COMPLEXITY OF DIAGRAMS OF COUNTABLE STRUCTURES

Valentina S. Harizanov
The George Washington University, Washington, DC
harizanv@gwu.edu

Julia F. Knight
University of Notre Dame, Notre Dame, IN
knight.1@nd.edu

Andrei S. Morozov
Sobolev Institute of Mathematics, Novosibirsk, Russia
morozov@math.nsc.ru
Fragments of Diagrams

Consider a *countable* structure $\mathcal{A}$ for a *computable* language $L$.

- **Open diagram of** $\mathcal{A}$, $D(\mathcal{A}) = D_0(\mathcal{A})$, is the set of all quantifier-free sentences of $L_\mathcal{A}$ true in $\mathcal{A}_\mathcal{A}$.

- **$n$-diagram of** $\mathcal{A}$, $D_n(\mathcal{A})$, is the set of all $\Sigma_n$ sentences of $L_\mathcal{A}$ that are true in $\mathcal{A}_\mathcal{A}$.

- **Complete (elementary) diagram of** $\mathcal{A}$, $D^c(\mathcal{A})$, is the set of all sentences of $L_\mathcal{A}$ that are true in $\mathcal{A}_\mathcal{A}$.

- **Turing degree** of $\mathcal{A}$ is the Turing degree of $D(\mathcal{A})$.

- $\mathcal{A}$ is *computable* if its Turing degree is 0.

- $\mathcal{A}$ is *$n$-decidable* if its $n$-diagram is computable.

- $\mathcal{A}$ is *decidable* if its complete diagram is computable.

- $B_n$ sentences are Boolean combinations of $\Sigma_n$ (and $\Pi_n$) sentences.

- $D_n(\mathcal{A}) = D^c(\mathcal{A}) \cap \Sigma_n \equiv_T D^c(\mathcal{A}) \cap B_n$

- $D_{n+1}(\mathcal{A})$ is c.e. in and above $D_n(\mathcal{A})$, uniformly in $n$. 
Examples

• $A$ effectively eliminates quantifiers
  
  $(\forall B \cong A)[D^c(B) \equiv_T D_0(B)]$

  (Intrinsic collapse of diagrams)

• (Moses)
  
  There is a linear order that is $n$-decidable, but has no
  $(n + 1)$-decidable copy.

• (Chisholm and Moses)
  
  There is a linear order that is $n$-decidable for every $n$, but has no decidable copy.

• $\mathcal{N} = (\omega, +, \cdot, S, 0)$
  
  $D_n(\mathcal{N}) \equiv_T \emptyset^{(n)}$, uniformly in $n$

• (Knight)
  
  $\mathcal{A}$ a nonstandard model of Peano Arithmetic ($PA$)
  
  $(\exists B \cong \mathcal{A})[D_0(B) <_T D_1(B) <_T D_2(B) <_T \ldots]$  

• (Knight)
  
  $\mathcal{A}$ any model of $PA$
  
  $(\exists B \cong \mathcal{A})[D^c(B) \equiv_T D_0(B)]$
• (Knight)
A structure $\mathcal{A}$ is trivial if there is sequence $\overrightarrow{c} \in A^{<\omega}$ such that every permutation of $\mathcal{A}$ that fixes $\overrightarrow{c}$ pointwise is an automorphism of $\mathcal{A}$.

(i) If $\mathcal{A}$ is trivial, then

$$(\forall \mathcal{B} \simeq \mathcal{A})[D_0(\mathcal{B}) \equiv_T D_0(\mathcal{A})]$$

(ii) If $\mathcal{A}$ is non-trivial, then

the set of Turing degrees of isomorphic copies of $\mathcal{A}$ is closed upwards.

• (Harizanov, Knight, Morozov)

For any structure $\mathcal{A}$, there exists $\mathcal{B} \simeq \mathcal{A}$ such that

$D^c(\mathcal{B}) \equiv_T D_0(\mathcal{B})$

(i) $\mathcal{A}$ is trivial, via $\overrightarrow{c}$.

For every formula $\psi(\overrightarrow{c}, \overrightarrow{x})$, effectively find a quantifier-free formula $\psi^*(\overrightarrow{c}, \overrightarrow{x})$, in the language with equality,

$\mathcal{A}_A \models [\psi(\overrightarrow{c}, \overrightarrow{x}) \iff \psi^*(\overrightarrow{c}, \overrightarrow{x})]$

(ii) $\mathcal{A}$ is non-trivial.

Let $X$ be such that $D^c(\mathcal{A}) \leq_T X$.

There are structure $\mathcal{B}$, isomorphism $f : \mathcal{A} \rightarrow \mathcal{B}$ with

$f \leq_T X \leq_T D_0(\mathcal{B})$

Thus,

$X \leq_T D_0(\mathcal{B}) \leq_T D^c(\mathcal{B}) \leq_T f \oplus D^c(\mathcal{A}) \leq_T X$
Intrinsic Collapse of Fragments of Diagrams

Questions

• (A) For fixed $n$, find syntactic conditions on $A$ such that $(\forall B \equiv A)[D^c(B) \equiv_T D_n(B)]$

• (B) For fixed $n$, find syntactic conditions on $A$ such that $(\forall B \equiv A)[D_{n+1}(B) \equiv_T D_n(B)]$.

(Harizanov, Knight, Morozov)

• (A) The following are equivalent:
  
  (i) For every structure $B \equiv A$ we have $D^c(B) \leq_T D_n(B)$

  (ii) There is $\overrightarrow{c} \in A^{<\omega}$ and a computable function $d$ assigning to every finitary formula $\gamma(\overrightarrow{x})$ a computable infinitary formula

  $d_\gamma(\overrightarrow{c}, \overrightarrow{x}) = \bigvee_{i \in W} \beta_i(\overrightarrow{c}, \overrightarrow{x})$, 

  where $W$ is c.e. and every $\beta_i$ is a finitary $\Sigma_{n+1}$ formula, such that

  $A_A \models (\forall \overrightarrow{x})[\gamma(\overrightarrow{x}) \iff d_\gamma(\overrightarrow{c}, \overrightarrow{x})]$

• (B) Similar characterization holds for $D_{n+1}(B) \leq_T D_n(B)$

In (ii), instead of an arbitrary finitary formula, $\gamma(\overrightarrow{x})$ is an arbitrary finitary $\Pi_{n+1}$ formula.
Proofs based of the following relativized Ash-Nerode theorem, similar to the one by Chisholm; Ash, Knight, Manasse and Slaman.

- (Harizanov, Knight, Morozov)
  For a sequence \((R_k)_{k \in \omega}\) of relations on a structure \(A\), the following are equivalent:

  \((i)\) For every isomorphism \(f\) from \(A\) to some structure \(B\), \(f(R_k)\) is c.e. relative to \(D_n(B)\), uniformly in \(k\).

  \((ii)\) For some \(\vec{c} \in A^{<\omega}\), for each \(k\), we can effectively find an index for a c.e. set of finitary \(\Sigma_{n+1}\) formulas with parameters \(\vec{c}\), whose disjunction defines \(R_k\).

- Examples
  \((i)\) For \(N \geq 1\), if \(B\) is a linear order of type \(\omega^N\), then
  \[D^c(B) \equiv_T D_{2N-1}(B)\]

  \((ii)\) If \(B\) is a linear order of type \(\omega^N \cdot \eta\), where \(\eta\) is the order type of rationals, then
  \[D^c(B) \equiv_T D_{2N}(B)\]
Turing Degrees of 1-Diagrams

• (Harizanov)
  Let \( Y \subseteq \omega \) be c.e. There is a computable linear order \( B \) of order type \( \omega \) whose successor relation \( S \) is Turing equivalent to \( Y \). Hence
  \[
  D_1(B) \equiv_T Y,
  \]
since Moses showed that \( D_1(B) \leq_T D_0(B) \oplus S \).

• (Harizanov, Knight, Morozov)
  Let \( Y \subseteq \omega \) be c.e. Let \( A \) be 1-decidable. Assume that for every \( \overrightarrow{c} \in A^{<\omega} \), we can effectively find a finitary \( \Pi_1 \) formula \( \theta(\overrightarrow{c}, \overrightarrow{u}) \) and \( \overrightarrow{a} \) such that
  \[
  \mathcal{A}_A \models \theta(\overrightarrow{c}, \overrightarrow{a}),
  \]
  and for every finitary \( \Sigma_1 \) formula \( \sigma(\overrightarrow{c}, \overrightarrow{u}) \),
  \[
  [\mathcal{A}_A \models \sigma(\overrightarrow{c}, \overrightarrow{a})] \Rightarrow (\exists a') [\mathcal{A}_A \models (\sigma(\overrightarrow{c}, a') \land \neg \theta(\overrightarrow{c}, \overrightarrow{a}'))].
  \]
  Then there is a computable structure \( B \simeq A \) such that
  \[
  D_1(B) \equiv_T Y.
  \]
Turing Degrees of the First Two Diagrams

- (Harizanov, Knight, Morozov)
  Let $X, Y \subseteq \omega$ be such that $Y$ is c.e. in and above $X$ and $D_0(A) \leq_T X$.

  Assume that $A$ satisfies the following conditions.

  (0) For every $\vec{c} \in A^{<\omega}$, we can effectively find a finitary $\Delta_1$ formula (given by a pair of finitary $\Sigma_1$, $\Pi_1$ formulas) $\chi(\vec{c}, \vec{x})$ such that

  \[
  A_A \models (\exists \vec{x})[\text{ran}(\vec{c}) \cap \text{ran}(\vec{x}) = \emptyset \land \chi(\vec{c}, \vec{x})]
  \]

  \[
  A_A \models (\exists \vec{x})[\text{ran}(\vec{c}) \cap \text{ran}(\vec{x}) = \emptyset \land \neg \chi(\vec{c}, \vec{x})].
  \]

  (1) For every $\vec{c} \in A^{<\omega}$, we can effectively find a finitary $\Pi_1$ formula $\theta(\vec{c}, \vec{x})$ and $\vec{a} \in A^{<\omega}$ such that

  \[
  A_A \models \theta(\vec{c}, \vec{a}),
  \]

  and for every finitary $\Sigma_1$ formula $\sigma(\vec{c}, \vec{x})$,

  \[
  [A_A \models \sigma(\vec{c}, \vec{a})] \Rightarrow (\exists \vec{a}')[A_A \models (\sigma(\vec{c}, \vec{a}')) \land \neg \theta(\vec{c}, \vec{a}'))]
  \]

  Then there is $B \simeq A$ such that

  \[
  D_0(B) \equiv_T X \land D_1(B) \equiv_T Y
  \]

- Example: $A = (\omega, <)$
  For a given $\vec{c} = (c_0, \ldots, c_{m-1})$, a corresponding $\Pi_1$ formula $\theta(\vec{c}, x, y)$ is

  \[
  [S(y, x) \land x > c_0 \land \ldots \land x > c_{m-1}],
  \]

  and a corresponding $\Delta_1$ formula $\chi(\vec{c}, x, y)$ is

  \[
  [(x < y) \land x > c_0 \land \ldots \land x > c_{m-1}].
  \]
Examples of $\mathcal{A}$ such that there exists $\mathcal{B} \cong \mathcal{A}$ with $D_0(\mathcal{B}) \equiv_T X$ and $D_1(\mathcal{B}) \equiv_T Y$, where $Y$ is c.e. in and above $X$

- Let $\mathcal{A}$ be the Boolean algebra $I(\omega)$
  A corresponding $\Pi_1$ formula uses a unary atom relation, and
  a corresponding $\Delta_1$ formula uses the binary relation of being disjoint.

- Let $\mathcal{A}$ be the Abelian group $\mathbb{Z}_p^\omega \oplus \mathbb{Z}_p^\omega$.
  A corresponding $\Pi_1$ formula uses the unary relation of not being divisible by $p$, and
  a corresponding $\Delta_1$ formula uses the binary relation of one element being a multiple by $p$ of the other element.
Turing Degrees of Sequences of $n$-Diagrams

• An $(N+1)$-$table$ is a sequence of sets $(C_n)_{n \leq N}$ such that $C_{n+1}$ is c.e. in and above $C_n$ for $n < N$.

• An $\omega$-$table$ is a sequence of sets $(C_n)_{n \in \omega}$, where $C_{n+1}$ is c.e. in and above $C_n$, uniformly in $n$.

• We say these tables are over $C_0$.

• For any structure $\mathcal{A}$, $(D_n(\mathcal{A}))_{n \in \omega}$ is an $\omega$-table.

• (Knight)
  
  For a sequence of Turing degrees $(d_n)_{n \in \omega}$, the following are equivalent:
  
  (i) There exists a nonstandard model $\mathcal{A}$ of $PA$ such that for all $n$,
      \[ \text{deg}(D_n(\mathcal{A})) = d_n. \]
  
  (ii) There exists an $\omega$-table $(C_n)_{n \in \omega}$, over a completion of $PA$, such that for all $n$,
       \[ \text{deg}(C_n) = d_n. \]
Questions

• For fixed $N \in \omega$, find conditions on $\mathcal{A}$ guaranteeing that for every $(N+1)$-table $(C_n)_{n \leq N}$ there exists $\mathcal{B} \cong \mathcal{A}$ such that
  
  \[(\forall n \leq N)[D_n(\mathcal{B}) \equiv_T C_n]\]

• Find conditions on $\mathcal{A}$ guaranteeing that for every $\omega$-table $(C_n)_{n \in \omega}$ there exists $\mathcal{B} \cong \mathcal{A}$ such that
  
  \[(\forall n \in \omega)[D_n(\mathcal{B}) \equiv_T C_n]\]
Back and Forth Relations

For simplicity, assume $A$ is a structure for a finite relational language.

**Convention:** If $\overrightarrow{a} \in A^{<\omega}$, assume the elements in $\overrightarrow{a}$ distinct. Assume concatenation $\overrightarrow{a} \overrightarrow{c}$ defined only if the elements in $\overrightarrow{a}$ distinct from the elements in $\overrightarrow{c}$.

(Barwise; Ash and Knight)

- Let $\overrightarrow{a}, \overrightarrow{b} \in A^{<\omega}$ be such that $lh(\overrightarrow{a}) \leq lh(\overrightarrow{b})$.
  
  (i) $\overrightarrow{a} \leq_0 \overrightarrow{b}$ iff the open formulas true of $\overrightarrow{a}$ are all true of $\overrightarrow{b}$.

  (ii) $\overrightarrow{a} \leq_{n+1} \overrightarrow{b}$ iff

  $$(\forall \overrightarrow{d})(\exists \overrightarrow{c})[\overrightarrow{b} \overrightarrow{d} \leq_n \overrightarrow{a} \overrightarrow{c}]$$

- $\overrightarrow{a} \leq_n \overrightarrow{b}$ iff the infinitary $\Pi_n$ formulas true of $\overrightarrow{a}$ are also true of $\overrightarrow{b}$. 
Independence of Formulas

(Harizanov, Knight, Morozov)

• Formula $\theta(\overrightarrow{u}, \overrightarrow{x})$ is 0-independent over $\overrightarrow{u}$ if it is open, and for every $\overrightarrow{c} \in A^{lh}(\overrightarrow{u})$, there exist $\overrightarrow{a}$ and $\overrightarrow{a}'$:

$$A_A \models \theta(\overrightarrow{c}, \overrightarrow{a}) \land \neg \theta(\overrightarrow{c}, \overrightarrow{a}')$$

• For $n > 0$, $\theta(\overrightarrow{u}, \overrightarrow{x})$ is $n$-independent over $\overrightarrow{u}$ if it is $\Pi_n$ and for every $\overrightarrow{c}$:

  (i) There exists $\overrightarrow{a}$ such that $A_A \models \theta(\overrightarrow{c}, \overrightarrow{a})$,

  (ii) For every $\overrightarrow{a}$ with $A_A \models \theta(\overrightarrow{c}, \overrightarrow{a'})$, and every $\overrightarrow{a_1}$, there exist $\overrightarrow{a'}$ and $\overrightarrow{a'_1}$ such that $A_A \models \neg \theta(\overrightarrow{c}, \overrightarrow{a'})$ and

$$\overrightarrow{c} \land \overrightarrow{a} \land \overrightarrow{a_1} \leq_{n-1} \overrightarrow{c} \land \overrightarrow{a'} \land \overrightarrow{a'_1}$$

• Examples: $\omega^\omega$, $\omega^\omega \cdot \eta$

  0-independent: $x$ is greater than elements in $\overrightarrow{u}$, and $x_0 < x_1$;

  1-independent: $\ldots$ and $S(x_1, x_0)$;

  2-independent: $\ldots$ and $x$ is a 1-limit (first in its copy of $\omega$);

  3-independent: $\ldots$ and $x_0$ and $x_1$ are 1-limits, where $x_1$ is the next one after $x_0$;

  4-independent: $\ldots$ and $x$ is a 2-limit (first in its copy of $\omega^2$), etc.

These successor relations and initial element relations were used by Moses.
Turing Degrees of Arbitrary Sequences of Diagrams

- (Harizanov, Knight, Morozov)
  Suppose $D_N(\mathcal{A})$ is computable, and the relations $\leq_n$ are c.e., for $n < N$. Suppose also that for each tuple $\vec{u}$ of distinct variables, and each $n \leq N$, we can find a formula that is $n$-independent over $\vec{u}$. Then for any $(N+1)$-table $(C_n)_{n \leq N}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that
  \[(\forall n \leq N)[D_n(\mathcal{B}) \equiv_T C_n]\]

- (Harizanov, Knight, Morozov)
  Suppose $D^c(\mathcal{A})$ is computable, and the relations $\leq_n$ are c.e., uniformly in $n$, for $n \in \omega$. Suppose also that for each tuple $\vec{u}$ of distinct variables, and each $n$, we can find a $\Pi_n$ formula that is $n$-independent over $\vec{u}$. Then for any $\omega$-table $(C'_n)_{n < \omega}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that
  \[D_n(\mathcal{B}) \equiv_T C_n, \text{ uniformly in } n.\]
  The uniformity implies that
  \[D^c(\mathcal{B}) \equiv_T \bigoplus_n C_n\]
Examples

• For any $2N$-table $(C_n)_{n \leq 2N - 1}$, there is a linear order $\mathcal{B}$ of type $\omega^N$ such that for all $n \leq 2N - 1$, $D_n(\mathcal{B}) \equiv_T C_n$.

• For any $(2N + 1)$-table $(C_n)_{n \leq 2N}$, there is a linear order $\mathcal{B}$ of type $\omega^N \cdot \eta$ such that for all $n \leq 2N$, $D_n(\mathcal{B}) \equiv_T C_n$.

• For any $\omega$-table $(C_n)_{n \in \omega}$, there is a linear order $\mathcal{B}$ of type $\omega^\omega$ or of type $\omega^\omega \cdot \eta$ such that for all $n$, $D_n(\mathcal{B}) \equiv_T C_n$, uniformly in $n$.

Open Problems

• Weaken the definition of $n$-independent formulas so that the general results still hold, but can be applied to Boolean algebras and other structures.

• What are the possible sequences $(\deg(D_n(\mathcal{B})))_{n \in \omega}$, for $\mathcal{B} \cong \mathcal{N}$?

• Characterize the sequences of Turing degrees of $n$-diagrams for models of a given completion of $PA$. 
Knight’s Conjecture

Let $S$ be a completion of $PA$ and let $(C_n)_{n \in \omega}$ be an $\omega$-table. Suppose that there exist an enumeration $R$ of a Scott set and a family of functions $(t_n)_{n \geq 1}$ such that:

- $R \leq_T C_0$,
- $t_n \leq_T C_{n-1}$ uniformly in $n$,
- $\lim_s t_n(s)$ is an $R$-index for $S \cap \Sigma_n$,
- for all $s$, $t_n(s)$ is an $R$-index for a subset of $S \cap \Sigma_n$.

Then there is a (nonstandard) model $A$ of $S$ such that

$$D_n(A) \equiv_T C_n,$$

uniformly in $n$.

Open Problem

- If $A$ is a non-standard model of $PA$, must there be $B \cong A$ such that $D_1(A) \leq_T D_1(B)$ and $D_0(B) <_T D_0(A)$?

Knight showed that for any nonstandard model of $PA$, there is an isomorphic copy whose atomic diagram has strictly lower Turing degree. The problem is to produce such a copy without lowering the degree of the 1-diagram.