On-Line Appendix for
“Portfolio Selection with a Drawdown Constraint”
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This Appendix provides proofs of Theorems 1–3 in the paper “Portfolio Selection with a Drawdown Constraint” published in the Journal of Banking and Finance 30, 3171–3189, November 2006.

Proof of Theorem 1. First, assume $D[r_{w_E}] \leq D$. The desired result follows from Eq. (1). This completes the first part of our proof.

Second, assume $D[r_{w_E}] > D$. By definition, $D[r_{w_E,D}] \leq D$. We claim that $D[r_{w_{E,D}}] = D$. Suppose by way of a contradiction that $D[r_{w_{E,D}}] < D$. Since $w_E$ is on the mean-variance boundary, $\sigma[r_{w_E}] < \sigma[r_{w_{E,D}}]$. Let $w^* \equiv \varepsilon w_E + (1 - \varepsilon)w_{E,D}$, where $\varepsilon > 0$ is arbitrarily small. Note that $E[r_{w^*}] = E$, $\sigma[r_{w^*}] < \sigma[r_{w_{E,D}}]$, and $D[r_{w^*}] < D$, a contradiction of the fact that $w_{E,D}$ is on the constrained mean-variance boundary. Hence, $D[r_{w_{E,D}}] = D$.

Let $s_3, ..., s_{K+2}$ be the states at which the constraint binds. Then, $w_{E,D}$ solves
\[
\min_{w \in \mathbb{R}^J} \frac{1}{2} w^\top \Sigma w \\
\text{s.t. } w^\top \iota = 1 \\
w^\top \mu = E \\
w^\top r_s = -D \quad \forall s \in \{s_3, ..., s_{K+2}\} \\
w^\top r_s \geq -D \quad \forall s \notin \{s_3, ..., s_{K+2}\}.
\]
Note that constraint (16) does not bind. First-order necessary and sufficient conditions for \( w_{E,D} \) to solve problem (12) subject to constraints (13)–(15) are

\[
\Sigma w_{E,D} - \lambda_1 \iota - \lambda_2 \mu - \sum_{k=3}^{K+2} \lambda_k r_{s_k} = 0 \quad (17)
\]

\[
w_{E,D}^\top \iota = 1 \quad (18)
\]

\[
w_{E,D}^\top \mu = E \quad (19)
\]

\[
w_{E,D}^\top r_s = -D, \ s = s_3, \ldots, s_{K+2}, \quad (20)
\]

where \( \lambda_1, \ldots, \lambda_{K+2} \) are Lagrange multipliers associated with constraints (13)–(15). Since \( \text{rank}(\Sigma) = J \), Eq. (17) implies that

\[
w_{E,D} = \lambda_1 (\Sigma^{-1} \iota) + \lambda_2 (\Sigma^{-1} \mu) + \sum_{k=3}^{K+2} \lambda_k (\Sigma^{-1} r_{s_k}). \quad (21)
\]

Eq. (21) implies that Eq. (3) holds with

\[
[\varphi_1 \ \varphi_2 \ \varphi_3 \ \cdots \ \varphi_{K+2}] \equiv [\lambda_1 \ c \ \lambda_2 a \ \lambda_3 e_{s_3} \ \cdots \ \lambda_{K+2} e_{s_{K+2}}].
\]

We now find the \((K + 2) \times 1\) vector \(L \equiv [\lambda_1 \ \cdots \ \lambda_{K+2}]^\top\). Premultiplying Eq. (21) by \(\iota^\top\) and using Eq. (18), we have

\[
\lambda_1 (\iota^\top \Sigma^{-1} \iota) + \lambda_2 (\iota^\top \Sigma^{-1} \mu) + \sum_{k=3}^{K+2} \lambda_k (\iota^\top \Sigma^{-1} r_{s_k}) = 1. \quad (22)
\]

Premultiplying Eq. (21) by \(\mu^\top\) and using Eq. (19), we obtain

\[
\lambda_1 (\mu^\top \Sigma^{-1} \iota) + \lambda_2 (\mu^\top \Sigma^{-1} \mu) + \sum_{k=3}^{K+2} \lambda_k (\mu^\top \Sigma^{-1} r_{s_k}) = E. \quad (23)
\]

Premultiplying Eq. (21) by \(r_s^\top\) and using Eq. (20), we have

\[
\lambda_1 (r_s^\top \Sigma^{-1} \iota) + \lambda_2 (r_s^\top \Sigma^{-1} \mu) + \sum_{k=3}^{K+2} \lambda_k (r_s^\top \Sigma^{-1} r_{s_k}) = -D, \ s = s_3, \ldots, s_{K+2}. \quad (24)
\]

Let the \((K + 2) \times 1\) vector \(A_{E,D}\) denote \([1 \ E \ -D \ \cdots \ -D]^\top\). Let \(M \equiv [\iota \ \mu \ r_{s_3} \ \cdots \ r_{s_{K+2}}]\) and \(N \equiv M^\top \Sigma^{-1} M\). Using Eqs. (22)–(24), \(NL = A_{E,D}\). Since \(\text{rank}(\Sigma) = J\) and
rank(M) = K + 2, rank(N) = K + 2. Hence, L = N^{-1}A_{E,D}. This completes the second part of our proof. ■

**Proof of Theorem 2.** First, assume \( D[r_{we}] \leq D \). The desired result follows from Eq. (4). This completes the first part of our proof.

Second, assume \( D[r_{we}] > D \). It follows from the argument used in the proof of Theorem 1 that \( D[r_{we,o}] = D \). Let \( s_3, ..., s_{K+2} \) be the states at which the constraint binds. Let \( \overline{w} \) be the vector that consists of the first \( J \) elements of \( w_{E,D} \). Then, \( \overline{w} \) solves

\[
\min_{w \in \mathbb{R}^J} \frac{1}{2} \mathbf{w}^\top \Sigma \mathbf{w}
\]

s.t. \( \mathbf{w}^\top \mathbf{\mu} + (1 - \mathbf{w}^\top \mathbf{\iota}) \mathbf{r}_f = E \)  
\[\mathbf{w}^\top \mathbf{r}_s + (1 - \mathbf{w}^\top \mathbf{\iota}) \mathbf{r}_f = -D \quad \forall s \in \{s_3, ..., s_{K+2}\} \]
\[\mathbf{w}^\top \mathbf{r}_s + (1 - \mathbf{w}^\top \mathbf{\iota}) \mathbf{r}_f \geq -D \quad \forall s \notin \{s_3, ..., s_{K+2}\}.\]

Note that constraint (28) does not bind. First-order necessary and sufficient conditions for \( \overline{w} \) to solve problem (25) subject to constraints (26) and (27) are

\[
\Sigma \overline{w} - \gamma_1 (\mathbf{\mu} - \mathbf{\iota} \mathbf{r}_f) - \sum_{k=3}^{K+2} \gamma_k (\mathbf{r}_s - \mathbf{\iota} \mathbf{r}_f) = 0
\]
\[
\overline{w}^\top (\mathbf{\mu} - \mathbf{\iota} \mathbf{r}_f) = E - \mathbf{r}_f
\]
\[
\overline{w}^\top (\mathbf{r}_s - \mathbf{\iota} \mathbf{r}_f) = -D - \mathbf{r}_f, \quad s = s_3, ..., s_{K+2},
\]

where \( \gamma_1, \gamma_3, ..., \gamma_{K+2} \) are Lagrange multipliers associated with constraints (26) and (27).

Since \( \text{rank}(\Sigma) = J \), Eq. (29) implies that

\[
\overline{w} = \gamma_1 \left[ \Sigma^{-1} (\mathbf{\mu} - \mathbf{\iota} \mathbf{r}_f) \right] + \sum_{k=3}^{K+2} \gamma_k \left[ \Sigma^{-1} (\mathbf{r}_s - \mathbf{\iota} \mathbf{r}_f) \right].
\]

Eq. (32) implies that Eq. (6) holds with

\[
[\theta_1 \quad \theta_2 \quad \theta_3 \quad \cdots \quad \theta_{K+2}] \equiv \left[ 1 - \sum_{k=2}^{K+2} \gamma_k (a - cr_f) \right] \gamma_3 (e_{s_3} - cr_f) \cdots \gamma_{K+2} (e_{s_{K+2}} - cr_f)]
\]
We now find the \((K+1) \times 1\) vector \(Q \equiv [\gamma_1 \, \gamma_3 \, \cdots \, \gamma_{K+2}]^\top\). Premultiplying Eq. (32) by \((\mu - \nu_f)^\top\) and using Eq. (30), we have
\[
\gamma_1 \left[ (\mu - \nu_f)^\top \Sigma^{-1} (\mu - \nu_f) \right] + \sum_{k=3}^{K+2} \gamma_k \left[ (\mu - \nu_f)^\top \Sigma^{-1} (r_{s_k} - \nu_f) \right] = E - r_f. \tag{33}
\]
Premultiplying Eq. (32) by \((r_s - \nu_f)^\top\) and using Eq. (31), we obtain
\[
\gamma_1 \left[ (r_s - \nu_f)^\top \Sigma^{-1} (\mu - \nu_f) \right] + \sum_{k=3}^{K+2} \gamma_k \left[ (r_s - \nu_f)^\top \Sigma^{-1} (r_{s_k} - \nu_f) \right] = -D - r_f, \tag{34}
\]
s = \(s_3, \ldots, s_{K+2}\). Let the \((K+1) \times 1\) vector \(B_{E,D}\) denote \([E - r_f \, -D - r_f \, \cdots \, -D - r_f]^\top\).

Let \(R \equiv [\mu - \nu_f \, r_{s_3} - \nu_f \, \cdots \, r_{s_{K+2}} - \nu_f]\) and \(T \equiv R^\top \Sigma^{-1} R\). Using Eqs. (33)–(34), \(TQ = B_{E,D}\). The fact that \(\text{rank}(M) = K+2\) implies that \(\text{rank}(R) = K+1\). Since \(\text{rank}(\Sigma) = J\) and \(\text{rank}(R) = K+1\), \(\text{rank}(T) = K+1\). Hence, \(Q = T^{-1} B_{E,D}\). This completes the second part of our proof. \(\blacksquare\)

**Proof of Theorem 3.** First, assume \(D^\varepsilon[r_{w_{E}^\varepsilon}] \leq D^\varepsilon\). The desired result follows from Eqs. (7) and (8). This completes the first part of our proof.

Second, assume \(D^\varepsilon[r_{w_{E}^\varepsilon}] > D^\varepsilon\). It follows from arguments similar to those used in the proof of Theorem 1 that \(D^\varepsilon[r_{w_{E,D}^\varepsilon}] = D^\varepsilon\). Let \(s_3, \ldots, s_{K+2}\) be the states at which the constraint binds. Then, \(\pi = w_{E,D}^\varepsilon - w_b\) solves

\[
\min_{x \in \mathbb{R}^J} \frac{1}{2} x^\top \Sigma x \tag{35}
\]
\[
s.t. \quad x^\top l = 0 \tag{36}
\]
\[
x^\top \mu = E - E[r_w] \tag{37}
\]
\[
x^\top r_s = -D^\varepsilon \quad \forall s \in \{s_3, \ldots, s_{K+2}\} \tag{38}
\]
\[
x^\top r_s \geq -D^\varepsilon \quad \forall s \notin \{s_3, \ldots, s_{K+2}\}. \tag{39}
\]
Note that constraint (39) does not bind. First-order necessary and sufficient conditions for \( \bar{x} \) to solve problem (35) subject to constraints (36)–(38) are

\[
\Sigma \bar{x} - \delta_1 \iota - \delta_2 \mu - \sum_{k=3}^{K+2} \delta_k r_{s_k} = 0
\]

\[
\bar{x}^\top \iota = 0
\]

\[
\bar{x}^\top \mu = E - E[r_{w_k}]
\]

\[
\bar{x}^\top r_s = -D^\varepsilon \forall s \in \{s_3, \ldots, s_{K+2}\},
\]

where \( \delta_1, \ldots, \delta_{K+2} \) are Lagrange multipliers associated with constraints (36)–(38). Since \( \text{rank}(\Sigma) = J \), Eq. (40) implies that

\[
\bar{x} = \delta_1 (\Sigma^{-1} \iota) + \delta_2 (\Sigma^{-1} \mu) + \sum_{k=3}^{K+2} \delta_k (\Sigma^{-1} r_{s_k}).
\]

Eq. (44) implies that Eqs. (10) and (11) hold with

\[
[\pi_1 ~ \pi_2 ~ \pi_3 ~ \cdots ~ \pi_{K+2}] \equiv [\delta_1 \ c \ \delta_2 a \ \delta_3 e_s \ \cdots \ \delta_{K+2} e_{s_{K+2}}],
\]

We now find the \( (K + 2) \times 1 \) vector \( U \equiv [\delta_1 \ \cdots \ \delta_{K+2}]^\top \). Premultiplying Eq. (44) by \( \iota^\top \) and using Eq. (41), we have

\[
\delta_1 (\mu^\top \Sigma^{-1} \iota) + \delta_2 (\mu^\top \Sigma^{-1} \mu) + \sum_{k=3}^{K+2} \delta_k (\mu^\top \Sigma^{-1} r_{s_k}) = 0.
\]

Premultiplying Eq. (44) by \( \mu^\top \) and using Eq. (42), we obtain

\[
\delta_1 (\mu^\top \Sigma^{-1} \mu) + \delta_2 (\mu^\top \Sigma^{-1} \mu) + \sum_{k=3}^{K+2} \delta_k (\mu^\top \Sigma^{-1} r_{s_k}) = E - E[r_{w_k}].
\]

Premultiplying Eq. (44) by \( r_s^\top \) and using Eq. (43), we have

\[
\delta_1 (r_s^\top \Sigma^{-1} \iota) + \delta_2 (r_s^\top \Sigma^{-1} \mu) + \sum_{k=3}^{K+2} \delta_k (r_s^\top \Sigma^{-1} r_{s_k}) = -D^\varepsilon, \ s = s_3, \ldots, s_{K+2}.
\]

Let the \( (K + 2) \times 1 \) vector \( C_{E,D^\varepsilon} \) denote \([0 \ E - E[r_{w_k}] \ -D^\varepsilon \ \cdots \ -D^\varepsilon]^\top \). Eqs. (45) and (46) imply that \( NU = C_{E,D^\varepsilon} \). Since \( \text{rank}(N) = K + 2 \), \( U = N^{-1}C_{E,D^\varepsilon} \). This completes the second part of our proof. \( \blacksquare \)