TWO-DIMENSIONAL TOPOLOGICAL QUANTUM FIELD THEORIES AND FROBENIUS ALGEBRAS

LOWELL ABRAMS
Johns Hopkins University
Department of Mathematics
3400 Charles Street
Baltimore, MD 21218

ABSTRACT

We characterize Frobenius algebras A as algebras having a comultiplication which is a map of A-modules. This characterization allows a simple demonstration of the compatibility of Frobenius algebra structure with direct sums. We then classify the indecomposable Frobenius algebras as being either "annihilator algebras" — algebras whose socle is a principal ideal — or field extensions. The relationship between two-dimensional topological quantum field theories and Frobenius algebras is then formulated as an equivalence of categories. The proof hinges on our new characterization of Frobenius algebras.

These results together provide a classification of the indecomposable two-dimensional topological quantum field theories.

Keywords: topological quantum field theory, frobenius algebra, two-dimensional cobordism, category theory

1. Introduction

Topological Quantum Field Theories (TQFT's) were first described axiomatically by Atiyah in [1]. Since then, much work has been done to understand the algebraic structures arising in the three and four-dimensional cases (see [2] and the references cited there.) In the two-dimensional case, the algebraic structure of lattice field theories are well discussed in [3], but the case of a two-dimensional theory not having distinguished zero-cells, or "corners," has not been completely understood. Of course, these two theories are not the same; the most immediately apparent difference between the lattice and regular cases is the lack of commutativity in the former. A classification of the two-dimensional case in terms of the spectrum of a specific linear operator has been offered in [4], but actually deals with a restricted case, as will be discussed below. Interest specifically in the two dimensional case goes back to such sources as Segal's presentation in [5] of two-dimensional conformal field theories and Witten's work in [6] relating the same to results in higher dimensions.

In [7] Voronov presents a "folk theorem" asserting that a two-dimensional TQFT "is equivalent to a Frobenius algebra" (FA), and sketches a proof. (See [8] for a physicist's account.) Nevertheless, there has been difficulty formulating this theorem precisely and filling in the details of its proof [9, 10]. Indeed, the existing literature on the structure of FA's (see [11] and [12]) does not seem sufficient to support such a theorem. In particular, a precise definition of the coalgebra structure of a FA, and an understanding of its relation to the multiplicative structure, has been lacking. In addition, no mention has been made of maps which preserve FA structure. These gaps are filled below by section 2. That section also contains a small result on the uniqueness of a Frobenius algebra structure for a given algebra. Section 3 continues the discussion of the structure of FA's, and in particular classifies the indecomposable FA's.

A second difficulty is the lack of a careful discussion of the category **2-Cobord** of two-dimensional cobordisms. To eliminate this, section 4 provides a description of **2-Cobord**'s two-category structure particularly convenient for our purposes, and then details its structure as given by generators and relations.

Section 5 clarifies the exact relationship between TQFT's and FA's by expressing it as an equivalence of monoidal categories. In other words, the correspondence respects the direct sum and tensor product, and continues to hold on the level of morphisms. A succinct yet rigorous proof is given. In section 6, certain difficult points regarding issues of orientation in **2-Cobord** are discussed, and it is shown how these lie at the foundation of the difference between the results here and in [4]. Section 7 presents a variety of examples of FA's and their corresponding TQFT's.

Many of the results discussed here were discovered independently by Sawin [13]. However, in addition to results not appearing in [13], the approach here features a number of advantages: The discussion of comultiplication and its compatibility with direct sums of algebras allows for a clear definition of direct sums of FA's, and therefore of direct sums of TQFT's. The relationship between comultiplication and multiplication highlighted here makes the correspondence between **2-Cobord** and FA's highly intuitive. The results about maps of FA's and natural transformations of TQFT's allows the correspondence theorem to be expressed as an equivalence of categories. Finally, the lack of any condition about algebraic closure of the ground field K broadens the possibilities for the structures of FA's over K.

2. Frobenius Algebras

Fix a field K. No assumption is made about K; it may be finite or infinite dimensional, algebraically-closed or not. All algebras A/K are assumed to be finite dimensional and commutative, and to contain a unit 1_A. Multiplication in A will be denoted by $\beta : A \otimes A \to A$, and $\overline{\beta} : A \to End(A)$ will denote the map taking $a \in A$ to "multiplication by a." The dual algebra A^* has an A-module structure $A \otimes A^* \to A^*$ given by $a \otimes \zeta \mapsto a \cdot \zeta := \zeta \circ \overline{\beta}(a)$.

Proposition 1. The following conditions on A are equivalent:

(i) There exists an A-module isomorphism $\lambda : A \cong A^*$.

(ii) There exists a linear form $f : A \to K$ whose kernel contains no non-trivial ideals.

(iii) There exists a nondegenerate linear form $\eta : A \otimes A \to K$ which is associative, i.e. $\eta(ab \otimes c) = \eta(a \otimes bc)$.

(iv) For all ideals $I \in A$, $\text{ann}(\text{ann}(I)) = I$ and $(I : K) + (\text{ann}(I) : K) = (A : K)$.

Proof. A complete proof is given in [11, pages 414-418]. The proof of the equivalence of the first three conditions rests on the following: Given $\lambda : A \cong A^*$ satisfying condition (i), the linear form $f = \lambda(1_A)$ satisfies condition (ii). Given form $f : A \to K$ satisfying condition (ii), the linear form $\eta = f \circ \beta$ satisfies condition (iii). Given $\eta : A \otimes A \to K$, the linear form $f = \eta(1_A \otimes \_)$ satisfies condition (ii), and the linear map $\lambda = f \circ \overline{\beta}$ satisfies condition (i). □

An algebra A satisfying these conditions is called a Frobenius algebra. When A is a FA, the maps f, η, λ which are guaranteed to exist by conditions (i), (ii), and (iii) will henceforth be presumed to satisfy the relationships mentioned above. When it is useful to emphasize the FA structure endowed by particular f, η, and λ, the algebra will be denoted by (A, f).

Proposition 2. If (A, f) is a FA, then all FA-structures on A are given by $(A, u \cdot f)$, where $u \in A$ may be any unit.

Proof. If $u \in A$ is a unit, then for any $a \in A$ such that $u \cdot f(ax) = f(uax) = 0$ for all $x \in A$, it must be that ua, and thus a, is 0. By proposition 1, $u \cdot f$ is a FA form.

Assume (A, g) is another FA structure on A. Now $g \in A^*$, so we have $g = \lambda(u) = u \cdot f$ for some $u \in A$. Since g is a FA form, the map $\lambda' := g \circ \overline{\beta}$ is an isomorphism $A \cong A^*$, as in the proof of proposition 1. Thus, there is a $v \in A$ such that $f = \lambda'(v) = v \cdot g = vu \cdot f$. But $\lambda(1) = f = vu \cdot f = \lambda(vu)$ implies that $1 = vu$, since λ is an isomorphism, so u is a unit. □

Proposition 3. If A is a FA, then so is A^*

Proof. If (A, f) is a FA then by proposition 1 all elements of A^* are of the form $a \cdot f := f \circ \overline{\beta}(a)$ for some $a \in A$. The isomorphism $A \cong A^*$ allows us to define multiplication in A^* by $(a \cdot f)(b \cdot f) := ab \cdot f$. Define $\tau : A^* \to K$ to be "evaluation at 1_A". Then the identity $\tau(f \circ \overline{\beta}(ax)) = f(ax)$ and proposition 1 shows that (A^*, τ) is a FA. □

The isomorphism of a FA A with its dual A^* endows A with a coalgebra structure. Define comultiplication $\alpha : A \to A \otimes A$ to be the map $(\lambda^{-1} \otimes \lambda^{-1}) \circ \beta^* \circ \lambda$:

$$\begin{tikzcd}
A \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & A \otimes A \\
A^* \arrow[r, "\beta^*"] & A^* \otimes A^* \arrow[u, "\lambda^{-1} \otimes \lambda^{-1}"']
\end{tikzcd}$$

It is clear from the definition of α that A is coassociative and cocommutative. Note also that α can be used to define the multiplication in A^* since $(a \cdot f)(b \cdot f) = [a \cdot f \otimes b \cdot f] \circ \alpha = ab \cdot f$. Let $g : K \to A$ denote the unit map $1_K \mapsto 1_A$, and let $I : A \to A$ denote the identity map. The commutativity of

$$\begin{tikzcd}
A^* \arrow[dr, "g^*"] & \\
A \arrow[u, "\lambda"] \arrow[r, "f"'] & K
\end{tikzcd}
\qquad
\begin{tikzcd}
f \circ \overline{\beta}(a) \arrow[dr] & \\
a \arrow[u] \arrow[r] & f(a) = f \circ \overline{\beta}(a)(1_A)
\end{tikzcd}$$

guarantees the commutativity of

$$\begin{tikzcd}
A^* \arrow[r, "\beta^*"] & A^* \otimes A^* \arrow[r, "g^* \otimes I^*"] & K \otimes A^* \arrow[d, "\lambda^{-1}"] \\
A \arrow[u, "\lambda"] \arrow[r, "\alpha"] & A \otimes A \arrow[u, "\lambda \otimes \lambda"] \arrow[r, "f \otimes I"] & K \otimes A.
\end{tikzcd}$$

Since the top row is nothing other than I^*, we see that the bottom row is I. Thus f is the counit in A.

For the next result, view A and $A \otimes A$ as A-modules via the usual module actions $\beta : A \otimes A \to A$ and $\beta \otimes I : A \otimes A \otimes A \to A \otimes A$ respectively.

Theorem 1. A finite dimensional commutative algebra A with multiplication $\beta : A \otimes A \to A$ and unit $g : K \to A$ is a FA if and only if it has a cocommutative comultiplication $\alpha : A \to A \otimes A$, with a counit, which is a map of A-modules.

Proof. Assume A is a FA with the comultiplication α as defined above. As discussed, α is coassociative, cocommutative, and has a counit. To show α is an A-module map, we must confirm commutativity of

$$\begin{tikzcd}
A \otimes A \arrow[r, "\beta"] \arrow[d, "I \otimes \alpha"'] & A \arrow[d, "\alpha"] \\
A \otimes A \otimes A \arrow[r, "\beta \otimes I"] & A \otimes A.
\end{tikzcd}$$

Choose a basis $e_1, \ldots, e_n$, and corresponding tensor representations β_{ij}^k and f_i for β, f respectively. Viewing $\overline{\beta}$ as a map $A \to A \otimes A^*$, the commutativity of

$$\begin{tikzcd}
A \arrow[r, "\overline{\beta}"] \arrow[d, "\lambda"] & A \otimes A^* \arrow[dd, "\lambda \otimes I^*"] \\
A^* \arrow[dr, "\beta^*"] & \\
& A^* \otimes A^*
\end{tikzcd}
\qquad
\begin{tikzcd}
e_i \arrow[r] \arrow[d] & \beta_{ij}^k e_k \otimes e_j^* \arrow[d] \\
\beta_{ik}^m f_m e_k^* \arrow[dr] & \beta_{ij}^k \beta_{kl}^m f_m e_l^* \otimes e_j^* \\
& \beta_{jl}^k \beta_{ik}^m f_m e_l^* \otimes e_j^*
\end{tikzcd}$$

follows from the commutativity and associativity of β and the fact that λ is the adjoint of $f \circ \beta$. By the definition of α it is immediately evident that the diagram

$$\begin{tikzcd}
A \arrow[r, "\lambda"] \arrow[d, "\alpha"'] & A^* \arrow[r, "\lambda^{-1}"] \arrow[d, "\beta^*"'] & A \arrow[d, "\overline{\beta}"'] \\
A \otimes A & A^* \otimes A^* \arrow[l, "\lambda^{-1} \otimes \lambda^{-1}"'] & A \otimes A^* \arrow[l, "\lambda \otimes I^*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[d, "\alpha"'] \arrow[dr, "\overline{\beta}"] & \\
A \otimes A & A \otimes A^* \arrow[l, "I \otimes \lambda^{-1}"']
\end{tikzcd}$$

commutes. Thus, commutativity of the diagram in question follows from the commutativity of the outer edge of the following diagram:

$$\begin{tikzcd}
 & A\otimes A \arrow[r, "\beta"] \arrow[dl, "I\otimes\overline{\beta}"'] \arrow[d, "I\otimes\alpha"] & A \arrow[d, "\alpha"] \arrow[dr, "\overline{\beta}"] & \\
A\otimes A\otimes A^* \arrow[r, "I\otimes I\otimes\lambda^{-1}"'] \arrow[dr, "\beta\otimes I^*"'] & A\otimes A\otimes A \arrow[r, "\beta\otimes I"] & A\otimes A \arrow[d, "I\otimes\lambda"] & A\otimes A^* \arrow[l, "I\otimes\lambda^{-1}"'] \arrow[dl, "I\otimes I^*"] \\
 & A\otimes A^* \arrow[r, "I\otimes I^*"] & A\otimes A^* &
\end{tikzcd}$$

Since the outer edge is nothing other than an expression of the associativity of β, it certainly commutes.

Now assume A has a comultiplication α satisfying the hypotheses, and let $f : A \to K$ be the counit. To show that A is actually a FA, it suffices to show that the linear form $\eta := f \circ \beta : A \otimes A \to K$ satisfies the conditions required by proposition 1. Associativity of η immediately follows from the associativity of β. It remains to show non-degeneracy.

Define $\psi := \alpha \circ g : K \to A \otimes A$. By assumption, the following diagram commutes:

$$\begin{tikzcd}
 & & A\otimes A\otimes A \arrow[dr, "I\otimes\beta"] & & \\
K\otimes A \arrow[r, "g\otimes I"] & A\otimes A \arrow[ur, "\alpha\otimes I"] \arrow[dr, "\beta"] & & A\otimes A \arrow[r, "I\otimes f"] & A\otimes K \\
 & & A \arrow[ur, "\alpha"] & &
\end{tikzcd}$$

By definition of g and f, composition along the lower edge of this diagram gives the identity. Thus, the top line shows that $(I \otimes \eta) \circ (\psi \otimes I)$ is the identity map on A. Choosing a basis $e_1, \ldots, e_n$ for A, this composition maps an arbitrary $a \in A$ as follows:

$$a \mapsto (\sum_j u_j \otimes e_j) \otimes a \mapsto \sum_j u_j \eta(e_j \otimes a) = a$$

where the u_j are some elements in A. In fact, these u_j form a basis for A, since they clearly span A, and there are at most $\dim(A)$ of them. Taking $a = u_i$, we see that $u_i = \sum_j u_j \eta(e_j \otimes u_i)$, so $\eta(e_j \otimes u_i) = \delta_{ij}$. Assume that for some $k_1, \ldots, k_n \in A$ we have $\eta(\sum_j k_j e_j \otimes x) = 0$ for all x. Plugging in $x = u_i$, we see that $k_i = 0$ for all i. In other words, η is non-degenerate, and A is a FA. $\Box$

A FA map $\phi : (A, f) \to (A', f')$ is a map of algebras which preserves the action of f, i.e. $f' \circ \phi = f$.

Proposition 4. All FA maps ϕ are injective. In addition, ϕ is also a map of coalgebras if and only if it is an isomorphism.

Proof. Injectivity of ϕ follows from the commutativity of

$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A^* \\ \phi \downarrow & & \uparrow \phi^* \\ A' & \xrightarrow{\lambda'} & A'^*. \end{array}$$

This diagram commutes because

$$\phi^*(f' \circ \overline{\beta}'(\phi(a))) = f' \circ \overline{\beta}'(\phi(a)) \circ \phi = f' \circ \phi \circ \overline{\beta}(a) = f \circ \overline{\beta}(a).$$

It also follows from this that ϕ^* is surjective. Because

$$\tau(g' \circ \phi) = g' \circ \phi(1_A) = g'(1_{A'}) = \tau'(g'),$$

the map ϕ^* preserves the action of τ'. If ϕ is comultiplicative, then ϕ^* is multiplicative, hence an FA map, and hence both injective and surjective. If ϕ is an isomorphism, then ϕ^* must be multiplicative, and thus ϕ is comultiplicative. $\square$

Let **Frob/K** be the category of FA's (A, f) and FA-isomorphisms. (The purpose of restriction to isomorphisms will become clear later.) Each FA (A, f) has a distinguished element $\omega = \omega(A) := \beta \circ \alpha(1_A)$.

Proposition 5. The distinguished element ω of (A, f) is $\sum_j^n e_j \lambda^{-1}(e_j^*)$, where $e_1, \ldots, e_n$ is a basis for A. If $u \in A$ is a unit, then $u^{-1}\omega$ is the distinguished element of $(A, u \cdot f)$.

Proof. It is easily verified that for each i, j, we have $\eta(e_i \otimes \lambda^{-1}(e_j^*)) = \delta_{i,j}$. Since λ is an isomorphism, by proposition 1, we see that $\lambda^{-1}(e_1^*), \ldots, \lambda^{-1}(e_n^*)$ is the dual basis of $e_1, \ldots, e_n$ relative to η. Note also that $(\lambda^{-1}(e_j^*))^* = \lambda(e_j)$. It now follows that

$$\begin{aligned} \alpha(1) &= (\lambda^{-1} \otimes \lambda^{-1}) \circ \beta^* \circ \lambda(1) \\ &= (\lambda^{-1} \otimes \lambda^{-1}) \circ \eta \\ &= (\lambda^{-1} \otimes \lambda^{-1}) \circ \textstyle\sum_j (e_j^* \otimes \lambda(e_j)) \\ &= \textstyle\sum_j (\lambda^{-1}(e_j^*) \otimes e_j), \end{aligned}$$

and the first claim follows.

View A^* as a module over itself, using the multiplication $\alpha^* : A^* \otimes A^* \to A^*$ defined above. We may also view $A^* \otimes A^*$ as an A^*-module, using $\alpha^* \otimes I^*$. It follows from theorem 1 that we have the equality $\beta^* \circ \alpha^* = (\alpha^* \otimes I^*) \circ (I^* \otimes \beta^*)$, and thus β^* is an A^* module map. Now, $\mathrm{Hom}_{A^*}(A^*, A^* \otimes A^*) \cong A^* \otimes A^*$, and there is a fixed element $\zeta \in A^* \otimes A^*$ such that β^* is "multiplication by ζ" [14, p. 203]. Clearly, $\zeta = \beta^*(1 \cdot f) = \eta$, and we therefore have

$$\begin{aligned} \beta^*(c \cdot f) &= (c \cdot f \otimes \eta) \circ (\alpha \otimes I) \\ &= \textstyle\sum_j ((c \cdot f) \otimes e_j^* \otimes \lambda(e_j)) \circ (\alpha \otimes I) \\ &= \textstyle\sum_j (c\lambda^{-1}(e_j^*) \cdot f \otimes \lambda(e_j)). \end{aligned}$$

Thus $\alpha(c) = \sum_j c\lambda^{-1}(e_j^*) \otimes e_j$.

Now let α_u and λ_u denote the appropriate maps of $(A, u \cdot f)$. We have

$$\begin{aligned}
\alpha_u(1_A) &= (\lambda_u^{-1} \otimes \lambda_u^{-1}) \circ \beta^*(u \cdot f) \\
&= (\lambda_u^{-1} \otimes \lambda_u^{-1}) \sum_j (u\lambda^{-1}(e_j^*) \cdot f \otimes \lambda(e_j)). \\
&= \sum_j (\lambda^{-1}(e_j^*) \otimes u^{-1} e_j).
\end{aligned}$$

Thus the distinguished element of $(A, u \cdot f)$ is $\beta \circ \alpha_u(1_A) = \sum_j \lambda^{-1}(e_j^*) u^{-1} e_j = u^{-1}\omega$. $\square$

3. Monoidal Structure and Decomposition of FA's

A direct sum of FA's is a direct sum of algebras, each of which is a FA.

Proposition 6. Comultiplication in A respects the direct sum structure.

Proof. Let $A = A' \oplus A''$. By definition of direct sum for algebras, multiplication in A is a map

$$A \otimes A \cong (A' \otimes A') \oplus (A' \otimes A'') \oplus (A'' \otimes A') \oplus (A'' \otimes A'')$$

$$\longrightarrow A' \oplus 0 \oplus 0 \oplus A'' \cong A' \oplus A'' = A$$

Thus the comultiplication map α as defined above will satisfy this diagram:

$$\begin{tikzcd}
A' \oplus A'' \arrow[r, "\alpha"] \arrow[d, "\lambda"] & (A' \oplus A'') \otimes (A' \oplus A'') \\
A'^* \oplus A''^* \arrow[d, "\beta^*"] & (A'^* \oplus A''^*) \otimes (A'^* \oplus A''^*) \arrow[u, "\lambda^{-1} \otimes \lambda^{-1}"] \\
(A'^* \otimes A'^*) \oplus (A''^* \otimes A''^*) \arrow[r, "\cong"] & (A'^* \otimes A'^*) \oplus 0 \oplus 0 \oplus (A''^* \otimes A''^*). \arrow[u, "\supseteq"]
\end{tikzcd}$$

The result easily follows. $\square$

It follows from this that the distinguished element $\omega(A)$ of a direct sum $A = A' \oplus A''$ is the direct sum $\omega(A') \oplus \omega(A'')$. The FA structure itself of A is also, up to a unit, determined by the FA structures of the direct summands of A.

Proposition 7. If $A = \bigoplus_i A_i$ then $(A, f) \in$ **Frob/K** for some f if and only if for each i there is an f_i such that $(A_i, f_i) \in$ **Frob/K** . Furthermore, if $(A, f), (A_i, f_i) \in$ **Frob/K** for all i then f and $\{f_i\}$ determine each other up to module-action by a unit of A.

Proof. If $A \in$ **Frob/K** then for each i define $f_i := f \mid_{A_i}$. Assume that for some i there is an element $a_i \in A_i$ such that $f_i(a_i x) = 0$ for all $x \in A$. Now, we may view

a_i as $(0, \ldots, 0, a_i, 0, \ldots, 0) \in A$. Thus $f(a_i A) = f(a_i A_i) = f_i(a_i A_i) = 0$. It follows that $a_i = 0$ and therefore (A_i, f_i) is a FA for each i.

Conversely, if $(A_i, f_i) \in$ **Frob/K** for all i, then define the form $f := \bigoplus_i f_i$ by $f(x_1, \ldots, x_n) := \sum_i f_i(x_i)$. Let $e_1, \ldots, e_n$ denote the canonical basis for A relative to the given decomposition of A. Assume that there is an element $b = (b_1, \ldots, b_n) \in A$ such that $f(bx) = 0$ for all $x \in A$. Then for all i, $f_i(b_i A_i) = f(bA_i e_i) = 0$, showing that all b_i, and hence b, are 0. Therefore, (A, f) is a FA.

If $(A, f), (A_i, f_i) \in$ **Frob/K** for all i, then by proposition 2 any FA form $f : A \to K$ must satisfy $f = u \cdot (\bigoplus_i f_i)$ for some unit $u \in A$. Conversely, for each i there is a unit $u_i \in A_i$ such that $f_i = u_i \cdot (f\,|_{A_i})$. But $u := u_1 \oplus \ldots \oplus u_n$ is a unit in A, and $f_i = (u \cdot f)\,|_{A_i}$ for each i. $\square$

FA's which are indecomposable under direct sum possess easily described structures, as we will now show. Let $\mathcal{N} = \mathcal{N}(A)$ denote the nilradical of A.

Proposition 8. If A is indecomposable then $\mathcal{N}$ consists of all non-units of A.

Proof. This follows from a series of results in [11, pp. 370–372] and the commutativity of A. $\square$

Since $\mathcal{N}$ is an ideal, it is a subspace of A. (Note that $\mathcal{N} \neq A$, since A has an identity element.) We may therefore choose a basis for A such that all basis elements not in $\mathcal{N}$ are units. Let $\mathcal{U}$ be the (non-trivial) subspace generated by the unit basis elements. Note that any non-unit in $\mathcal{U}$ lies in $\mathcal{N}$ as well, so the only non-unit in $\mathcal{U}$ is 0. It follows that $(\mathcal{N} : K) + (\mathcal{U} : K) = (A : K)$. Assume now that A is a FA. Since $\mathcal{N}$ is an ideal, proposition 1 shows that $(\mathcal{N} : K) + (\mathrm{ann}(\mathcal{N}) : K) = (A : K)$, and thus $(\mathrm{ann}(\mathcal{N}) : K) = (\mathcal{U} : K) \neq 0$. Let $\mathcal{S}$ denote the ideal $\mathrm{ann}(\mathcal{N})$. This is the socle of A.

Proposition 9. $\mathcal{S}$ is a principal ideal, any of whose elements is a generator.

Proof. Choose a basis of units $u_1, \ldots, u_n$ for $\mathcal{U}$, and let a be any non-zero element of $\mathcal{S}$. Assume that $\sum_{i=1}^n s_i(au_i) = 0$ for some $s_1, \ldots, s_n \in K$ not all zero. Now $u = \sum_{i=1}^n s_i u_i$ is a unit, so we have $a = 0u^{-1} = 0$, a contradiction. Thus the elements $au_1, \ldots, au_n$ of $\mathcal{S}$ are linearly independent. Since $(\mathcal{S} : K) = (\mathcal{U} : K)$, we see that $\mathcal{S} = a\mathcal{U} = aA$. $\square$

If A is indecomposable and $\mathcal{N}(A) = 0$, then A contains only units and 0, so is just a field extension of K. If $\mathcal{N}(A) \neq 0$, we will refer to A as an "annihilator algebra."

Proposition 10. If algebra A is a field, any nonzero $f \in A^*$ is a FA form. If A is an annihilator algebra, any $f \in A^*$ such that $f(a) \neq 0$, where a is a generator of $\mathcal{S}$, is a FA form.

Proof. Assume A is a field and that $f(x) \neq 0$. Given any $b \in A$, we have $f(b(b^{-1}x)) \neq 0$. Now assume A is an annihilator algebra with $\mathcal{S} = aA$, and that $f(a) \neq 0$. As mentioned in [13], the fact that A is finite dimensional guarantees that each element in A divides a. Given any $b \in A$, let $b' \in A$ be an element such that $bb' = a$. Then $f(bb') \neq 0$. Both cases of the proposition now follow from proposition 1. $\square$

Combining the discussion above with propositions 2 and 7, we have proven:

Theorem 2. Every FA A decomposes into a direct sum of fields and indecomposable annihilator algebras, and the FA form of A is determined, up to a unit, by its indecomposable constituents.

In addition to a direct sum operation, the category **Frob/K** also has a tensor product.

Proposition 11. If $(A, f), (A', f') \in$ **Frob/K**, then $(A \otimes A', f \otimes f') \in$ **Frob/K** also.

A detailed proof is given in [12, pp. 203-204]. The direct sum and the tensor product each endow **Frob/K** with a monoidal structure. In other words, we have associative bifunctors **Frob/K** $\times$ **Frob/K** $\to$ **Frob/K** with identity K for the tensor product and identity 0 (i.e. the zero-dimensional algebra) for the direct sum. Note that we appeal here, and in the sequel without explicit mention, to Mac Lane's coherence theorem [15, Chapter 7] in order to assure associativity. Intuitively, this theorem allows us to work with natural equivalence in a monoid as if it were identity.

4. The Category of Two-Dimensional Cobordisms

Let **Pre2-Cobord** denote the two-category defined as follows:

- Objects are disjoint unions of labelled, oriented, compact one manifolds. Specifically, define $B_k : [0, 2\pi) \to \mathbb{C}$ by $B_k(t) := 3k + \cos t + i \sin t$. For each k, orient the image of B_k in accordance with the parametrization, and label the image with the index k. The objects are taken to be the empty manifold $\mathbf{0}$ and the disjoint unions $\mathbf{n} := \bigcup_{k=1}^{n} B_k([0, 2\pi))$ for all $n \in \mathbb{N} - \{0\}$.

- Morphisms $\Sigma : \mathbf{n} \to \mathbf{m}$ are oriented topological surfaces (not necessarily connected) equipped with an orientation preserving homeomorphism from the boundary $\partial\Sigma$ to the disjoint union $\mathbf{n}^* \cup \mathbf{m}$. Here, $\mathbf{n}^*$ indicates reversal of orientation. In other words, the orientation induced by Σ on the portion of $\partial\Sigma$ corresponding to $\mathbf{n}$ is the opposite of the orientation that portion inherits from $\mathbf{n}$. Each boundary component is given the labelling induced by its homeomorphic image. Composition of morphisms consists of gluing correspondingly-labelled boundaries in an orientation-preserving manner.

- Two-morphisms are orientation-preserving homeomorphisms $T : \Sigma \to \Sigma'$ of morphisms such that the following diagram commutes:

$$\begin{tikzcd}
\partial\Sigma \arrow[r, "\cong"] \arrow[d, "T|_{\partial\Sigma}"'] & \mathbf{n} \cup \mathbf{m}^* \\
\partial\Sigma' \arrow[ur, "\cong"'] &
\end{tikzcd}$$

  Note that $T\,|_{\partial\Sigma}$ must preserve labelling.

The two-morphisms of **Pre2-Cobord** form a topological space X. With the exception of those path components of X consisting of $T : \Sigma \to \Sigma'$, where $\Sigma : \mathbf{0} \to \mathbf{0}$ is of genus zero (the sphere) or genus one (the torus), the path components of X are contractible [16]. The group $\pi_0(X)$ is the direct sum of the mapping class groups of the morphisms of **Pre2-Cobord** . For a discussion of the mapping class group, see [17].

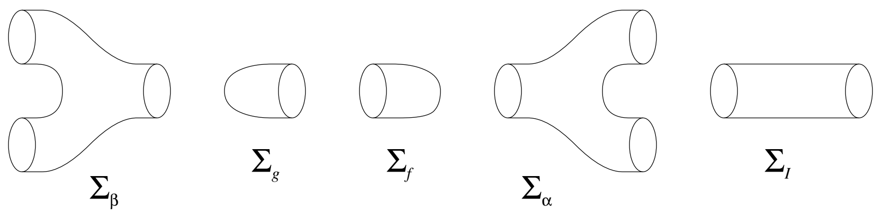

Figure 1: The generators of **2-Cobord**. Incompatibly-oriented boundaries are shown to the left of each component, compatibly-oriented boundaries are shown to the right.

Define **2-Cobord** to be the category whose objects are those of **Pre2-Cobord**, but whose morphisms are the equivalence classes of morphisms induced by the two-category structure of **Pre2-Cobord**. In other words, two morphisms Σ, Σ' are equivalent in **2-Cobord** if there is a two-morphism $T : \Sigma \to \Sigma'$ in **Pre2-Cobord**. Because boundary-preserving homeomorphisms of surfaces are homotopy equivalences, the morphisms of **2-Cobord** are distinguished only up to homotopy class.

Note that **2-Cobord** is in fact the "topological skeleton" of the category originally studied in [5].

2-Cobord also has a monoidal structure induced by disjoint union. When referring to the monoidal structure, disjoint union will be termed "tensor product." The equivalence relation induced by the two-morphisms guarantees well-definedness and hence associativity of this tensor product.

Proposition 12. The morphisms in **2-Cobord** are generated by gluing copies of the five basic surfaces shown in figure 1, subject to the five sets of relations shown in figure 2.

Proof. According to the classification theorem for two-dimensional surfaces with boundary, each connected morphism $\Sigma \in$ **2-Cobord** is determined up to homotopy class by a triple (m, g, n), where g is genus, m is the number of incompatibly-oriented boundaries, and n is the number of compatibly-oriented boundaries. Thus, each such Σ with $m, g, n > 0$ may be decomposed as shown in figure 3. If this Σ has $m = 0$ ($n = 0$) then the left (right) portion of the shown decomposition will be replaced with Σ_g (Σ_f). If $g = 0$ then the central portion will be deleted. It is clear that the five basic shapes of figure 1 generate all Σ, whether connected or not, via composition and tensor product. Completeness of the relations follows easily by inspection. $\square$

5. TQFT's and FA's

Let **Vect/K** denote the category consisting of finite dimensional vector spaces over K and linear maps, with the monoid structure given by tensor products. A topological quantum field theory is a monoidal functor Z:**2-Cobord**$\to$ **Vect/K** taking $\mathbf{0} \mapsto$ K and $\mathbf{n} \mapsto V^{\otimes n}$. As in [1], the functor Z is normalized so that $Z(\Sigma_I)$ $:= \mathrm{id}_{Z(1)}$.

Proposition 13. Each TQFT Z induces a FA structure on $Z(\mathbf{1})$.

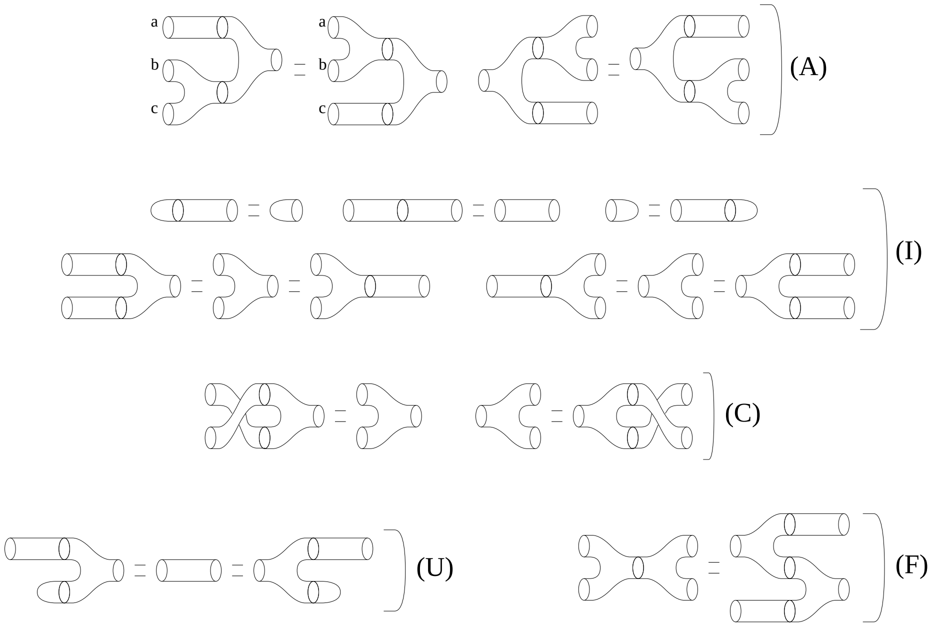

Figure 2: The relations of **2-Cobord**. Within each relation, correspondingly oriented boundaries are labelled consistently from top to bottom.

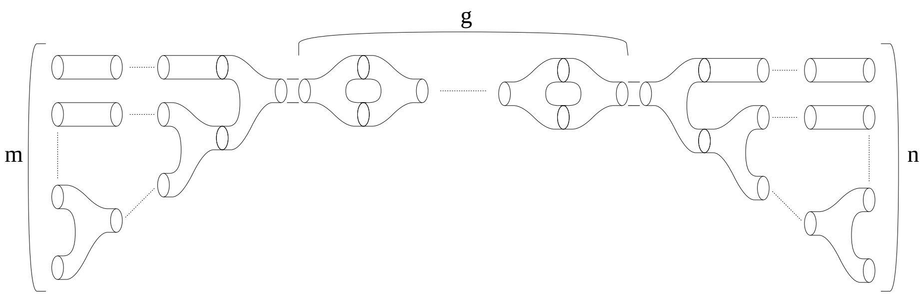

Figure 3: Decomposition of a generic morphism.

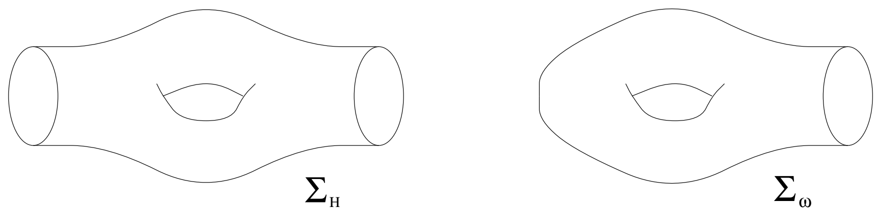

Figure 4: The surfaces corresponding to "the handle operator" and "the distinguished element."

Note: The following proof uses the functoriality of Z repeatedly, although no explicit reference will be made.

Proof. Denote $Z(\mathbf{1})$ by V. The relations (C), (A) and (U) from proposition 12 guarantee that the operators $\beta := Z(\Sigma_\beta)$ and $g := Z(\Sigma_g)$ define a commutative algebra structure, with identity, on V. These same relations guarantee that the operators $\alpha := Z(\Sigma_\alpha)$ and $f := Z(\Sigma_f)$ define a cocommutative coalgebra structure, with counit, on V. Relation (F) implies that we have commutativity of the diagram

$$\begin{tikzcd} V \otimes V \arrow[r, "\beta"] \arrow[d, "I\otimes\alpha"'] & V \arrow[d, "\alpha"] \\ V \otimes V \otimes V \arrow[r, "\beta\otimes I"] & V \otimes V. \end{tikzcd}$$

It follows from theorem 1 that Z induces a FA structure on V. $\square$

Any TQFT Z sends the morphism $\Sigma_H = \Sigma_\beta \circ \Sigma_\alpha$ depicted in figure 4 to a map $H := \beta \circ \alpha : A \to A$ called a "handle operator." Here, we use A to denote $Z(\mathbf{1})$. That this H is in fact a module homomorphism follows from the commutativity of

$$\begin{tikzcd} A \otimes A \arrow[r, "\alpha\otimes I"] \arrow[d, "\beta"'] & A \otimes A \otimes A \arrow[r, "\beta\otimes I"] \arrow[d, "I\otimes\beta"'] & A \otimes A \arrow[d, "\beta"'] \\ A \arrow[r, "\alpha"] & A \otimes a \arrow[r, "\beta"] & A. \end{tikzcd}$$

Commutativity of the left hand square follows from theorem 1. The right hand square simply expresses associativity of β. Thus, $H = \overline{\beta}(a)$ for some $a \in A$ [14, *ibid.*]. Since $H(1_A) = \beta \circ \alpha(1_A) = \omega$, it follows that $H = \overline{\beta}(\omega)$. Note that Z sends the morphism Σ_ω shown in figure 4 to $\omega \in A$. Gluing the surface Σ_H to a boundary corresponds to the addition of a loop in a Feynman diagram, and therefore the distinguished element corresponds to Planck's constant $\hbar$.

A map $\Phi : Z \to Z'$ of TQFT's is a monoidal natural transformation. Explicitly, Φ consists of a collection of linear maps $\Phi_n := \Phi_1^{\otimes n} : A^{\otimes n} \to A'^{\otimes n}$, where $\Phi_1 : A \to A'$, $A^{\otimes 0} := K$, $\Phi_0 :=\mathrm{id}_K$, and A, A' are $Z(\mathbf{1})$, $Z'(\mathbf{1})$ respectively, such that the following diagram commutes for all n and any $\Sigma : \mathbf{n} \to \mathbf{m} \in$ **2-Cobord**:

$$\begin{tikzcd}
A^{\otimes n} \arrow[r, "\Phi_n"] \arrow[d, "Z(\Sigma)"'] & A'^{\otimes n} \arrow[d, "Z'(\Sigma)"'] \\
A^{\otimes m} \arrow[r, "\Phi_m"] & A^{\otimes m}
\end{tikzcd}$$

Note that Φ satisfies all three of the following commutative diagrams:

$$\begin{tikzcd}
A \arrow[r, "\Phi_1"] \arrow[dr, "f'"] & A' \arrow[d, "f"'] \\
& K
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\Phi_1"] \arrow[d, "\alpha"'] & A' \arrow[d, "\alpha'"'] \\
A \otimes A \arrow[r, "\Phi_2"] & A' \otimes A'
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[r, "\Phi_2"] \arrow[d, "\beta"'] & A' \otimes A' \arrow[d, "\beta'"'] \\
A \arrow[r, "\Phi_1"] & A'
\end{tikzcd}$$

It follows from proposition 4 that Φ_1 is a FA isomorphism.

Let **TQFT/K** be the category whose objects are TQFT's and whose morphisms are the maps Φ defined above. We now describe two senses in which Φ is monoidal. Given $Z, Z' \in \mathbf{TQFT/K}$, define the tensor product $Z \otimes Z'$ by $(Z \otimes Z')(\mathbf{n}) := Z(\mathbf{n}) \otimes Z'(\mathbf{n})$. This is well defined, since we have the isomorphism

$$\overbrace{(A \otimes A') \otimes \cdots \otimes (A \otimes A')}^{n} = (\overbrace{A \otimes \cdots \otimes A}^{n}) \otimes (\overbrace{A' \otimes \cdots \otimes A'}^{n}).$$

This in turn shows that for morphisms $\Sigma \in \mathbf{TQFT/K}$, it is consistent to define $(Z \otimes Z')(\Sigma) = Z(\Sigma) \otimes Z'(\Sigma)$. For example, $(Z \otimes Z')(\Sigma_\beta)$ is defined by the composition

$$(A \otimes A') \otimes (A \otimes A') \xrightarrow{\cong} (A \otimes A) \otimes (A' \otimes A') \xrightarrow{\beta \otimes \beta'} A \otimes A'.$$

Note that $(Z \otimes Z')(\Sigma_f) = f \otimes f'$, where $f = Z(\Sigma_f)$ and $f' = Z'(\Sigma_f)$. It is clear from proposition 11 that $Z \otimes Z' \in \mathbf{TQFT/K}$.

Given $Z, Z' \in \mathbf{TQFT/K}$, it is also possible to define the direct sum $Z \oplus Z'$ by $(Z \oplus Z')(\mathbf{n}) := Z(\mathbf{n}) \oplus Z'(\mathbf{n})$. It is not obvious that this is well defined since, in general, $(A \oplus A') \otimes (A \oplus A')$ is not isomorphic to $(A \otimes A) \oplus (A' \otimes A')$. However, proposition 12 shows that it suffices to insure that the images under $Z \oplus Z'$ of the five basic morphisms satisfy the relations induced by the relations of **2-Cobord**. To this end, define the images of the five basic morphisms to act "componentwise." For example, if $\beta = Z(\Sigma_\beta)$, we have

$$\beta\left((a \oplus b) \otimes (c \oplus d)\right) = \beta(a \otimes c) \oplus \beta(b \otimes d).$$

In other words, β gives $Z(\mathbf{1})$ the structure of a direct sum of algebras. Consistency with the desired relations follows from propositions 6 and 7 and the following theorem:

Theorem 3. The functor F:**TQFT/K**$\to$ **Frob/K** which maps objects by $Z \mapsto (Z(\mathbf{1}), Z(\Sigma_f))$ and morphisms by $\Phi \mapsto \Phi_1$ is an equivalence of categories which respects tensor products and direct sums.

Note: Proposition 13 and the remarks preceding this theorem show that F is well defined.

Proof. It is necessary to construct a TQFT for an arbitrary (A, f) in **Frob/K** . Define $Z \in$ **TQFT/K** by $\mathbf{n} \cong \bigcup_{i=1}^{n} \mathbf{1} \mapsto A^{\otimes n}$ and

$$\begin{array}{lll} \Sigma_\beta \mapsto \beta & \Sigma_\alpha \mapsto \alpha & \Sigma_I \mapsto I \\ \Sigma_f \mapsto f & \Sigma_g \mapsto g & \end{array}$$

Since β, α, f, the unit g, and I already satisfy the conditions of (co)associativity, (co)commutatitivity, (co)unit and identity, we need only check that the relation in A corresponding to relation (F) of proposition 12 holds. In other words, we must confirm commutativity of

$$\begin{tikzcd} A \otimes A \arrow[r, "\beta"] \arrow[d, "I\otimes\alpha"'] & A \arrow[d, "\alpha"] \\ A \otimes A \otimes A \arrow[r, "\beta\otimes I"] & A \otimes A. \end{tikzcd}$$

However, this has already been shown in theorem 1.

Given a morphism ϕ of **Frob/K**, there is a unique morphism Φ of **TQFT/K** having $\Phi_1 = \phi$. Thus, we have successfully constructed, up to isomorphism, an inverse G for F.

Obviously, if TQFT Z is a tensor product $Z' \otimes Z''$ then $\mathrm{F}(Z) = (Z'(\mathbf{1}) \otimes Z''(\mathbf{1}), Z'(\Sigma_f) \otimes Z''(\Sigma_f))$ is a tensor product in **Frob/K**. To show the converse, assume that $Z(\mathbf{1}) = (A' \otimes A'', f' \otimes f'') \in$ **Frob/K**. Denote by Z', Z'' the functors $\mathrm{G}(A', f')$, $\mathrm{G}(A'', f'')$ respectively. We have $Z \cong Z' \otimes Z''$.

Similarly, if TQFT Z is a direct sum, then $F(Z)$ is clearly a direct sum as well. Because direct sums in **Frob/K** are direct sums of algebras, propositions 6 and 7 show that if $Z(\mathbf{1})$ is a direct sum, then Z can be written as a direct sum in **TQFT/K**. $\Box$

The equivalence of the categories **Frob/K** and **TQFT/K** shows that theorem 2 may be viewed as a decomposition theorem for TQFT's.

6. Clarification

Durhuus and Jonsson [4] classify two-dimensional TQFT's with K=$\mathbb{C}$ in terms of the spectrum of the "handle operator" $Z(\Sigma_H)$ mentioned above. The relationship of that classification to the results in this paper bears clarification. The most obvious difference is the choice in [4] of a particular base field. More subtle differences, however, arise from issues regarding orientation of morphisms and duals of algebras.

If Σ is a morphism in **2-Cobord**, let Σ^* denote Σ with orientation reversed. Because the two-morphisms in **2-Cobord** were defined using only orientation-preserving maps, there is no *a priori* reason to assume that $Z(\Sigma) = Z(\Sigma^*)$, as is assumed in [4]. Of course, there is necessarily a relationship between $Z(\Sigma)$ and $Z(\Sigma^*)$, because any such Σ can be decomposed as $\Sigma' \circ (\Sigma_I^{\otimes m} \otimes \Sigma^* \otimes \Sigma_I^{\otimes n}) \circ \Sigma''$ for some Σ', Σ'', m, n. See figure 5 for an example of such a decomposition of Σ_β. For a given choice of FA (A, f) and corresponding TQFT $Z = G(A, f)$, these relationships

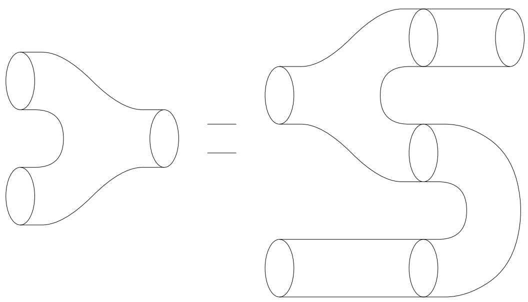

Figure 5: An example of the "orientation relations."

determine a unique isomorphism $\lambda : A \cong A^*$ — namely the adjoint of $Z(\Sigma_f \circ \Sigma_\beta)$. This isomorphism determines the effect in **Vect/K** of the reversal of orientation in **2-Cobord**.

In [1], Atiyah specifically leaves the possible axiom $Z(\Sigma^*) = Z(\Sigma)^*$, where the latter use of * indicates the vector space dual, as an open issue. It is important to note that this axiom is assumed in [4]. The two assumptions made there yield the strong identity $Z(\Sigma) = Z(\Sigma)^*$, which forces λ and the "handle operator" to be simultaneously diagonalizable. Durhuus and Jonsson use the phrase "unitary topological field theory" to indicate these particular assumptions.

7. Examples

In this section, connected morphisms of **2-Cobord** will be denoted by triples (m, g, n) as in the proof of proposition 12.

Example 1. Truncated polynomial algebras $P_n := K[x]/(x^n)$.

Let $w = x^{n-1}$. Take the standard basis for P_n, and let $f = w^*$. Since every basis element divides w, we see that w^* is a FA form. In fact, P_n contains no idempotents, so is indecomposable, and is thus an annihilator algebra with $\mathcal{S} = wA$. By the proof of proposition 5, comultiplication is determined by $\alpha(1) = \sum_k (x^k \otimes x^{n-1-k})$, and $\omega = nx^{n-1}$. Since $\omega^2 = 0$, the TQFT $Z = G(P_n, w^*)$ sends any morphism (m, g, n) with $g > 1$ to the 0-map.

Example 2. Commutative cohomology rings $H^* := H^*(M; K)$ of n-dimensional connected K-orientable manifolds.

By Poincaré duality, $H^q \cong H_{n-q}$. By the universal coefficient theorem, since the coefficients are in a field, H^{n-q} is naturally isomorphic to the vector space dual of H_{n-q}. It follows that for each q there is a non-degenerate bilinear pairing $\eta' : H^q \otimes H^{n-q} \to K$ defined by $u \otimes v \mapsto [u \cup v, \mu_M]$, where $[\cdot, \cdot]$ denotes the Kronecker index, and $\mu_M \in H_n$ is the fundamental orientation class of M. Define a non-degenerate linear form $\eta : H^* \otimes H^* \to K$ by $\eta(u \cup v) := \eta'(u \cup v)$ if $| u | + | v | = n$, and 0 otherwise. By proposition 1, H^* is a FA with FA form Λ^*, where Λ denotes the generator of H^n. The grading of H^* and the connectivity of M guarantee that H^* is indecomposable, and $\mathcal{S} = H^n$ is an annihilator ideal.

Calculation of $\alpha(1)$, and hence ω, is simplified by the fact that we can choose bases $a_1^q, \ldots, a_{m_q}^q$ for each H^q such that $a_j^q \cup a_k^{n-q} = \delta_{jk}\Lambda$. In fact, $\alpha(1) = \sum_q \sum_j^{m_q}(a_j^q \otimes a_j^{n-q})$, and thus $\omega = (H^* : K)\Lambda$. Note that $\omega \cup \omega = 0$ so that, as with P_n, $G(H^*, \Lambda^*)$ "kills" all genus greater than one.

Note that even if H^* is not commutative, as long as n is even the sub-algebra consisting of the elements of even degree in H^* is an annihilator FA. In addition, if K is chosen to be $Z/2Z$, then M will not only be K-orientable for any M, but H^* will automatically be commutative.

It is also possible to work with the entirety of a non-commutative cohomology ring, even though this does not strictly determine a TQFT as defined here. In order to do so, specify the A-module action on A^* by $a \otimes g \mapsto g \circ \overline{\beta}_L(a)$, where $\overline{\beta}_L$ denotes multiplication on the left. To see what happens, take a basis element $a \in H^*$ of odd degree, and let b be the basis element such that $ab = \Lambda$. Then $\lambda^{-1}(a^*) = -b$, and $\lambda^{-1}(b^*) = a$. Assuming $a \neq b$, these basis elements contribute $\beta(a \otimes (-b) + b \otimes a) = -2\Lambda$ to the distinguished element ω. Of course, elements of even degree will contribute copies of Λ with positive coefficients. It follows that the distinguished element of this special FA structure is the Euler class, i.e. $\chi\Lambda$, where χ is the Euler characteristic of M.

The FA structure exhibited by a noncommutative H^* corresponds nicely to its structure as a super-algebra, i.e. a vector space $V = V^0 \oplus V^1$ with a $\mathbb{Z}/2\mathbb{Z}$-graded multiplication. A linear mapping of super-algebras is called even (odd) if it maps an element of degree i to an element of degree i $(i+1)$; this distinction gives $\mathrm{End}(V)$ a super-algebra structure. Super-algebras have an even endomorphism $\epsilon : V \to V$ which is the identity on V^0 and multiplication by -1 on V^1. Note that even (odd) endomorphisms commute (anti-commute) with ϵ. The super-trace Tr_s:$\mathrm{End}(V) \to K$, corresponding to the usual linear algebra trace Tr of endomorphisms, is defined by $\mathrm{Tr}_s(g) := \mathrm{Tr}(\epsilon \circ g)$. Tr_s vanishes on odd endomorphisms, and on even endomorphisms gives the difference of the traces on V^0 and V^1.

In the case of cohomology, take V^0 (V^1) to be the space of elements of even (odd) degree in H^*. Clearly, the identity map $I : H^* \to H^*$ is even, and we have $\mathrm{Tr}_s(I) = \chi$. This is consistent with the fact that $f(\omega) = \chi$, and corresponds to the equality $\mathrm{Tr}(I)= (A : K) = f(\omega)$ for (A, f) with the usual FA structure.

The next example is similar to cohomology; it includes the case of a truncated polynomial algebra.

Example 3. Finite dimensional (graded) commutative connected Hopf algebras.

Margolis [18] shows that if A is such an algebra then it is a Poincaré algebra, i.e. there is an isomorphism $A_q \cong A_{n-q}$, where $n = (A : K)$. It follows that A is an annihilator algebra with socle generated by the highest degree element.

This same argument applies to the K-theory of a compact spin-manifold, which is also a Poincaré algebra. As in cohomology, the distinguished element is the Euler class, but in this case need not be self-annihilating; the associated TQFT would therefore be able to detect genus greater than one.

As an application of the results in section 3, we offer the following:

Example 4. Finite dimensional local rings (gradient algebras) $Q(h_0) := C_0^\infty(\mathbb{R}^n)/(h_0)$, where $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is a smooth map, h_0 is the germ of h at 0, (h_0) is the ideal generated by the components of h_0, and $C_0^\infty(\mathbb{R}^n)$ denotes the ring of germs at 0 of smooth functions $\mathbb{R}^n \to \mathbb{R}^n$.

Let J be the Jacobian of h, and let J_0 be the residue class of J in $Q(h_0)$. It is shown in [19] that any linear functional $f : Q(h_0) \to \mathbb{R}$ such that $f(J_0) > 0$ is a FA form. In fact, $Q(h_0)$ is local, and hence an annihilator algebra. It follows from the discussion in section 3 that $Q(h_0)$ has annihilator ideal $\mathcal{S} = J_0 Q(h_0)$.

A similar structure is exhibited by topological Landau-Ginzburg models in physics [20]. This is the study of the ring of states $\mathbb{C}[x_i]/\partial W$, where x_i denote chiral superfields, and $W(x_i)$ is a quasi-homogenous superpotential.

The next example is perhaps the best known example of FA's, and is a rich source of TQFT's:

Example 5. Group algebras $K[H]$, where H is a finite abelian group.

These are actually just ungraded Hopf algebras.

Let $f = 1_H^*$. Since each basis element has an inverse, f is clearly a FA form. Moreover, because the basis elements form a group, we have $\alpha(1_H) = \sum_{h \in H} h \otimes h^{-1}$, and thus $\omega = \mid H \mid 1_H$. The associated TQFT $Z := G(K[H], 1_H^*)$ sends a morphism (m, g, n) to the map $\mid H \mid^g Z(m, 0, n)$.

If f is adjusted by a unit, then more interesting things can happen. For instance, if $h' \in H$ is an element of order d, then the FA form $\mid H \mid^{-1} h' \cdot f$ yields the comultiplication given by $\alpha_{h'}(1_H) = \mid H \mid^{-1} \sum_{h \in H} hh'^{-1} \otimes h^{-1}$, and $\omega = h'^{-1}$. The associated TQFT will now send a morphism (m, g, n) to the map $Z(m, g \bmod d, n)$; in other words, it will only distinguish genus "mod d."

Group algebras can be used, at least in some cases, to distinguish morphisms of **2-Cobord** both in terms of genus and number of components. Assume K has characteristic 0 and that $\mathbb{Z}/2\mathbb{Z}$ is given as a multiplicative group by the elements e_0, e_1, satisfying $\{e_0 e_1 = e_1, e_0^2 = e_0 = e_1^2\}$. Define $(A, f) := (K[e_0, e_1], e_1^*)$. Let $\Sigma := (0, g, 1) \otimes (0, h, 1)$ and $\Sigma' := (0, g + h, 2)$. The following table shows the value of $G(A, f)$ when evaluated on Σ and Σ', for various classes of genus.

	Σ	Σ'
g, h both odd:	$2^{g+h} e_1 \otimes e_1$	$2^{g+h}(e_1 \otimes e_0 + e_0 \otimes e_1)$
g, h both even:	$2^{g+h} e_0 \otimes e_0$	" "
g even, h odd:	$2^{g+h} e_0 \otimes e_1$	$2^{g+h}(e_0 \otimes e_0 + e_1 \otimes e_1)$

Example 6. The character ring $R(H)$ of representations of finite or compact groups H, tensored with $\mathbb{Q}$.

Let $V_0, \ldots, V_n$ denote the irreducible representations of H, and let $\chi_0, \ldots, \chi_n$ respectively denote their characters. Assume χ_0 is the trivial character. The bilinear form η defined by $< \chi_i, \chi_j >:=\dim \mathrm{Hom}_H(V_i, V_j) = \delta_{ij}$ is non-degenerate. It is associative because $\dim \mathrm{Hom}_H(V_i, V_j)$ equals the dimension of the space of H-invariant bilinear forms, i.e. $\dim(V_i^* \otimes V_j^*)^H$; associativity follows from the (non-canonical) isomorphism $V_i \cong V_j^*$ and the associativity of the tensor product. We see that η defines a FA structure having FA form χ_0^*. Since each basis element of $R(H)$ is self-dual relative to η, we have $\omega = \sum_{i=1}^n \chi_i^2$. For $h \in H$ let $c(h)$ denote the number of members of the conjugacy class of h in H. The virtual character ω then has the following well known definition [21, p. 20]:

$$\omega(h) = \begin{cases} \mid H \mid / c(h) & \text{if } h \text{ and } h^{-1} \text{ are conjugates} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that if every element of H is conjugate to its inverse, which is equivalent to saying that all representations are real and also that ω is invertible, then $R(H)$ must be a direct sum (as algebras) of field extensions.

Example 7. Fusion algebras and quantum cohomology rings.

Fusion algebras are the representation rings of loop groups. See [22] for details. The (non-commutative) FA structure of quantum cohomology, and its applications, are discussed in [23, chapter 8]. The examples of representations and cohomology are actually most naturally defined over $\mathbb{Z}$, so we ask:

- What effects does the additional structure of a "Frobenius ring" have on its associated "TQFT's"?

Example 8. Algebraic number field extensions L/K.

The usual trace $\mathrm{Tr}_{L/K}$ is in fact a FA form, since $\mathrm{Tr}_{L/K}(1) = (L : K)$. In this case, η is the "trace form" sending $(a, b) \mapsto \mathrm{Tr}_{L/K}(ab)$. The discriminant of L/K is the discriminant of this η, which is just $\det(\mathrm{Tr}_{L/K}(e_i e_j))$, where $\{e_i\}$ is a basis for L/K. The study of trace forms goes hand-in-hand with the study of the Witt ring $W(K)$. See [24] for a discussion of these matters. It is clear that the definition of discriminant generalizes to all FA's, and from [24] we see that the Witt ring corresponds to a certain quotient of **Frob/K** .

- What information can the generalized discriminant and Witt ring give about FA's and their associated TQFT's?

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