\sqrt{D/C}) \) and \( H_m \leq H_{\alpha_m} \) for any account \( m \in \{1, ..., M\} \). First, we show the 'if' part. Suppose that \( \Psi_a = \delta_1 \mathbf{1} + \delta_2 \mu \) for some constants \( \delta_1 \) and \( \delta_2 \). Using arguments similar to those in the proof of Corollary 2, we have:

\[
\mathbf{w}_a = \mathbf{w}_0 + (\eta_a - A\delta_2) (\mathbf{w}_1 - \mathbf{w}_0).
\]

(74)

It follows from Eqs. (59) and (74) that portfolio \( \mathbf{w}_a \) is on the mean-variance frontier. This completes the first part of our proof.

Second, we show the 'only if' part. Suppose that \( \mathbf{w}_a \) is on the mean-variance frontier. Using Eqs. (14) and (59), \( \mathbf{w}_a \) is also on this frontier. Hence, Eq. (59) implies that:

\[
\mathbf{w}_a = \theta_a \mathbf{w}_0 + (1 - \theta_a) \mathbf{w}_1
\]

(75)

for some \( \theta_a \in \mathbb{R} \). Using arguments similar to those used in the proof of Corollary 2, we have:

\[
\Psi_a = \frac{1 + \mathbf{1}' \Sigma^{-1} \Psi_a - \theta_a}{\mathbf{C}} \mathbf{1} - \frac{1 - \theta_a}{\mathbf{A}} \mu.
\]

(76)

It follows from Eq. (76) that \( \Psi_a = \delta_1 \mathbf{1} + \delta_2 \mu \) for some constants \( \delta_1 \) and \( \delta_2 \). This completes the second part of our proof. \(\square\)

Proof of Corollary 6. Suppose that \( \alpha_m < \Phi(-\sqrt{D/C}) \) and \( H_m \leq H_{\alpha_m} \) for any account \( m \in \{1, ..., M\} \), and \( \Psi_a = \delta_1 \mathbf{1} + \delta_2 \mu \) for some constants \( \delta_1 \) and \( \delta_2 \). It follows that Eq. (74) holds. Eqs. (64) and (74) imply the desired result. \(\square\)