ON THE LEARNABILITY OF VECTOR SPACES

by

Valentina S. Harizanov
George Washington University
Washington, D.C., USA

and

Frank Stephan
Mathematisches Institut
Universität Heidelberg
Heidelberg, Germany
1 Computably enumerable vector spaces

- $V_\infty$
  computable $\aleph_0$-dimensional vector space over a computable field (will assume $\mathbb{Q}$) with (uniformly computable) dependence relations $(D_n)_{n \in \omega}$ (dependence algorithm)

- $(L(V_\infty), \subseteq, \cap, +)$
  the lattice of c.e. vector subspaces of $V_\infty$
  nondistributive, modular

- $V_0, V_1, V_2, \ldots$
  computable enumeration of all c.e. subspaces of $V_\infty$
  $I_0, I_1, I_2, \ldots$
  computable enumeration of all c.e. independent subsets of $V_\infty$
  $V_e = cl(I_e)$
• $V_\infty$ is computable (c.e.) iff $V$ is computable (c.e.)

• $L(V) = \{W : V \subseteq W \subseteq V_\infty \land (W \text{ is c.e.})\}$ for c.e. $V$

principal filter
Algorithmic learning theory: inductive inference

E. M. Gold (1967): "Language identification in the limit"

Learning is a dialogue between a teacher and a learner.

A learner is a Turing machine that receives more and more data of a c.e. set $W$ to be learned, and outputs a sequence of indices $e_0, e_1, e_2, ...$ that converges to the “description” of $W$.

- $EX$ class (syntactic convergence)
  explanatory learning
  For almost every $n$, $e_n$ is the same hypothesis $e$, which is an index of $W$. 
• **BC** class (semantic convergence)

  *behaviorally correct* learning

  For almost every \( n \), the hypothesis \( e_n \) is an index of \( W \). (These indices are permitted to be distinct.)

• **Txt**

  Learning from *text*: the learner requests only *positive* (type 1) information, and the teacher eventually provides all elements in the set \( W \).

• **Inf**

  Learning from *informant*: the learner alternately requests information of type 0 and type 1, and the teacher eventually provides every \( x \in \omega \), after some request of the type \( W(x) \).

• **Sw**

  Learning from *switching* type of information: when the learner almost always requests information of the same type \( y \), the teacher eventually gives all \( x \) with \( W(x) = y \).
• Learning from text is much more restrictive than learning from informant. (For example, the collection consisting of an infinite c.e. set together with all of its finite subsets can be learned from informant, but not from text.)

• Switching type of information provides more learning power than giving positive information only; it is still much weaker than providing information from informant.
3 Learning principal filters

Theorem 1 (Harizanov, Stephan)

TFAE for $V \in L(V_\infty)$:

(a) $\dim \frac{V_\infty}{V} < \infty$;

(b) The class $L(V)$ is $T_{txEx}$-learnable;

(c) The class $L(V)$ is $T_{txBC}$-learnable.

(d) The class $L(V)$ is $S_{wEx}$-learnable.

Sketch of the proof

• (a) $\Rightarrow$ (b)
  There is an algorithm that checks for every finite set $D \subset V_\infty$ and a vector $x$, whether $x$ is in the linear closure of $V \cup D$.

• (b) $\Rightarrow$ [(c) and (d)]
  This follows directly from the definitions.
• (c) ⇒ (a)

$L(V)$ is $TxtBC$-learnable $⇒ dim \frac{V}{V_\infty} < \infty$

Assume $V \neq V_\infty$.

$V_\infty = \{v_0, v_1, \ldots\}$ computable enumeration

$U_n = cl(V \cup \{v_0, v_1, \ldots, v_n\})$

$V_\infty$ is the ascending union of $U_n$, $n \in \omega$.

Chain must be finite.

• (d) ⇒ (a)

$L(V)$ is $SwEx$-learnable $⇒ dim \frac{V}{V_\infty} < \infty$

Assume $L(V)$ is $SwEx$-learnable and

$dim(V_\infty/V) = \infty$.

We can show that $V$ is computable.

Let $\{w_0, w_1, \ldots\}$ be a computable basis of a vector space $U$ with $U \cap V = \{0\}$.

$W =_{def} cl(V \cup \{w_x : x \in K\})$ not computable.

$L(W)$ and, hence, $L(V)$ not $SwEx$-learnable.
4    Maximal, $k$-thin and strongly supermaximal spaces

Let $V \in L(V_\infty)$ and $\dim \frac{V_\infty}{V} = \infty$.

The space $V$ is maximal iff for every c.e. subspace $W$,

$$V \subseteq W \subseteq V_\infty \Rightarrow [\dim \frac{W}{V} < \infty \lor \dim \frac{V_\infty}{W} < \infty].$$

**Theorem 2** (Metakides-Nerode) There exists a maximal subspace of $V_\infty$.

**Theorem 3** (Shore) Every maximal subset $M$ of a computable basis $B$ of $V_\infty$ generates a maximal subspace of $V_\infty$. 
• $V \in L(V_{\infty})$ is called supermaximal (0-thin) iff for every c.e. subspace $W$,

$$V \subseteq W \subseteq V_{\infty} \Rightarrow [\dim \frac{W}{V} < \infty \lor W = V_{\infty}].$$

• $V \in L(V_{\infty})$ is $k$-thin iff $\dim(V_{\infty}/V) = \infty$ and for every c.e. $W \supseteq V$,

either $\dim(V_{\infty}/W) \leq k$ or $\dim(W/V) < \infty$,

and there exists $U \in L(V_{\infty})$ such that $U \supseteq V$ and $\dim(V_{\infty}/U) = k$.

**Theorem 4** *(Kalantari-Retzlaaff)* There exists a supermaximal subspace of $V_{\infty}$.
• The space $V$ is called strongly supermaximal if for every c.e. subset $X \subseteq V_\infty - V$,

$$(\exists a_0, \ldots, a_{n-1} \in V_\infty)[X \subseteq cl(V \cup \{a_0, \ldots, a_{n-1}\})].$$

**Theorem 5 (Hird-Downey)** There exists a strongly supermaximal subspace of $V_\infty$.

(Hird, Kalantari-Retzlaff)

$$SSMAX \subset SMAX \subset MAX$$
5 Characterizing $SwBC$ learnable principal filters

Theorem 6 (Harizanov, Stephan)

TFAE for $V \in L(V_\infty)$:

(a) The class $L(V)$ is $SwBC$-learnable;

(b) $(\dim V_\infty < \infty) \lor (V$ is 0-thin $(SM)) \lor (V$ is 1-thin).

- Sketch of the proof

(a) $\Rightarrow$ (b) Uses the following general theorem.
Theorem 7 (Harizanov, Stephan) Let $L$ be a set of c.e. sets. Assume that there is $W \in L$ such that for every finite set $D$, there are $U, U' \in L$ with

$U \subset W \subset U'$ and

$D \cap U = D \cap U'$.

Then $L$ cannot be $SwBC$-learned.
6 Learning from informant

Theorem 8 (Harizanov, Stephan)

(i) There is a strongly supermaximal (and hence 0-thin) vector space $V$ such that $\overline{K}$, the complement of the halting set, is uniformly enumerable relative to every c.e vector space $W$ with $V \subseteq W \subset V_\infty$. Hence, $L(V)$ is InfEx-learnable.

(ii) There is a strongly supermaximal vector space $V$ such that the class $L(V)$ is not InfEx-learnable.

- For $i \in \omega$, let $l_i$ be the linear mapping defined by $l_i(\epsilon_j) = \epsilon_{\langle i,j \rangle}$ for every $j$, where $\{\epsilon_0, \epsilon_1, \epsilon_2, \ldots\}$ is a standard basis for $V_\infty$.

$$V_{lp} = \text{def} \text{ cl}( \ l_0(V) \cup l_1(V) \cup \ldots$$
Theorem 9 (Harizanov, Stephan)

(i) There is a strongly supermaximal (hence 0-thin) vector space $V$ such that the class $L(V_{lp})$ is $\text{InfEx}$-learnable.

(ii) There is a strongly supermaximal space $V$ such that $L(V_{lp})$ is not $\text{InfBC}$-learnable.

Theorem 10 (Harizanov, Stephan) Assume that $V_\infty$ is over a finite field $F$. Let $V \in L(V_\infty)$.

(i) $L(V)$ is $\text{SwEx}$-learnable iff $\dim V_\infty < \infty$.

(ii) $L(V)$ is $\text{SwBC}$-learnable iff $(\dim \frac{V_\infty}{V} < \infty \text{ or } V \text{ is } k\text{-thin})$.

(iii) $L(V)$ is not $\text{SwBC}$-learnable iff there is a c.e space $W$ such that $V \subset W \subset V_\infty$ and $(\dim \frac{V_\infty}{W} = \infty \land \dim \frac{W}{V} = \infty)$. 
An infinite set $X$ and an operator $\Phi$ form a *computable matroid* $(X, \Phi)$ if $\Phi$ maps c.e. subsets of $X$ to c.e. subsets of $X$ and satisfies:

- $Y \subseteq Z \implies \Phi(Y) \subseteq \Phi(Z)$,
- $Y \subseteq \Phi(Y) \land \Phi(\Phi(Y)) = \Phi(Y)$,
- $(\forall x \in \Phi(Y))(\exists$ finite $D \subseteq Y)[x \in \Phi(D)]$,
- $[Y = \Phi(Y) \land x, y \notin Y \land x \in \Phi(Y \cup \{y\})] \implies y \in \Phi(Y \cup \{x\})$,
- There is a computable function $f$ with $\Phi(W_e) = W_{f(e)}$ for all $e$. 
Theorem 11 (Harizanov, Stephan) Let $L$ be the class of c.e. submatroids of $(X, \Phi)$. TFAE:

(a) the class $L$ is $SwEx$-learnable relative to $K$;
(b) the class $L$ is $SwBC$-learnable relative to $K$;
(c) it is not true that there is $W \in L$ such that for every finite set $D$, there are $U, U' \in L$ with $U \subset W \subset U'$ and $D \cap U = D \cap U'$.

• There is a computable matroid such that every c.e. submatroid is either finite or cofinite. The class of these submatroids can be $SwEx$-learned, but not with any bound on the number of switches.

• There is a computable matroid whose class of all c.e. submatroids can be $SwBC$-learned with oracle $K$, but not with any oracle $A \not\equiv_T K$. 