Countable Models, Computability, and Enumerations, Part I

Valentina Harizanov
George Washington University
Washington, DC
harizanv@gwu.edu
http://home.gwu.edu/~harizanv/

Joint work with Sergei Goncharov, Julia Knight, Charlie McCoy, Russell Miller, and Reed Solomon.
Computable models

- Consider a *countable* structure $A$ for a *computable* language $L$.

- *Turing degree* of $A$ is the Turing degree of the atomic diagram of $A$, $D(A)$. $A$ is *computable* (*recursive*) if its Turing degree is 0.

- $D(A)$ may be of much lower Turing degree than $Th(A)$. $N$, the standard model of arithmetic, is computable. *True Arithmetic*, $TA = Th(N)$, is of Turing degree $0^{(\omega)}$.

- *(Tennenbaum)* There is no computable nonstandard model of $PA$. 

Computable categoricity

Let $A$ be a computable structure.

- $A$ is computably categorical if for all computable $B \cong A$, there is a computable isomorphism $f$ from $A$ onto $B$.

- $A$ is relatively computably categorical if for all $B \cong A$, there is an isomorphism $f$ from $A$ onto $B$, which is computable relative to $D(B)$.

- $A$ is relatively computably categorical \(\Rightarrow\) $A$ is computably categorical.
Examples

- $(\omega, <)$ is not computably categorical.

- $(Q, <)$ is relatively computably categorical (usual back-and-forth argument). It is not computably stable.

- A computable structure $A$ is *computably stable* if for all computable $B \cong A$, every isomorphism from $A$ onto $B$ is computable.

- (R. Miller) No computable tree $(T, \prec)$ of infinite height is computably categorical.
(LaRoche, Goncharov-Dzgoev, Remmel)

• A computable linear order is computably categorical iff it has \textit{finitely many successors}.

• A computable Boolean algebra is computably categorical iff it has \textit{finitely many atoms}.

(Goncharov, Smith)

• Computably categorical abelian \(p\)-groups are those that can be written in one of the following forms:
  \((Z(p^{\infty}))^l \oplus G\) for \(l \in \omega \cup \{\infty\}\) and \(G\) is finite, or
  \((Z(p^{\infty}))^n \oplus G \oplus (Z(p^k))^\infty\), where \(n, k \in \omega\) and \(G\) is finite.
Computably categorical does not imply relatively computably categorical

• (Goncharov, 1980) There is a rigid computable graph that is computably categorical, but *not relatively* computably categorical.

• *Our goal:* Generalize Goncharov’s result to higher levels of hyperarithmetical hierarchy.

• $X$ is $\Sigma^0_n \iff X$ is c.e. relative to $\emptyset^{(n-1)}$, for $1 \leq n < \omega$

• $X$ is $\Sigma^0_\alpha \iff X$ is c.e. relative to $H(a)$, for $|a| = \alpha \geq \omega$
Classification of computable formulas

• A computable $\Sigma_0$ ($\Pi_0$) formula is a finitary quantifier-free formula. A computable $\Sigma_\alpha$ formula, $\alpha > 0$, is a c.e disjunction of formulas

$$\exists \overline{u} \psi(\overline{x}, \overline{u}),$$

where $\psi$ is computable $\Pi_\beta$ for some $\beta < \alpha$.

A computable $\Pi_\alpha$ formula, $\alpha > 0$, is a c.e. conjunction of formulas

$$\forall \overline{u} \psi(\overline{x}, \overline{u}),$$

where $\psi$ is computable $\Sigma_\beta$ for some $\beta < \alpha$. 
$\Delta^0_\alpha$ categoricity

Let $A$ be a *computable* structure, $\alpha$ a computable ordinal.

- $A$ is $\Delta^0_\alpha$ *categorical* if for all computable $B \cong A$, there is a $\Delta^0_\alpha$ isomorphism $f$ from $A$ onto $B$.

- $A$ is *relatively* $\Delta^0_\alpha$ *categorical* if for all $B \cong A$, there is an isomorphism $f$ from $A$ onto $B$, which is $\Delta^0_\alpha$ relative to $D(B)$. 
Scott families of formulas

Let $A$ be a countable structure.

- A Scott family for $A$ is a set $\Phi$ of formulas, with a fixed finite tuple of parameters $\bar{c}$ in $A$, such that each tuple in $A$ satisfies some $\psi \in \Phi$, if $\bar{a}$, $\bar{b}$ are tuples in $A$ satisfying the same formula $\psi \in \Phi$, then there is an automorphism of $A$ taking $\bar{a}$ to $\bar{b}$.

- If $A$ rigid, Scott family replaced by defining family of formulas with a single free variable:
every $a \in A$ satisfies some $\psi(x) \in \Phi$, no $\psi$ in $\Phi$ satisfied by more than one element in $A$. 
Effective Scott families

• A formally $\Sigma^0_\alpha$ Scott family is a $\Sigma^0_\alpha$ Scott family consisting of computable $\Sigma_\alpha$ formulas.

• A formally c.e. Scott family is a c.e. Scott family consisting of finitary existential formulas.

• If a computable structure $A$ has a formally c.e. Scott family, then it is relatively computably categorical.
• *Proof sketch.*

Let \((A, \overline{c}) \cong (B, \overline{d})\).

Will construct an isomorphism \(f\) computable in \(D(B)\).

\[ f = \bigcup_{s} f_{s}, \quad f_{s} \subset f_{s+1}. \]

Assume \(f_{s}\) maps \(\overline{c} \overline{a} \rightarrow \overline{d} \overline{b}\),
\(a' \in A\), where \(a' \notin \overline{c} \overline{a}\).

Find \(\psi(\overline{c}, \overline{x}, \overline{y}) \in \Phi\) and \(b' \in B\) such that

\[ A \models \psi(\overline{c}, \overline{a}, a') \land B \models \psi(\overline{d}, \overline{b}, b'). \]

Let \(f_{s+1}(a') = b'\).
Equivalence of semantic and syntactic conditions

• (Ash-Knight-Manasse-Slaman, Chisholm) Let $A$ be a computable structure.

  $A$ is relatively $\Delta^0_\alpha$ categorical iff
  $A$ has a formally $\Sigma^0_\alpha$ Scott family iff
  $A$ has a c.e. Scott family consisting of computable $\Sigma_\alpha$ formulas.

  In particular, $A$ is relatively computably categorical iff
  it has a formally c.e. Scott family.

• (Goncharov) Assume that the $\exists\forall$-diagram of $A$ is computable.

  If $A$ is computably categorical, then it has a formally c.e. Scott family.

• (Ash) Under some additional decidability on $A$,

  if $A$ is $\Delta^0_\alpha$ categorical, then it has a formally $\Sigma^0_\alpha$ Scott family.
General result

(Goncharov-Harizanov-Knight-McCoy-Miller-Solomon, *APAL* 2005)

- For every computable successor ordinal $\alpha$, there is a computable structure $A$, which is $\Delta^0_\alpha$ categorical, but $A$ does not have a formally $\Sigma^0_\alpha$ Scott family (not relatively $\Delta^0_\alpha$ categorical).
Relations on structures

• Let $R$ be an additional relation on a structure $A$. Let $\mathbb{P}$ be a computability-theoretic complexity class.

• (Ash-Nerode) $R$ is intrinsically $\mathbb{P}$ on a computable $A$ if in all computable isomorphic copies of $A$, the image of $R$ is $\mathbb{P}$.

• $R$ is relatively intrinsically $\mathbb{P}$ on a computable $A$ if in all isomorphic copies $B$ of $A$, the image of $R$ is $\mathbb{P}$ relative to $D(B)$.

• Examples: (i) Successor is intrinsically $\Pi^0_1$ on a computable linear order. (ii) Dependence is intrinsically c.e. on a computable vector space.
Definability versus complexity of relations

- (Kueker) The following are equivalent for a relation $R$ on a countable $A$:
  
  (i) $R$ has fewer than $2^{|\mathbb{N}|}$ different images under automorphisms of $A$;

(ii) $R$ is definable in $A$ by an $L_{\omega_1 \omega}$ formula with finitely many parameters.

Assume (i). There exists $\bar{c}$ such that for every $a \in R$
there is a formula $\psi_a(x, \bar{c})$ satisfied by $a$ but not by any $a' \notin R$.
Hence, $R$ is defined by $\bigvee_{a \in R} \psi_a(x, \bar{c})$.

- (Harizanov) **Turing degree spectrum** of a relation $R$ on $A$,
  $DgSp(R)$, the set of Turing degrees of images of $R$
in computable isomorphic copies of $A$. 
Σ_α definability of relations

- (Ash) A relation defined in a countable structure A by a computable Σ_α (Π_α) formula is Σ_α (Π_α) relative to D(A).

- The relation R is formally Σ_0^0 on A if it is definable by a computable Σ_0^0 formula with finitely many parameters.

- (Ash-Nerode) Under some effectiveness condition (enough to have the existential diagram of (A, R) computable), R is intrinsically c.e. on a computable A iff R is formally c.e. on A.

- (Barker) Under some effectiveness conditions, R is intrinsically Σ_0^0 on a computable A iff R is formally Σ_0^0 on A.
• (Harizanov)

Under some effectiveness condition
( enough to have the existential diagram of \((A, R)\) computable):

\((i)\) If \(R\) is \textit{not intrinsically computable},
then \(DgSp(R)\) includes all c.e. degrees.

At least one of \(R\), \(\neg R\) is not definable in \(A\) by a computable \(\Sigma_1\) formula.

Example: \(A = (\omega, <)\), \(R = Succ\)

\((ii)\) If \(\neg R\) is not definable in \((A, R)\) by a computable \(\Sigma_1\) formula
in which the symbol \(R\) occurs only positively,
then \(DgSp(R)\) includes all c.e. degrees realized via c.e. sets.
• Degrees coarser than Turing degrees:

\[ X \leq_{\Delta^0_{\alpha}} Y \iff X \leq_T Y \oplus \Delta^0_{\alpha} \]

\[ X \equiv_{\Delta^0_{\alpha}} Y \iff (X \leq_{\Delta^0_{\alpha}} Y \land Y \leq_{\Delta^0_{\alpha}} X) \]

\[ (\equiv_{\Delta^0_1} \text{ is } \equiv_T) \]

• (Ash-Knight) Under some effectiveness condition, if \( R \) is not intrinsically \( \Delta^0_{\alpha} \) on a computable \( A \), then for every \( \Sigma^0_{\alpha} \) set \( C \), there is an isomorphism \( f \) from \( A \) onto a computable structure such that \( f(R) \equiv_{\Delta^0_{\alpha}} C \).

Not possible to replace these by Turing degrees.
Equivalence of semantic and syntactic conditions

• (Ash-Knight-Manasse-Slaman, Chisholm)
  \( R \) is relatively intrinsically \( \Sigma^0_\alpha \) on \( A \) \iff \( R \) is formally \( \Sigma^0_\alpha \) on \( A \).

• (Soskov) TFAE:
  (i) \( R \) is relatively intrinsically \( \Delta^1_1 \) on \( A \);
  (ii) \( R \) is definable in \( A \) by a computable formula with finitely many parameters;
  (iii) \( R \) is intrinsically \( \Delta^1_1 \) on \( A \).

• (Soskov, Goncharov-Harizanov-Knight-Shore) TFAE:
  (i) \( R \) is relatively intrinsically \( \Pi^1_1 \) on \( A \);
  (ii) \( R \) is definable in \( A \) by a \( \Pi^1_1 \) disjunction of computable formulas with finitely many parameters;
  (iii) \( R \) is intrinsically \( \Pi^1_1 \) on \( A \).
Intrinsically effective does not imply relatively intrinsically effective

• (Manasse)
  There is a computable structure with an intrinsically c.e., but not relatively intrinsically c.e. relation.

• (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon)
  For every computable successor ordinal $\alpha$, there is a computable structure $A$ with a relation $R$, such that $R$ is intrinsically $\Sigma^0_\alpha$ on $A$, but $R$ is not definable in $A$ by a $\Sigma_\alpha$ formula with finitely many parameters (not relatively intrinsically $\Sigma^0_\alpha$).
Turing degree spectrum of a structure

- \(DgSp(A) = \{\text{deg}(B) : B \cong A\}\).

- (Knight) \(A\) is automorphically trivial if it has a finite tuple \(\bar{c}\) such that every permutation of \(A\) that fixes \(\bar{c}\) pointwise is an automorphism of \(A\).
  (i) \(A\) is automorphically trivial \(\Rightarrow |DgSp(A)| = 1\).
  (ii) \(A\) is automorphically nontrivial \(\Rightarrow DgSp(A)\) is closed upwards.

- (Harizanov-Knight-Morozov) (i) If \(A\) is automorphically trivial, then \((\forall B \cong A)[D^e(B) \equiv_T D(B)]\).
  (ii) If \(A\) is automorphically nontrivial, and \(X \geq_T D^e(A)\), there exists \(B \cong A\) such that
  \[D^e(B) \equiv_T D(B) \equiv_T X\.
• $\mathcal{D}$=the set of all Turing degrees

• (Wehner, Slaman)
  There is a structure $A$ such that
  \[ DgSp(A) = \mathcal{D} - \{0\}. \]

• (Hirschfeldt)
  There is a complete decidable theory, with all types computable, whose prime model $A$ has no computable copy, but has an $X$-decidable copy for every noncomputable $X$. 
• (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon)
  For each computable successor ordinal \( \alpha \),
  there is a structure \( A \) whose
  \( DgSp(A) \) consists of the Turing degrees of sets \( X \)
  such that \( \Delta^0_\alpha(X) \) is not \( \Delta^0_\alpha \).

• In particular, for every \( n \in \omega \), there is a structure \( A \) such that

\[
DgSp(A) = \{ c \in \mathcal{D} : c^{(n)} > 0^{(n)} \}.
\]

A degree \( c \) is non-\( low_n \) if \( c^{(n)} > 0^{(n)} \).
Computable dimension of a structure

- **Computable dimension** of $A$ is the number of computable isomorphic copies of $A$, up to computable isomorphism.

- (Metakides-Nerode, Nurtazin, Goncharov, Goncharov-Dzgoev, Remmel, LaRoche) The following classes have computable dimension $1$ or $\omega$: algebraically closed fields, and real closed fields, abelian groups, linear orders, Boolean algebras, $\Delta^0_2$ categorical structures.

- (Goncharov) There are structures of computable dimension $n$ for every finite $n \geq 1$. 
\( \Delta^0_\alpha \) dimension of a structure

- \( \Delta^0_\alpha \) dimension of \( A \) is the number of computable copies of \( A \), up to \( \Delta^0_\alpha \) isomorphism.

- (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon) For each computable successor ordinal \( \alpha \) and every finite \( n \geq 1 \), there is a computable structure \( A \) such that the \( \Delta^0_\alpha \) dimension of \( A \) is \( n \).
Enumerations

- An *enumeration* of $S \subseteq P(\omega)$ is a binary relation $\nu$:

  $$S = \{\nu(i) : i \in \omega\}, \text{ where } \nu(i) = \{x : (i, x) \in \nu\}.$$

  $\nu$ is *computable (c.e.)* if it is computable (c.e.) as a binary relation.

- $\nu$ is *Friedberg* if it is 1-1: $i \neq j \Rightarrow \nu(i) \neq \nu(j)$.

- (Wehner)
  There is a family $S$ such that for every noncomputable $X$, $S$ has an enumeration computable in $X$ (c.e. relative to $X$), but $S$ has no computable (c.e.) enumeration.
Equivalent enumerations

• $\nu \leq \mu$ if there is a computable function $f$ such that:

$$ (\forall i)[\nu(i) = \mu(f(i))]. $$

• $\nu$ and $\mu$ are **computably equivalent** if $\mu \leq \nu$ and $\nu \leq \mu$.

• (Goncharov)
  For every finite $n \geq 1$, there is a family of sets with exactly $n$ c.e. Friedberg enumerations, up to computable equivalence.

• (Marchenkov)
  Not true for computable Friedberg enumerations if $n > 1$. 
Discrete families

- $S \subseteq P(\omega)$ is discrete if for each $A \in S$, there exists $\sigma \in 2^{<\omega}$ such that for all $B \in S$, $\sigma \subseteq B \iff B = A$.

- $S$ is effectively discrete if there is a c.e. set $E \subseteq 2^{<\omega}$ such that:
  \[(\forall A \in S')(\exists \sigma \in E)[\sigma \subseteq A];\]
  \[(\forall \sigma \in E)(\forall A, B \in S)[(\sigma \subseteq A \land \sigma \subseteq B) \Rightarrow A = B].\]

- (Selivanov)
  There exists $S$ with unique computable Friedberg enumeration, (in fact, c.e. since it consists of the graphs of functions) up to computable equivalence, such that $S$ is discrete but not effectively discrete.
Transforming $S$ into a graph

- Assign to $A \in S$, a *daisy graph* $G_A$ consisting of one index point $a$ at the center with $a \rightarrow a$, and for each $n \in A$ a petal (disjoint from other petals)

  $$a \rightarrow a_0 \rightarrow \cdots \rightarrow a_n \rightarrow a$$

- $G(S)$ is the union of a disjoint family of $G_A$ for each $A \in S$. $G(S)$ is a rigid graph.

- $S^+ = \text{def} \{ A \oplus \overline{A} : A \in S \}$.

- If $S$ has $n$ c.e. (computable) Friedberg enumerations, up to computable equivalence, then $G(S)$ ($G(S^+)$) has computable dimension $n$. 
Discrete families and defining families of graphs

- Assume $S$ is discrete. Every element of $G(S^+)$ has a finitary existential definition without parameters.

- Assume $S$ is discrete but not effectively discrete. Assume $S$ has a computable Friedberg enumeration. $G(S^+)$ does not have a formally c.e. defining family.

- If $S$ is a Selivanov’s family, then $G(S^+)$ is computably categorical, but not relatively.
• The *cardinal sum* $B_0 \oplus B_1$ of disjoint structures $B_0, B_1$ in the same relational language: take the disjoint union of the structures and add predicates $P_0$ and $P_1$ which hold of the elements of $B_0$ and $B_1$, respectively.

• Let $A = G \oplus G$ for $G = G(S^+)$, where $S$ a Selivanov family. Let $R$ be the unique isomorphism. $R$ is intrinsically c.e. (since $G$ is computably categorical).

• $R$ is not relatively intrinsically c.e.

Assume otherwise. For any copy $H$ of $G$, we take the disjoint union of the universes, and form a copy of $A$. There is an isomorphism from $G$ onto $H$, computable in $H$. However, $G$ is not relatively computably categorical.
• $S \subseteq P(\omega)$
  $G^\infty(S)$ consists of infinitely many copies of $G_A$ for each $A \in S$.
  $G^\infty(S)$ is not rigid.
  Copies of $G^\infty(S)$ correspond to enumerations of $S$.

• $X \subseteq \omega$
  There is an enumeration of $S$ c.e. in $X$ (computable in $X$) iff there is an isomorphic copy of $G^\infty(S)$ ($G^\infty(S^+)$) computable in $X$.
(Goncharov-Harizanov-Knight-McCoy-Miller-Solomon)

Let $\alpha \geq 2$ be a computable successor ordinal. There is a structure with copies in exactly the Turing degrees of sets $X$ such that $\Delta^0_\alpha(X)$ is not $\Delta^0_\alpha$.

**Proof sketch.** Relativize the proof for $\Delta^0_1$ to $\Delta^0_\alpha$.

Get a graph $G$ such that the degrees of copies of $G$ are just the degrees of sets that are not $\Delta^0_\alpha$.

Code a directed graph $G$ in a structure $G^*$ such that:

$G$ has a $\Delta^0_\alpha$ copy iff $G^*$ has a computable copy.

More generally, for any $X \subseteq \omega$,

$G$ has a $\Delta^0_\alpha(X)$ copy iff $G^*$ has an $X$-computable copy.
• **Proof sketch continued.** Code $\Delta^0_\alpha$ directed graph $G$ in a (computable) structure $G^*$, using a pair of structures $B_0$, $B_1$ such that $B_0$ codes $G \models a \rightarrow b$ and $B_1$ codes $G \models \neg(a \rightarrow b)$.

• $G^* = (G \cup U, G, U, Q, \ldots)$, where $G$ and $U$ are disjoint, $Q$ (a ternary relation) assigns to $a, b \in G$ an infinite set $U_{(a,b)}$: $(x \in U_{(a,b)} \iff Qabx)$, the sets $U_{(a,b)}$ form a partition of $U$,

\[
(U_{(a,b)}, \ldots) \overset{\Delta}{=} \begin{cases} 
B_0, & \text{if } G \models a \rightarrow b, \\
B_1, & \text{if } G \models \neg(a \rightarrow b).
\end{cases}
\]
• (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon)
  Let $\alpha \geq 2$ be a computable successor ordinal.
  There is a computable structure that is $\Delta^0_\alpha$ categorical but not relatively $\Delta^0_\alpha$ categorical.

• Proof sketch. Relativize the proof for $\Delta^0_1$ to $\Delta^0_\alpha$.
  There is a rigid $\Delta^0_\alpha$ directed graph $G$ such that:
  (i) $G$ has exactly one $\Delta^0_\alpha$ isomorphic copy, up to $\Delta^0_\alpha$ isomorphism,
  (ii) $G$ does not have $\Sigma^0_\alpha$ Scott family of finitary existential formulas.

• Code $G$ in a computable structure $G^* = (G \cup U, G, U, Q, \ldots)$, using a pair of structures $B_0$, $B_1$ such that
  $B_0$ codes $G \models a \rightarrow b$ and
  $B_1$ codes $G \models \neg(a \rightarrow b)$. 
• $B_0$ and $B_1$ are computable structures, for which the standard back-and-forth relations $\leq_\beta$ for $\beta < \alpha$ are uniformly c.e. (Pair $\{B_0, B_1\}$ is $\alpha$-friendly.)

• $B_0$ and $B_1$ satisfy the same infinitary $\Pi_\beta$ sentences for $\beta < \alpha$. (If $\alpha$ were a limit ordinal, then $B_0$ and $B_1$ would also satisfy the same $\Pi_\alpha$ sentences.)

• $B_0$ satisfies some computable $\Pi_\alpha$ sentence that is not true in $B_1$, and vice versa.

• Then for any $\Delta^0_\alpha$ set $S$, there is a uniformly computable sequence $(C_n)_{n \in \omega}$ such that

$$C_n \models \begin{cases} B_0, & \text{if } n \in S, \\ B_1, & \text{if } n \notin S. \end{cases}$$
Back-and-forth relations

• \( \leq_{\beta} \) on the set of pairs \( \{(i, \bar{b}) : \bar{b} \in B_i\} \), are defined inductively as follows:

  (i) \( (i, \bar{b}) \leq_1 (j, \bar{c}) \) iff the existential formulas true of \( \bar{c} \) in \( B_j \) are true of \( \bar{b} \) in \( B_i \);

  (ii) if \( \beta > 1 \), \( (i, \bar{b}) \leq_{\beta} (j, \bar{c}) \) iff for all \( \bar{c}' \) in \( B_j \), and all \( \gamma \) with \( 1 \leq \gamma < \beta \), there exists \( \bar{b}' \) in \( B_i \) such that

  \[
  (j, \bar{c}, \bar{c}') \leq_{\gamma} (i, \bar{b}, \bar{b}').
  \]

• \( (i, \bar{b}) \leq_{\beta} (j, \bar{c}) \) iff all \( \Pi_{\beta} \) formulas of \( L_{\omega_1 \omega} \) true of \( \bar{b} \) in \( B_i \) are true of \( \bar{c} \) in \( B_j \).
Existence of structures $B_0$ and $B_1$

- Case $\alpha = 2$: orders $\omega$ and $\omega^*$.

- Can be distinguished by finitary $\Pi_2$ sentences saying that there is no first, or last, element.

- $\omega \leq_1 \omega^*$ and $\omega^* \leq_1 \omega$ (since both orders are infinite).

- Each order is rigid, with a c.e. defining family consisting of finitary $\Sigma_2$ formulas $\psi_n(x)$ saying that there are exactly $n$ elements to the left, or right, of $x$. Any tuple of elements $\bar{x}$ in $\omega$ or $\omega^*$ can be defined by a conjunction of such formulas. Formally $\Sigma^0_2$ Scott family without parameters.
\{\omega, \omega^*\} is 2-friendly

Facts about linear orders $L_0, L_1$.

- $L_0 \leq_1 L_1$ iff either both orders are infinite or $L_0$ is at least as big as $L_1$.

- $(L_0, \overline{a}) \leq_\gamma (L_1, \overline{b})$ iff for $L_0 = A_0 + a_1 + A_1 + \ldots + a_n + A_n$ and $L_1 = B_0 + b_1 + B_1 + \ldots + b_n + B_n$, we have $A_i \leq_\gamma B_i$ for $i = 0, \ldots, n$.

- Can enumerate the $\leq_1$ relation between tuples in our orders.
Uniformly relatively $\Delta^0_\alpha$ categorical structures

- $B$ is uniformly relatively $\Delta^0_\alpha$ categorical if given an $X$-computable index for $C$ with $C \cong B$, we can find a $\Delta^0_\alpha(X)$ index for an isomorphism from $B$ onto $C$.

- $B$ has a formally $\Sigma^0_\alpha$ Scott family with no parameters $\implies B$ is uniformly relatively $\Delta^0_\alpha$ categorical.

- Assume, in addition, $B_0$ and $B_1$ are uniformly relatively $\Delta^0_\alpha$ categorical. Can show that:
  - (i) $G^*$ is $\Delta^0_\alpha$ categorical ($G$ had exactly one $\Delta^0_\alpha$ isomorphic copy, up to $\Delta^0_\alpha$ isomorphism);
  - (ii) $G^*$ does not have formally $\Sigma^0_\alpha$ Scott family ($G$ did not have $\Sigma^0_\alpha$ Scott family of finitary existential formulas).