

Research Notes for Anisotropic Willmore problem

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1 Isotropic Willmore

Consider the isotropic Willmore problem:

$$\text{Min}_{\mathcal{A}} \int_{\Gamma} H^2 dA$$

with area and volume constraints.

For the axis-symmetric case, we can simplify above problem into:

$$\text{Min}_{\mathcal{A}} \int_0^\pi H^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

subject to

$$\begin{aligned} x\sqrt{\dot{x}^2 + \dot{y}^2} &= \sin t, \\ \pi \int_0^\pi x^2 \dot{y} dt &= V. \end{aligned}$$

with \mathcal{A} given as the following:

$$\mathcal{A} := \left\{ (x, y) \in C^4[0, \pi] \mid \begin{array}{l} x(0) = y(0) = \dot{y}(0) = x(\pi) = \dot{y}(\pi) = 0, \\ x(t) \geq 0 \end{array} \right\} \quad (1)$$

This problem is the same as the following non-constraint problem:

$$\text{Min}_{\mathcal{A}} \int_0^\pi \underbrace{H^2 x \sqrt{\dot{x}^2 + \dot{y}^2}}_f + \mu \underbrace{(x \sqrt{\dot{x}^2 + \dot{y}^2} - \sin t)}_g + p \underbrace{(x^2 \dot{y})}_h dt \quad (2)$$

where μ, p are Lagrange multipliers. p is constant, and $\mu = \mu(t)$. In order to obtain the Euler-Lagrange equation of (2), let us do the following variations.

1.1 Euler-Lagrange equation along the normal direction

First of all, we do the variation for

$$\int_0^\pi f dt, \int_0^\pi g dt, \int_0^\pi h dt,$$

along the normal direction. for the part of f , note that

$$\begin{aligned} \delta(x\sqrt{\dot{x}^2 + \dot{y}^2}) &= x\delta(\sqrt{\dot{x}^2 + \dot{y}^2}) + \delta x(\sqrt{\dot{x}^2 + \dot{y}^2}) \\ &= x \frac{\dot{x}\dot{x}' + \dot{y}\dot{y}'}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_{\epsilon=0} + (-u \sin \phi \sqrt{\dot{x}^2 + \dot{y}^2}) \\ &= -ux\dot{\phi} - u\dot{y} \\ &= -2uHx\sqrt{\dot{x}^2 + \dot{y}^2} \end{aligned}$$

$$\begin{aligned}
& \delta \int_0^\pi f dt \\
&= \delta \int_0^\pi H^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\
&= \int_0^\pi 2H\delta H x \sqrt{\dot{x}^2 + \dot{y}^2} + H^2 \delta(x \sqrt{\dot{x}^2 + \dot{y}^2}) dt \\
&= \int_0^\pi \left(-\dot{u} \frac{\ddot{y}\dot{y} + \ddot{x}\dot{x}}{(\dot{x}^2 + \dot{y}^2)^2} + \frac{\ddot{u}}{\dot{x}^2 + \dot{y}^2} + \frac{\dot{u} \cos \phi}{x \sqrt{\dot{x}^2 + \dot{y}^2}} + u k_1^2 + u k_2^2 - 2uH^2 \right) H x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\
&= \int_0^\pi \left(\frac{\dot{u}x}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) H + 2uH(H^2 - K)x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\
&= \frac{\dot{u}xH}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^\pi + \int_0^\pi -\frac{\dot{u}x\dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + 2uH(H^2 - K)x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\
&= \frac{\dot{u}xH}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^\pi - \frac{ux\dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^\pi + \int_0^\pi u \left(\frac{x\dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) + 2uH(H^2 - K)x \sqrt{\dot{x}^2 + \dot{y}^2} dt.
\end{aligned}$$

For the boundary terms,

$$\frac{\dot{u}xH}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^\pi - \frac{ux\dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^\pi$$

to varnish them, we apply the essential boundary condition where

$$x(0) = 0, x(\pi) = 0.$$

However, we need \dot{H} bounded at the two ending points. Let us set the following boundary conditions on \dot{H} :

$$\dot{H}(0) = 0, \dot{H}(\pi) = 0.$$

For the part of g ,

$$\begin{aligned}
\delta \int_0^\pi \mu g dt &= \delta \int_0^\pi \mu(x \sqrt{\dot{x}^2 + \dot{y}^2} - \sin t) dt \\
&= \int_0^\pi \mu \delta(x \sqrt{\dot{x}^2 + \dot{y}^2}) dt \\
&= \int_0^\pi u(-2\mu H x \sqrt{\dot{x}^2 + \dot{y}^2}) dt
\end{aligned}$$

For the part of h ,

$$\begin{aligned}
& \delta \int_0^\pi p h dt \\
&= p \int_0^\pi \delta(x^2 \dot{y}) dt \\
&= p \int_0^\pi 2x \dot{y} \delta x + x^2 \delta \dot{y} dt \\
&= p \int_0^\pi 2x \dot{y} (-u \sin \phi) + x^2 (\dot{u} \cos \phi - u \sin \phi \cdot \dot{\phi}) dt \\
&= upx^2 \cos \phi \Big|_0^\pi + p \int_0^\pi u (-2x \dot{y} \sin \phi - 2x \dot{x} \cos \phi + \cancel{x^2 \sin \phi \dot{\phi}} - \cancel{x^2 \sin \phi \dot{\phi}}) dt \\
&= \int_0^\pi u (-2px \sqrt{\dot{x}^2 + \dot{y}^2}) dt
\end{aligned}$$

Finally, we combine all three parts together. It turns out that the Euler-Lagrange equation along the normal direction will be:

$$\left[\frac{1}{x \sqrt{\dot{x}^2 + \dot{y}^2}} \left(\frac{x \dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) \dot{\cdot} + 2H(H^2 - K) - (2\mu H + 2p) \right] x \sqrt{\dot{x}^2 + \dot{y}^2} = 0$$

1.2 Euler-Lagrange Equation along the Tangential Direction

In this subsection, we do the variation for

$$\int_0^\pi f dt, \int_0^\pi g dt, \int_0^\pi h dt,$$

along the tangential direction. for the part of f , note that

$$\begin{aligned}
\delta(x \sqrt{\dot{x}^2 + \dot{y}^2}) &= x \delta(\sqrt{\dot{x}^2 + \dot{y}^2}) + \delta x(\sqrt{\dot{x}^2 + \dot{y}^2}) \\
&= x \frac{\dot{x} \dot{x}' + \dot{y} \dot{y}'}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_{\epsilon=0} + (u \cos \phi \sqrt{\dot{x}^2 + \dot{y}^2}) \\
&= \dot{u}x + u \dot{x} \\
&= (ux) \cdot
\end{aligned}$$

$$\begin{aligned}
& \delta \int_0^\pi f dt \\
&= \delta \int_0^\pi H^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\
&= \int_0^\pi 2H \delta H x \sqrt{\dot{x}^2 + \dot{y}^2} + H^2 \delta(x \sqrt{\dot{x}^2 + \dot{y}^2}) dt \\
&= \int_0^\pi \left(u \frac{\dot{k}_1}{\sqrt{\dot{x}^2 + \dot{y}^2}} + u \frac{\cos \phi}{x} (k_1 - k_2) \right) H x \sqrt{\dot{x}^2 + \dot{y}^2} + H^2 (ux) \cdot dt \\
&= \cancel{uxH^2} \Big|_0^\pi + \int_0^\pi u H \left[\dot{k}_1 + \dot{x}(k_1 - k_2) - 2x \dot{H} \right] dt \\
&= \int_0^\pi \dot{x} k_1 - (xk_2) \cdot dt \\
&= 0
\end{aligned}$$

For the part of g ,

$$\begin{aligned}
\delta \int_0^\pi \mu g dt &= \delta \int_0^\pi \mu (x \sqrt{\dot{x}^2 + \dot{y}^2} - \sin t) dt \\
&= \int_0^\pi \mu \delta (x \sqrt{\dot{x}^2 + \dot{y}^2}) dt \\
&= \int_0^\pi \mu (ux) \cdot dt
\end{aligned}$$

For the part of h ,

$$\begin{aligned}
& \delta \int_0^\pi p h dt \\
&= p \int_0^\pi \delta(x^2 \dot{y}) dt \\
&= p \int_0^\pi 2x \dot{y} \delta x + x^2 \delta \dot{y} dt \\
&= p \int_0^\pi 2x \dot{y} (u \cos \phi) + x^2 (\dot{u} \sin \phi + u \cos \phi \cdot \dot{\phi}) dt \\
&= \cancel{upx^2 \sin \phi} \Big|_0^\pi + p \int_0^\pi u (2x \dot{y} \cos \phi - 2x \dot{x} \sin \phi - \cancel{x^2 \cos \phi \dot{\phi}} + \cancel{x^2 \cos \phi \dot{\phi}}) dt \\
&= \int_0^\pi u \cdot 0 dt \\
&= 0
\end{aligned}$$

Finally, we combine all three parts together. If we want to make the tangential variation equal 0, we need

$$\int_0^\pi \mu (ux) \cdot dt = \cancel{\mu ux} \Big|_0^\pi - \int_0^\pi \pi ux \dot{\mu} dt = 0$$

which implies $\mu(t) \equiv \text{constant}$.

1.3 Euler-Lagrange Equation with Boundary Conditions

Combining the derivation along the normal and tangential direction, we get the Euler-Lagrange equation for the isotropic Willmore problem:

$$\left[\frac{1}{x\sqrt{\dot{x}^2 + \dot{y}^2}} \left(\frac{x\dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) + 2H(H^2 - K) - (2\mu H + 2p) \right] x\sqrt{\dot{x}^2 + \dot{y}^2} = 0$$

or if we ignore the degenerated case, it will be simplified as

$$\frac{1}{x\sqrt{\dot{x}^2 + \dot{y}^2}} \left(\frac{x\dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) + 2H(H^2 - K) - (2\mu H + 2p) = 0.$$

Coupling it with the local area constraint $x\sqrt{\dot{x}^2 + \dot{y}^2} = \sin t$, and let

$$Q := \frac{x^2}{\sin^2 t} \dot{H}$$

then we obtain the following ODE system

$$\begin{aligned} \dot{Q} &= -\frac{\cos t}{\sin t} Q - 2H \left(H - \frac{\sin \phi}{x} \right)^2 + (2\mu H + 2p) \\ \dot{H} &= \frac{\sin^2 t}{x^2} Q \\ \dot{\phi} &= \left(2H - \frac{\sin \phi}{x} \right) \frac{\sin t}{x} \\ \dot{x} &= \cos \phi \frac{\sin t}{x} \\ \dot{y} &= \sin \phi \frac{\sin t}{x} \\ \dot{V} &= \pi x \sin \phi \sin t \\ \dot{\mu} &= 0 \end{aligned}$$

with boundary conditions

$$\begin{aligned} Q(0) &= 0, \phi(0) = 0, x(0) = 0, y(0) = 0, V(0) = 0, \\ Q(\pi) &= 0, \phi(\pi) = \pi, \quad V(\pi) = \text{Volume}. \end{aligned}$$

Here we have 7 ODEs with one unknown parameter p , that is why we impose 8 boundary conditions.

2 Anisotropic Willmore

We focus on the anisotropic Willmore problem:

$$\text{Min}_{\mathcal{A}} \int_{\Gamma} (\alpha_1 k_{\max} + \alpha_2 k_{\min})^2 dA$$

with area and volume constraint.

For the axis-symmetric case, we can simplify above problem into:

$$\text{Min}_{\mathcal{A}} \int_0^\pi (\alpha_1 k_{\max} + \alpha_2 k_{\min})^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt$$

subject to

$$\begin{aligned} x \sqrt{\dot{x}^2 + \dot{y}^2} &= \sin t \\ \int_0^\pi x^2 \dot{y} dt &= V \end{aligned}$$

where the admissible set \mathcal{A} is given by

$$\mathcal{A} := \left\{ (x, y) \in C^4[0, \pi] \mid \begin{array}{l} x(0) = y(0) = \dot{y}(0) = x(\pi) = \dot{y}(\pi) = 0 \\ k_1 \leq k_2 \quad \text{if } x \in [0, t_1] \cup [t_2, \pi] \\ k_1 \geq k_2 \quad \text{if } x \in [t_1, t_2] \end{array} \right\}$$

Assume the $(x(t), y(t)) \in \mathcal{A}$ is a minimizer, then we take any perturbed curve $(\bar{x}, \bar{y}) \in \mathcal{A}$ as the following:

$$\begin{aligned} \bar{x}(t) &= x(t) + (-\epsilon u \sin \phi) + (\eta v \cos \phi), \\ \bar{y}(t) &= y(t) + (+\epsilon u \cos \phi) + (\eta v \sin \phi). \end{aligned} \tag{3}$$

However, we cannot pick u, v as arbitrary smooth functions, instead, we have several restrictions on them. Note that

$$\begin{aligned} \bar{x}(0) &= x(0) + \eta v(0) \Rightarrow v(0) = 0, \\ \bar{y}(0) &= x(0) + \epsilon u(0) \Rightarrow u(0) = 0, \\ \bar{x}(\pi) &= x(\pi) - \eta v(\pi) \Rightarrow v(\pi) = 0. \end{aligned}$$

Since

$$\dot{y}(t) = \dot{y}(t) + (\epsilon \dot{u} \cos \phi - \epsilon u \sin \phi \dot{\phi}) + (\eta \dot{v} \sin \phi + \eta v \cos \phi \dot{\phi}),$$

we have

$$\begin{aligned} \dot{\bar{y}}(0) &= \dot{y}(0) + \epsilon \dot{u}(0) + \eta v(0) \dot{\phi}(0) \Rightarrow \dot{u}(0) = 0, \\ \dot{\bar{y}}(\pi) &= \dot{y}(\pi) - \epsilon \dot{u}(\pi) + \eta v(\pi) \dot{\phi}(\pi) \Rightarrow \dot{u}(\pi) = 0. \end{aligned}$$

From the area constraint $x\sqrt{\dot{x}^2 + \dot{y}^2} = \sin t$, we obtain

$$\begin{aligned} \text{at } t = 0, \sqrt{\dot{x}^2 + \dot{y}^2} &= \frac{\sin t}{x} = \frac{\cos t}{\dot{x}} \Rightarrow \dot{x}(0) = 1; \\ \text{at } t = \pi, \sqrt{\dot{x}^2 + \dot{y}^2} &= \frac{\sin t}{x} = \frac{\cos t}{\dot{x}} \Rightarrow \dot{x}(\pi) = -1; \end{aligned}$$

combining

$$\dot{\bar{x}}(t) = \dot{x}(t) + (-\epsilon \dot{u} \sin \phi - \epsilon u \cos \phi \dot{\phi}) + (\eta \dot{v} \cos \phi - \eta v \sin \phi \dot{\phi}),$$

we have

$$\begin{aligned} \dot{\bar{x}}(0) &= \dot{x}(0) - \epsilon u(0) \dot{\phi}(0) + \eta \dot{v}(0) \Rightarrow \dot{v}(0) = 0, \\ \dot{\bar{x}}(\pi) &= \dot{x}(\pi) + \epsilon u(\pi) \dot{\phi}(\pi) + \eta \dot{v}(\pi) \Rightarrow \dot{v}(\pi) = 0 \quad + \begin{cases} \dot{\phi}(\pi) = 0 \\ \text{or } u(\pi) = 0 \end{cases}. \end{aligned}$$

One more important constraint on u, v is from role-exchanging condition

$$\begin{cases} k_1 \leq k_2 & \text{if } x \in [0, t_1] \cup [t_2, \pi], \\ k_1 \geq k_2 & \text{if } x \in [t_1, t_2]. \end{cases} \quad (4)$$

A necessary condition for (4) is that

$$k_1 = k_2, \quad \text{if } t = t_1, t_2,$$

which implies

$$\dot{\phi} = \frac{\dot{y}}{x} \Leftrightarrow \frac{\ddot{y}\dot{x} - \dot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} = \frac{\dot{y}}{x} \Leftrightarrow \ddot{y}\dot{x}x - \ddot{x}\dot{y}x = \dot{x}^2\dot{y} + \dot{y}^3 \quad (5)$$

To make sure that $(\bar{x}, \bar{y}) \in \mathcal{A}$, the condition (5) should hold for (\bar{x}, \bar{y}) , which means

$$\ddot{\bar{y}}\dot{\bar{x}}\bar{x} - \ddot{\bar{x}}\dot{\bar{y}}\bar{x} = \dot{\bar{x}}^2\dot{\bar{y}} + \dot{\bar{y}}^3$$

Notice that for the anisotropic bending energy,

$$\begin{aligned} &\int_{\Gamma} (\alpha_1 k_{\max} + \alpha_2 k_{\min})^2 dA \\ &= \int_{\Gamma} (\alpha_1(k_{\max} + k_{\min}) + (\alpha_2 - \alpha_1)k_{\min})^2 dA \\ &= \int_{\Gamma} \left(\alpha_1^2(k_{\max} + k_{\min})^2 + 2\alpha_1(\alpha_2 - \alpha_1)(k_{\max}k_{\min} + k_{\min}^2) + (\alpha_2 - \alpha_1)^2 k_{\min}^2 \right) dA \\ &= 4\alpha_1^2 \int_{\Gamma} H^2 dA + 2\alpha_1(\alpha_2 - \alpha_1) \int_{\Gamma} K dA + (\alpha_2^2 - \alpha_1^2) \int_{\Gamma} k_{\min}^2 dA \\ &= 4\alpha_1^2 \int_{\Gamma} H^2 dA + 2\alpha_1(\alpha_2 - \alpha_1) \int_{\Gamma} K dA + (\alpha_2 - \alpha_1) \int_{\Gamma} k_{\min}^2 dA \end{aligned}$$

then if we do the variation for the axis-symmetric case, we have

$$\begin{aligned} & \delta \int_0^\pi (\alpha_1 k_{\max} + \alpha_2 k_{\min})^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\ &= 4\alpha^2 \delta \int_0^\pi H^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt + (\alpha_2 - \alpha_1) \cdot \\ & \quad \delta \left(\underbrace{\int_0^{t_1} k_1^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt}_{\text{II}} + \underbrace{\int_{t_1}^{t_2} k_2^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt}_{\text{III}} + \underbrace{\int_{t_2}^\pi k_1^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt}_{\text{IV}} \right) \end{aligned}$$

2.1 Euler-Lagrange Equation along Normal Direction

First of all, let us do the variation along normal direction. Since

$$\begin{aligned} & \delta \int_0^\pi (\alpha_1 k_{\max} + \alpha_2 k_{\min})^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\ &= 4\alpha^2 \delta \underbrace{\int_0^\pi H^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt}_{\text{I}} + (\alpha_2 - \alpha_1) \cdot \\ & \quad \delta \left(\underbrace{\int_0^{t_1} k_1^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt}_{\text{II}} + \underbrace{\int_{t_1}^{t_2} k_2^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt}_{\text{III}} + \underbrace{\int_{t_2}^\pi k_1^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt}_{\text{IV}} \right) \end{aligned}$$

and from section 1.2 we have known that the variation of I is

$$\left(\frac{x \dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right)' + 2H(H^2 - K)x \sqrt{\dot{x}^2 + \dot{y}^2} = 0,$$

so now we focus on the variation of II, III, and IV.

For II, we refer to the appendix II,

$$\begin{aligned} & \delta \int_0^{t_1} k_1^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\ &= \int_0^{t_1} 2k_1 \left(-\dot{u} \frac{\ddot{y}\dot{y} + \ddot{x}\dot{x}}{(\dot{x}^2 + \dot{y}^2)^2} + \frac{\ddot{u}}{\dot{x}^2 + \dot{y}^2} + u k_1^2 \right) x \sqrt{\dot{x}^2 + \dot{y}^2} + k_1^2 (-2uHx \sqrt{\dot{x}^2 + \dot{y}^2}) dt \\ &= \int_0^{t_1} 2k_1 x \left(\frac{\dot{u}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right)' + u(2k_1^3 - 2k_1^2 H) x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\ &= \frac{2k_1 x \dot{u}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^{t_1} - \frac{(2k_1 x) \dot{u}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^{t_1} + \int_0^{t_1} 2 \left(\frac{k_1 \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right)' u + u \cdot 2k_1^2 (k_1 - H) x \sqrt{\dot{x}^2 + \dot{y}^2} dt \end{aligned}$$

which implies

$$\begin{aligned} & \delta \int_0^{t_1} k_1^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt / \delta l \Big|_{\text{normal}} \\ &= \frac{2k_1 x \dot{u}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^{t_1} - \frac{(2k_1 x) \cdot u}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^{t_1} + \int_0^{t_1} 2 \left(\frac{k_1 \dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) \cdot u + u \cdot 2k_1^2 (k_1 - H) x \sqrt{\dot{x}^2 + \dot{y}^2} dt \end{aligned}$$

For III, we have

$$\begin{aligned} & \delta \int_{t_1}^{t_2} k_2^2 x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\ &= \int_{t_1}^{t_2} 2k_2 \left(\frac{\dot{u} \cos \phi}{x \sqrt{\dot{x}^2 + \dot{y}^2}} + u k_2^2 \right) x \sqrt{\dot{x}^2 + \dot{y}^2} + k_2^2 (-2uHx \sqrt{\dot{x}^2 + \dot{y}^2}) dt \\ &= \int_{t_1}^{t_2} 2k_2 \cos \phi \dot{u} + u(2k_2^3 - 2k_2^2 H) x \sqrt{\dot{x}^2 + \dot{y}^2} dt \\ &= 2k_2 \cos \phi u \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} -2(k_2 \cos \phi) \cdot u + u \cdot 2k_2^2 (k_2 - H) x \sqrt{\dot{x}^2 + \dot{y}^2} dt \end{aligned}$$

If we pay more attention to the boundary terms,

$$4\alpha^2 \left\{ \frac{\dot{u}xH}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^\pi - \frac{ux\dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^\pi \right\} + 2(\alpha_2 - \alpha_1) \cdot \left\{ \frac{k_1 x \dot{u}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^{t_1} - \frac{(k_1 x) \cdot u}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^{t_1} + k_2 \cos \phi u \Big|_{t_1}^{t_2} + \frac{k_1 x \dot{u}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_{t_2}^\pi - \frac{(k_1 x) \cdot u}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_{t_2}^\pi \right\}$$

we can find out that in order to vanish all the boundary terms, we just need play with

$$\frac{k_1 x \dot{u}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^{t_1} - \frac{(k_1 x) \cdot u}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_0^{t_1} + k_2 \cos \phi u \Big|_{t_1}^{t_2} + \frac{k_1 x \dot{u}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_{t_2}^\pi - \frac{(k_1 x) \cdot u}{\sqrt{\dot{x}^2 + \dot{y}^2}} \Big|_{t_2}^\pi$$

Now let us combine II, III and IV. The Euler-Lagrange equation is

$$4\alpha^2 \left\{ \left(\frac{x \dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) \cdot + 2H(H^2 - K)x \sqrt{\dot{x}^2 + \dot{y}^2} \right\} +$$

2.2 Asymptotic Analysis for the Extreme Case: $k_1 \leq k_2$ for $t \in [0, \pi]$

One extreme case actually is that $k_1 \leq k_2$ globally. For this case, the Euler-Lagrange equation will be:

$$\begin{aligned} & \dot{Q} + \frac{\cos t}{\sin t} Q + 2H(H^2 - K) + (2\mu H + 2p) \\ & - 4\epsilon \left[\dot{Q} + \frac{\cos t}{\sin t} Q + \frac{(k_2 \cos \phi)^\cdot}{\sin t} - \frac{1}{2}(k_1^2 + k_2^2)(k_2 - H) \right] \\ & + 4\epsilon^2 \left[\dot{Q} + \frac{\cos t}{\sin t} Q + 2H(H^2 - K) \right] = 0 \end{aligned}$$

or equivalently it yields the ODE system:

$$\begin{aligned} \dot{Q} &= -\frac{\cos t}{\sin t} Q - 2H \left(H - \frac{\sin \phi}{x} \right)^2 + \frac{2\mu H + 2p}{(1-2\epsilon)^2} - \frac{4\epsilon}{(1-2\epsilon)^2} \left[\frac{(k_2 \cos \phi)^\cdot}{\sin t} - \frac{1}{2}(k_1^2 + k_2^2)(k_2 - H) \right] \\ \dot{H} &= \frac{\sin^2 t}{x^2} Q \\ \dot{\phi} &= \left(2H - \frac{\sin \phi}{x} \right) \frac{\sin t}{x} \\ \dot{x} &= \cos \phi \frac{\sin t}{x} \\ \dot{y} &= \sin \phi \frac{\sin t}{x} \\ \dot{V} &= \pi x \sin \phi \sin t \\ \dot{\mu} &= 0 \\ \dot{p} &= 0 \end{aligned}$$

with boundary conditions

$$\begin{aligned} Q(0) &= 0, \phi(0) = 0, x(0) = 0, y(0) = 0, V(0) = 0, \\ Q(\pi) &= 0, \phi(\pi) = \pi, \quad V(\pi) = \text{Volume}. \end{aligned}$$

Since

$$\begin{aligned} (k_2 \cos \phi)^\cdot &= \left(\frac{\sin \phi \cos \phi}{x} \right)^\cdot = \frac{1}{2} \left(\frac{\sin 2\phi}{x} \right)^\cdot \\ &= \frac{\cos 2\phi}{x} \cdot \frac{\sin t}{x} \left(2H - \frac{\sin \phi}{x} \right) - \frac{\sin \phi \cos^2 \phi}{x^2} \cdot \frac{\sin t}{x} \\ &= 2H \frac{\cos 2\phi \sin t}{x^2} - \frac{\cos 2\phi \sin \phi \sin t}{x^3} - \frac{\sin \phi \cos^2 \phi \sin t}{x^3} \end{aligned}$$

we have

$$\frac{(k_2 \cos \phi)^\cdot}{\sin t} = 2H \frac{\cos 2\phi}{x^2} - \frac{\cos 2\phi \sin \phi}{x^3} - \frac{\sin \phi \cos^2 \phi}{x^3}$$

To do the asymptotic analysis, let us asymptotically expand all variables as the following (refer to the appendix for the details):

$$f(t) = f_0(t) + \epsilon f_1(t) + \epsilon^2 f_2(t) + o(\epsilon^2)$$

where $f = Q, H, \phi, x, y, V, \mu, p$. Then the $O(1)$ ODE system is

$$\begin{aligned}\dot{Q}_0 &= -\frac{\cos t}{\sin t}Q_0 - 2H_0\left(H_0 - \frac{\sin \phi_0}{x}\right)^2 + (2\mu_0 H_0 + 2p_0) \\ \dot{H}_0 &= \frac{\sin^2 t}{x_0^2}Q_0 \\ \dot{\phi}_0 &= \left(2H_0 - \frac{\sin \phi_0}{x_0}\right)\frac{\sin t}{x_0} \\ \dot{x}_0 &= \cos \phi_0 \frac{\sin t}{x_0} \\ \dot{y}_0 &= \sin \phi_0 \frac{\sin t}{x_0} \\ \dot{V}_0 &= \pi x_0 \sin \phi_0 \sin t \\ \dot{\mu}_0 &= 0 \\ \dot{p}_0 &= 0\end{aligned}$$

with boundary conditions

$$\begin{aligned}Q_0(0) &= 0, \phi_0(0) = 0, x_0(0) = 0, y_0(0) = 0, V_0(0) = 0, \\ Q_0(\pi) &= 0, \phi_0(\pi) = \pi, \quad V_0(\pi) = \text{Volume.}\end{aligned}$$

This ODE system provide us a typical solution for the isotropic case which satisfies $k_1 \leq k_2$ globally.

The $O(\epsilon)$ ODE system is:

$$\begin{aligned}\dot{Q}_1 &= -\frac{\cos t}{\sin t}Q_1 - 2H_1\left(H_0 - \frac{\sin \phi_0}{x_0}\right)^2 - 4H_0\left(H_0 - \frac{\sin \phi_0}{x_0}\right)\left(H_1 - \frac{\phi_1 \cos \phi_0}{x_0} + \frac{x_1 \sin \phi_0}{x_0^2}\right) \\ &\quad - 4\left[\dot{Q}_0 + \frac{\cos t}{\sin t}Q_0 + \frac{(k_{2,0} \cos \phi_0)}{\sin t} - \frac{1}{2}(k_{1,0}^2 + k_{2,0}^2)(k_{2,0} - H_0)\right] - (2\mu_0 H_1 + 2\mu_1 H_0 + p_1) \\ \dot{H}_1 &= \sin^2 t\left(\frac{Q_1}{x_0^2} - \frac{2x_1 Q_0}{x_0^3}\right) \\ \dot{\phi}_1 &= \left(2H_1 - 2H_0 \frac{x_1}{x_0}\right)\frac{\sin t}{x_0} - \left(\phi_1 \cos \phi_0 - 2\frac{x_1}{x_0} \sin \phi_0\right)\frac{\sin t}{x_0^2} \\ \dot{x}_1 &= -\sin \phi_0 \frac{\phi_1 \sin t}{x_0} - \cos \phi_0 \frac{x_1 \sin t}{x_0^2} \\ \dot{y}_1 &= +\cos \phi_0 \frac{\phi_1 \sin t}{x_0} - \sin \phi_0 \frac{x_1 \sin t}{x_0^2} \\ \dot{V}_1 &= \pi x_1 \sin \phi_0 \sin t + \pi x_0 \phi_1 \cos \phi_0 \sin t \\ \dot{\mu}_1 &= 0 \\ \dot{p}_1 &= 0\end{aligned}$$

2.3 Extreme Case: $k_1 \geq k_2$ for $t \in [0, \pi]$

One extreme case actually is that $k_1 \geq k_2$ globally. For this case, the Euler-Lagrange equation will be:

$$\begin{aligned} & \dot{Q} + \frac{\cos t}{\sin t} Q + 2H(H^2 - K) + (2\mu H + 2p) \\ & + 4\epsilon \left[\dot{Q} + \frac{\cos t}{\sin t} Q + \frac{(k_2 \cos \phi)'}{\sin t} - \frac{1}{2}(k_1^2 + k_2^2)(k_2 - H) \right] \\ & + 4\epsilon^2 \left[\dot{Q} + \frac{\cos t}{\sin t} Q + 2H(H^2 - K) \right] = 0 \end{aligned}$$

or equivalently it yields the ODE system:

$$\begin{aligned} \dot{Q} &= -\frac{\cos t}{\sin t} Q - \frac{2(1+4\epsilon^2)}{(1+2\epsilon)^2} H \left(H - \frac{\sin \phi}{x} \right)^2 + \frac{2\mu H + 2p}{(1+2\epsilon)^2} - \frac{4\epsilon}{(1+2\epsilon)^2} \left[\frac{(k_2 \cos \phi)'}{\sin t} - \frac{1}{2}(k_1^2 + k_2^2)(k_2 - H) \right] \\ \dot{H} &= \frac{\sin^2 t}{x^2} Q \\ \dot{\phi} &= \left(2H - \frac{\sin \phi}{x} \right) \frac{\sin t}{x} \\ \dot{x} &= \cos \phi \frac{\sin t}{x} \\ \dot{y} &= \sin \phi \frac{\sin t}{x} \\ \dot{V} &= \pi x \sin \phi \sin t \\ \dot{\mu} &= 0 \end{aligned}$$

with boundary conditions

$$\begin{aligned} Q(0) &= 0, \phi(0) = 0, x(0) = 0, y(0) = 0, V(0) = 0, \\ Q(\pi) &= 0, \phi(\pi) = \pi, \quad V(\pi) = \text{Volume}. \end{aligned}$$

A Appendix I: Principal Curvatures for Axis-symmetric surfaces

Let us consider surface

$$\vec{r}(\theta, t) = (x(t) \cos \theta, x(t) \sin \theta, y(t)).$$

Calculate the first and second fundamental forms g and h . Obviously,

$$\begin{aligned}\vec{r}_\theta &= (-x \sin \theta, x \cos \theta, 0); \\ \vec{r}_t &= (\dot{x} \cos \theta, \dot{x} \sin \theta, \dot{y}); \\ \vec{r}_{\theta\theta} &= (-x \cos \theta, -x \sin \theta, 0); \\ \vec{r}_{\theta t} &= (-\dot{x} \sin \theta, \dot{x} \cos \theta, 0); \\ \vec{r}_{tt} &= (\ddot{x} \cos \theta, \ddot{x} \sin \theta, \ddot{y}); \\ \vec{n} &= (\dot{y} \cos \theta, \dot{y} \sin \theta, -\dot{x}) / \sqrt{\dot{x}^2 + \dot{y}^2};\end{aligned}$$

Then

$$g = \begin{pmatrix} g_{\theta\theta} & g_{\theta t} \\ g_{t\theta} & g_{tt} \end{pmatrix} = \begin{pmatrix} x^2 & 0 \\ 0 & \dot{x}^2 + \dot{y}^2 \end{pmatrix}$$

and

$$h = \begin{pmatrix} h_{\theta\theta} & h_{\theta t} \\ h_{t\theta} & h_{tt} \end{pmatrix} = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \begin{pmatrix} -x\dot{y} & 0 \\ 0 & \dot{x}\dot{y} - \dot{y}\dot{x} \end{pmatrix}$$

If we assume the angle between the tangential direction and x direction is ϕ , sometimes we call ϕ the inclination angle, then we can get

$$g = \begin{pmatrix} x^2 & 0 \\ 0 & \dot{x}^2 + \dot{y}^2 \end{pmatrix}, h = \begin{pmatrix} -x \sin \phi & 0 \\ 0 & -\dot{\phi} \sqrt{\dot{x}^2 + \dot{y}^2} \end{pmatrix}$$

Since

$$\begin{aligned}H &= -\frac{1}{2} \sum g^{ij} h_{ij} = \frac{1}{2} \left(\frac{\sin \phi}{x} + \frac{\dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) \\ K &= \frac{|h|}{|g|} = \frac{\sin \phi \cdot \dot{\phi}}{x \sqrt{\dot{x}^2 + \dot{y}^2}}\end{aligned}$$

We obtain that

$$k_1 = -\frac{\dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, k_2 = -\frac{\sin \phi}{x}$$

and

$$\dot{x} = \cos \phi \sqrt{\dot{x}^2 + \dot{y}^2}, \dot{y} = \sin \phi \sqrt{\dot{x}^2 + \dot{y}^2}, \dot{\phi} = \frac{\dot{y}\dot{x} - \dot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}.$$

B Appendix II: Variation of k_1, k_2 along Normal Direction

Assume (x, y) is a minimizer and (\bar{x}, \bar{y}) is arbitrary perturbed curve

$$\begin{aligned}\bar{x}(t) &= x(t) + (-\epsilon u \sin \phi) + (\eta v \cos \phi), \\ \bar{y}(t) &= y(t) + (+\epsilon u \cos \phi) + (\eta v \sin \phi).\end{aligned}\tag{6}$$

First of all, let us compute the variation of k_1, k_2 along the normal direction which means in (6) we set $\eta = 0$. In this case the perturbed curve degenerates as:

$$\begin{aligned}\bar{x}(t) &= x(t) + (-\epsilon u \sin \phi), \\ \bar{y}(t) &= y(t) + (+\epsilon u \cos \phi).\end{aligned}\tag{7}$$

Let $A = -u \sin \phi, B = u \cos \phi$, then

$$\begin{cases} \bar{x} = x + \epsilon A, \\ \bar{y} = y + \epsilon B, \end{cases} \quad \begin{cases} \dot{\bar{x}} = \dot{x} + \epsilon \dot{A}, \\ \dot{\bar{y}} = \dot{y} + \epsilon \dot{B}, \end{cases} \quad \begin{cases} \ddot{\bar{x}} = \ddot{x} + \epsilon \ddot{A}, \\ \ddot{\bar{y}} = \ddot{y} + \epsilon \ddot{B}. \end{cases}$$

with

$$\begin{aligned}A &= -u \sin \phi, \\ B &= u \cos \phi, \\ \dot{A} &= -\dot{u} \sin \phi - u \cos \phi \dot{\phi}, \\ \dot{B} &= \dot{u} \cos \phi - u \sin \phi \dot{\phi}, \\ \ddot{A} &= -\ddot{u} \sin \phi - 2\dot{u} \cos \phi \dot{\phi} + u \sin \phi \dot{\phi}^2 - u \cos \phi \ddot{\phi}, \\ \ddot{B} &= \ddot{u} \cos \phi - 2\dot{u} \sin \phi \dot{\phi} - u \cos \phi \dot{\phi}^2 - u \sin \phi \ddot{\phi}, \\ \dot{A}\ddot{y} - \dot{B}\ddot{x} &= -u\dot{\phi}^2 \sqrt{\dot{x}^2 + \dot{y}^2} - \dot{u} \frac{\ddot{y}\dot{y} + \ddot{x}\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \\ \ddot{B}\dot{x} - \ddot{A}\dot{y} &= \ddot{u}\sqrt{\dot{x}^2 + \dot{y}^2} - u\dot{\phi}^2 \sqrt{\dot{x}^2 + \dot{y}^2} \\ \dot{A}\dot{x} + \dot{B}\dot{y} &= -u\dot{\phi}\sqrt{\dot{x}^2 + \dot{y}^2}, \\ \dot{A}^2 + \dot{B}^2 &= \dot{v}^2 + v^2 \dot{\phi}^2.\end{aligned}$$

B.1 Variation of k_2 along Normal Direction

Since

$$\delta k_2 = \delta\left(\frac{\sin \phi}{x}\right) = \frac{\delta(\sin \phi)}{x} + \sin \phi \cdot \delta\left(\frac{1}{x}\right),$$

and

$$\begin{aligned}\delta(\sin \phi) &= \frac{\partial}{\partial \epsilon} \left. \sin \bar{\phi} \right|_{\epsilon=0} = \frac{\partial}{\partial \epsilon} \left. \frac{\dot{\bar{y}}}{\sqrt{\dot{\bar{x}}^2 + \dot{\bar{y}}^2}} \right|_{\epsilon=0} \\ &= \frac{\dot{\bar{y}}'}{\sqrt{\dot{\bar{x}}^2 + \dot{\bar{y}}^2}} - \dot{\bar{y}} \left. \frac{\dot{\bar{x}}\dot{\bar{x}}' + \dot{\bar{y}}\dot{\bar{y}}'}{(\dot{\bar{x}}^2 + \dot{\bar{y}}^2)^{3/2}} \right|_{\epsilon=0} \\ &= \frac{\dot{u} \cos \phi}{\sqrt{\dot{x}^2 + \dot{y}^2}} - \frac{u \sin \phi \dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + \frac{u \dot{y} \dot{\phi}}{\dot{x}^2 + \dot{y}^2} \\ &= \frac{\dot{u} \cos \phi}{\sqrt{\dot{x}^2 + \dot{y}^2}},\end{aligned}$$

plus

$$\delta\left(\frac{1}{x}\right) = \frac{\partial}{\partial\epsilon} \left. \frac{1}{\bar{x}} \right|_{\epsilon=0} = \frac{u \sin \phi}{(x - \epsilon u \sin \phi)^2} \Big|_{\epsilon=0} = \frac{u \sin \phi}{x^2}$$

hence

$$dk_2 = \delta\left(\frac{\sin \phi}{x}\right) = \frac{\dot{u} \cos \phi}{x \sqrt{\dot{x}^2 + \dot{y}^2}} + uk_2^2.$$

B.2 Variation of k_1 along Normal Direction

Since

$$\delta k_1 = \delta\left(\frac{\dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = \frac{\delta \dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + \dot{\phi} \cdot \delta\left(\frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right),$$

and

$$\begin{aligned} \delta \dot{\phi} &= \frac{\partial}{\partial\epsilon} \left. \dot{\phi} \right|_{\epsilon=0} = \frac{\partial}{\partial\epsilon} \left. \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \right|_{\epsilon=0} \\ &= \frac{\partial}{\partial\epsilon} \left. \frac{(\ddot{y} + \epsilon \ddot{B})(\dot{x} + \epsilon \dot{A}) - (\ddot{x} + \epsilon \dot{A})(\dot{y} + \epsilon \dot{B})}{(\dot{x} + \epsilon \dot{A})^2 + (\dot{y} + \epsilon \dot{B})^2} \right|_{\epsilon=0} \\ &= \frac{\ddot{y}\dot{A} + \dot{x}\ddot{B} - \dot{y}\ddot{A} - \dot{x}\dot{B}}{\dot{x}^2 + \dot{y}^2} - 2 \frac{(\ddot{y}\dot{x} - \ddot{x}\dot{y})(\dot{x}\dot{A} + \dot{y}\dot{B})}{(\dot{x}^2 + \dot{y}^2)^2} \\ &= \frac{\ddot{y}\dot{A} + \dot{x}\ddot{B} - \dot{y}\ddot{A} - \dot{x}\dot{B}}{\dot{x}^2 + \dot{y}^2} - 2 \frac{\dot{x}\dot{A} + \dot{y}\dot{B}}{\dot{x}^2 + \dot{y}^2} \cdot \dot{\phi} \\ &= \frac{-2u\dot{\phi}^2 \sqrt{\dot{x}^2 + \dot{y}^2} - \dot{u} \frac{\dot{y}\dot{y} + \dot{x}\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + \ddot{u} \sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}^2 + \dot{y}^2} + 2 \frac{u\dot{\phi} \sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}^2 + \dot{y}^2} \dot{\phi} \end{aligned}$$

so

$$\frac{\delta \dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = -\dot{u} \frac{\dot{y}\dot{y} + \dot{x}\dot{x}}{(\dot{x}^2 + \dot{y}^2)^2} + \frac{\ddot{u}}{\dot{x}^2 + \dot{y}^2}.$$

Plus,

$$\delta \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{\partial}{\partial\epsilon} \left. \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right|_{\epsilon=0} = - \frac{\dot{x}\dot{x}' + \dot{y}\dot{y}'}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \Big|_{\epsilon=0} = \frac{u\dot{\phi} \sqrt{\dot{x}^2 + \dot{y}^2}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{u\dot{\phi}}{\dot{x}^2 + \dot{y}^2},$$

so

$$\dot{\phi} \cdot \delta \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{u\dot{\phi}^2}{\dot{x}^2 + \dot{y}^2} = uk_1^2.$$

Finally,

$$\delta k_1 = \delta\left(\frac{\dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = -\dot{u} \frac{\dot{y}\dot{y} + \dot{x}\dot{x}}{(\dot{x}^2 + \dot{y}^2)^2} + \frac{\ddot{u}}{\dot{x}^2 + \dot{y}^2} + uk_1^2.$$

C Appendix III: Variation of k_1, k_2 along Tangential Direction

Assume (x, y) is a minimizer and (\bar{x}, \bar{y}) is arbitrary perturbed curve

$$\begin{aligned}\bar{x}(t) &= x(t) + (-\epsilon u \sin \phi) + (\eta v \cos \phi), \\ \bar{y}(t) &= y(t) + (+\epsilon u \cos \phi) + (\eta v \sin \phi).\end{aligned}\quad (8)$$

Now we compute the variation of k_1, k_2 along the tangential direction which means in (8) we set $\epsilon = 0$. In this case the perturbed curve degenerates as:

$$\begin{aligned}\bar{x}(t) &= x(t) + (\eta v \cos \phi), \\ \bar{y}(t) &= y(t) + (\eta v \sin \phi).\end{aligned}\quad (9)$$

Let $A = v \cos \phi, B = v \sin \phi$, then

$$\begin{cases} \bar{x} = x + \epsilon A, \\ \bar{y} = y + \epsilon B, \end{cases} \quad \begin{cases} \dot{\bar{x}} = \dot{x} + \epsilon \dot{A}, \\ \dot{\bar{y}} = \dot{y} + \epsilon \dot{B}. \end{cases} \quad \begin{cases} \ddot{\bar{x}} = \ddot{x} + \epsilon \ddot{A}, \\ \ddot{\bar{y}} = \ddot{y} + \epsilon \ddot{B}. \end{cases}$$

with

$$\begin{aligned}A &= v \cos \phi, \\ B &= v \sin \phi \\ \dot{A} &= \dot{v} \cos \phi - v \sin \phi \dot{\phi}, \\ \dot{B} &= \dot{v} \sin \phi + v \cos \phi \dot{\phi}, \\ \ddot{A} &= \ddot{v} \cos \phi - 2\dot{v} \sin \phi \dot{\phi} - v \cos \phi \dot{\phi}^2 - v \sin \phi \ddot{\phi}, \\ \ddot{B} &= \ddot{v} \sin \phi + 2\dot{v} \cos \phi \dot{\phi} - v \sin \phi \dot{\phi}^2 + v \cos \phi \ddot{\phi}, \\ \dot{A}\ddot{y} - \dot{B}\ddot{x} &= \dot{v}\dot{\phi}\sqrt{\dot{x}^2 + \dot{y}^2} - v\dot{\phi}\frac{\ddot{y}\dot{y} + \ddot{x}\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}, \\ \ddot{B}\dot{x} - \ddot{A}\dot{y} &= 2\dot{v}\dot{\phi}\sqrt{\dot{x}^2 + \dot{y}^2} + v\ddot{\phi}\sqrt{\dot{x}^2 + \dot{y}^2} \\ \dot{A}\dot{x} + \dot{B}\dot{y} &= \dot{v}\sqrt{\dot{x}^2 + \dot{y}^2}, \\ \dot{A}^2 + \dot{B}^2 &= \dot{v}^2 + v^2\dot{\phi}^2.\end{aligned}$$

C.1 Variation of k_2 along Tangential Direction

Since

$$\delta k_2 = \delta\left(\frac{\sin \phi}{x}\right) = \frac{\delta(\sin \phi)}{x} + \sin \phi \cdot \delta\left(\frac{1}{x}\right),$$

and

$$\begin{aligned}
\delta(\sin \phi) &= \frac{\partial}{\partial \epsilon} \left|_{\epsilon=0} \right. \sin \bar{\phi} = \frac{\partial}{\partial \epsilon} \left|_{\epsilon=0} \right. \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \\
&= \frac{\dot{y}'}{\sqrt{\dot{x}^2 + \dot{y}^2}} - \dot{y} \frac{\dot{x}\dot{x}' + \dot{y}\dot{y}'}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \Big|_{\epsilon=0} \\
&= \frac{\dot{u} \sin \phi}{\sqrt{\dot{x}^2 + \dot{y}^2}} - \frac{u \cos \phi \dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}} - \frac{\dot{u}\dot{y}}{\dot{x}^2 + \dot{y}^2} \\
&= \frac{u \cos \phi \dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}},
\end{aligned}$$

plus

$$\delta\left(\frac{1}{x}\right) = \frac{\partial}{\partial \epsilon} \left|_{\epsilon=0} \right. \frac{1}{\dot{x}} = -\frac{u \cos \phi}{(x - \epsilon u \cos \phi)^2} \Big|_{\epsilon=0} = -\frac{u \cos \phi}{x^2}$$

hence

$$\delta k_2 = \delta\left(\frac{\sin \phi}{x}\right) = u \frac{\cos \phi}{x} k_1 - u \frac{\cos \phi}{x} k_2 = u \frac{\cos \phi}{x} (k_1 - k_2).$$

C.2 Variation of k_1 along Tangential Direction

Since

$$\delta k_1 = \delta\left(\frac{\dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) = \frac{\delta \dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + \dot{\phi} \cdot \delta\left(\frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right),$$

and

$$\begin{aligned}
\delta \dot{\phi} &= \frac{\partial}{\partial \epsilon} \left|_{\epsilon=0} \right. \dot{\phi} = \frac{\partial}{\partial \epsilon} \left|_{\epsilon=0} \right. \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \\
&= \frac{\partial}{\partial \epsilon} \left|_{\epsilon=0} \right. \frac{(\ddot{y} + \epsilon \ddot{B})(\dot{x} + \epsilon \dot{A}) - (\ddot{x} + \epsilon \ddot{A})(\dot{y} + \epsilon \dot{B})}{(\dot{x} + \epsilon \dot{A})^2 + (\dot{y} + \epsilon \dot{B})^2} \\
&= \frac{\ddot{y}\dot{A} + \dot{x}\ddot{B} - \dot{y}\ddot{A} - \ddot{x}\dot{B}}{\dot{x}^2 + \dot{y}^2} - 2 \frac{(\ddot{y}\dot{x} - \ddot{x}\dot{y})(\dot{x}\dot{A} + \dot{y}\dot{B})}{(\dot{x}^2 + \dot{y}^2)^2} \\
&= \frac{\ddot{y}\dot{A} + \dot{x}\ddot{B} - \dot{y}\ddot{A} - \ddot{x}\dot{B}}{\dot{x}^2 + \dot{y}^2} - 2 \frac{\dot{x}\dot{A} + \dot{y}\dot{B}}{\dot{x}^2 + \dot{y}^2} \cdot \dot{\phi} \\
&= \frac{\dot{v}\dot{\phi}\sqrt{\dot{x}^2 + \dot{y}^2} - v\dot{\phi}\frac{\dot{y}\dot{y} + \dot{x}\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} + 2\dot{v}\dot{\phi}\sqrt{\dot{x}^2 + \dot{y}^2} + v\ddot{\phi}\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}^2 + \dot{y}^2} - 2 \frac{\dot{v}\sqrt{\dot{x}^2 + \dot{y}^2}}{\dot{x}^2 + \dot{y}^2} \dot{\phi}
\end{aligned}$$

so

$$\frac{\delta \dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}} = -v\dot{\phi} \frac{\dot{y}\dot{y} + \dot{x}\dot{x}}{(\dot{x}^2 + \dot{y}^2)^2} + \frac{\dot{u}\dot{\phi} + u\ddot{\phi}}{\dot{x}^2 + \dot{y}^2}.$$

Plus,

$$\delta \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} = \frac{\partial}{\partial \epsilon} \left|_{\epsilon=0} \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} = - \frac{\dot{x}\dot{x}' + \dot{y}\dot{y}'}{(\dot{x}^2 + \dot{y}^2)^{3/2}} \right|_{\epsilon=0} = \frac{-\dot{u}\sqrt{\dot{x}^2 + \dot{y}^2}}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{-\dot{u}}{\dot{x}^2 + \dot{y}^2},$$

so

$$\dot{\phi} \cdot \delta \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2}} = - \frac{\dot{u}\dot{\phi}}{\dot{x}^2 + \dot{y}^2}.$$

Finally,

$$\boxed{\delta k_1 = \delta \left(\frac{\dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = -v\dot{\phi} \frac{\ddot{y}\dot{y} + \ddot{x}\dot{x}}{(\dot{x}^2 + \dot{y}^2)^2} + \frac{u\ddot{\phi}}{\dot{x}^2 + \dot{y}^2} = u \frac{\ddot{k}_1}{\sqrt{\dot{x}^2 + \dot{y}^2}}.}$$

D Appendix IV: Calculation of the Equivalence between $x(\pi) = 0$ and $\dot{H}(\pi) = 0$

The Euler-Lagrange Equation we derive is:

$$\left[\frac{1}{x\sqrt{\dot{x}^2 + \dot{y}^2}} \left(\frac{x\dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right)' + 2H(H^2 - K) - (2\mu H + 2p) \right] x\sqrt{\dot{x}^2 + \dot{y}^2} = 0$$

or if we ignore the degenerated case, it will be simplified as

$$\frac{1}{x\sqrt{\dot{x}^2 + \dot{y}^2}} \left(\frac{x\dot{H}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right)' + 2H(H^2 - K) - (2\mu H + 2p) = 0. \quad (10)$$

Coupling it with the local area constraint $x\sqrt{\dot{x}^2 + \dot{y}^2} = \sin t$, and let

$$Q := \frac{x^2}{\sin^2 t} \dot{H}$$

then we have

$$\dot{Q} + \frac{\cos t}{\sin t} Q + 2H(H^2 - K) - (2\mu H + 2p) = 0. \quad (11)$$

Our goal here is to prove that

$$\boxed{x(0) = 0, \dot{H}(0) = 0, \dot{H}(\pi) = 0} \Rightarrow x(\pi) = 0.$$

We can rewrite (11) into a form of total derivative. Actually,

$$\begin{aligned} & \dot{Q} + \frac{\cos t}{\sin t} Q + 2H(H^2 - K) - (2\mu H + 2p) = 0 \\ & \Rightarrow \left[\dot{Q} + \frac{\cos t}{\sin t} Q + 2H(H^2 - K) - (2\mu H + 2p) \right] \sin t \cos \phi = 0 \\ & \Rightarrow \underbrace{\dot{Q} \sin t \cos \phi}_{\text{I}} + \underbrace{Q \cos t \cos \phi}_{\text{II}} + \underbrace{2H^3 \sin t \cos \phi}_{\text{III}} - \underbrace{2HK \sin t \cos \phi}_{\text{IV}} - \underbrace{2\mu H \sin t \cos \phi}_{\text{V}} - \underbrace{2p \sin t \cos \phi}_{\text{VI}} = 0 \end{aligned}$$

After simple calculation, it yields

$$\begin{aligned}
I &= (Q \sin t \cos \phi) \cdot + \underbrace{Q \sin t \sin \phi \cdot \dot{\phi}}_{VI} \\
III &= 2H \frac{\dot{\phi}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \cdot \frac{\sin \phi}{x} \sin t \cos \phi = 2H \frac{x \dot{\phi}}{\sin t} \cdot \frac{\sin \phi}{x} \sin t \cos \phi = H(\sin^2 \phi) \cdot \\
VI &= \frac{x^2}{\sin^2 t} \dot{H} \sin t \sin \phi (2H - \frac{\sin \phi}{x}) \frac{\sin t}{x} = x \dot{H} \sin \phi (2H - \frac{\sin \phi}{x}) \\
IV &= \mu \left(\frac{x \dot{\phi}}{\sin t} + \frac{\sin \phi}{x} \right) \sin t \cos \phi = \mu(x \cos \phi \cdot \dot{\phi} + \dot{x} \sin \phi) = (\mu x \sin \phi) \cdot \\
V &= 2px \dot{x} = (px^2) \cdot .
\end{aligned}$$

Then (11) turns into

$$\begin{aligned}
&(Q \sin t \cos \phi - \mu x \sin \phi - px^2) \cdot + x \dot{H} \sin \phi (2H - \frac{\sin \phi}{x}) + 2H^3 \sin t \cos \phi - H(\sin^2 \phi) \cdot = 0 \\
\Rightarrow &(Q \sin t \cos \phi - \mu x \sin \phi - px^2) \cdot + \underbrace{2xH \dot{H} \sin \phi}_{VII} - \dot{H} \sin^2 \phi + \underbrace{2H^3 \sin t \cos \phi}_{II} - H(\sin^2 \phi) \cdot = 0 \\
\Rightarrow &(Q \sin t \cos \phi - \mu x \sin \phi - px^2 - H \sin^2 \phi) \cdot + \underbrace{2xH \dot{H} \sin \phi}_{VII} + \underbrace{2H^3 \sin t \cos \phi}_{II} = 0
\end{aligned}$$

In addition,

$$\begin{aligned}
VII+II &= (xH^2 \sin \phi) \cdot - \dot{x}H^2 \sin \phi - \cancel{xH^2 \cos \phi \dot{\phi}} + H^2 \cos \phi \left(\cancel{x \dot{\phi}} + \frac{\sin \phi \sin t}{x} \right) \\
&= (xH^2 \sin \phi) \cdot - \dot{x}H^2 \sin \phi + H^2 \cos \phi \frac{\sin \phi \sin t}{x} \\
&= (xH^2 \sin \phi) \cdot
\end{aligned}$$

Finally, from (11) we obtain the following total derivative

$$(Q \sin t \cos \phi - \mu x \sin \phi - px^2 - H \sin^2 \phi + xH^2 \sin \phi) \cdot = 0. \quad (12)$$

If we take integral from 0 to π on both sides of (12), it gives us

$$(Q \sin t \cos \phi - \mu x \sin \phi - px^2 - H \sin^2 \phi + xH^2 \sin \phi) \Big|_{t=0}^{\pi} = 0.$$

Since

$$\boxed{x(0) = 0, \dot{H}(0) = 0, \dot{H}(\pi) = 0} \Rightarrow Q(0) = 0, Q(\pi) = 0,$$

then we end up with:

$$-p[x(\pi)]^2 = 0 \Rightarrow x(\pi) = 0.$$