Degrees of Structures

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• Consider countable structures $A$ for computable languages.

  *Turing degree* of $A$ is the Turing degree of the *atomic diagram of* $A$, $D(A)$. $A$ is *computable (recursive)* if its Turing degree is 0. $D(A)$ may be of much lower Turing degree than $Th(A)$.

• (Tennenbaum) If $A$ is a nonstandard model of $PA$, then $A$ is not computable.

• (Harrington, Knight) There is a nonstandard model $A$ of $PA$ such that $A$ is low and $Th(A) \equiv_T \emptyset(\omega)$.

• (Downey and Jockusch) Every Boolean algebra of low Turing degree has a computable copy.
• The *Turing degree spectrum* of $A$ is

$$DgSp(A) = \{\deg(B) : B \cong A\}.$$ 

• (Knight) A structure $A$ is *automorphically trivial* if there is a sequence $\vec{c} \in A^{<\omega}$ such that every permutation of $A$ that fixes $\vec{c}$ pointwise is an automorphism of $A$.

(i) If $A$ is automorphically trivial, then

$$|DgSp(A)| = 1.$$ 

(ii) If $A$ is automorphically nontrivial, then $DgSp(A)$ is closed upwards.
(Harizanov, Knight and Morozov)

(i) If $A$ is automorphically trivial, then

$$(\forall B \simeq A)[D^e(B) \equiv_T D(B)].$$

(ii) If $A$ is automorphically nontrivial, and $X \geq_T D^e(A)$, there exists $B \cong A$ such that

$$D^e(B) \equiv_T D(B) \equiv_T X.$$

(Harizanov and R. Miller) If the language of $A$ is finite, then $A$ is trivial iff and $DgSp(A) = \{0\}$. 

• (Hirschfeldt, Khoussainov, Shore and Slinko) For every automorphically nontrivial structure $A$, there is a structure $B$, which can be:

- a symmetric irreflexive graph,
- a partial order,
- a lattice,
- a ring,
- an integral domain of arbitrary characteristic,
- a commutative semigroup,
- a 2-step nilpotent group, such that

$$DgSp(A) = DgSp(B).$$
* $\mathcal{D}$ = the set of all Turing degrees

* (Wehner; Slaman)
  There is a structure $A$ such that
  \[
  DgSp(A) = \mathcal{D} - \{0\}.
  \]

* (Hirschfeldt)
  There is a complete decidable theory, with all types computable, whose prime model $A$ has no computable copy, but has an $X$-decidable copy for every $X \geq_T \emptyset$. 
• (Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon)
  For each computable successor ordinal $\alpha$, there is a structure $A$ such that
  $DgSp(A)$ consists of the Turing degrees of sets $X$ such that $\Delta^0_\alpha(X)$ is not $\Delta^0_\alpha$.

• In particular, for every $n \in \omega$, there is a structure $A$ such that
  $$DgSp(A) = \{ c \in \mathcal{D} : c^{(n)} > 0^{(n)} \}.$$ 

A degree $c$ is non-$\text{low}_n$ if $c^{(n)} > 0^{(n)}$. 
Enumerations

• An enumeration of $S \subseteq P(\omega)$ is a binary relation $\nu$:

$$S = \{\nu(i) : i \in \omega\}, \text{ where } \nu(i) = \{x : (i, x) \in \nu\}.$$  

$\nu$ is computable (c.e.) if it is computable (c.e.) as a binary relation.

• (Wehner)

There is a family $S$ such that for every $X >_T \emptyset$, $S$ has an enumeration computable in $X$, but $S$ has no computable enumeration.

• There is a family $S$ such that for every $X >_T \emptyset$, $S$ has an enumeration c.e. relative to $X$, but $S$ has no c.e. enumeration.
Transforming $S$ into a graph

- Assign to $A \in S$, a *daisy graph* $G_A$ consisting of one
  *index* point $a$ at the center with $a \to a$, and for each $n \in A$
  a *petal* (disjoint from other petals)

  $$a \to a_0 \to \cdots \to a_n \to a$$

- $G(S)$ is the union of a disjoint family of $G_A$ for each $A \in S$.
  $G(S)$ is a rigid graph.

- $G^\infty(S)$ consists of infinitely many copies of $G_A$ for each $A \in S$.
  $G^\infty(S)$ is not rigid.
  Copies of $G^\infty(S)$ correspond to enumerations of $S$. 
Let $S \subseteq P(\omega)$, $X \subseteq \omega$.

- There is an enumeration of $S$ c.e. in $X$ iff there is a copy of $G^\infty(S)$ computable in $X$.

- $S^+ =_{def} \{ A \oplus \overline{A} : A \in S \}$.

- There is an enumeration of $S$ computable in $X$ iff there is a copy of $G^\infty(S^+)$ computable in $X$. 
• (Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon)

Let $\alpha \geq 2$ be a computable successor ordinal.
There is a structure with copies in exactly the Turing degrees of sets $X$ such that $\Delta^0_\alpha(X)$ is not $\Delta^0_\alpha$.

• Proof sketch. Relativize the proof for $\Delta^0_1$ to $\Delta^0_\alpha$.

Get a graph $G$ such that the degrees of copies of $G$ are just the degrees of sets that are not $\Delta^0_\alpha$.

• Code a directed graph $G$ in a structure $G^*$ such that:

$G$ has a $\Delta^0_\alpha$ copy iff $G^*$ has a computable copy.
More generally, for any $X \subseteq \omega$,
$G$ has a $\Delta^0_\alpha(X)$ copy iff $G^*$ has an $X$-computable copy.
Proof sketch. Code a directed graph $G$ in a structure $G^*$, using a pair of structures $B_0, B_1$ such that

- $B_0$ codes $G \models a \rightarrow b$ and
- $B_1$ codes $G \models \neg (a \rightarrow b)$.

$G^* = (G \cup U, G, U, Q, \ldots)$, where

- $G$ and $U$ are disjoint,
- $Q$ (a ternary relation) assigns to $a, b \in G$ an infinite set $U_{(a,b)}$:
  $(x \in U_{(a,b)} \iff Qabx)$,
- the sets $U_{(a,b)}$ form a partition of $U$,

$$(U_{(a,b)}, \ldots) \cong \begin{cases} 
  B_0, & \text{if } G \models a \rightarrow b, \\
  B_1, & \text{if } G \models \neg (a \rightarrow b). 
\end{cases}$$
Assume

- Pair \( \{B_0, B_1\} \) is \( \alpha \)-friendly.

- \( B_0 \) and \( B_1 \) satisfy the same infinitary \( \Pi_\beta \) sentences for \( \beta < \alpha \).

- \( B_0 \) satisfies some computable \( \Pi_\alpha \) sentence that is not true in \( B_1 \), and *vice versa*.

Then for any \( \Delta_\alpha^0 \) set \( S \), there is a uniformly computable sequence \( (C_n)_{n \in \omega} \) such that

\[
C_n \simeq \begin{cases} 
B_0, & \text{if } n \in S, \\
B_1, & \text{if } n \notin S.
\end{cases}
\]
• (R. Miller)
  There is a linear order $A$ such that
  \[ DgSp(A) \cap \Delta^0_2 = \Delta^0_2 - \{0\}. \]

• (Harizanov and R. Miller)
  There exists a structure $A$ such that $DgSp(A)$ consists of the degrees that are high-or-above:
  \[ DgSp(A) = \{c \in \mathcal{D}: c' \geq 0''\}. \]

• A degree $c$ is high if $c' = 0''$. 

• \((\omega, \prec)\) computable linear order
  Computable isomorphism \(f : L = (\omega, \prec) \rightarrow (\mathbb{Q}, \prec)\).

• (Harizanov and R. Miller)
  For any relation \(R\) on \(L\), there exists a structure \(A\) such that
  \[
  DgSp(A) = DgSp_L(R).
  \]

• Define a relation \(R\) on \(L\) by:
  \[
  f(R) = \left( -1, -\frac{1}{2} \right) \cup \left( \bigcup_{n \in \mathbb{N}} \left[ n, n + \frac{1}{2} \right) \right) \cup \left( \bigcup_{n \notin \mathbb{N}} \left( n - \frac{1}{\pi}, n + \frac{1}{2} \right) \right)
  \]
• $DgSp_L(R) = \{ c \in D : c' \geq 0'' \}$.

• Proof sketch. Show
  
  $c \in DgSp_L(R)$ iff $\emptyset''' \leq_1 Fin^C$

  for some set $C$ with $deg(C) = c$

• $Fin^C = \{ e : W^C_e \text{ is finite} \}$

• $\emptyset''' \leq_1 Fin^C \iff \emptyset'' \leq_T C'$
• (Jockusch) The (Turing) degree of the isomorphism type of $A$, if it exists, is the least Turing degree in $DgSp(A)$.

• (Richter) Assume that a structure $A$ satisfies the effective extendability condition. If the degree of the isomorphism type of $A$ exists, then it must be $0$. ($DgSp(A)$ will contain a minimal pair of degrees.)

• **Effective Extendability Condition for $A$**

  For every finite structure $C$ isomorphic to a substructure of $A$, and every embedding $f$ of $C$ into $A$, there is an algorithm that determines whether a given finite structure $D$ extending $C$ can be embedded into $A$ by an embedding extending $f$. 
• (Richter)
  (i) A linear order without a computable copy does not have the isomorphism type degree.

  (ii) A tree without a computable copy does not have the isomorphism type degree.

• Abelian $p$-group $G$

  $$x \in (G - \{0\}) \Rightarrow (\exists n)[\text{order}(x) = p^n]$$

• (A. Khisamiev)

  An abelian $p$-group without a computable copy does not have the isomorphism type degree.
**Richter’s Combination Method**

Let $T$ be a theory in a finite language $L$ such that there is a computable sequence $A_0, A_1, A_2, \ldots$ of finite structures for $L$, which are pairwise nonembeddable. Assume that for every $X \subseteq \omega$, there is a model $A_X$ of $T$ such that

$$A_X \preceq_T X,$$

and for every $i \in \omega$,

$$A_i \text{ is embeddable in } A_X \iff i \in X.$$

Then for every Turing degree $d$, there is a model of $T$ whose isomorphism type has degree $d$.

For every Turing degree $d$, there is an *abelian group* whose isomorphism type has degree $d$. 
• (Calvert, Harizanov, Shlapentokh)
  For every Turing degree $d$, there are various fields whose isomorphism types have degree $d$.

• *Proof sketch.* Let $M_0 = F$ be any computable finitely generated field.
  $\tilde{F}$ the algebraic closure of $F$.
  $\{f_i(t) \in F(t)\}_{i \geq 1}$ computable sequence of monic irreducible polynomials (over $F$).
  $\alpha_i$ a root of $f_i$, and $M_i = F(\alpha_i)$.
  Assume further that the sequence $\{M_i\}_i$ is *totally linearly disjoint* over $F$,
  and is *stable* with respect to $F$.

• Let $A_X = \prod_{i \in X} M_i$, where $X = D \oplus \overline{D}$.
  $DgSp(A_X) = \{c \in D : c \geq \text{deg}(D)\}$. 
• Let $F$ be a field, $\{L_i\}_{i \in \omega}$ a sequence of extensions of $F$. Let $L = \prod_{i \in \omega} L_i$.

• $\{L_i\}_{i \in \omega}$ is totally linearly disjoint over $F$ if the extensions are finite, and for all $i$, $L_i$ and $\prod_{j \in \omega \setminus \{i\}} L_j$ are linearly disjoint over $F$:

$$[L_i : F] = [L : \prod_{j \in \omega \setminus \{i\}} L_j] > 1.$$ 

• $\{L_i\}_{i \in \omega}$ is stable with respect to $F$ if for any embedding $\sigma : L \rightarrow \tilde{F}$ ($\tilde{F}$ is the algebraic closure of $F$), such that $\sigma|_F = id$, then for all $i$,

either $\sigma(L_i) = L_i$ or $\sigma(L_i) \not\subset L$.

$\{L_i\}_{i \in \omega}$ is stable if $F = \mathbb{Q}$, or $F$ is a finite field.
• Let $F = \mathbb{Q}$.
  
  $\{p_i\}_i$ listing of rational primes.
  
  $f_i(t) = t^2 - p_i$
  
  $M_i = \mathbb{Q}(\sqrt{p_i})$
  
  (Sequence $\{M_i\}_i$ is stable, and totally linearly disjoint over $\mathbb{Q}$.)

• Let $F = \mathbb{Q}(x)$, where $x$ is not algebraic over $\mathbb{Q}$.
  
  $M_i = \mathbb{Q}(x, \sqrt{p_i}, \sqrt[3]{x^2 + 1})$
  
  (Sequence $\{M_i\}_i$ is stable with respect to $\mathbb{Q}(x)$, and totally linearly disjoint over $\mathbb{Q}(x)$.)
• Let $F = \mathbb{F}_p$ be a field of $p$ elements for some rational prime $p$.
  Let $\alpha_i$ be of degree $p_i$ over $\mathbb{F}_p$.
  $M_i = \mathbb{F}_p(\alpha_i)$.
  (Sequence $\{M_i\}_i$ is stable, and totally linearly disjoint over $\mathbb{F}_p$.)

• Let $F = \mathbb{F}_p(x)$, where $x$ is not algebraic over $\mathbb{F}_p$.
  Let $M_i = \mathbb{F}_p(\sqrt{x^2 + i})$.
  (Sequence $\{M_i\}_i$ is stable with respect to $\mathbb{F}_p(x)$,
  and totally linearly disjoint over $\mathbb{F}_p(x)$.)