

THE WHITNEY ALGEBRA OF A MATROID

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1. INTRODUCTION

The concept of *matroid*, with its companion concept of *geometric lattice*, was distilled by Hassler Whitney [19], Saunders Mac Lane [10] and Garrett Birkhoff [2] from the common properties of linear and algebraic dependence. The inverse problem, how to *represent* a given abstract matroid as the matroid of linear dependence of a specified set of vectors over some field (or as the matroid of algebraic dependence of a specified set of algebraic functions) has already prompted fifty years of intense effort by the leading researchers in the field: William Tutte, Dominic Welsh, Tom Brylawski, Neil White, Bernt Lindstrom, Peter Vamos, Joseph Kung, James Oxley, and Geoff Whittle, to name only a few. (A goodly portion of this work aimed to provide a proof or refutation of what is now, once again, after a hundred or so years, the 4-color theorem.)

One way to attack this inverse problem, the representation problem for matroids, is first to study the ‘play of coordinates’ in vector representations. In a vector representation of a matroid M , each element of M is assigned a vector in such a way that dependent (resp., independent) subsets of M are assigned dependent (resp., independent) sets of vectors. The coefficients of such linear dependencies are computable as minors of the matrix of coordinates of the dependent sets of vectors; this is Cramer’s rule. For instance, if three points a, b, c are represented in \mathbb{R}^4 (that is, in real projective 3-space) by the dependent vectors forming the rows of the matrix

$$C = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{pmatrix} 1 & 4 & 0 & 6 \\ -2 & 3 & 1 & -5 \\ -4 & 17 & 3 & -3 \end{pmatrix} \end{matrix},$$

they are related by a linear dependence, which is unique up to an overall scalar multiple, and is computable by forming ‘complementary minors’ in any pair of independent columns of the matrix C . For example, choosing columns 3 and 4, we find

$$a(bc)_{34} - b(ac)_{34} + c(ab)_{34} = 12a + 18b - 6c = 0,$$

while in columns 1 and 3,

$$a(bc)_{13} - b(ac)_{13} + c(ab)_{13} = -2a - 3b + c = 0.$$

The six minors 2×2 minors $(bc)_{jk}$, of the matrix C are the coordinates of the wedge product of vectors $b \wedge c$ with respect to the standard basis $\{e_j \wedge e_k\}$ of the exterior power $\Lambda^2 = \Lambda^2(\mathbb{R}^4)$, and similarly for $a \wedge c$ and $a \wedge b$. Hence the

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corresponding six linear relations satisfied by vectors a, b, c may be summed up most efficiently by the equation

$$(1.1) \quad a \otimes bc - b \otimes ac + c \otimes ab = 0$$

in the tensor product $\mathbb{R}^4 \otimes \Lambda^2 = \Lambda^1 \otimes \Lambda^2$, where we have denoted the wedge product by juxtaposition.

On the other hand, by taking coordinates of the vectors a, b and c , we see that Equation 1.1 is equivalent to the set of four linear relations:

$$\begin{aligned} bc - 2ac - 4ac &= 0 \\ 4bc + 3ac + 17ab &= 0 \\ ac + 3ab &= 0 \\ 6bc - 5ac - 3ab &= 0 \end{aligned}$$

in Λ^2 . The space of all linear relations satisfied by wedge products bc, ac, ab has rank 2 (spanned by any two of the above relations), because bc, ac, ab correspond to the same projective line (rank 2), and are thus scalar multiples of one another.

Another way of interpreting the tensor equation (1.1), that better explains how it arises, is to observe that the scalar expression

$$a_i \otimes (bc)_{jk} - b_i \otimes (ac)_{jk} + c_i \otimes (ab)_{jk}$$

is a Laplace expansion of $(abc)_{ijk}$, the ijk -coordinate of the wedge product abc , and since the set $\{a, b, c\}$ is linearly dependent, the product abc is zero and thus all its coordinates (3×3 minors of the matrix C) are zero. Equation 1.1 is thus keeping track of those algebraic relations among *non-zero* coordinates of vectors and wedge products of vectors that follow from the fact that $abc = 0$.

The origins of tensor equations such as (1.1) are most clearly revealed, however, by the Hopf algebra structure of the exterior algebra $\Lambda = \bigoplus \Lambda^k$. Recall that the coproduct $\delta: \Lambda \rightarrow \Lambda \otimes \Lambda$ is the multiplicative map determined by $\delta(a) = a \otimes 1 + 1 \otimes a$, for all vectors $a \in \Lambda^1$; for example,

$$\begin{aligned} \delta(abc) &= \delta(a) \delta(b) \delta(c) \\ &= abc \otimes 1 + ab \otimes c - ac \otimes b + bc \otimes a \\ &\quad + c \otimes ab - b \otimes ac + a \otimes bc + 1 \otimes abc, \end{aligned}$$

for vectors a, b, c (where the signs are determined by anticommutativity). Now if the set $\{a, b, c\}$ is dependent, then the wedge product abc is equal to zero in Λ , and hence the coproduct $\delta(abc)$ is also zero. Since Λ is graded by the nonnegative integers \mathbb{N} , the tensor product $\Lambda \otimes \Lambda$ is thus graded by $\mathbb{N} \times \mathbb{N}$, and an element of $\Lambda \otimes \Lambda$ is equal to zero if and only if all its $(\mathbb{N} \times \mathbb{N})$ -homogeneous components are zero. Hence, in particular, if $\{a, b, c\}$ is linearly dependent, then the degree, or *shape*, $(1, 2)$ homogeneous component $a \otimes bc - b \otimes ac + c \otimes ab$ of the coproduct $\delta(abc)$ is equal to zero; in other words, Equation 1.1 holds. We obtain similar relations in each component $T^k(\Lambda) = \Lambda \otimes \cdots \otimes \Lambda$ of the tensor algebra $T(\Lambda) = \bigoplus T^k(\Lambda)$ from the fact that the iterated coproduct $\delta^k(a_1 \cdots a_r)$ is zero for any dependent set of vectors $\{a_1, \dots, a_r\}$.

These observations suggest a symbolic calculus based directly on a matroid $M = M(S)$, a calculus of independent sets for M that is the analogue of the exterior algebra of a vector space. The idea is to begin with the free exterior algebra E , over the integers, generated by the set of points S ; hence E consists of \mathbb{Z} -linear

combinations of anticommutative words on S , and is a graded Hopf algebra, with coproduct determined just as for the exterior algebra of a vector space. We then construct the tensor algebra $T(E) = \bigoplus T^k(E)$, consisting of linear combinations of tensor products of anticommutative words on S and, finally, divide out the ideal generated by all words formed from dependent sets in M and all homogeneous components of coproducts of such words. In this manner we impose those algebraic relations on $T(E)$ that necessarily would hold if ‘dependence in M ’ meant ‘linear dependence over \mathbb{Z} ’, but without imposing specific \mathbb{Z} -linear relations on the points of M . We call the resulting structure the *Whitney algebra* of the matroid M .

The Whitney algebra $W = \bigoplus W^k$ is graded, with each homogeneous component W^k equal to the image of $T^k(E)$ under the canonical surjection. In addition to the product (which we denote by \circ , rather than \otimes) that W inherits as a quotient of $T(E)$, each component has an *internal product*, induced by the product on $T^k(E)$. For example, in W^3 ,

$$(ad \circ cdf \circ a)(be \circ ae \circ b) = -abde \circ acdef \circ ab.$$

The coproduct on E induces a map $\delta: W^1 \rightarrow W^2$, to which we also refer as a coproduct, which is coassociative in the appropriate sense and respects internal products, that is, $\delta(uv) = \delta(u)\delta(v)$ in W^2 , for all $u, v \in W^1$. In fact, W has precisely the same algebraic structure as the tensor algebra of a commutative Hopf algebra H , where W^k plays the role of the tensor power $T^k(H)$, but with the crucial distinction that W^k is not equal to $T^k(W^1)$.

Before beginning a serious study of the Whitney algebra, indeed before we can even state its most basic properties, we must first determine axioms for its algebraic structure and describe the morphisms between such structures. This we do in Section 3, where we construct the category of *lax Hopf algebras*, which generalize commutative Hopf algebras in precisely the sense required for the definition of the Whitney algebra. Adopting the appropriate categorical point of view, we see that lax Hopf algebras are simply ‘weakened’ versions of commutative Hopf algebras (just as lax monoidal functors are weakened versions of monoidal functors) and hence are their natural generalizations. We describe (in Proposition 3.14) how to construct a lax Hopf algebra from a Hopf algebra by factoring out an ideal that is not necessarily a coideal, and we characterize (in Propositions 3.11 and 3.13) morphisms between lax Hopf algebras in an important special case.

In order to prepare the reader for the material on lax Hopf algebras we first present, in Section 2, some background on categories of graded algebras and Hopf algebras. Section 4 is also mainly background, and consists primarily of well-known facts about free exterior algebras and exterior algebras of finite-dimensional vector spaces, assembled here for the convenience of the reader. The material near the end of the section (Definitions 4.11 through 4.13 and Proposition 4.15) may be less familiar, though, even to readers well-versed in multilinear algebra. Section 5 is devoted to the proof of our main technical result, the *Zipper lemma* (Theorem 5.7), which is an identity satisfied by homogeneous components of coproducts in an exterior (Hopf) algebra. As far as we know, the Zipper is a new theorem about exterior algebra, and thus may be of independent interest to algebraists; we use it here for the proof of the fundamental *exchange relations* (Theorem 7.4) in the Whitney algebra. These relations, in particular, generalize the ‘two definitions of meet’ in the Cayley algebra of a Peano space (see [4]).

In Section 6, we finally define the Whitney algebra of a matroid, and start to investigate its basic properties. In particular, we show that the construction determines a functor from the category of matroids and weak maps to the category of lax Hopf algebras, and that it is universal with respect to representations of matroids; specifically: any representation of a matroid in a vector space extends to a unique lax Hopf algebra morphism from the Whitney algebra of the matroid to the exterior algebra of the vector space. We prove the exchange relations and examine the commutativity properties of the Whitney algebra in Section 7, and in Section 8, we prove a Whitney algebra version of Bazin’s theorem for compound determinants (Theorem 8.2) and use it to study the subring of the Whitney algebra generated by the bases of the matroid. In particular, we show that this *basis ring* is isomorphic, modulo nilpotent elements, to White’s bracket ring, and thus the two rings have the same properties with respect to matroid representations.

It is our conviction that the present paper merely scratches the surface of a deep subject, which, with a little time and good luck, will reach maturity some years hence. We have done our best to lay firm foundations for the subject; we heartily invite others now to join in the work.

This is of course not the first mathematical effort in this direction. The bracket ring of a matroid, introduced by Neil White in his doctoral thesis [16], and analyzed in detail in subsequent papers, [17] and [18], is an especially important predecessor of the Whitney algebra. Indeed, as we have mentioned above, it is essentially a subring of the Whitney algebra. Vamos [15], Fenton [7], and Dress and Wenzel [5], also have associated algebras to matroids that reflect their representation properties, and these must bear interesting comparison with the Whitney algebra as well. Still other constructions of algebras for matroids, not directly related to questions of representability, have been made by Graves [8] and Orlik and Solomon [14].

Our article is dedicated to the memory of Gian-Carlo Rota. We wish in this way to convey our appreciation for the profound influence he had on our lives and on our mathematical work, to convey the sense of loss we endure with his parting. He was a dear friend. He was also an active and enthusiastic participant in the work that led up to the present paper. We extract some of his thinking on the subject from electronic mail dated 1995-96, and include it as a closing section to our paper. We take pleasure in recording Gian-Carlo’s comments and advice, since they convey so keenly his enthusiasm for the subject, and provide yet another proof of his uncanny intuition for algebraic structures in combinatorics. As with so many of Rota’s long range predictions, this one has taken years to sort out, but we can assure the reader that his assessment of the situation was brutally correct. The abstract play of coordinates on a matroid indeed points us to a natural algebraic structure – a lax Hopf algebra – that is “not quite a Hopf algebra, but a new object closely related to it”, and that may find wide use far from its birthplace in matroid theory.

2. GRADED ALGEBRAS AND HOPF ALGEBRAS

Throughout this paper we will be working with modules (usually equipped with additional algebraic structure) over a commutative ring R with unit, that are graded by various monoids. We refer to modules graded by the nonnegative integers \mathbb{N} simply as *graded*, and those graded by any other monoid G as *G-graded*. Given a graded R -module $M = \bigoplus_{k \geq 0} M_k$, we denote by $|x|$ the degree of a homogeneous

element $x \in M$; whenever we use this notation, we assume that x is homogeneous. The k -fold tensor product $T^k(M) = M \otimes_R \cdots \otimes_R M$ is graded in the usual fashion, with homogeneous components

$$T_r^k(M) = \sum_{r_1 + \cdots + r_k = r} M_{r_1} \otimes \cdots \otimes M_{r_k},$$

for all $r \geq 0$. In particular, the empty tensor product $T^0(M)$ is equal to the ring R , with the trivial grading in which all elements have degree 0.

For graded R -modules M and N , the *twist map* $\tau = \tau_{M,N}: M \otimes_R N \rightarrow N \otimes_R M$ is the isomorphism defined by

$$\tau(x \otimes y) = (-1)^{|x||y|} y \otimes x.$$

The tensor product operation and twist maps equip the category \mathbf{Mod}_R of graded R -modules and degree zero homogeneous linear maps with a symmetric monoidal structure (we refer the reader to Mac Lane's book [11] for all category theory terminology that we shall use). It follows that the category \mathbf{Alg}_R of graded R -algebras is also symmetric monoidal, with the product $\mu_{A \otimes B}$ and unit $\eta_{A \otimes B}$ on the tensor product of algebras $A \otimes_R B$ defined as the compositions

$$\mu_{A \otimes B} = (\mu_A \otimes \mu_B)(1_A \otimes \tau \otimes 1_B): A \otimes_R B \otimes_R A \otimes_R B \rightarrow A \otimes_R B$$

and

$$\eta_{A \otimes B} = (\eta_A \otimes \eta_B) \kappa: R \rightarrow A \otimes_R B,$$

respectively, where $\kappa: R \rightarrow R \otimes_R R$ is the canonical isomorphism. In more familiar notation, the multiplication on $A \otimes_R B$ is given by

$$(x \otimes y)(x' \otimes y') = (-1)^{|x'||y|} xx' \otimes yy',$$

and the unit element is $1_{A \otimes B} = 1_A \otimes 1_B$.

Suppose that $A = \bigoplus_{g \in G} A_g$ is a G -graded R -algebra. For all $x \in A$ and $g \in G$, we denote by x_g the image of x under the projection map $A \rightarrow A_g$. Homogeneity of the unit and product maps of A means that $1_A \in A_{1_G}$ and

$$(2.1) \quad (xy)_g = \sum_{hk=g} x_h y_k,$$

for all $x, y \in A$ and $g \in G$. An ideal I of A is *homogeneous* if $I = \bigoplus_{g \in G} (I \cap A_g)$ or, equivalently, if I is generated by homogeneous elements. If I is homogeneous, then the quotient algebra A/I is also G -graded, with homogeneous components $A_g/(I \cap A_g)$.

If M is any R -module, the tensor algebra $T(M) = \bigoplus_{k \geq 0} T^k(M)$ is a graded R -algebra with homogeneous components $T^k(M)$. If M itself is graded, then $T(M)$ is also graded by the free monoid $\langle \mathbb{N} \rangle$ on \mathbb{N} , whose elements are finite sequences of nonnegative integers, or *words* on \mathbb{N} , and whose product is given by concatenation of words. The homogeneous components of $T(M)$ are given by

$$T_\alpha(M) = M_{a_1} \otimes_R \cdots \otimes_R M_{a_k},$$

for all $\alpha = (a_1, \dots, a_k) \in \langle \mathbb{N} \rangle$; we refer to the degree $(|x_1|, \dots, |x_k|)$ of homogeneous $x = x_1 \otimes \cdots \otimes x_k \in T(M)$ as the *shape* of x .

If A is a graded R -algebra then the tensor power $T^k(A)$, also graded by shape, is an \mathbb{N}^k -graded algebra, and hence has homogeneous components $T_\alpha^k(A) = T_\alpha(A)$, for all $\alpha \in \mathbb{N}^k \subseteq \langle \mathbb{N} \rangle$ (the additive monoid \mathbb{N}^k is not a submonoid of $\langle \mathbb{N} \rangle$, which has

concatenation as product). The product of homogeneous elements $x = x_1 \otimes \cdots \otimes x_k$ and $y = y_1 \otimes \cdots \otimes y_k$ in $T^k(H)$ of shapes $\alpha = (a_1, \dots, a_k)$ and $\beta = (b_1, \dots, b_k)$, respectively, is given by the *internal multiplication*:

$$(2.2) \quad xy = (-1)^{N(\alpha, \beta)} x_1 y_1 \otimes \cdots \otimes x_k y_k,$$

where $N(\alpha, \beta) = \sum_{j < i} a_i b_j$.

A graded R -algebra A is *commutative* if $\mu_A \tau = \mu_A$; in other words, if $xy = (-1)^{|x||y|}yx$, for all homogeneous $x, y \in A$. (In general, the notion of commutativity for algebras in a symmetric monoidal category depends on the choice of twist map; the above form of commutativity, familiar to topologists, is referred to in many contexts as *anticommutativity*.) The category \mathbf{ComAlg}_R of graded commutative R -algebras, and degree zero homogeneous R -algebra maps, is not only symmetric monoidal, but has finite sums given by the tensor product operation. (Note: what we mean by ‘sum’ here is what most category theorists would call ‘coproduct’, namely, the dual of a product; we choose the former term to avoid confusion with the coproduct *map* of a co- or Hopf algebra.) For all objects A and B of \mathbf{ComAlg}_R , the injections $j_1: A \rightarrow A \otimes_R B$ and $j_2: B \rightarrow A \otimes_R B$ are given by $a \mapsto a \otimes 1$ and $b \mapsto 1 \otimes b$, respectively. Given morphisms $f: A \rightarrow C$ and $g: B \rightarrow C$, the corresponding map $h = \langle f, g \rangle: A \otimes_R B \rightarrow C$ is determined by $h(x \otimes y) = f(x) \cdot g(y)$.

We will need to consider mappings between algebras having different rings of scalars, hence we define the category \mathbf{ComAlg} having as objects all pairs (R, A) such that R is a trivially graded commutative ring and A is a commutative R -algebra, with morphisms $(R, A) \rightarrow (S, B)$ given by pairs of ring homomorphisms $g: R \rightarrow S$, $f: A \rightarrow B$ satisfying $f \eta_A = \eta_B g$. The tensor product operation $(R, A) \otimes (S, B) = (R \otimes_{\mathbb{Z}} S, A \otimes_{\mathbb{Z}} B)$ equips the category \mathbf{ComAlg} with finite sums; the injections $(R, A) \rightarrow (R, A) \otimes (S, B)$ and $(S, B) \rightarrow (R, A) \otimes (S, B)$ are given by $(r, a) \mapsto (r \otimes 1, a \otimes 1)$ and $(s, b) \mapsto (1 \otimes s, 1 \otimes b)$, respectively.

Suppose that $(g, f): (R, A) \rightarrow (S, B)$ and $(g, f'): (R, A') \rightarrow (S, B')$ are algebra morphisms having the same scalar map g . The tensor product of ring maps $f \otimes f': A \otimes_{\mathbb{Z}} A' \rightarrow B \otimes_{\mathbb{Z}} B'$ satisfies

$$(f \otimes f')(x \otimes ry - rx \otimes y) = f(x) \otimes g(r)f'(y) - g(r)f(x) \otimes f'(y),$$

for all $r \in R$, and so $f \otimes f'$ induces a ring homomorphism $A \otimes_R A' \rightarrow B \otimes_S B'$, that we also denote by $f \otimes f'$; furthermore, the pair $(g, f \otimes f')$ is a morphism $(R, A \otimes_R A) \rightarrow (S, B \otimes_S B)$ in \mathbf{ComAlg} .

We regard \mathbf{ComAlg}_R as a subcategory of \mathbf{ComAlg} , identifying an R -algebra homomorphism f with the morphism $(1_R, f)$ in \mathbf{ComAlg} . Note that \mathbf{ComAlg}_R is not a full subcategory of \mathbf{ComAlg} , and the inclusion functor $\mathbf{ComAlg}_R \rightarrow \mathbf{ComAlg}$ does not preserve sums, since the sum of (R, A) and (R, B) in \mathbf{ComAlg}_R is $(R, A \otimes_R B)$, while in \mathbf{ComAlg} it is $(R \otimes_{\mathbb{Z}} R, A \otimes_{\mathbb{Z}} B)$. However, there is a naturally defined epimorphism $(R \otimes_{\mathbb{Z}} R, A \otimes_{\mathbb{Z}} B) \rightarrow (R, A \otimes_R B)$, given by (μ_R, γ) , where $\gamma: A \otimes_{\mathbb{Z}} B \rightarrow A \otimes_R B$ is the canonical surjection.

From now on, we usually will write simply $f: A \rightarrow B$ for an algebra morphism $(g, f): (R, A) \rightarrow (S, B)$, omitting explicit mention of the rings R and S , and the scalar map $g: R \rightarrow S$.

We turn next to graded commutative Hopf algebras, beginning with the more general notion of cogroup object in a category equipped with finite sums.

Suppose that \mathcal{C} is a category having finite sums. To be consistent with our primary example of such a category, \mathbf{ComAlg}_R , we denote the sum of objects

X and Y in \mathcal{C} by $X \otimes Y$. For each object X , the universal property of sums guarantees the existence of a unique morphism $\mu: X \otimes X \rightarrow X$, called the *product* of X , making the diagram

$$\begin{array}{ccc} X \otimes X & \longleftarrow & X \\ \uparrow & \searrow \mu & \downarrow 1 \\ X & \longrightarrow & X \end{array}$$

commute. (In this diagram, and all following diagrams in this section, all unlabeled arrows are canonical injections into sums). The empty sum is the initial object 1 of \mathcal{C} ; for each object X we refer to the unique morphism $\eta: 1 \rightarrow X$ as the *unit* of X . Whenever $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$, there is a unique morphism $f \otimes f': X \otimes X' \rightarrow Y \otimes Y'$ making the diagram

$$\begin{array}{ccccc} X & \longrightarrow & X \otimes X' & \longleftarrow & X' \\ f \downarrow & & \downarrow f \otimes f' & & \downarrow f' \\ Y & \longrightarrow & Y \otimes Y' & \longleftarrow & Y' \end{array}$$

commute.

In particular, if A is an object of \mathbf{ComAlg}_R , then $\mu = \mu_A$ and $\eta = \eta_A$ are the usual product and unit maps of the graded commutative R -algebra A , and if f and f' are morphisms in \mathbf{ComAlg}_R , then $f \otimes f'$ is the tensor product of algebra homomorphisms.

Definition 2.3. A *cogroup object* in a category \mathcal{C} with finite sums is a quadruple $(X, \delta, \epsilon, \chi)$ where X is an object of \mathcal{C} , and $\delta: X \rightarrow X \otimes X$, $\epsilon: X \rightarrow 1$ and $\chi: X \rightarrow X$ are morphisms such that the diagrams

$$\begin{array}{ccc} X & \xrightarrow{\delta} & X \otimes X \\ \delta \downarrow & & \downarrow \delta \otimes 1 \\ X \otimes X & \xrightarrow{1 \otimes \delta} & X \otimes X \otimes X \end{array} \quad \begin{array}{ccc} X \otimes X & \xrightarrow{1 \otimes \epsilon} & X \otimes 1 \\ \epsilon \otimes 1 \downarrow & \swarrow \delta & \uparrow \cong \\ 1 \otimes X & \xrightarrow{\cong} & X \end{array}$$

and

$$\begin{array}{ccccc} X \otimes X & \xleftarrow{\delta} & X & \xrightarrow{\delta} & X \otimes X \\ \downarrow \chi \otimes 1 & & \downarrow \epsilon & & \downarrow 1 \otimes \chi \\ X \otimes X & \xrightarrow{\mu} & X & \xleftarrow{\mu} & X \otimes X \end{array}$$

commute.

The cogroup objects in \mathcal{C} form a category, with morphisms

$$(X, \delta, \epsilon, \chi) \rightarrow (X', \delta', \epsilon', \chi')$$

given by morphisms $f: X \rightarrow X'$ in \mathcal{C} that commute with the structure morphisms, that is, such that

$$(f \otimes f) \delta = \delta' f, \quad \epsilon = \epsilon' f, \quad \text{and} \quad f \chi = \chi' f.$$

In particular, a cogroup object $(H, \delta, \epsilon, \chi)$ in \mathbf{ComAlg}_R is a graded commutative R -Hopf algebra, with coproduct δ , counit ϵ and antipode χ . The category of cogroup objects in \mathbf{ComAlg}_R is thus the category $\mathbf{ComHopf}_R$ of graded commutative R -Hopf algebras and degree zero homogeneous Hopf algebra maps.

For any graded R -Hopf algebra $H = \bigoplus_{k \geq 0} H_k$, the homogeneity of the coproduct $\delta: H \rightarrow H \otimes_R H$ and counit $\epsilon: H \rightarrow K$ mean that

$$(2.4) \quad \delta(H_k) \subseteq \sum_{r+s=k} H_r \otimes_R H_s,$$

for all $k \geq 0$, and $\epsilon(H_k) = 0$ for all $k > 0$. For all $r \geq 0$, we write $\delta^{(r)}$ for the iterated coproduct $H \rightarrow T^r(H)$; in this notation, $\delta^{(2)}: H \rightarrow H \otimes_R H$ is the ordinary coproduct, $\delta^{(1)}: H \rightarrow H$ the identity map and, by convention, $\delta^{(0)}: H \rightarrow K$ is the counit. For each $\alpha = (a_1, \dots, a_r) \in \mathbb{N}^r$, we define the *coproduct slice*

$$\delta_\alpha: H \rightarrow T_\alpha^r(H) = H_{a_1} \otimes_R \cdots \otimes_R H_{a_r}$$

by setting $\delta_\alpha(x)$ equal to $(\delta^{(r)}(x))_\alpha$, the homogeneous part of $\delta^{(r)}(x)$ in degree α . Hence, in particular, $\delta^{(r)} = \sum_{\alpha \in \mathbb{N}^r} \delta_\alpha$. Note also that the homogeneity of the coproduct, expressed by Equation 2.4, is equivalent to requirement that

$$\delta(x_k) = \sum_{r+s=k} \delta_{(r,s)}(x),$$

for all $x \in H$ and $k \geq 0$. The fact that the iterated coproducts $\delta^{(r)}: H \rightarrow T^r(H)$ are algebra maps, together with the homogeneity of the product (2.2), implies that

$$\delta_\alpha(xy) = \sum_{\beta+\gamma=\alpha} \delta_\beta(x) \delta_\gamma(y),$$

for all $x, y \in H$ and $\alpha \in \mathbb{N}^r$.

We employ the *Sweedler notation*:

$$\delta^{(r)}(x) = \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(r)}$$

for the iterated coproduct of $x \in H$, and use the modified Sweedler notation

$$\delta_\alpha(x) = \sum_{(x)_\alpha} x_{(1)} \otimes \cdots \otimes x_{(k)}$$

for coproduct slices. We also shall need to consider sums of coproduct slices $\delta_\alpha(x)$, where some of the components of α are fixed and others range over all of \mathbb{N} . To denote such sums, we use dots in place of components that are to be summed over; for example

$$\delta_{(r,s,\cdot)}(x) = \sum_{(x)_{(r,s,\cdot)}} x_{(1)} \otimes x_{(2)} \otimes x_{(3)},$$

is equal to the sum $\sum_{k \in \mathbb{N}} \delta_{(r,s,k)}(x)$.

3. LAX HOPF ALGEBRAS

In this section we introduce a generalization of the notion of commutative Hopf algebra, called a *lax Hopf algebra*, that is a graded algebra having the same structure as the tensor algebra of a commutative Hopf algebra, but without the requirement that all homogeneous components are tensor powers of the component of degree one. We begin by reformulating the definition of cogroup object, as a certain type of functor, which then may be suitably generalized.

We denote by \mathcal{T} the free category with finite sums generated by a single cogroup object X . Hence \mathcal{T} has objects X^k , for $k \geq 0$, where X^k is the k -fold sum $X \otimes \dots \otimes X$, and $(X, \delta, \epsilon, \chi)$ is a cogroup object. In addition to the structure morphisms δ, ϵ, χ , and the canonical injections $i_1: X^r \rightarrow X^r \otimes X^s = X^{r+s}$ and $i_2: X^s \rightarrow X^r \otimes X^s$, for all $r, s \geq 0$, the category \mathcal{T} contains all morphisms forced by the universal property of sums; in particular, the product and unit morphisms $\mu = \mu_k: X^{2k} \rightarrow X^k$ and $\eta = \eta_k: 1 = X^0 \rightarrow X^k$, for all $k \geq 0$, and for all $f: X^k \rightarrow X^r$ and $g: X^\ell \rightarrow X^s$ in \mathcal{T} , the morphism $f \otimes g: X^{k+\ell} \rightarrow X^{r+s}$ commuting with the appropriate injections. Also, \mathcal{T} contains the *twist* morphisms $\tau = \tau_{r,s}: X^{r+s} \rightarrow X^{s+r}$, defined for all $r, s \geq 0$ by

$$\begin{array}{ccc} X^{r+s} & \xleftarrow{i_2} & X^s \\ i_1 \uparrow & \searrow \tau & \downarrow i_1 \\ X^r & \xrightarrow{i_2} & X^{s+r} \end{array}$$

The twist morphisms satisfy $\tau_{r,s}^2 = 1$ and $\tau_{r,s} \neq 1$, for all $r, s > 0$.

Proposition 3.1. *The product μ_k is commutative, that is, $\mu_k \tau = \mu_k$, for all $k \geq 0$.*

Proof. By the definition of μ_k and τ , the diagram

$$\begin{array}{ccccc} & & X^{2k} & & \\ & i_1 \nearrow & \downarrow \tau & \nwarrow i_2 & \\ X^k & \xrightarrow{i_2} & X^{2k} & \xleftarrow{i_1} & X^k \\ & \searrow 1 & \downarrow \mu_k & \swarrow 1 & \\ & & X^k & & \end{array}$$

commutes. Since μ_k is the unique morphism $X^{2k} \rightarrow X^k$ such that $\mu_k i_1 = \mu_k i_2 = 1$, it follows that $\mu_k \tau = \mu_k$. \square

The category \mathcal{T} is free in the sense that it contains all compositions of the above morphisms, and these compositions satisfy as few relations as possible in order that \mathcal{T} is a category having X as a cogroup object; more precisely, \mathcal{T} is characterized by the following property: for any cogroup object $(X', \delta', \epsilon', \chi')$ in a category \mathcal{C} having finite sums, there exists a unique (up to natural isomorphism) sum-preserving functor $F: \mathcal{T} \rightarrow \mathcal{C}$ such that $F(X) = X'$, $F(\delta) = \delta'$, $F(\epsilon) = \epsilon'$ and $F(\chi) = \chi'$.

On the other hand, if $F: \mathcal{T} \rightarrow \mathcal{C}$ is any sum-preserving functor, then the quadruple $(F(X), F(\delta), F(\epsilon), F(\chi))$ is a cogroup object in \mathcal{C} . Hence there is a one-to-one correspondence between cogroup objects in \mathcal{C} and sum-preserving functors $\mathcal{T} \rightarrow \mathcal{C}$, and so we may *define* a cogroup object in \mathcal{C} as such a functor. According to this

definition, a morphism of cogroup objects is simply a natural transformation of functors; hence the category of cogroup objects in \mathcal{C} is isomorphic to the full subcategory of the functor category $\mathcal{C}^{\mathcal{T}}$ having sum-preserving functors as objects.

In particular, we identify the category of graded commutative R -Hopf algebras with the category of all sum-preserving functors $\mathcal{T} \rightarrow \mathbf{ComAlg}_R$. Adopting this point of view allows us to generalize the notion of commutative Hopf algebra in the following succinct manner:

Definition 3.2. A *lax R -Hopf algebra* is a functor $H: \mathcal{T} \rightarrow \mathbf{ComAlg}_R$ that satisfies $H(1) = R$.

Our reason for this choice of terminology is the following: while sum-preserving functors between categories equipped with finite sums are special instances of monoidal functors between monoidal categories, arbitrary functors between such categories preserve the monoidal structure in a weaker sense, and are examples of what are often called *lax monoidal functors*. (This terminology is not standard, however; Mac Lane refers to the former as *strong monoidal functors* and the latter as *monoidal functors* in [11].) Hence a Hopf algebra is a kind of monoidal functor and a lax Hopf algebra is a kind of lax monoidal functor.

Given a lax Hopf algebra $H: \mathcal{T} \rightarrow \mathbf{ComAlg}_R$, we write H^k , or $H \circ \dots \circ H$, for $H(X^k)$, and denote the homomorphisms $H(\delta)$, $H(\epsilon)$, $H(\chi)$, $H(\mu_k)$, $H(\eta_k)$ simply by δ , ϵ , χ , μ_k , η_k . For any pair of morphisms α and β in \mathcal{T} , we write $\alpha \circ \beta$ for the image under H of $\alpha \otimes \beta$. Thus H has *structure maps*

$$\delta: H^1 \rightarrow H^1 \circ H^1, \quad \epsilon: H^1 \rightarrow R, \quad \chi: H^1 \rightarrow H^1,$$

all homomorphisms of R -algebras, to which we refer as the coproduct, counit and antipode of H , respectively; and for all $k \geq 1$, we have homomorphisms

$$\mu_k: H^k \circ H^k \rightarrow H^k \quad \text{and} \quad \eta_k: R \rightarrow H^k.$$

We will see shortly that μ_k is closely related to the product μ_{H^k} , and that η_k is equal to the unit η_{H^k} of the algebra H^k . The structure maps make commute all of the usual diagrams in the definition of a Hopf algebra, but with the symbol \circ replacing the tensor product \otimes_R throughout. For example, the coassociativity and counitary axioms are given by

$$\begin{array}{ccc} H^1 & \xrightarrow{\delta} & H^1 \circ H^1 \\ \delta \downarrow & & \downarrow \delta \circ 1 \\ H^1 \circ H^1 & \xrightarrow{1 \circ \delta} & H^1 \circ H^1 \circ H^1 \end{array} \qquad \begin{array}{ccc} H^1 \circ H^1 & \xrightarrow{1 \circ \epsilon} & H^1 \circ R \\ \epsilon \circ 1 \downarrow & \swarrow \delta & \uparrow = \\ R \circ H^1 & \xleftarrow{=} & H^1. \end{array}$$

We denote by **LaxHopf** the category whose objects are all pairs (R, H) , where R is a commutative ring and H is a lax R -Hopf algebra, or equivalently, whose objects are functors $\mathcal{T} \rightarrow \mathbf{ComAlg}$ that factor through the inclusion $\mathbf{ComAlg}_R \hookrightarrow \mathbf{ComAlg}$, for some R . A morphism $f: H \rightarrow L$ in **LaxHopf** is a natural transformation of functors $UH \Rightarrow UL$, where $U: \mathbf{ComAlg} \rightarrow \mathbf{ComRng}$ is the functor that forgets the scalars; hence $f: H \rightarrow L$ consists of a sequence of ring homomorphisms $f^k: H^k \rightarrow L^k$ that commute with the structure maps of H and L . We write **LaxHopf $_R$** for the subcategory of **LaxHopf** having as objects all lax R -Hopf algebras, and as morphisms all $f: H \rightarrow L$ in **LaxHopf** such the component $f^0: H^0 \rightarrow L^0$ is the identity map on R . We will see shortly that every morphism

$H \rightarrow L$ in **LaxHopf** corresponds to a unique natural transformation $H \Rightarrow L$, and thus **LaxHopf** is a subcategory of the functor category \mathbf{ComAlg}^T , and $\mathbf{LaxHopf}_R$ is equal to \mathbf{ComAlg}_R^T . Unless we say otherwise, we always will assume that morphisms between lax R -Hopf algebras belong to $\mathbf{LaxHopf}_R$.

Since a lax Hopf algebra H is not a sum-preserving functor, the algebra $H^r \circ H^s$ is not, in general, equal to the tensor product $H^r \otimes_R H^s$; however there is an algebra homomorphism

$$\pi_{r,s}: H^r \otimes_R H^s \rightarrow H^r \circ H^s,$$

defined as the unique map making the diagram on the right commute:

$$\begin{array}{ccc} & & H^r \otimes_R H^s \\ & \nearrow j_1 & \downarrow \pi_{r,s} \\ X^r \xrightarrow{i_1} X^{r+s} \xleftarrow{i_2} X^s & & H^r \circ H^s \\ & \searrow j_2 & \downarrow H(i_2) \\ & & H^r \end{array}$$

Similarly, there are homomorphisms

$$\pi_\alpha: H^{a_1} \otimes_R \cdots \otimes_R H^{a_k} \rightarrow H^{a_1 + \cdots + a_k},$$

for all $\alpha = (a_1, \dots, a_k) \in \langle \mathbb{N} \rangle$.

Proposition 3.3. *The direct sum $\widehat{H} = \bigoplus_{k \geq 0} H^k$ is a graded R -algebra, with product $\hat{\pi}: \widehat{H} \otimes_R \widehat{H} \rightarrow \widehat{H}$ given by $\hat{\pi} = \bigoplus_{r,s \geq 0} \pi_{r,s}$ and unit $\hat{\eta}$ given by the inclusion of $R = H^0$ into \widehat{H} .*

Proof. Repeated use of the definition of the maps $\pi_{i,j}$ shows that the diagram

$$\begin{array}{ccc} H^r \otimes_R H^s \otimes_R H^t & \xrightarrow{1 \otimes \pi_{s,t}} & H^r \otimes_R H^{s+t} \\ \downarrow \pi_{r,s} \otimes 1 & \searrow \pi_{r,s,t} & \downarrow \pi_{r,s+t} \\ H^{r+s} \otimes_R H^t & \xrightarrow{\pi_{r+s,t}} & H^{r+s+t} \end{array}$$

commutes, for all $r, s, t \geq 0$, and thus $\hat{\pi}$ is associative. The unit property of $\hat{\eta}$ is trivial. \square

We refer to the product on \widehat{H} as the *external* product to avoid confusion with the *internal* product that each H^k has as an object of \mathbf{ComAlg}_R . We usually don't distinguish between the graded algebra \widehat{H} , equipped with all of its structure maps, and the functor H itself; hence we refer to \widehat{H} , as well as H , as a lax Hopf algebra.

For all $k \geq 0$, we denote by π_k the homomorphism $\pi_{(1, \dots, 1)}: T^k(H^1) \rightarrow H^k$.

Proposition 3.4. *For any $\alpha = (a_1, \dots, a_k) \in \langle \mathbb{N} \rangle$, with $a_1 + \cdots + a_k = a$, the diagram*

$$\begin{array}{ccc} T^r(H^1) & \xrightarrow{\pi_{a_1} \otimes \cdots \otimes \pi_{a_k}} & H^{a_1} \otimes_R \cdots \otimes_R H^{a_k} \\ & \searrow \pi_\alpha & \downarrow \pi_\alpha \\ & & H^r \end{array}$$

commutes.

Proof. The result follows from the definition of the maps π_α . \square

Corollary 3.5. *The direct sum $\pi = \bigoplus_k \pi_k$ is a homomorphism from the tensor algebra $T(H^1)$ to the algebra \widehat{H} .*

Proof. By Proposition 3.4, the square

$$\begin{array}{ccc} T^r(H^1) \otimes_R T^s(H^1) & \xrightarrow{\cong} & T^{r+s}(H^1) \\ \pi_r \otimes \pi_s \downarrow & & \downarrow \pi_{r+s} \\ H^r \otimes_R H^s & \xrightarrow{\pi_{r,s}} & H^{r+s} \end{array}$$

commutes, for all $r, s \geq 0$, and hence π preserves products. It is clear that π preserves units. \square

In the special case that the functor H preserves sums (i.e., is a Hopf algebra), each of the maps $\mu_k : H^k \circ H^k \rightarrow H^k$ is equal to the product on H^k . If H is an arbitrary lax Hopf algebra then μ_k is related to the product on H^k as follows:

Proposition 3.6. *For all k , the product $\mu_{H^k} : H^k \otimes H^k \rightarrow H^k$ factors through the homomorphism $\mu_k : H^k \circ H^k \rightarrow H^k$ as $\mu_k \pi_{k,k} = \mu_{H^k}$.*

Proof. By the definition of $\pi_{k,k}$ and μ_k , the diagram

$$\begin{array}{ccccc} & & H^k \otimes_R H^k & & \\ & j_1 \nearrow & \downarrow \pi_{k,k} & \nwarrow j_2 & \\ H^k & \xrightarrow{H(i_1)} & H^k \circ H^k & \xleftarrow{H(i_2)} & H^k \\ & \searrow 1 & \downarrow \mu_k & \swarrow 1 & \\ & & H^k & & \end{array}$$

commutes. Since $H^k \otimes_R H^k$ is the sum of H^k and H^k in \mathbf{ComAlg}_R , the product μ_{H^k} is the unique map $H^k \otimes_R H^k \rightarrow H^k$ such that $\mu_{H^k} j_1 = \mu_{H^k} j_2 = 1$. Hence $\mu_k \pi_{k,k} = \mu_{H^k}$. \square

Corollary 3.7. *For each $k \geq 0$, the map $\eta_k = H(\eta_k) : R \rightarrow H^k$ is equal to the unit η_{H^k} of the algebra H^k .*

Proof. Proposition 3.6 allows us to infer the commutativity of the diagram

$$\begin{array}{ccccc} R \otimes_R H^k & \xrightarrow{\eta_k \otimes 1} & H^k \otimes_R H^k & \xleftarrow{1 \otimes \eta_k} & H^k \otimes_R R \\ & \searrow \cong & \downarrow \mu_{H^k} & \swarrow \cong & \\ & & H^k & & \end{array}$$

from that of the diagram

$$\begin{array}{ccccc} R \circ H^k & \xrightarrow{\eta_k \circ 1} & H^k \circ H^k & \xleftarrow{1 \circ \eta_k} & H^k \circ R \\ & \searrow = & \downarrow \mu_k & \swarrow = & \\ & & H^k & & \end{array}$$

which is immediate from the definitions, by embedding the two in the appropriate larger commutative diagram. It follows that η_k is the unit of H^k . \square

We now turn our attention to morphisms of lax Hopf algebras. Suppose that H is a lax R -Hopf algebra and L is a lax S -Hopf algebra. A morphism $f: H \rightarrow L$ in $\mathbf{LaxHopf}$ consists of a sequence of ring homomorphisms $f^k: H^k \rightarrow L^k$ that commute with the structure maps of H and L . In particular, we have $f^k \eta_k = \eta_k f^0$ or, by Corollary 3.7, $f^k \eta_{H^k} = \eta_{L^k} f^0$, and hence f^k is an algebra morphism, with scalar map $f^0: R \rightarrow S$, for all $k \geq 0$. It is therefore equivalent to define a morphism $f: H \rightarrow L$ in $\mathbf{LaxHopf}$ as a natural transformation of functors $\mathcal{T} \rightarrow \mathbf{ComAlg}$ such that each algebra homomorphism $f^k: H^k \rightarrow L^k$ has the same scalar map, and a morphism in $\mathbf{LaxHopf}_R$ as an arbitrary natural transformation of functors $\mathcal{T} \rightarrow \mathbf{ComAlg}_R$.

The next proposition tells us, in particular, how the maps f^k and $f^1 \otimes \dots \otimes f^1$ are related for a morphism f of lax Hopf algebras.

Proposition 3.8. *For any morphism $f: H \rightarrow L$ of lax Hopf algebras, the square*

$$\begin{array}{ccc} H^{a_1} \otimes_R \dots \otimes_R H^{a_k} & \xrightarrow{f^{a_1} \otimes \dots \otimes f^{a_k}} & L^{a_1} \otimes_S \dots \otimes_S L^{a_k} \\ \pi_\alpha \downarrow & & \downarrow \pi_\alpha \\ H^{a_1 + \dots + a_k} & \xrightarrow{f^{a_1 + \dots + a_k}} & L^{a_1 + \dots + a_k} \end{array}$$

commutes, for all $\alpha = (a_1, \dots, a_k) \in \langle \mathbb{N} \rangle$.

Proof. First we note that, since the maps f^k all have the same scalar map, namely f^0 , it follows from the discussion in Section 2 that $f^{a_1} \otimes \dots \otimes f^{a_k}$ is a well-defined algebra homomorphism from $H^{a_1} \otimes_R \dots \otimes_R H^{a_k}$ to $L^{a_1} \otimes_S \dots \otimes_S L^{a_k}$, having f^0 as scalar map. The result for $\alpha = (r, s)$ follows by contemplating the commutative cube:

$$\begin{array}{ccccc} & & H^r \otimes_R H^s & \xleftarrow{\quad} & H^s \\ & & \downarrow \pi_{r,s} & \searrow & \downarrow H(i_2) \\ & & H^r & \xrightarrow{H(i_2)} & H^{r+s} \\ & & \downarrow f^r & \downarrow f^s & \downarrow f^{r+s} \\ & & L^r \otimes_S L^s & \xleftarrow{\quad} & L^s \\ & & \downarrow \pi_{r,s} & \searrow & \downarrow L(i_2) \\ & & L^r & \xrightarrow{L(i_1)} & L^{r+s} \end{array}$$

The proof of the general result is essentially the same, but requires a much bigger diagram. \square

We now examine a special class of lax Hopf algebras; morphisms having domain belonging to this class are particularly simple to describe.

Definition 3.9. A lax Hopf algebra H is of *quotient type* if the homomorphism $\pi: T(H^1) \rightarrow \widehat{H}$, defined in Corollary 3.5, is surjective.

Proposition 3.10. *If a lax Hopf algebra H is of quotient type then the homomorphisms*

$$\pi_\alpha: H^{a_1} \otimes_R \cdots \otimes_R H^{a_k} \rightarrow H^{a_1 + \cdots + a_k}$$

are surjective, for all $\alpha = (a_1, \dots, a_k) \in \mathbb{N}^k$.

Proof. The result follows directly from Proposition 3.4. \square

Proposition 3.11. *Suppose that H and L are lax Hopf algebras, with H of quotient type. A morphism $f: H \rightarrow L$ is determined by the maps $f^0: H^0 \rightarrow L^0$ and $f^1: H^1 \rightarrow L^1$; in this case we write $f^1 \circ \cdots \circ f^1$ for f^k .*

Proof. The result follows by from Proposition 3.10 and Proposition 3.8, taking $\alpha = (1, \dots, 1)$. \square

If H and L happen to be Hopf algebras (i.e., sum-preserving functors) then, according to Proposition 3.11, a morphism $f: H \rightarrow L$ is determined by $f^0: H^0 \rightarrow L^0$ and $f^1: H^1 \rightarrow L^1$, with each f^k equal to the k -fold tensor product $f^1 \otimes \cdots \otimes f^1$. If H and L are Hopf algebras over the same ring R and f is a morphism in $\mathbf{LaxHopf}_R$, then f^0 must be the identity map on R , and f is thus determined by $f^1: H^1 \rightarrow L^1$, which is a Hopf algebra map in the usual sense. Hence we have the following corollary.

Corollary 3.12. *The categories $\mathbf{ComHopf}_R$ and $\mathbf{ComHopf}$ of commutative R -Hopf algebras and all commutative Hopf algebras, respectively, are full subcategories of $\mathbf{LaxHopf}_R$ and $\mathbf{LaxHopf}$.*

It was already apparent before Proposition 3.11 that $\mathbf{ComHopf}_R$ and $\mathbf{ComHopf}$ are subcategories of $\mathbf{LaxHopf}_R$ and $\mathbf{LaxHopf}$; the point of Corollary 3.12 is the fullness of these subcategories, in other words, the fact that any lax Hopf algebra morphism between Hopf algebras is actually a Hopf algebra morphism.

Proposition 3.13. *Suppose that H and L are lax Hopf algebras, with H of quotient type. If $h: H^1 \rightarrow L^1$ is an algebra morphism, with scalar map $g: H^0 \rightarrow L^0$, such that $\delta_L(h) = (h \circ h) \delta_H$, $\epsilon_L h = g \epsilon_H$ and $\chi_L h = h \chi_H$, then (g, h) determines a morphism $f: H \rightarrow L$ by setting $f^0 = g$ and $f^k = h \circ \cdots \circ h$, for all $k \geq 1$.*

The lax Hopf algebras we will meet in Section 6 will be of a special variety of quotient type, constructed from commutative Hopf algebras as in the following proposition.

Proposition 3.14. *Suppose that H is a graded commutative R -Hopf algebra and that $I \subseteq H$ is a homogeneous ideal (not necessarily a coideal) such that $\epsilon(I) = 0$ and $\chi(I) \subseteq I$. For each $k \geq 0$, let $I_k \subseteq T^k(H)$ be the ideal generated by the set of coproduct slices $\bigcup_{|\alpha|=k} \delta_\alpha(I)$. Then $I_\infty = \bigoplus_{k \geq 0} I_k$ is an ideal of $T(H)$, and the quotient algebra*

$$H_I = T(H)/I_\infty = \bigoplus_{k \geq 0} T^k(H)/I_k$$

is a lax R -Hopf algebra of quotient type, with structure maps δ , ϵ , χ induced by those of H . The projection map $T(H) \rightarrow H_I$ is a morphism of lax Hopf algebras.

Proof. We verify that I_∞ is an ideal; the rest of the proposition follows readily. Suppose that $z \in I_k$ and $x \in T^r(H)$, for some k and r . We may assume that $z = w \delta_\alpha(u)$ for some $u \in I$ and $w \in T^k(H)$, where $|\alpha| = k$. Letting

$$x' = x \otimes \overbrace{1 \otimes \cdots \otimes 1}^k, \quad w' = \overbrace{1 \otimes \cdots \otimes 1}^r \otimes w, \quad \text{and} \quad \alpha' = (\overbrace{0, \dots, 0}^r, \alpha_1, \dots, \alpha_k),$$

we have that the tensor product $x \otimes z$ is equal to the internal product $x'w'\delta_{\alpha'}(u)$ in $T^{r+k}(H)$, and thus belongs to I_{r+k} . Hence I_∞ is a left ideal of $T(H)$. Similarly, I_∞ is a right ideal. \square

Proposition 3.15. *Suppose that H and L are commutative Hopf algebras over rings R and S , respectively, and that $I \subseteq H$ and $J \subseteq L$ are ideals. If $h: H \rightarrow L$ is a Hopf algebra morphism with scalar map $g: R \rightarrow S$, such that $h(I) \subseteq J$, then the pair (g, h) induces a morphism of lax Hopf algebras $f: H_I \rightarrow L_J$, such that $f^0 = g$ and f^1 is equal to the map $H/I \rightarrow L/J$ induced by h .*

Proof. Set $f^0 = g$ and let f^1 be the induced map $H/I \rightarrow L/J$. Now, since H_I is of quotient type, we can apply Proposition 3.13, and let $f^k = h \circ \cdots \circ h$, for all $k > 1$. \square

4. EXTERIOR ALGEBRA

We collect here some basic facts about free exterior algebras and exterior algebras of finite-dimensional vector spaces. Much of this material is well-known, and may be found in any algebra text containing a good section on multilinear algebra (e.g., [6]). However, definitions 4.11 through 4.13 and Proposition 4.15 may be less familiar to many readers.

If $w = a_1 \cdots a_k$ is a word on a set S , then a word of the form $v = a_{i_1} \cdots a_{i_r}$, for some $1 \leq i_1 < \cdots < i_r \leq k$ is a *subword* of w , and $\{i_1, \dots, i_r\}$ is the *position-set* of v . A *partition* of w is a sequence (v_1, \dots, v_s) of subwords of w such that the position-sets of the v_i are pairwise disjoint and have union equal to the position-set of w . The *shape* of the partition (v_1, \dots, v_s) is the sequence of word-lengths $(|v_1|, \dots, |v_s|) \in \mathbb{N}^s$. We write $v \subseteq w$ to indicate that v is a subword of w , and $(v_1, \dots, v_s) \vdash w$ to indicate that (v_1, \dots, v_s) is a partition of w .

From now on, we assume that S is a linearly ordered set. A word $w = a_1 \cdots a_k$ on S is *monotone* if $a_1 \leq \cdots \leq a_k$, and *strictly monotone* if $a_1 < \cdots < a_k$. The *free commutative monoid* on S , denoted by $[S]$, is the quotient of $\langle S \rangle$ by the congruence generated by the relation $\{ab \sim ba : a, b \in S\}$. We regard $[S]$ either as consisting of commutative words (i.e., monomials) in the set of variables S under ordinary product of monomials, or as multisets on S under the operation of multiset union. The power-set 2^S identifies with the subset of $[S]$ consisting of all multisets with no repeated elements. We let $c: \langle S \rangle \rightarrow [S]$ denote the canonical surjection, and refer to $c(w)$ as the *content* of w , for all $w \in \langle S \rangle$. The content function has a right inverse $m: [S] \rightarrow \langle S \rangle$, defined by letting $m(p)$ be the unique monotone word with content p , for $p \in [S]$. The composition $\langle S \rangle \xrightarrow{c} [S] \xrightarrow{m} \langle S \rangle$ is the idempotent function on $\langle S \rangle$ that rearranges words monotonically. For any word w , we write w_\leq for the monotone word $mc(w)$.

The *free R -module* $R\{S\}$ on S is the set of all R -linear combinations of elements of S , together with the obvious module structure. The *free R -algebra* on S , denoted by $R\langle S \rangle$, is the monoid algebra of $\langle S \rangle$ over R , that is, the free R -module $R\{S\}$,

with product operation induced by that of the monoid $\langle S \rangle$. As is usually done, we identify S and $\langle S \rangle$ with their respective images under the natural injections into $R\{S\}$ and $R\langle S \rangle$. Note that the correspondence $a_1 \cdots a_k \leftrightarrow a_1 \otimes \cdots \otimes a_k$ defines an algebra isomorphism between $R\langle S \rangle$ and the tensor algebra of $R\{S\}$. The algebra $R\langle S \rangle$ is graded (by length of words) and is also $[S]$ -graded. For all $p \in [S]$, the homogeneous component $R\langle S \rangle_p$ is spanned by all words on S having content equal to p .

The *free exterior algebra on S (over R)* is the quotient algebra $E = E(S) = R\langle S \rangle/I$, where I is the ideal of $R\langle S \rangle$ generated by all words of the form aa and expressions of the form $ab+ba$, for $a, b \in S$. We will refer to elements of E always in terms of their representatives in $R\langle S \rangle$. Thus, in particular, we write $w = a_1 \cdots a_k$ to denote either a word in $\langle S \rangle$, or its image in E , depending on the context. This convention allows us to refer to such things as the i th letter of a word $w \in E$, which only makes sense if it is understood that w is a word representing an element of E , and is not itself an element of E .

As an R -module, E is generated by the set $\langle S \rangle$, with relations

$$a_1 \cdots a_k = \operatorname{sgn}(\tau) a_{\tau(1)} \cdots a_{\tau(k)},$$

for all words $a_1 \cdots a_k \in \langle S \rangle$ and permutations τ of $\{1, \dots, k\}$, where $\operatorname{sgn}(\tau) = \pm 1$ denotes the sign τ . It follows that E has as basis the set of all strictly monotone words on S .

The ideal I is homogeneous with respect to the length and content gradings on $R\langle S \rangle$ and so the algebra E is also graded by length and content of words. We write E_k and E_p , respectively, for the homogeneous components of E generated by words of length k and those of content $p \in [S]$. Note that $E_p = \{0\}$ whenever the multiset p contains repeated elements.

For any word $w = a_1 \cdots a_k \in \langle S \rangle$, the *sign* $\sigma(w)$ of w is zero if w contains repeated letters, and otherwise is equal to $\operatorname{sgn}(\tau)$, where τ is the permutation of $\{1, \dots, k\}$ such that $a_{\tau(1)} \cdots a_{\tau(k)}$ is monotone. It follows that

$$(4.1) \quad w = \sigma(w) w_{\leq}$$

in E , for all words w , and the product in E is determined on the basis of strictly monotone words by

$$vw = \sigma(vw) (vw)_{\leq}.$$

Note that there are different products appearing in this formula; on the left is the product in E , while on the right is concatenation of words, which may be inferred from the fact that σ and $u \mapsto u_{\leq}$ are functions defined on the free monoid $\langle S \rangle$.

Example 4.2. Suppose that $a < b < \cdots < e$ in S ; then

$$(bde)(ac) = \sigma(bdeac) abcde = -abcde$$

in E .

The free exterior algebra is a Hopf algebra, with coproduct $\delta: E \rightarrow E \otimes E$ determined by $\delta(a) = 1 \otimes a + a \otimes 1$, for any single element $a \in S$. By multiplicativity, δ is determined on the basis of monotone words by

$$(4.3) \quad \delta(u) = \sum_{(v,w) \vdash u} \sigma(vw) v \otimes w.$$

Example 4.4. If $a < b < c$ in S , then the coproduct of ab is given by

$$\begin{aligned}\delta(ab) &= \sigma(ab)(ab \otimes 1 + a \otimes b + 1 \otimes ab) + \sigma(ba)b \otimes a \\ &= ab \otimes 1 + a \otimes b - b \otimes a + 1 \otimes ab,\end{aligned}$$

and the coproduct slice $\delta_{(2,1)}(abc)$ is given by

$$\begin{aligned}\delta_{(2,1)}(abc) &= \sigma(abc)ab \otimes c + \sigma(acb)ac \otimes b + \sigma(bca)bc \otimes a \\ &= ab \otimes c - ac \otimes b + bc \otimes a.\end{aligned}$$

The counit ϵ of E satisfies

$$(4.5) \quad \epsilon(w) = \begin{cases} 1 & \text{if } |w| = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the antipode χ , also multiplicative, is determined by its values $\chi(a) = -a$, for all $a \in S$. It follows that

$$(4.6) \quad \chi(w) = (-1)^{|w|} w$$

for all $w \in \langle S \rangle$.

In the next proposition we state, without proof, the universal mapping properties characterizing the free exterior algebras on a set, over R and \mathbb{Z} , in the category of R -Hopf algebras and the category of all Hopf algebras, respectively.

Proposition 4.7. *Suppose that S is a set, H is an R -Hopf algebra, and $f: S \rightarrow H$ is a map such that $\delta(f(a)) = 1 \otimes f(a) + f(a) \otimes 1$ (that is, $f(a)$ is primitive) and $(f(a))^2 = 0$, for all $a \in S$. Let E_R and $E_{\mathbb{Z}}$ denote the free exterior algebras on S over R and \mathbb{Z} , respectively. Then f extends to a unique R -Hopf algebra map $\bar{f}: E_R \rightarrow H$, and a unique morphism of Hopf algebras $\hat{f}: E_{\mathbb{Z}} \rightarrow H$.*

Note that, since \bar{f} is multiplicative in particular, it is determined by

$$\bar{f}(a_1 \cdots a_k) = f(a_1) \cdots f(a_k),$$

for all words $a_1 \cdots a_k$ on S , and similarly for \hat{f} . The scalar map of \hat{f} is the unique homomorphism $\mathbb{Z} \rightarrow R$.

If $w = w_1 \otimes \cdots \otimes w_k \in T^k(E)$ is $[S]$ -homogeneous, then its content $c(w)$ is equal to the content of the product $w_1 \cdots w_k$. Letting $k = 2$, we thus see that the product and coproduct on E are homogeneous maps, and therefore E is a $[S]$ -graded Hopf algebra.

We now recall the definition and basic properties of the exterior algebra of a vector space. Throughout the remainder of this section, V will denote a vector space of dimension n over a field K .

The *exterior algebra* of V is the quotient algebra $\Lambda(V) = T(V)/I$, where I is the ideal of the tensor algebra $T(V)$ generated by all $v \otimes v$, for $v \in V$. Since I is a homogeneous ideal, it follows that $\Lambda(V) = \bigoplus_{k \geq 0} \Lambda^k(V)$ is a graded algebra, with homogeneous components $\Lambda^k(V) = T^k(V)/(I \cap T^k(V))$. We refer to the vector space $\Lambda^k(V)$ as the k th *exterior power* of V . We denote the product of x and y (that is, the image of $x \otimes y$) in $\Lambda(V)$ by $x \vee y$, and refer to $x \vee y$ as the *join* of x and y . We remark that the product in exterior algebra is usually denoted by \wedge and referred to as the *wedge product*.

The exterior algebra $\Lambda(V)$ is a graded commutative, cocommutative Hopf algebra, with coproduct δ , counit ϵ and antipode χ , determined by $\delta(v) = 1 \otimes v + v \otimes 1$, $\epsilon(v) = 0$ and $\chi(v) = -v$, for all $v \in V$.

Proposition 4.8. *Suppose that V is a vector space with basis $S = \{v_1, \dots, v_n\}$. Then, for $1 \leq k \leq n$, the vector space $\Lambda^k(V)$ has basis $\{v_{i_1} \vee \dots \vee v_{i_k} : 1 \leq i_1 < \dots < i_k \leq n\}$.*

Hence, in particular, $\Lambda^k(V)$ has dimension $\binom{n}{k}$ and $\Lambda(M)$ has dimension 2^n . Also, $\Lambda^0(V) = K$, and $\Lambda^k(V) = 0$, for $k > n$.

Suppose that $S = \{v_1, \dots, v_n\}$ is a basis for V , and $E(S)$ is the free exterior algebra on S over the field K . The inclusion $S \rightarrow V$ extends to a K -Hopf algebra map $E(S) \rightarrow \Lambda(V)$, given by $v_{i_1} \cdots v_{i_k} \mapsto v_{i_1} \vee \dots \vee v_{i_k}$. By Proposition 4.8, this is Hopf algebra isomorphism.

The linearly independent, and hence also the linearly dependent, subsets of V are characterized by the following proposition.

Proposition 4.9. *A set of vectors $\{v_1, \dots, v_k\}$ in V is linearly independent if and only if $v_1 \vee \dots \vee v_k \neq 0$.*

An element $x \in \Lambda^k(V)$ is called *decomposable* if $x = v_1 \vee \dots \vee v_k$, for some $v_1, \dots, v_k \in V$.

Proposition 4.10. *Suppose that $x = v_1 \vee \dots \vee v_k$ and $y = w_1 \vee \dots \vee w_k$ are nonzero decomposable elements of $\Lambda^k(V)$. The spaces spanned by $\{v_1, \dots, v_k\}$ and $\{w_1, \dots, w_k\}$ are the same if and only if $x = cy$, for some nonzero constant $c \in K$.*

Hence every decomposable $x \in \Lambda^k(V)$ has associated to it a unique k -dimensional subspace of V , that is, the subspace spanned by $\{v_1, \dots, v_k\}$, for any expression of x as a join of vectors $v_1 \vee \dots \vee v_k$.

Definition 4.11. The exterior power $\Lambda^n(V)$ of an n -dimensional vector space V is called the *pseudoscalar space* of V .

The pseudoscalar space $\Lambda^n(V)$ is one-dimensional and hence is isomorphic to K as a vector space. It is important to note, however, that there is no natural isomorphism $\Lambda^n(V) \cong K$, and so, in particular, the pseudoscalar space $\Lambda^n(V)$ does not have a canonical multiplicative structure. In order to have a place in which we can multiply pseudoscalars in a natural manner, we need to make the following definition.

Definition 4.12. The *pseudoscalar algebra* $\Psi(V)$ of an n -dimensional vector space V is the tensor algebra $T(\Lambda^n(V))$.

Definition 4.13. A *Peano space* (see [1]) is a vector space V together with a linear isomorphism $[\]$, called a *bracket on V* , from the pseudoscalar space $\Lambda^n(V)$ onto the field K .

By the universal property of exterior powers, specifying a bracket on V is equivalent to giving a nondegenerate alternating n -linear map $V \times \dots \times V \rightarrow K$. Since $\Lambda^n(V)$ is one-dimensional, a bracket on V is determined by the choice of $[v_1 \vee \dots \vee v_n]$, for any nonzero pseudoscalar $v_1 \vee \dots \vee v_n$. The bracket $[\] : \Lambda^n(V) \rightarrow K$ on a Peano space V extends to a unique algebra map $\beta : \Psi(V) \rightarrow K$, given by

$$(4.14) \quad \beta(w_1 \otimes \dots \otimes w_k) = [w_1] \cdots [w_k],$$

for all $w_1, \dots, w_k \in \Lambda^n(V)$.

Proposition 4.15. *Suppose that $A = \bigoplus_{k \geq 0} A_k$ is a graded algebra and that V is a Peano space with bracket $[]: \Lambda^n(V) \rightarrow K$. Let $\mathbf{Alg}'(A, K)$ and $\mathbf{Alg}(A, \Psi(V))$ denote the set of all algebra maps from A to K and the set of all homogeneous degree-zero algebra maps from A to $\Psi(V)$, respectively. The correspondence $f \mapsto \beta f$ is a bijection from $\mathbf{Alg}(A, \Psi(V))$ onto $\mathbf{Alg}'(A, K)$.*

Proof. It is clear that the composition βf is an algebra map $A \rightarrow K$, for each homogeneous degree-zero $f: A \rightarrow \Psi(V)$.

Now, for each $k \geq 0$, let β_k denote the restriction of β to $\Psi_k(V) = T^k(\Lambda^n(V))$. Since $\beta_1 = []$ is an isomorphism, and $\Psi_1(V) = \Lambda^n(V)$ is one-dimensional, it follows that each β_k is an isomorphism. Hence if we are given any algebra map $g: A \rightarrow K$, we obtain $\bar{g}: A \rightarrow \Psi(V)$ by setting $\bar{g}|_{A_k}$ equal to the composition $\beta_k^{-1}g$. \square

5. THE ZIPPER

In this section, we prove the *Zipper lemma*, Theorem 5.7, which is an identity satisfied by coproduct slices in the Hopf algebra $E(S)$, or $\Lambda(V)$, that we need for the proof of the exchange relations in Section 7.

For the purposes of the proof of Theorem 5.7, we extend the definition of the binomial coefficients $\binom{n}{k}$ to all $n, k \in \mathbb{Z}$ by setting

$$\binom{n}{k} = \begin{cases} \#\{k\text{-element subsets of an } n\text{-element set}\} & \text{if } n \geq 0, \\ (-1)^k \cdot \#\{k\text{-element multisets on an } |n|\text{-element set}\} & \text{if } n \leq 0. \end{cases}$$

It follows that

$$(5.1) \quad \binom{n}{k} = \begin{cases} \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots(1)} & \text{for } k > 0, \\ 1 & \text{for } k = 0, \\ 0 & \text{for } k < 0. \end{cases}$$

This definition is consistent with the defining relation for binomial coefficients:

Proposition 5.2. *For all integers n, k ,*

$$(5.3) \quad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Proof. The result follows from the combinatorial definition of the binomial coefficients. Alternatively, using the formula (5.1), we see that all terms in this equation are zero for $k < 0$, and for $k = 0$ we have $\binom{n-1}{k-1} = 0$ and $\binom{n}{k} = \binom{n-1}{k} = 1$. If $k > 0$ then

$$\begin{aligned} \binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{n(n-1)\cdots(n-k+1)}{k(k-1)\cdots(1)} \left(\frac{k}{n} + \frac{n-k}{n} \right) \\ &= \binom{n}{k}. \end{aligned}$$

\square

Proposition 5.4. *For all integers n, k, p , with $p \geq 0$,*

$$(5.5) \quad \sum_{i=0}^p (-1)^i \binom{n}{k+i} = \binom{n-1}{k-1} + (-1)^p \binom{n-1}{k+p}.$$

Proof. Iterating the defining relation (5.3) yields

$$\begin{aligned} 0 &= \sum_{i=0}^p \left[\binom{n-1}{k-1+i} - \binom{n}{k+i} + \binom{n-1}{k+i} \right] \\ &= \binom{n-1}{k-1} - \left[\sum_{i=0}^p \binom{n}{k+i} \right] + (-1)^p \binom{n-1}{k+p}. \end{aligned}$$

□

In the following lemma, we show that for any partition (t, u, v) of a strictly monotone word w , the expression $\sigma(tuv) \delta(t)(u \otimes v)$ expands to that portion of the coproduct $\delta(w)$ consisting of monomials $\pm x \otimes y$ such that $u \subseteq x$ and $v \subseteq y$ (recall that the $\sigma(tuv)$ is the sign of the word tuv , determined by Equation 4.1). What is significant in this result is that the sign of each term $\pm x \otimes y$ occurring in the expansion of $\sigma(tuv) \delta(t)(u \otimes v)$ is independent of the choice of t, u, v ; it is simply $\sigma(xy)$.

Lemma 5.6. *For any partition (t, u, v) of a strictly monotone word w ,*

$$\sum_{\substack{(w) \\ u \subseteq w(1), v \subseteq w(2)}} w(1) \otimes w(2) = \sigma(tuv) \delta(t)(u \otimes v).$$

Proof. Let $z = (uv)_{\leq}$, the monotone rearrangement of uv . Since $w = \sigma(tz)tz$ and the coproduct is multiplicative,

$$\delta(w) = \sigma(tz) \delta(t) \delta(z).$$

Selecting from these expressions those terms $\pm x \otimes y$ for which $u \subseteq x$ and $v \subseteq y$, we find

$$\begin{aligned} \sum_{\substack{(w) \\ u \subseteq w(1), v \subseteq w(2)}} w(1) \otimes w(2) &= \sigma(tz) \sigma(uv) \delta(t)(u \otimes v) \\ &= \sigma(tuv) \delta(t)(u \otimes v). \end{aligned}$$

□

For any subword $u = a_{i_1} \cdots a_{i_k}$ of a monotone word $w = a_1 \cdots a_r$, let

$$\max_u = \begin{cases} i_k & \text{if } u \neq 1, \\ -1 & \text{if } u = 1, \end{cases}$$

and if v is any other subword of w , write $u \prec v$ to indicate that $v = a_{i_k+1} \cdots a_r$ is the final word in w , following all elements of u .

Theorem 5.7 (The Zipper Lemma). *For $0 \leq k \leq r$, and for any word w of length r ,*

$$(5.8) \quad \sum_{\substack{(w)_{(r-k+1, \cdot, \cdot)} \\ w(1) \prec w(2)}} \delta(w(1)) (\chi(w(2)) \otimes w(3)) = \sum_{j=0}^r \binom{j-1}{k-1} \delta_{r-j,j}(w).$$

Proof. Regrouping terms according to the value $i = |w_{(2)}|$ and using the expression (4.6) for the antipode χ , the left-hand side of (5.8) may be written

$$\sum_{i=0}^r (-1)^i \sum \sigma(tuv) \delta(t) (u \otimes v),$$

where the inner sum is taken over all partitions (t, u, v) of w having shape $(r - k + 1, i, k - i - 1)$, with $\max_t = r - i$ and $u = a_{r-i+1} \cdots a_r$. By Lemma 5.6, the left-hand side of (5.8) is thus equal to

$$\sum_{(x,y) \vdash w} \left(\sum_i (-1)^i b_i(x, y) \right) \sigma(xy) (x \otimes y),$$

where $b_i(x, y)$ is the number of partitions (t, u, v) of w having shape $(r - k + 1, i, k - i - 1)$, with

$$(5.9) \quad \begin{aligned} \max_t &= r - i, \quad u = a_{r-i+1} \cdots a_r, \\ u &\subseteq x, v \subseteq y. \end{aligned}$$

Since $u \subseteq y$ and $i \leq k - 1$, the integer i is constrained to lie in the interval

$$0 \leq i \leq \min(r, r - \max_y).$$

Let $m = |y|$. It remains to prove that the sum

$$(5.10) \quad \sum_{i=0}^{\min(k-1, r-\max_y)} (-1)^i b_i(x, y) = \binom{m-1}{k-1}.$$

For any monomial $x \otimes y$, the calculation of $b_i(x, y)$ splits into two cases: (a) for $i = r - \max_y$; and (b) for $0 \leq i < r - \max_y$. Note that the calculation for the monomial $w \otimes 1$ falls into case (b), because $i \leq r < r - \max_y$. In case (a), we may therefore assume that $m > 0$.

Case (a): To construct all partitions (t, u, v) of shape $(r - k + 1, i, k - i - 1)$, with $i = r - \max_y$ and having properties (5.9):

- let $u = a_{r-i+1} \cdots a_r$ (the largest suffix of w contained in x),
- place the remaining $r - i$ elements of x in t ,
- place a_{r-i} , an element of y , in t ,
- choose v to be an arbitrary $(k - i - 1)$ -element subset of $y \setminus a_{r-i}$; place the complementary subset of $y \setminus a_{r-i}$ in t .

There are thus $\binom{m-1}{k-i-1} = \binom{m-1}{m-k+i}$ such choices, for k in the interval $i + 1 \leq k \leq i + m$, where $0 \leq m - k + i \leq m - 1$, and no choices for values of k outside this interval. Since $m > 0$ in this case, $b_i(x, y) = \binom{m-1}{m-k+i}$.

Case (b): For $0 \leq i < r - \max_y$,

- let $u = a_{r-i+1} \cdots a_r$,
- place the remaining $r - i$ elements of x (including the element a_{r-i}) in t ,
- choose v to be an arbitrary $(k - i - 1)$ -element subset of y , and place the complementary subset of y in t .

There are thus $\binom{m}{k-i-1} = \binom{m}{m-k+i+1}$ such choices, for k in the interval $i + 1 \leq k \leq m + i + 1$, where $0 \leq m - k + i + 1 \leq m$, and no choices for values of k outside this interval, except for one choice (the empty set) when $m = 0$, $k = i + 1$, and $\binom{m}{m-k+i+1} = \binom{0}{0} = 1$. We have $b_i(x, y) = \binom{m}{m-k+i+1}$.

When $x \otimes y = w \otimes 1$, the only possible partition (t, u, v) of shape $(r - k + 1, i, k - i - 1)$ that satisfies conditions (5.9) is

$$(a_1 \cdots a_{r-k+1}, a_{r-k+2} \cdots a_r, 1),$$

so $i = k - 1$. Thus the sum (5.10) is $(-1)^{k-1} = \binom{-1}{k-1}$, as required.

If $k - 1 \leq r - \max_y$, and $m \neq 0$, the sum (5.10) may be written in the form

$$\sum_{i=0}^{k-1} (-1)^i \binom{m}{m-k+i+1}.$$

Taking $d = m$, $e = m - k + 1$, and $p = k - 1$ in formula (5.5) this simplifies to

$$\binom{m-1}{m-k} + (-1)^{k-1} \binom{m-1}{m},$$

the second summand being zero when $m \neq 0$.

If $r - \max_y < k - 1$, take $d = m$, $e = m - k + 1$, and $p = r - \max_y - 1$ in formula (5.5) to obtain that the sum (5.10) is equal to

$$\begin{aligned} & \sum_{i=0}^{r-\max_y-1} (-1)^i \binom{m}{m-k+i+1} + (-1)^{(r-\max_y)} \binom{m-1}{m-k+r-\max_y} \\ &= \binom{m-1}{m-k}. \end{aligned}$$

□

Corollary 5.11. *For $0 \leq k \leq r$, and for any word w of length r ,*

$$\begin{aligned} \sum_{\substack{(w)_{(r-k+1, \dots)} \\ w_{(1)} \prec w_{(2)}}} \delta(w_{(1)}) (\chi(w_{(2)}) \otimes w_{(3)}) &= (-1)^{k-1} w \otimes 1 + \delta_{r-k,k}(w) \\ &+ \{ \text{terms of shape } (r-l, l) \text{ for } l > k \}. \end{aligned}$$

Proof. The result follows by evaluating the extended binomial coefficients in Equation 5.8. □

We have much abused the reader's patience by recording the Zipper lemma in such a condensed form, with so little possible appeal to intuition. A few examples will illustrate the extensive cancellation which there occurs, and the reason for the appearance of the binomial coefficients. It may then be clear why the Zipper (itself a theorem in exterior algebra) becomes the key to exchange properties in the Whitney algebra, where the terms of shape $(r-l, l)$, for $l > k$, in Corollary 5.11 will all be zero.

The simplest non-trivial case of Theorem 5.7 is that for $r = k = 2$, where, for $w = ab$, we have

$$\begin{aligned} \delta(a)(b \otimes 1) + \delta(b)(1 \otimes a) &= (ab \otimes 1 - b \otimes a) + (b \otimes a - 1 \otimes ab) \\ &= ab \otimes 1 - 1 \otimes ab \end{aligned}$$

For $r = 4$, $k = 3$, and $w = abcd$, all monomial terms cancel except for those of shapes $(4, 1)$, $(1, 3)$, and $(1, 4)$; the sum

$$\begin{aligned} & \delta(ab)(cd \otimes 1) - \delta(ac)(d \otimes b) + \delta(bc)(d \otimes a) + \delta(ad)(1 \otimes bc) \\ & - \delta(bd)(1 \otimes ac) + \delta(cd)(1 \otimes ab) \end{aligned}$$

is equal to

$$\begin{aligned} & abcd \otimes 1 + (a \otimes bcd - b \otimes acd + c \otimes abd - d \otimes abc) + 3(1 \otimes abcd) \\ & = (\delta_{(4,1)} + \delta_{(1,3)} + 3\delta_{(1,4)})(abcd). \end{aligned}$$

The full extent of cancellation in the Zipper lemma is best revealed in tabular form, below. Take $r = 5$, $k = 4$, and $w = abcde$. The columns of the table below are labelled by the terms of the sum on the left side of Equation 5.8, abbreviated as follows:

$$\begin{array}{ll} ab : & +\delta(ab)(cde \otimes 1) & ac : & +\delta(ac)(de \otimes b) \\ bc : & -\delta(bc)(de \otimes a) & ad : & +\delta(ad)(e \otimes bc) \\ bd : & -\delta(bd)(e \otimes ac) & cd : & +\delta(cd)(e \otimes ab) \\ ae : & +\delta(ae)(1 \otimes bcd) & be : & -\delta(be)(1 \otimes acd) \\ ce : & +\delta(ce)(1 \otimes abd) & de : & -\delta(de)(1 \otimes abc), \end{array}$$

and contain the signs of the various monomials (given as row labels) occurring in the expansions of those expressions. The total coefficient of each monomial is shown in the final column T .

	ab	ac	bc	ad	bd	cd	ae	be	ce	de	T
$abcde \otimes 1$	+										+1
$acde \otimes b$	-	+									0
$bcde \otimes a$	+		-								0
$ade \otimes bc$		-		+							0
$bde \otimes ac$			+		-						0
$cde \otimes ab$	+	-	-			+					0
$ae \otimes bcd$				-			+				0
$be \otimes acd$					+			-			0
$ce \otimes abd$						-			+		0
$de \otimes abc$		-	-	+	+	+				-	0
$a \otimes bcde$							-				-1
$b \otimes acde$								+			+1
$c \otimes abde$									-		-1
$d \otimes abce$										+	+1
$e \otimes abcd$				+	+	+	-	-	-	-	-1
$1 \otimes abcde$							-	-	-	-	-4

so the sum is equal to

$$(\delta_{(5,1)} - \delta_{(1,4)} - 4\delta_{(1,5)})(abcde).$$

Notice that the non-zero signs in any row, for columns labelled by terms in which the coproduct acts on words with the same last letter, are constant, and that the number of such signs is a binomial coefficient.

6. THE WHITNEY ALGEBRA OF A MATROID

We now introduce our primary object of study, the Whitney algebra of a matroid. We show in this section that the construction is functorial, and satisfies a universal property with respect to representations of matroids.

From now on, we restrict our attention to the case in which R is the ring of integers \mathbb{Z} , and our set S is the underlying set of points of some matroid $M = M(S)$. We apply the usual matroid terminology for subsets of S to words in the free exterior algebra $E = E(S)$; for example $w = a_1 \dots, a_k$ is *dependent* if the set $\{a_1, \dots, a_k\}$ is dependent in M , and w is *independent* if it is not dependent. A minimal dependent word is a *circuit*, and a maximal independent word is a *basis* of M . We write $\rho(w)$ for the rank of w , that is, the length of a maximal independent subword of w , and we write \bar{w} for the closure of w ; hence $\bar{w} = \{a \in S: \rho(wa) = \rho(w)\}$.

Let $I = I_M$, $D = D_M$ and $C = C_M$ denote the submodules of E generated by the collections of independent words, dependent words, and circuits of M , respectively. We observe that D is an ideal of E , generated by C , but is not in general a coideal, I is a coideal (in fact, a subcoalgebra) of E , but typically not an ideal, and $E = I \oplus D$ as abelian groups. For all $k \geq 0$, we denote by D_k the ideal of the ring $T^k(E)$ generated by the set $\bigcup_{|\alpha|=k} \delta_\alpha(D)$. Note that D_k is also generated by the set of degree k coproduct slices of circuits.

By Proposition 3.14, the direct sum $D_\infty = \bigoplus_{k \geq 0} D_k$ is an ideal in the tensor algebra $T(E)$ and the quotient ring

$$W = T(E)/D_\infty = \bigoplus_{k \geq 0} (T^k(E)/D_k)$$

is a lax \mathbb{Z} -Hopf algebra of quotient type. Note that the tensor algebra $T(E)$ has \mathbb{N} -, $\langle \mathbb{N} \rangle$ - and $[S]$ -gradings, corresponding to degree, shape and content, respectively; for example, the product $ab \otimes ac \otimes abc$ has degree 3, shape $(2, 2, 3)$ and content $a^3 b^2 c^2$. The ideal D_∞ is homogeneous with respect to each of these gradings, and so W is also graded by degree, shape and content.

We denote the product in W by \circ , and write $W^k = W^1 \circ \dots \circ W^1$ for the degree k homogeneous component $T^k(E)/D_k$, which is a ring with product induced by the internal multiplication (2.2). We also write W_α for the homogeneous component of shape α , which is the image of $T_\alpha(E)$ under the canonical map $T(E) \rightarrow W$. The coproduct $\delta: W^1 \rightarrow W^2$, counit $\epsilon: W^1 \rightarrow \mathbb{Z}$ and antipode $\chi: W^1 \rightarrow W^1$ of W are induced by those of the Hopf algebra E ; hence δ is determined by Equation 4.3, with the symbol \circ replacing the tensor product, and ϵ and χ are given by Equations 4.5 and 4.6, respectively.

We refer to the lax Hopf algebra W as the *Whitney algebra* of the matroid M .

Observe that $W^0 = \mathbb{Z}$ and $W^1 = E/D$. Hence W^1 has as a basis the set of all monotone independent words on S .

Before verifying the functoriality of the Whitney algebra construction, we recall the definition of a weak map between matroids. We adjoin a *zero* to a matroid $M = M(S)$ by taking the direct sum of M with the rank zero matroid on a singleton set $\{o\}$, where $o \notin S$. We denote the resulting matroid on $S \cup \{o\}$ by M_o . A *weak map* from M to a matroid $N = N(T)$ consists of a function $f: S \cup \{o\} \rightarrow T \cup \{o\}$ such that $f(o) = o$, and the multiset $f(U)$ is dependent in N_o , whenever $U \subset S$ is dependent in M . (Note: we call a multiset on the point set of matroid dependent if it is a dependent set, or if it contains repeated elements.)

Proposition 6.1. *The Whitney algebra construction is a functor from the category of matroids and weak maps to the category of lax Hopf algebras.*

Proof. Suppose that $M = M(S)$ and $N = N(T)$ are matroids and that $f: M \rightarrow N$ is a weak map. Extend the domain of the inclusion map $j: T \rightarrow E(T)$ to $T \cup \{o\}$ by setting $j(o) = 0$. By the universal property of the free exterior algebra, the restriction of the composition jf to S extends to a unique Hopf algebra map $f': E(S) \rightarrow E(T)$. If $u = a_1 \cdots a_k$ is a dependent word on S then, since f is a weak map, $f'(u) = f'(a_1) \cdots f'(a_k)$ is either a dependent word on T or equal to zero in $E(T)$; hence $f'(D_M) \subseteq D_N$. According to Proposition 3.15, f' induces a morphism of lax Hopf algebras $\hat{f}: W(M) \rightarrow W(N)$. It follows that W is a functor, with $W(f) = \hat{f}$, for all weak maps f . \square

Suppose that U is a subset of S . Recall that the *restriction* $M|U$ is the matroid on U whose independent sets are the independent sets of M that are contained in U . We write $W(M)|U$ for the sublax Hopf algebra of $W(M)$ generated by all words on U .

Proposition 6.2. *For all subsets U of S , the Whitney algebra of $M|U$ is equal to $W(M)|U$.*

Proof. The inclusion of U in S is a weak map $j: M|U \rightarrow M$, and hence extends to a lax Hopf algebra morphism $W(j): W(M|U) \rightarrow W(M)$, by the previous proposition. Since a word on U is dependent in $M|U$ if and only if it is dependent in M , and all relations in $W(M|U)$ and $W(M)|U$ are generated by coproduct slices of such words, it follows that $W(j)$ is simply the inclusion map of $W(M|U) = W(M)|U$ into $W(M)$. \square

Suppose that V is a vector space over a field K . Recall that a *representation* of a matroid $M = M(S)$ over V is a function $g: S \rightarrow V$ such that the rank of any subset $U \subseteq S$ is equal to the dimension of the subspace of V spanned by $g(U)$.

Proposition 6.3. *Suppose that $M = M(S)$ is a matroid of rank n , and that V is an n -dimensional K -vector space, for some field K . A representation $g: S \rightarrow V$ of M induces a unique morphism of lax Hopf algebras $\hat{g}: W(M) \rightarrow T(\Lambda(V))$ such that $\hat{g}(a) = g(a)$, for all $a \in S$.*

Proof. The representation g extends to a Hopf algebra morphism $g': E(S) \rightarrow \Lambda(V)$, having the unique ring homomorphism $\mathbb{Z} \rightarrow K$ as scalar map, and satisfying $g'(a) = g(a)$, for all $a \in S$. It is clear that $g'(w) = 0$, for all dependent words w on S , and so $g'(D_M) \subseteq J$, where J is the zero ideal of $\Lambda(V)$. Hence by Proposition 3.15, the map g' induces a morphism of lax Hopf algebras $\hat{g} = \{g^k\}$ from $W(M) = E(S)_{D_M}$ to $T(\Lambda(V)) = \Lambda(V)_J$, such that g^1 is equal to the map $W^1 = E(S)/D_M \rightarrow \Lambda(V)$ induced by g' . \square

The determination of the morphism \hat{g} in terms of g is straightforward; since the maps $g^k: W^k \rightarrow T(\Lambda(V))$, and in particular $g^1: W^1 \rightarrow \Lambda(V)$, are algebra homomorphisms, we have

$$\hat{g}(a_1 \cdots a_k) = g(a_1) \vee \cdots \vee g(a_k),$$

for all $a_1, \dots, a_k \in S$. It follows that $\hat{g}(w) \neq 0$ in $\Lambda(V)$, for all independent words w on S and, more generally, $\hat{g}(m) \neq 0$ in $T(\Lambda(V))$ for all products of independent

words $m = w_1 \circ \cdots \circ w_k$, in W . Hence we have the following result (the converse of which also holds, and is given in Corollary 8.9).

Corollary 6.4. *If a matroid M is representable, then no product of independent words is equal to zero in the Whitney algebra W .*

Another immediate consequence of Proposition 6.3 is the following:

Corollary 6.5. *Suppose that there exists a product m of independent words in $W(M)$ and some integer $r > 1$ such that $rm = 0$. If M is representable over some field K , then the characteristic of K divides r .*

Example 6.6. The matroid whose points are represented by the seven nonzero vectors in a vector space of rank 3 over the two-element field $GF(2)$ is called the *Fano matroid*, and denoted by F_7 . Let

$$\begin{array}{cccccc} a & b & c & d & e & f & g \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

be the matrix whose columns represent the points of F_7 . There are then 14 circuits:

$$\begin{aligned} & abc, ade, afg, bdg, bef, cdf, ceg, \\ & defg, bcfg, bcde, acef, acdg, abeg, abdf. \end{aligned}$$

Consider the three syzygies:

$$\begin{aligned} \gamma_1: \quad 0 &= (1 \circ bc \circ de) \delta_{(1,1,1)}(afg) = +a \circ bcf \circ deg - a \circ bcf \circ def, \\ \gamma_2: \quad 0 &= (a \circ b \circ eg) \delta_{(0,2,1)}(cdf) = -a \circ bcd \circ efg - a \circ bcf \circ deg, \\ \gamma_3: \quad 0 &= (a \circ c \circ ef) \delta_{(0,2,1)}(bdg) = -a \circ bcd \circ efg + a \circ bcf \circ def. \end{aligned}$$

Note that many monomials potentially occurring in these syzygies are zero because of their inclusion of dependent words. Syzygies γ_1 and γ_3 establish that the three monomials

$$a \circ bcf \circ deg, \quad a \circ bcd \circ efg, \quad a \circ bcf \circ def$$

are equal in the Whitney algebra of the Fano matroid. Syzygy $-\gamma_2$ then establishes that $a \circ bcf \circ deg$, and thus each of these monomials, becomes zero when multiplied by 2. Hence, by Corollary 6.5, the Fano matroid is only representable over fields of characteristic two.

Given the high degree of symmetry in this matroid, up to automorphism of the Whitney algebra $W(F_7)$, there is only one non-zero monomial m of shape $(1, 3, 3)$ and content $abcdefg$. Hence, any such monomial m satisfies $2m = 0$.

The next result shows how the notion of dependence in a matroid translates to that of linear dependence in the Whitney algebra. For a word $w = b_1 \cdots b_k$ and a letter $a \in S$, we write $w_{i,a}$ for the word $b_1 \cdots b_{i-1} a b_{i+1} \cdots b_k$.

Proposition 6.7. *If the letter $a \in S$ is in the closure of the word $w = b_1 \cdots b_k$, then the equality*

$$w \circ a = \sum_{i=1}^k w_{i,a} \circ b_i$$

holds in W .

Proof. Since $a \in \bar{w}$, the word wa is dependent, and thus we have

$$\begin{aligned} 0 &= \delta_{(k,1)}(wa) = w \circ a + \sum_{i=1}^k (-1)^{k-i+1} b_1 \cdots \hat{b}_i \cdots b_k a \circ b_i \\ &= w \circ a - \sum_{i=1}^k w_{i,a} \circ b_i. \end{aligned}$$

□

In the representable case we may apply Proposition 6.3 to obtain:

Corollary 6.8. *Suppose that $g: S \rightarrow V$ is a representation of M and that V is a Peano space with bracket $[\]: \Lambda^n(V) \rightarrow K$. For any basis $w = b_1 \cdots b_n$ of M , and $a \in S$, the equality*

$$g(a) = \frac{1}{[\hat{g}(w)]} \sum_{i=1}^k [\hat{g}(w_{i,a})] g(b_i)$$

holds in V .

7. THE GEOMETRIC PRODUCT AND EXCHANGE RELATIONS

In this section, we prove the fundamental *exchange relations*, and examine the commutativity properties of the Whitney algebra. We begin by introducing an operator on pairs of words that generalizes the join and meet operations in the Cayley algebra of a Peano space.

Definition 7.1. For words $u, v \in W^1$, with $|u| = r$, $|v| = s$, let $k = r + s - \rho(uv)$. The *geometric product* of u and v in W , written $u \diamond v$, is given by the expression

$$(7.2) \quad u \diamond v = \sum_{(u)_{r-k,k}} u_{(1)} v \circ u_{(2)}$$

Note that $u \diamond v$ has shape $(\rho(uv), k)$. Recall that, in the lattice of subspaces of a vector space V , the join $W \vee U$ of spaces is the subspace generated by the union $W \cup U$, and the meet $W \wedge U$ is the intersection $W \cap U$. The geometric significance of the operation \diamond is given in the following Proposition.

Proposition 7.3. *Suppose that $g: S \rightarrow V$ is a representation of the matroid M . Let $u, v \in W^1$ be independent words with $|u| = r$ and $|v| = s$, and let $t = \rho(uv)$ and $k = r + s - t$. If W and U are the subspaces of V associated to the decomposables $\hat{g}(u) \in \Lambda^r(V)$ and $\hat{g}(v) \in \Lambda^s(V)$, then $\hat{g}(u \diamond v) = x \otimes y$, where $x \in \Lambda^t(V)$ and $y \in \Lambda^k(V)$ are decomposable, with associated subspaces $W \vee U$ and $W \wedge U$, respectively.*

Proof. Let $T = \{t_1, \dots, t_k\}$ be a basis for $W \cap U$, and extend to bases

$$\{t_1, \dots, t_k, u_1, \dots, u_{r-k}\} \quad \text{and} \quad \{t_1, \dots, t_k, u_1, \dots, u_{s-k}\}$$

for W and U , respectively. Since \hat{g} is a lax Hopf algebra morphism, we have

$$\begin{aligned} \hat{g}(u \diamond v) &= (-1)^{sk} \hat{g}(\delta_{r-k,k}(u)(v \circ 1)) \\ &= (-1)^{sk} \delta_{r-k,k}(\hat{g}(u))(\hat{g}(v) \otimes 1) \\ &= c \cdot \delta_{r-k,k}(t_1 \vee \cdots \vee t_k \vee u_1 \vee \cdots \vee u_{r-k}) \\ &\quad \cdot (t_1 \vee \cdots \vee t_k \vee u_1 \vee \cdots \vee u_{s-k} \otimes 1), \end{aligned}$$

where $c \neq 0$ is a constant. The only nonzero term is $cx \otimes y$, where

$$x = t_1 \vee \cdots \vee t_k \vee u_1 \vee \cdots \vee u_{r-k} \vee u_1 \vee \cdots \vee u_{s-k},$$

and

$$y = t_1 \vee \cdots \vee t_k.$$

□

The following theorem, which provides an alternative expression for the geometric product, is the Whitney algebra analogue of the basic properties of the meet operation in a Cayley algebra (see [4], Section 4).

Theorem 7.4 (Exchange Relations). *For words $u, v \in W^1$, with $|u| = r$, $|v| = s$, let $k = r + s - \rho(uv)$; then*

$$(7.5) \quad \sum_{(u)_{r-k,k}} u_{(1)}v \circ u_{(2)} = \sum_{(v)_{k,s-k}} uv_{(2)} \circ v_{(1)}.$$

Proof. Since $0 \leq k \leq r$, Theorem 5.7 applies. Multiplying both sides of Equation 5.8 (with the word u replacing w) by $\delta(v)$, and projecting from the tensor algebra $T(E(S))$ to the Whitney algebra W , we obtain

$$\sum_{\substack{(u)_{(r-k+1, \dots)} \\ u_{(1)} \prec u_{(2)}}} \delta(u_{(1)}) (\chi(u_{(2)}) \circ u_{(3)}) \delta(v) = \sum_{j=0}^r \binom{j-1}{k-1} \delta_{r-j,j}(u) \delta(v).$$

Each term on the left side contains a factor $\delta(u_{(1)}) \delta(v) = \delta(u_{(1)}v)$, which is equal to zero in W , since $u_{(1)}v$ has length $s + r - k + 1 = \rho(uv) + 1$. Hence, in particular, the homogeneous component of the right side of shape $(\rho(uv), k)$ is zero; that is,

$$(7.6) \quad 0 = \sum_{j=0}^r \binom{j-1}{k-1} \delta_{r-j,j}(u) \delta_{s-k+j,k-j}(v).$$

The above binomial coefficient is zero unless $j = 0$ or $j \geq k$, and the coproduct slice $\delta_{s-k+j,k-j}(v)$ is zero unless $j \leq k$; so (7.6) becomes

$$\begin{aligned} 0 &= (-1)^{k-1} \delta_{r,0}(u) \delta_{s-k,k}(v) + \delta_{r-k,k}(u) \delta_{s,0}(v) \\ &= (-1)^{k-1} (u \circ 1) \delta_{s-k,k}(v) + \delta_{r-k,k}(u) (v \otimes 1) \\ &= (-1)^{k-1} \sum_{(v)_{s-k,k}} uv_{(1)} \circ v_{(2)} + (-1)^{sk} \sum_{(u)_{r-k,k}} u_{(1)}v \circ u_{(2)} \\ &= (-1)^{k-1+(s-k)k} \sum_{(v)_{k,s-k}} uv_{(2)} \circ v_{(1)} + (-1)^{sk} \sum_{(u)_{r-k,k}} u_{(1)}v \circ u_{(2)}. \end{aligned}$$

Since $k - 1 + (s - k)k \equiv sk + 1$ modulo 2, multiplying by $(-1)^{sk}$, gives expression (7.5). □

The next proposition is a simple rephrasing of Theorem 7.4, and expresses the commutativity of the geometric product. This is the Whitney algebra analogue of Theorem 1 of [4].

Proposition 7.7. *For words u, v , with $|u| = r$, $|v| = s$, let $k = r + s - \rho(uv)$. Then*

$$(7.8) \quad u \diamond v = (-1)^{(r-k)(s-k)} v \diamond u$$

Proof. By Theorem 7.4,

$$\begin{aligned} (-1)^{(r-k)(s-k)} v \diamond u &= (-1)^{(r-k)(s-k)} \sum_{(u)_{k,r-k}} v u_{(2)} \circ u_{(1)} \\ &= \sum_{(u)_{r-k,k}} u_{(1)} v \circ u_{(2)} = u \diamond v. \end{aligned}$$

□

Proposition 7.9. *If words u and v span the same flat of M , then $u \circ v = v \circ u$ in W .*

Proof. If either u or v is dependent, then $u \circ v = v \circ u = 0$, and so u and v commute. Suppose that u and v are independent. Since $\bar{u} = \bar{v}$, it follows that $|u| = |v| = \rho(u) = \rho(v) = \rho(uv)$, and thus by Proposition 7.7, $u \diamond v = v \diamond u$. According to the definition (7.2), $u \diamond v = u \circ v$ and $v \diamond u = v \circ u$, and so the result follows. □

Proposition 7.10. *Suppose that M is such that $u \circ v \neq 0$ in W whenever u and v are independent words on S . Then two words commute in W if and only if they span the same flat of M .*

Proof. If u and v span the same flat of M then they commute by the previous proposition. Suppose that u and v do not span the same flat. Then, without loss of generality, there is a point $a \in S$ such that $a \in \bar{v}$ but $a \notin \bar{u}$, and so $av = 0$ and $au \neq 0$. But then

$$(a \circ 1)(u \circ v) = au \circ v \neq 0,$$

while

$$(a \circ 1)(v \circ u) = av \circ u = 0,$$

and hence u and v do not commute. □

In particular, if the matroid M is representable, then no product of independent words in W is equal to zero, and hence pairs of commuting words in W are characterized by Proposition 7.10.

8. THE BASIS RING

We now consider the subring of the Whitney algebra generated by the bases of a matroid, and describe its properties with respect to representations of the matroid and its relationship to White's bracket ring. Throughout this section, M will denote a matroid of rank n on the set S , and V an n -dimensional vector space over the field K .

Definition 8.1. The *basis ring* of M is the subring

$$B = B(M) = W_{(n)} \oplus W_{(n,n)} \oplus W_{(n,n,n)} \oplus \cdots$$

of the Whitney algebra $W = W(M)$.

Note that B is commutative, by Proposition 7.9.

As we have seen in Proposition 6.3, a representation $g: S \rightarrow V$ of M extends to a unique lax Hopf algebra morphism $\hat{g}: W \rightarrow T(\Lambda(V))$. The restriction $f = \hat{g}|_B$ takes values in the pseudoscalar algebra $\Psi(V) = T(\Lambda^n(V))$ and thus, in particular, g induces an algebra homomorphism $f: B \rightarrow \Psi(V)$. In Proposition 8.7 below, we

prove the converse: any homomorphism $f: B \rightarrow \Psi(V)$ that is nonzero on all bases of M determines, up to choice of ordered bases for M and V , a unique representation g of M such that $\hat{g}|_B = f$.

The following theorem, which generalizes Proposition 6.7, is the Whitney algebra version of Bazin's theorem for compound determinants (see [13], part II, pp. 206-208, and [12]) and provides the essential step in the proof of the key Lemma 8.5 below.

Theorem 8.2. *Suppose that $w = b_1 \dots b_k$ and $v = a_1 \dots a_k$ are words in W , with $a_1, \dots, a_k \in \overline{w}$. For $1 \leq i, j \leq k$, let $w_{i,j} = w_{i,a_j} = b_1 \dots b_{i-1} a_j b_{i+1} \dots b_k$. Then the equality*

$$\overbrace{w \circ \dots \circ w}^{k-1} \circ v = \det(w_{i,j})_{1 \leq i, j \leq k}$$

holds in W .

Proof. We first note that, since each a_i belongs to the closure of w , it follows that all of the nonzero $w_{i,j}$ span the same flat of M . Hence, by Proposition 7.9, the $w_{i,j}$ generate a commutative subring of W and so the above determinant is well-defined.

For all $r \geq 0$, let 1_r denote the r -fold product $1 \circ \dots \circ 1$ in W , where 1 denotes the empty word; in particular, $1_0 = 1_z$ is the identity element of W . For $1 \leq j \leq k-1$, we have the equality

$$1_{k-j-1} \circ w \circ 1_{j-1} \circ a_j = \sum_{i=1}^k 1_{k-j-1} \circ w_{i,j} \circ 1_{j-1} \circ b_i$$

in $W_{(k, \dots, k, 1)} \subseteq W^k$, corresponding to the equation $w \circ a_j = \sum_{i=1}^k w_{i,j} \circ b_i$, given by Proposition 6.7. Taking the internal product in W^k , we have

$$\prod_{j=1}^{k-1} 1_{k-j-1} \circ w \circ 1_{j-1} \circ a_j = \prod_{j=1}^{k-1} \sum_{i=1}^k 1_{k-j-1} \circ w_{i,j} \circ 1_{j-1} \circ b_i,$$

from which we obtain the following equation (with the same power of -1 multiplying each side):

$$w \circ \dots \circ w \circ a_1 \dots a_{k-1} = \sum_{i_1=1}^k \dots \sum_{i_{k-1}=1}^k w_{i_1,1} \circ \dots \circ w_{i_{k-1},k-1} \circ b_{i_1} \dots b_{i_{k-1}}.$$

Multiplying (internally) by $1_{k-1} \circ a_k$, we obtain

$$\begin{aligned} w \circ \dots \circ w \circ v &= \sum_{\tau \in S_k} w_{\tau(1),1} \dots w_{\tau(k-1),k-1} \circ b_{i_1} \dots b_{i_{k-1}} a_k \\ &= \det(w_{i,j})_{1 \leq i, j \leq k}. \end{aligned}$$

□

Example 8.3. Suppose that $a, b, c, d \in S$ and $c, d \in \overline{ab}$; Then

$$ab \circ cd = \begin{vmatrix} cb & db \\ ac & ad \end{vmatrix} = cb \circ ad - ac \circ db$$

in W ; and if $a_1, a_2, a_3, b_1, b_2, b_3 \in S$, with $a_1, a_2, a_3 \in \overline{b_1 b_2 b_3}$, then

$$\begin{aligned}
b_1 b_2 b_3 \circ b_1 b_2 b_3 \circ a_1 a_2 a_3 &= \begin{vmatrix} a_1 b_2 b_3 & a_2 b_2 b_3 & a_3 b_2 b_3 \\ b_1 a_1 b_3 & b_1 a_2 b_3 & b_1 a_3 b_3 \\ b_1 b_2 a_1 & b_1 b_2 a_2 & b_1 b_2 a_3 \end{vmatrix} \\
&= a_1 b_2 b_3 \circ b_1 a_2 b_3 \circ b_1 b_2 a_3 - a_1 b_2 b_3 \circ b_1 b_2 a_2 \circ b_1 a_3 b_3 \\
&\quad + b_1 a_1 b_3 \circ b_1 b_2 a_2 \circ a_3 b_2 b_3 - b_1 a_1 b_3 \circ a_2 b_2 b_3 \circ b_1 b_2 a_3 \\
&\quad + b_1 b_2 a_1 \circ a_2 b_2 b_3 \circ b_1 a_3 b_3 - b_1 b_2 a_1 \circ b_1 a_2 b_3 \circ a_3 b_2 b_3.
\end{aligned}$$

Suppose that $f: B \rightarrow \Psi(V)$ is a nonzero homomorphism and $w = b_1 \cdots b_n$ is a basis of M such that $f(w) \neq 0$. Given any basis $T = \{v_1, \dots, v_n\}$ for V , we define a bracket $[\]: \Lambda^n(V) \rightarrow K$ by setting $[v_1 \vee \cdots \vee v_n] = 1$, thus making V a Peano space. Since f is a morphism of graded algebras, $f(w)$ belongs to $\Lambda^n(V)$ and is thus equal to $c v_1 \vee \cdots \vee v_n$, for some nonzero $c \in K$. It follows by taking brackets that $c = [f(w)]$. We define a map $f_{w,T}: S \rightarrow V$ by

$$(8.4) \quad f_{w,T}(a) = \frac{1}{[f(w)]} \sum_{i=1}^n [f(w_{i,a})] v_i,$$

for all $a \in S$. Note that, since $w_{i,b_i} = w$ and $w_{i,b_j} = 0$ for $i \neq j$, the function $f_{w,T}$ maps b_i to v_i , for $1 \leq i \leq n$.

Lemma 8.5. *Suppose that $f: B(M) \rightarrow \Psi(V)$ is a nonzero homomorphism, that $w = b_1 \cdots b_n$ is a basis of M such that $f(w) \neq 0$, and $T = \{v_1, \dots, v_n\}$ is a basis for V . For any word $u = a_1 \cdots a_n$ of length n on S , the equation*

$$(8.6) \quad f(u) = c \cdot \bar{f}(a_1) \vee \cdots \vee \bar{f}(a_n),$$

holds, where $\bar{f} = f_{w,T}: S \rightarrow V$ is defined by Equation 8.4, and c is the constant determined by $f(w) = c v_1 \vee \cdots \vee v_n$.

Proof. Define a bracket $[\]: \Lambda^n(V) \rightarrow K$ by setting $[v_1 \vee \cdots \vee v_n] = 1$, and let $\beta: \Psi(V) \rightarrow K$ be the multiplicative extension of $[\]$ to the pseudoscalar algebra given by Equation 4.14. Since β and f are algebra maps and $\beta f(w) = [f(w)] = c$, we have $\beta f(w \circ \cdots \circ w \circ u) = c^{n-1} [f(u)]$, and hence by Theorem 8.2,

$$c^{n-1} [f(u)] = \beta f(\det(w_{i,j})_{1 \leq i,j \leq n}) = \det([f(w_{i,j})])_{1 \leq i,j \leq n}.$$

On the other hand, computing in $\Lambda^n(V)$ yields

$$\begin{aligned}
\bar{f}(a_1) \vee \cdots \vee \bar{f}(a_n) &= c^{-n} \cdot \left(\sum_{i_1=1}^n [f(w_{i_1,1})] v_1 \right) \vee \cdots \vee \left(\sum_{i_n=1}^n [f(w_{i_n,n})] v_n \right) \\
&= c^{-n} \det(w_{i,j}) \cdot v_1 \vee \cdots \vee v_n \\
&= c^{-1} [f(u)] \cdot v_1 \vee \cdots \vee v_n \\
&= c^{-1} f(u),
\end{aligned}$$

and hence the result follows. \square

Proposition 8.7. *Suppose that f is a homomorphism from the basis ring B to the pseudoscalar algebra $\Psi(V)$ such that $f(u) \neq 0$ for all bases u of M . For any choice of bases $w = b_1 \cdots b_n$ and $T = \{v_1, \dots, v_n\}$ for M and V , there exists a unique representation $g: S \rightarrow V$ of M and nonzero scalar d such that $g(b_1) = d v_1$ and*

$g(b_i) = v_i$ for $i \geq 2$, and such that the restriction of the induced lax Hopf algebra morphism $\hat{g}: W \rightarrow T(\Lambda(V))$ to B is equal to f .

Proof. Let d be the scalar (necessarily nonzero) satisfying $f(w) = d v_1 \vee \cdots \vee v_n$, let T' be the basis $\{d v_1, v_2, \dots, v_n\}$ of V , and let $g: S \rightarrow V$ be the map $f_{w, T'}$ defined by Equation 8.4, using the basis T' instead of T . It follows immediately from Lemma 8.5 that, for any a_1, \dots, a_n in S , the set $\{g(a_1), \dots, g(a_n)\}$ is linearly independent in V if and only if a_1, \dots, a_n is a basis for M . Hence g is a representation of M such that $g(b_1) = d v_1$ and $g(b_i) = v_i$ for $i \geq 2$. Since \hat{g} is an extension of g and is a lax Hopf algebra morphism we have, for all words $a_1 \cdots a_n$,

$$\hat{g}(a_1 \cdots a_n) = \hat{g}(a_1) \vee \cdots \vee \hat{g}(a_n) = g(a_1) \vee \cdots \vee g(a_n),$$

which is equal to $f(a_1 \cdots a_n)$ by Lemma 8.5 (since the scalar c appearing in Equation 8.6 is equal to one in this case). Hence the restriction $\hat{g}|_B$ is equal to f .

Corollary 6.8 implies that the representation g is uniquely determined by the values $g(b_1) = d v_1$ and $g(b_i) = v_i$, for $i \geq 2$. The requirement that $\hat{g}|_B = f$ determines the scalar d uniquely. \square

Corollary 8.8. *Suppose that h is a homomorphism from B to the field K such that $h(u) \neq 0$ for all bases u of M . For any choice of bases $w = b_1 \cdots b_n$ and $T = \{v_1, \dots, v_n\}$ for M and V , there exists a unique representation $g: S \rightarrow V$ of M and nonzero scalar d such that $g(b_1) = d v_1$ and $g(b_i) = v_i$ for $i \geq 2$, and such that $h(u)$ is equal to the determinant of $(g(a_1), \dots, g(a_n))$ with respect to the basis T , for all bases $u = a_1 \cdots a_n$ of M .*

Proof. Define a bracket on V by setting $[v_1 \vee \cdots \vee v_n] = 1$. By Proposition 4.15, there exists a unique homomorphism of graded rings $f: B \rightarrow \Psi(V)$ such that $\beta f = h$, where $\beta: \Psi(V) \rightarrow K$ is the multiplicative extension of $[\]$ given by Equation 4.14. Note that, in particular, $f(w) = d v_1 \vee \cdots \vee v_n$, where $d = h(w)$. Let g be the map $f_{w, T'}: S \rightarrow V$ defined by Equation 8.4, corresponding to f , w and $T' = \{d v_1, v_2, \dots, v_n\}$. The result then follows from Proposition 8.7 and the fact that $[g(a_1) \vee \cdots \vee g(a_n)] = \det(g(a_1), \dots, g(a_n))$. \square

Proposition 8.7 and Corollary 8.8 may be summarized by the following commutative diagram

$$\begin{array}{ccccc}
 B & \xrightarrow{\quad} & W & \xleftarrow{\quad} & S \\
 & \searrow h & & & \downarrow \bar{f}=g \\
 & & & & V \\
 & \swarrow f & & & \downarrow \\
 K & \xleftarrow{\beta} & \Psi(V) & \xrightarrow{\quad} & T(\Lambda(V)) \\
 & & & & \downarrow \hat{g} \\
 & & & & W
 \end{array}$$

where the unlabelled arrows are inclusions.

Corollary 8.9. *A matroid M is representable if and only if no product of independent words is equal to zero in W .*

Proof. If M is representable, then no product of independent words in W is equal to zero, by Corollary 6.4. Conversely, suppose that no product of independent words is zero in W , and let N be the multiplicative semigroup in B consisting of all products of bases. The set of ideals J in B such that $J \cap N = \emptyset$ is nonempty, since

it contains (0) , and hence contains a maximal element P , which must be prime by basic ring theory arguments. Let F be the field of fractions of B/P , and let f be the composition $B \rightarrow B/P \hookrightarrow F$. Then $f(u) \neq 0$ for all bases u of M , and thus M is representable by Corollary 8.8. \square

We now recall the definition of White's bracket ring of a matroid M . For any word $u = a_1 \cdots a_r$ on S , subword $a_{i_1} \cdots a_{i_k}$ of u , and word $b_1 \cdots b_k$, we write $(u/a_{i_1} \rightarrow b_i, \dots, a_{i_k} \rightarrow b_k)$ for the word u with the letters a_{i_1}, \dots, a_{i_k} replaced by b_1, \dots, b_k , respectively. We denote by $Sym(E_n)$ the symmetric algebra on the degree n homogeneous component E_n of the free exterior algebra $E(S)$ (hence $Sym(E_n)$ is the polynomial ring generated by monotone words on S of length n). For all $1 \leq k \leq n$, and words $u = a_1 \cdots a_n$ and $v = b_1 \cdots b_n$ on S , the syzygy $\zeta_k(u, v)$ is the element of $Sym(E_n)$ given by

$$u \cdot v - \sum (u/a_{i_1} \rightarrow b_i, \dots, a_{i_k} \rightarrow b_k) \cdot (a_{i_1} \cdots a_{i_k} b_{k+1} \cdots b_n),$$

where the sum is over all sequences $1 \leq i_1 < \cdots < i_k \leq n$.

Definition 8.10 ([16]). The *bracket ring* B' of M is the quotient ring $Sym(E_n)/I$, where I is the ideal of $Sym(E_n)$ generated by the set of all dependent words of length n on S and all syzygies $\zeta_k(u, v)$, where u, v are words of length n and $1 \leq k \leq n$.

We write $\langle u \rangle$ for the image of the word u under the surjection $Sym(E_n) \rightarrow B'$ (we use this notation instead of White's $[u]$ in order to avoid confusion with the bracket of a Peano space).

Proposition 8.11. *There is a surjective ring homomorphism φ from the bracket ring B' onto the basis ring B , satisfying $\varphi(\langle u \rangle) = u$, for all bases u of M .*

Proof. Since B is commutative, there is homomorphism $\alpha: Sym(E_n) \rightarrow B$ satisfying $\alpha(u) = u$, for all words u of length n . It is clear $\alpha(u) = 0$, if u is dependent; it remains to show that all syzygies $\zeta_k(u, v)$ are also contained in the kernel of α .

Suppose that $u = a_1 \cdots a_n$ and $v = b_1 \cdots b_n$ are words on S , and that $x = b_1 \cdots b_k$ and $y = b_{k+1} \cdots b_n$, for some k . Then

$$\begin{aligned} \alpha(\zeta_k(u, v)) &= u \circ v - \sum (u/a_{i_1} \rightarrow b_i, \dots, a_{i_k} \rightarrow b_k) \circ (a_{i_1} \cdots a_{i_k} b_{k+1} \cdots b_n) \\ &= \left(u \circ x - \sum (u/a_{i_1} \rightarrow b_i, \dots, a_{i_k} \rightarrow b_k) \circ a_{i_1} \cdots a_{i_k} \right) (1 \circ y) \\ &= \left(u \circ x - \sum_{(u)_{n-k, k}} u_{(1)} x \circ u_{(2)} \right) (1 \circ y), \end{aligned}$$

which is equal to zero by the Exchange relations (7.5). Hence the ideal I is contained in $\ker \alpha$, and so α induces a homomorphism $\varphi: B' = Sym(E_n)/I \rightarrow B$. \square

Recall that a *rank-preserving weak map image* of M is a matroid N on the same set, such that every basis of N is also a basis of M . (In other words, N is a matroid on S such that the identity map is a rank-preserving weak map from M to N). The following result, due to White (see [17], Proposition 4.2 and 4.3, and [18], Proposition 2.1), expresses the universal property of the bracket ring.

Proposition 8.12. *For any homomorphism h from the bracket ring B' to the field K , the set*

$$\{ w : w \text{ is a basis for } M \text{ and } h(\langle w \rangle) \neq 0 \}$$

is the set of bases of a rank-preserving weak map image N of M , and for any choice of bases $w = b_1 \cdots b_n$ and $T = \{v_1, \dots, v_n\}$ for N and V , there exists a unique representation $g: S \rightarrow V$ of N and nonzero scalar d such that $g(b_1) = d v_1$ and $g(b_i) = v_i$ for $i \geq 2$, and such that $h(u)$ is equal to the determinant of $(g(a_1), \dots, g(a_n))$ with respect to the basis T , for all bases $u = a_1 \cdots a_n$ of M .

Comparing Proposition 8.12 and Corollary 8.8, we see that the bracket ring B' and the basis ring B have the same universal property with respect to representations of M ; this allows us to say considerably more about the surjective homomorphism $\varphi: B' \rightarrow B$.

Proposition 8.13. *The kernel of the homomorphism $\varphi: B' \rightarrow B$, from the bracket ring to the basis ring, is contained in the radical of B' .*

Proof. Suppose that $h: B' \rightarrow K$ is a nonzero homomorphism, for some field K , that w is a basis of M with $h(\langle w \rangle) \neq 0$, and $T = \{v_1, \dots, v_n\}$ is a basis for V . Let N and $g: S \rightarrow V$ be the rank-preserving weak map image of M and representation of N corresponding to h, w and T , as in Proposition 8.12, and let $\hat{g}: W(N) \rightarrow T(\Lambda(V))$ be the lax Hopf algebra morphism extending g . Define a bracket on V by setting $[v_1 \vee \cdots \vee v_n] = 1$, and let $\beta: \Psi(V) \rightarrow K$ be the multiplicative extension of $[\]$ to the pseudoscalar algebra. For any basis $u = a_1 \cdots a_n$ of M we have

$$\begin{aligned} h(\langle u \rangle) &= \det(g(a_1), \dots, g(a_n)) \\ &= [g(a_1) \vee \cdots \vee g(a_n)] \\ &= [\hat{g}(u)] \\ &= \beta f(u), \end{aligned}$$

where we have written f for the restriction $\hat{g}|_B$. Hence $h(\langle u \rangle) = \beta f(u) = \beta f \varphi(\langle u \rangle)$, for all bases u , and so $h = \beta f \varphi$. Since any homomorphism h from B' into a field thus factors through φ , it follows that $\ker \varphi$ is contained in the intersection of all prime ideals of B' , that is, $\ker \varphi \subseteq \text{rad } B'$. \square

The following commutative diagram may serve as a useful guide to the preceding proof:

$$\begin{array}{ccccccc} B' & \xrightarrow{\varphi} & B & \xrightarrow{\quad} & W & \xleftarrow{\quad} & S \\ \downarrow h & & \downarrow f & & \downarrow \hat{g} & & \downarrow g \\ K & \xleftarrow{\beta} & \Psi(V) & \xrightarrow{\quad} & T(\Lambda(V)) & \xleftarrow{\quad} & V \end{array}$$

9. THE VIEW FROM VIGEVANO

Gian-Carlo Rota's views on the Whitney algebra are fortunately recorded in a series of messages by electronic mail, and in the notebooks he filled during discussions in his Cambridge apartment. In this final section we include extracts from this material, not only because they convey his enthusiasm for the project and his ideas for its future course, but also because he sketched in detail an important development that we have not been able to carry out in the present article: the superalgebraic

expression of the Whitney algebra of a matroid via the Feynman entangling operator.

In the November 1995, after several years of intermittent discussions on the resolving bracket [3] and on the combinatorial structure of higher order syzygies, Crapo and Rota began an exchange of electronic mail on what Rota soon dubbed *the Whitney algebra of a matroid*. Work began in earnest over the winter holidays, and reached a climax in January 1966, thanks to a heavy snowfall that stranded Rota in Cambridge for several days. By October of that year, the correspondence ran to some 100 pages of text.

18 November, 1995 — Telephone call from Rota. He finds that the ‘tensor-product’ approach to non-spanning syzygies is correct, that is, that

$$a \otimes bc - bc \otimes ac + c \otimes ab$$

is the zero *tensor* whenever a, b, c are collinear points (dependent vectors) in *any* space, and gives a Hopf-algebra structure on an arbitrary matroid, potentially replacing the ‘bracket ring’, which had the disadvantage of being commutative. Idea: in an exterior algebra generated by formally independent points, set to zero all joins of dependent sets of points, and their coproducts.

22 November, 1995 — I just read your fax, it is exactly what I was thinking. I have gone a little further in the formalization of the Hopf algebra of a matroid, so far everything checks beautifully. The philosophical meaning of all this is that every matroid has a natural coordinatization ring, which is the infinite product of copies of a certain quotient of the free exterior algebra generated by the points of the matroid (loops and links allowed, of course). This infinite product is endowed with a coproduct which is not quite a Hopf algebra, but a new object closely related to it. Roughly, it is what one obtains when one mods out all coproducts of minimal dependent sets, and this, remarkably, give all the exchange identities. I now believe that everything that can be done with the Grassmann-Cayley algebra can also be done with this structure, especially meets.

28 November, 1995 — I will send you material as soon as I physically can. Everything works beautifully, and we have defined a new concept of independent algebraic interest: Whitney algebras, which generalize Hopf algebra in a way that is so natural that it will make the Hopf algebraists envious. Your latest fax was very helpful, but I will have to explain to you the main idea. I think there may be even an interpretation of the critical problem for general matroids! This is an idea of yours that is really bearing fruit.

29 November, 1995 — I will try to write down something tonight and send it to you by latex. I still think this is the best idea we have been working on in years, and all your past work on syzygies will fit in beautifully.

20 December, 1995 — I am working on your ideas, trying to recast them in letterplace language. I tried to write down something last night, but I was too tired. Things are getting quite rough around here.

9 January, 1996 — Thanks for the message. I am snowbound in Cambridge, and won't be leaving for Washington until Friday, at least, so I hope to redraft the remarks on Whitney algebras I have been collecting. It seems that we will have to translate Tutte's homotopy theorem into the language of Whitney algebras, using circuits instead of copoints. Has the theorem been restated in terms of circuits (as it can, by taking complements)? If it has, I would appreciate your sending me the statement, it will save me quite a bit of work. Neil White has a translation into the language of brackets, and I am working with his translation.

Here are some philosophical remarks. First, all of linear algebra should be done with the Whitney algebra, no scalars ever mentioned. Second, there is a new theorem to be stated and proved preliminarily, which seems to be a vast generalization of the second fundamental theorem of invariant theory (Why, Oh why, did I not see this before?!).

[Here, Rota suggests a comparison between the Whitney algebra of a vector space V , when viewed as a matroid, and the exterior algebra of V .]

I think this is the first step towards proving the big theorem. It is already difficult, and I would appreciate your help.

Another priority is to see following your lead how to completely get rid of meets, using Whitney algebra techniques. The point is to prove classical determinant identities, such as Jacobi's identity, using only Whitney algebra methods (with an eye towards their quantum generalizations!) Only by going through the Whitney algebra proofs will we see how to carry out a quantum generalization of all this stuff.

It is of the utmost importance that you familiarize yourself with the letterplace representation of the Whitney algebra, through the Feynman operators, and I will write up this stuff first and send it to you.

1 November, 1996 — Still snowbound in Cambridge, Rota composed a long text proposing two projects: first, the description of a module derived from a Whitney algebra $W(M)$, second, a faithful representation of a Whitney algebra as

a quotient of a supersymmetric letter-place algebra.

(1) **The Whitney module.** Let $W = W(M)$ be the Whitney algebra of a matroid M . In order to extend the internal multiplication on each $W^{(k)}$ to all of W , Rota replaced each monomial $w_1 \circ w_2 \circ \cdots \circ w_k$ by an infinite tensor product $w_1 \circ w_2 \circ \cdots \circ w_k \circ 1 \circ 1 \cdots$ obtained by appending infinitely many 1s on the right. For instance,

$$\begin{aligned} (a \circ bc \circ d \circ 1 \circ \cdots) (1 \circ e \circ 1 \circ fg \circ 1 \circ \cdots) \\ = - (a \circ bce \circ d \circ fg \circ 1 \circ \cdots) \end{aligned}$$

The submodule $W_{(1, \dots)}$ generated by monomials $w_1 \circ w_2 \circ \cdots \circ w_k \circ 1 \circ \cdots$ for various values of k , but with $|w_1| = 1$, can be considered as a $W_{(0, \dots)}$ -module, where $W_{(0, \dots)}$ is generated by monomials of the form $1 \circ w_2 \circ \cdots \circ w_k \circ 1 \circ \cdots$, again, for all k . The idea was to compare linear dependence in this module with linear dependence in the matroid M .

(2) **The Feynman entangling operator.** Rota proposed that the Whitney algebra $W(M)$ of any matroid $M = M(S)$ can be faithfully represented as a quotient of the supersymmetric algebra $Super[S^- | P^+]$. He mapped each monomial $w_1 \otimes w_2 \otimes \cdots \otimes w_k$ in $W(M)$ to the product

$$(w_1 | p_1^{(|w_1|)}) (w_2 | p_2^{(|w_2|)}) \cdots (w_k | p_k^{(|w_k|)}),$$

where the words $p_i^{(|w_i|)}$ are divided powers of positive letters representing the different possible positions in the tensor product. (The letter-place pairs $(a | p)$ are thus anticommutative.) The linear extension of this definition to a map on $W(M)$, Rota termed the *Feynman entangling operator*.

It was a busy Spring, with many visitors arriving at M.I.T. for the RotaFest. Schmitt, enroute for a fall term visit at M.I.T., made a stop-over in Paris in September. At the conclusion of this visit, Crapo and Schmitt proposed to collaborate with Rota in an effort to develop the theory of Whitney algebras, and in particular, to settle that first question: in precisely what sense is the Whitney algebra a generalization of a Hopf algebra? They met in Rota's Cambridge apartment late in October, to map out the project. This was regrettably to be their only three-way discussion of the subject. On that occasion, Rota reiterated his view that the exchange relations should be proved using properties of the Feynman entangling operator, along the lines of the simple proof of the superalgebra exchange property, Theorem 10 of [9], noting in passing that the coproduct operators of the Whitney algebra correspond, under entangling, to polarizations of positive places. (We would be delighted to share the original texts of these messages and notes on the Feynman entangling operator with any reader willing to pursue this research. We encourage those skilled in superalgebraic methods to press on!)

It was not until the summer of 1997 that the authors of the present text had the occasion to work together over an extended period. They found the Zipper lemma and exchange relations, and began planning the present article. The categorical setting, and in particular the concept of lax Hopf algebra, here presented, are quite recent developments.

We would have taken great pleasure in presenting this paper to Gian-Carlo Rota in person. We're sure he would have liked it. It is with the fondest of recollections, and gratitude for his guidance and friendship, that we now send it to press.

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