Hopf Algebra Methods in Graph Theory[1]

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1. Introduction

In this paper we introduce a Hopf algebraic framework for studying invariants of graphs, matroids, and other combinatorial structures. We begin by defining a category of objects, called Whitney systems, which are set systems having the minimum amount of structure necessary in order to have a sensible notion of connected subset, and which generalize graphs and matroids in several ways. Associated to any family $\mathcal{P}$ of Whitney systems, which is closed under restrictions and disjoint unions, is a certain Hopf algebra, generalizing the multiplicative formal group law, whose dual is isomorphic to the algebra of invariants defined on $\mathcal{P}$, and whose continuous dual is isomorphic to an important subalgebra of invariants, called restriction invariants on $\mathcal{P}$. We prove a structure theorem, namely, that the continuous dual of such a Hopf algebra is isomorphic to the polynomial algebra having the set of connected isomorphism types in $\mathcal{P}$ as indeterminates. We also introduce a general transformation, essentially the transpose of the logarithm map, which maps the class of restriction invariants onto the set of additive invariants on $\mathcal{P}$.

In a fundamental paper ([13]), H. Whitney showed that the chromatic polynomial of a graph G could be determined by examining only the doubly connected subgraphs of G; thus, in principle, it could be computed using less information than required by previously known techniques (see [4] and [12]). This result was derived independently in [10] by W. Tutte, who gave applications to the reconstruction problem in [11]. Some extensions were given by P. Erdös, L. Lovász and J. Spencer in [5]. Considerable clarification of Whitney's work was given in [1] by N. Biggs, who also discussed its applications to statistical mechanics in [2] and [3].

In the second half of his paper [13] Whitney introduced a transformation from a set of graph invariants $\{m_{ij}\}$, which determine the chromatic polynomial and satisfy a kind of multiplicative property, to a set of invariants $\{f_{ij}\}$, which are additive (see section 6 of this paper). This transformation is invertible, thus the study of chromatic polynomials reduces to the study of the additive invariants f_{ij}. The program which he then set forth was to attack the four-color problem by systematically studying inequalities involving the f_{ij}. Unfortunately, the proofs in his paper were very obscure and, in particular, he gave no explicit technique for computing the m_{ij} or f_{ij} in terms of doubly connected subgraphs. Consequently, one is left without any clear approach to the problem of investigating the relations satisfied by the f_{ij}.

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As an application of the theory introduced, we show that Whitney's theorem about doubly connected subgraphs is equivalent to a special case of our structure theorem for Hopf algebras corresponding to Whitney systems. Furthermore, the transformation from the m_{ij} to the f_{ij} is a special case of the general transformation involving additive invariants mentioned above, and we thus obtain formulas for m_{ij} and f_{ij} in terms of doubly connected subgraphs, whose coefficients have clear, combinatorial significance.

Whitney's results on graph polynomials reflect deep algebraic properties of families of graphs, yet his work was presented using elementary algebraic language which is inadequate for the clear expression of these ideas. We believe that the Hopf algebraic framework introduced here provides an appropriate context for understanding these results and, we hope, will provide some of the tools necessary for this line of investigation in graph theory to continue once again.

2. Whitney Systems

Definition 1. *A* **Whitney system** *is a pair $H = (S, \mathcal{C})$, where S is a set and $\mathcal{C}$ is a collection of non-empty subsets of S such that if U and V belong to $\mathcal{C}$ and $U \cap V \neq \emptyset$ then $U \cup V$ belongs to $\mathcal{C}$.*

If $H = (S, \mathcal{C})$ is a Whitney system then sometimes we write $S(H)$ for the underlying set S, and $\mathcal{C}(H)$ for the family of subsets $\mathcal{C}$. A **morphism** $\phi : H_1 \to H_2$ of Whitney systems is a function ϕ from $S(H_1)$ to $S(H_2)$ such that $\phi(U) \in \mathcal{C}(H_2)$ whenever $U \in \mathcal{C}(H_1)$. The isomorphism class of an object H in the category of Whitney systems is denoted by $[H]$. The **sum** of Whitney systems $H_1 = (S_1, \mathcal{C}_1)$ and $H_2 = (S_2, \mathcal{C}_2)$ is the Whitney system having the disjoint union of S_1 and S_2 as underlying set, and the disjoint union of $\mathcal{C}_1$ and $\mathcal{C}_2$ as family of distinguished subsets. The **restriction** of a Whitney system H to a subset U of $S(H)$ is the Whitney system given by

$$H|U = (U, \{V \in \mathcal{C}(H) \,:\, V \subseteq U\}).$$

Definition 2. *A Whitney system $H = (S, \mathcal{C})$ is* **connected** *if and only if $S \in \mathcal{C}$, or if $|S| = 1$.*

Hence, a Whitney system H is connected if and only if it is non-empty and cannot be written as the sum of two non-empty Whitney systems. A subset U of $S(H)$ is defined to be connected if and only if $H|U$ is connected, that is, if and only if $U \in \mathcal{C}(H)$ or $|U| = 1$. If $|S(H)| = 1$ and $S(H) \in \mathcal{C}(H)$, then H is a **loop**; and $U \subseteq S(H)$ is a loop if and only if $H|U$ is a loop.

Given a Whitney system $H = (S, \mathcal{C})$, where S is finite, let σ_H denote the collection of maximal connected subsets of S. It follows that the elements of σ_H are disjoint sets whose union is S, hence σ_H is a partition of S. The elements of σ_H are called the **connected components** of H, and the restrictions $H|U$, for $U \in \sigma_H$, are called the **blocks** of H. The set of all blocks of H is denoted by $\beta(H)$. Thus any Whitney system H has the unique decomposition into blocks $H = \sum_{B \in \beta(H)} B$.

Examples:

1) Suppose G is a graph with vertex set V and edge set E. If $U \subseteq V$ then $G|U$

denotes the induced subgraph of G having vertex set U, and if $T \subseteq E(G)$, then $G|T$ denotes the subgraph of G having edge set T and vertices which are incident to edges in T. Consider the following collections of sets:

$$\begin{aligned}
\mathcal{C}_v &= \{U \subseteq V \,:\, G|U \text{ is connected}\} \\
\mathcal{C}_e &= \{T \subseteq E \,:\, G|T \text{ is connected}\} \\
\mathcal{C}_d &= \{T \subseteq E \,:\, G|T \text{ is doubly connected and } |U| > 1\} \cup \{\{e\} \,:\, e \text{ is a loop of } G\}.
\end{aligned}$$

The pairs $G_v = (V, \mathcal{C}_v)$, $G_e = (E, \mathcal{C}_e)$, and $G_d = (E, \mathcal{C}_d)$ are Whitney systems. The correspondences $G \to G_v$, $G \to G_e$, and $G \to G_b$ define functors from the categories of graphs with vertex, edge, and cycle homomorphisms, respectively, into the category of Whitney systems. Each of these functors preserves sums and restrictions.
2) Let M be a matroid on a set S. If $U \subseteq S$, then $S|U$ and $S \cdot U$ denote, respectively, the restriction and contraction of M to U. Define

$$\mathcal{C}_r = \{U \subseteq S \,:\, M|U \text{ is connected and } |U| > 1\} \cup \{\{x\} \,:\, x \text{ is a loop of } M\}$$

and

$$\mathcal{C}_c = \{U \subseteq S \,:\, M \cdot U \text{ is connected and } |U| > 1\} \cup \{\{x\} \,:\, x \text{ is a loop of } M\}.$$

Then the pairs $M_r = (S, \mathcal{C}_r)$ and $M_c = (S, \mathcal{C}_c)$ are Whitney systems. The matroid M is uniquely determined by either M_r or M_c. For, the circuits of M are the minimal elements of $\mathcal{C}_r$, and the cocircuits of M are the minimal elements of $\mathcal{C}_c$. The correspondence $M \to M_r$ preserves sums and restrictions, and the correspondence $M \to M_c$ preserves sums and contractions.
3) Let $\mathcal{T}$ be a topology on a set S, and let $\mathcal{T}_c$ be the collection of connected subsets of S. The pair $(S, \mathcal{T}_c)$ is a Whitney system. The correspondence $(S, \mathcal{T}) \to (S, \mathcal{T}_c)$ defines a functor from the category of topological spaces and continuous maps to the category of Whitney systems.

3. Hopf Algebras of Whitney Systems

Suppose $\mathcal{P}$ is a set of isomorphism types of finite Whitney systems which is closed under formation of sums and restrictions. $\mathcal{P}$ is thus a commutative monoid with product defined by

$$[H_1][H_2] = [H_1 + H_2],$$

for all $[H_1], [H_2] \in \mathcal{P}$. The identity of $\mathcal{P}$ is equal to the type of the empty Whitney system. Let $\mathcal{P}_\circ \subseteq \mathcal{P}$ denote the set of types of connected Whitney systems in $\mathcal{P}$. Because $\mathcal{P}$ is closed under restrictions, it follows that $\mathcal{P}$ is the free commutative monoid on the set $\mathcal{P}_\circ$.

Suppose K is a field of characteristic zero, which will be fixed from now on. Let $C(\mathcal{P})$ denote the monoid algebra of $\mathcal{P}$ over K (note that $C(\mathcal{P})$ is isomorphic to the polynomial algebra $K[\mathcal{P}_\circ]$). Define linear maps $\Delta : C(\mathcal{P}) \to C(\mathcal{P}) \otimes C(\mathcal{P})$ and $\epsilon : C(\mathcal{P}) \to K$ by

$$\Delta[H] = \sum_{U_1 \cup U_2 = S(H)} [H|U_1] \otimes [H|U_2]$$

and

$$\epsilon[H] = \begin{cases} 1 & \text{if } S(H) = \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

for all $[H] \in \mathcal{P}$. The sum in the above expression for Δ is taken over all ordered pairs of (not necessarily disjoint) sets U_1 and U_2, whose union is equal to $S(H)$. It is straightforward to verify that $C(\mathcal{P})$ is a bialgebra with coproduct Δ and counit ϵ.

For each $n \geq 1$, let I_n be the ideal of $C(\mathcal{P})$ generated by the set $\{[H] \in \mathcal{P} : |S(H)| \geq n\}$. The set of ideals $\{I_n : n \geq 1\}$ forms a local base at 0 for a topology on $C(\mathcal{P})$. Thus, ignoring coalgebra structure for the moment, $C(\mathcal{P})$ is a topological algebra, the completion of which, $\widehat{C(\mathcal{P})}$, is isomorphic to the algebra of formal power series $K[[\mathcal{P}_\circ]]$ and contains $C(\mathcal{P})$ as a dense subalgebra. The composition

$$C(\mathcal{P}) \xrightarrow{\Delta} C(\mathcal{P}) \otimes C(\mathcal{P}) \hookrightarrow \widehat{C(\mathcal{P})} \otimes \widehat{C(\mathcal{P})}$$

is continuous, and thus extends uniquely to a continuous (and coassociative) map $\hat{\Delta} : \widehat{C(\mathcal{P})} \to \widehat{C(\mathcal{P})} \otimes \widehat{C(\mathcal{P})}$. Also, the counit $\epsilon : C(\mathcal{P}) \to K$ is continuous (where K has the discrete topology), and hence extends to a continuous map $\hat{\epsilon} : \widehat{C(\mathcal{P})} \to K$. Therefore $\widehat{C(\mathcal{P})}$ is a bialgebra, with coproduct $\hat{\Delta}$ and counit $\hat{\epsilon}$.

Furthermore, $\widehat{C(\mathcal{P})}$ is a Hopf algebra. The antipode $S : \widehat{C(\mathcal{P})} \to \widehat{C(\mathcal{P})}$ is the continuous linear map defined by $S(1) = 1$ and

$$S[H] = \sum_{n \geq 1} \sum (-1)^n \prod_{i=1}^{n} [H|U_i], \tag{1}$$

for all non-empty $[H] \in \mathcal{P}$, where the inner sum is over all ordered n-tuples $(U_1, U_2, \ldots, U_n)$ of non-empty subsets of $S(H)$, whose union is equal to $S(H)$.

If f is an element of the dual algebra $\widehat{C(\mathcal{P})}^*$ and $[H] \in \mathcal{P}$, let $\langle f \, , \, [H] \rangle$ denote the value of f on $[H]$. For any $[H]$ in $\mathcal{P}$, define $\delta_H \in \widehat{C(\mathcal{P})}^*$ by

$$\langle \delta_H \, , \, [G] \rangle = \begin{cases} 1 & \text{if } [G] = [H] \\ 0 & \text{otherwise,} \end{cases}$$

for all $[G] \in \mathcal{P}$. For all Whitney systems $H, H_1, H_2, \ldots, H_k$, define $\binom{H}{H_1, \ldots, H_k}$ to be the number of ordered k-tuples $(U_1, \ldots, U_k)$ of subsets of $S(H)$, with union $S(H)$, such that $[H|U_i] = [H_i]$, for $1 \leq i \leq k$. One then has the following product formula in $\widehat{C(\mathcal{P})}^*$:

$$\prod_{i=1}^{k} \delta_{H_i} = \sum_{[H] \in \mathcal{P}} \binom{H}{H_1, \ldots, H_k} \delta_H, \tag{2}$$

whenever $[H_1], \ldots, [H_k]$ belong to $\mathcal{P}$.

An element f of the dual $\widehat{C(\mathcal{P})}^*$ is continuous if and only if there exists $n \in \mathbb{N}$ such that $f[H] = 0$ whenever $|S(H)| \geq n$, that is, if and only if f is a finite linear combination of the functions δ_H. It follows from equation 2 that the product of continuous elements of $\widehat{C(\mathcal{P})}^*$ is also continuous. The continuous dual algebra of $\widehat{C(\mathcal{P})}$, denoted by $C(\mathcal{P})'$, is thus the subalgebra of $\widehat{C(\mathcal{P})}^*$ generated by the set $\{\delta_H : [H] \in \mathcal{P}\}$.

If G and H are Whitney systems where H has blocks $B_1, \ldots, B_k$, let $c(G, H)$ denote the number $\binom{G}{B_1, \ldots, B_k}$. In particular, if H has exactly k_B blocks which are isomorphic to B, for all $[B] \in \mathcal{P}_\circ$, then $c(H, H) = \prod_{[B] \in \mathcal{P}_\circ} k_B!$.

Proposition 1. *The set of types $\mathcal{P}$ is partially ordered by the relation $[G] \leq [H]$ if and only if $c(G, H) \neq 0$, or if $G = \emptyset$, the empty Whitney system.*

Proof. If G and H are Whitney systems, then $c(G, H) \neq 0$ if and only if there exists a surjection $f : S(H) \to S(G)$, such that $[H|U] = [G|f(U)]$, for all $U \in \mathcal{C}(H)$. The fact that $\leq$ is a partial order follows directly from this observation. □

If Q is a partially ordered set and $x \leq y$ in Q, the **interval** $[x, y]$ is the set $\{z \in Q : x \leq z \leq y\}$. Q is **locally finite** if all of its intervals are finite. The **incidence algebra**, over K, of a locally finite partially ordered set Q is the collection $I(Q)$ of all functions from the set of intervals in Q into the field K, with pointwise addition and scalar multiplication, and product, or **convolution**, of f and g in $I(Q)$ defined by

$$f * g(x, y) = \sum_{x \leq z \leq y} f(x, z) g(z, y),$$

for all $x \leq y$ in Q, where we write $f(x, y)$ as shorthand for the more cumbersome $f([x, y])$ (see [7]). The identity element e of $I(Q)$ is defined by $e(x, y) = \delta_{x,y}$, the Kronecker delta, for all $x \leq y$ in Q. A function $f \in I(Q)$ has a convolution inverse if and only if $f(x, x) \neq 0$ for all $x \in Q$, in which case

$$(3) \qquad f^{-1}(x, y) = \sum_{k \geq 0} \sum_{x = x_0 < \cdots < x_k = y} (-1)^k \frac{f(x_0, x_1) f(x_1, x_2) \cdots f(x_{k-1}, x_k)}{f(x_0, x_0) f(x_1, x_1) \cdots f(x_k, x_k)}.$$

This formula, which is not difficult to verify, follows from the general formula for the antipode of an incidence Hopf algebra given in [8].

The inverse of f is also given recursively by $f^{-1}(x, x) = 1/f(x, x)$, for all $x \in Q$, and

$$(4) \qquad f^{-1}(x, y) = \frac{1}{f(y, y)} \sum_{x \leq z < y} f^{-1}(x, z) f(z, y),$$

for all $x < y$ in Q.

We now have the following structure theorem for $C(\mathcal{P})'$.

Theorem 1. *The linear map from $C(\mathcal{P})'$ onto the polynomial algebra $K[\mathcal{P}_\circ]$ defined by $\delta_B \to [B]$, for all $[B] \in \mathcal{P}_\circ$, is an algebra isomorphism.*

Proof. Define a linear map $\varphi : C(\mathcal{P})' \to C(\mathcal{P})'$ by

$$\varphi(\delta_H) = \prod_{B \in \beta(H)} \delta_B,$$

for all $[H] \in \mathcal{P}$. It follows from equation 2 that

$$\varphi(\delta_H) = \sum_{[G] \in \mathcal{P}} c(G,H)\delta_G,$$

for all $[H] \in \mathcal{P}$.

Let $I(\mathcal{P})$ denote the incidence algebra of $\mathcal{P}$ over $C(\mathcal{P})'$. For simplicity of notation, we identify the rational number s with $s \cdot \epsilon \in C(\mathcal{P})'$, where ϵ is the unit of $C(\mathcal{P})'$, and for any $f \in I(\mathcal{P})$ and $[G] \leq [H]$ in $\mathcal{P}$, we write $f(G,H)$ instead of $f([G],[H])$.

Define $d \in I(\mathcal{P})$ by $d(G,H) = \delta_H$, for all $[G] \leq [H]$ in $\mathcal{P}$. Thus, using the convolution in $I(\mathcal{P})$, we have

$$\varphi(\delta_H) = d * c(\emptyset, H),$$

for all $[H] \in \mathcal{P}$. Therefore φ is bijective, with inverse given by

$$\varphi^{-1}(\delta_H) = d * c^{-1}(\emptyset, H),$$

for all $[H] \in \mathcal{P}$, where c^{-1} is the inverse of the function $c \in I(\mathcal{P})$, which exists because $c(H,H) \neq 0$, for all $[H] \in \mathcal{P}$. The theorem follows immediately from the fact that φ is bijective and the set $\{\delta_H : [H] \in \mathcal{P}\}$ is a basis for $C(\mathcal{P})'$. $\square$

Using equation 3 to compute c^{-1}, one can find the explicit expression for any element of $C(\mathcal{P})'$ as a polynomial in the functions δ_B, for B connected. For, if $[H] \in \mathcal{P}$ then

$$\varphi^{-1}(\delta_H) = \sum_{[G] \leq [H]} c^{-1}(G,H)\delta_G,$$

and by applying the map φ, we obtain

$$\delta_H = \sum_{[G] \leq [H]} c^{-1}(G,H) \prod_{B \in \beta(G)} \delta_B. \tag{5}$$

The following recursive formula for δ_H can be deduced by using equation 4 to determine c^{-1}:

$$\delta_H = \frac{1}{c(H,H)} \left[\prod_{B \in \beta(H)} \delta_B - \sum_{[G] < [H]} c(G,H)\delta_G \right]. \tag{6}$$

The dual Hopf algebra of $\widehat{C(\mathcal{P})}$, denoted by $\widehat{C(\mathcal{P})}^{\circ}$, is the subalgebra of all elements of $\widehat{C(\mathcal{P})}^{*}$ which vanish on a cofinite ideal of $\widehat{C(\mathcal{P})}$, with coproduct Ψ given by restriction of the transpose of the multiplication $\mu : \widehat{C(\mathcal{P})} \otimes \widehat{C(\mathcal{P})} \to \widehat{C(\mathcal{P})}$ (see [9]). The algebra $C(\mathcal{P})'$ is contained in $\widehat{C(\mathcal{P})}^{\circ}$ because, for each $[H] \in \mathcal{P}$, δ_H vanishes on the ideal generated by the set $\{[B] : [B] \in \mathcal{P}_\circ, B \notin \beta(H)\}$, which is cofinite. Thus $C(\mathcal{P})'$ is a Hopf algebra, with coproduct determined by

$$\begin{aligned} \langle \Psi(\delta_H) \, , \, [G_1] \otimes [G_2] \rangle &= \langle \delta_H \, , \, [G_1][G_2] \rangle \\ &= \sum_{[H_1][H_2]=[H]} \langle \delta_{H_1} \, , \, [G_1] \rangle \langle \delta_{H_2} \, , \, [G_2] \rangle. \end{aligned}$$

Therefore

$$\Psi(\delta_H) = \sum_{[H_1][H_2]=[H]} \delta_{H_1} \otimes \delta_{H_2},$$

for all $[H] \in \mathcal{P}$. In particular, $\Psi(\delta_B) = \delta_B \otimes \epsilon + \epsilon \otimes \delta_B$, that is, δ_B is primitive, whenever $[B] \in \mathcal{P}_\circ$. It follows that the antipode S' of $C(\mathcal{P})'$ is determined by $S'(\delta_B) = -\delta_B$, for all $[B] \in \mathcal{P}_\circ$.

Hence $C(\mathcal{P})'$ is isomorphic to the polynomial Hopf algebra $K[\mathcal{P}_\circ]$, where the indeterminates are primitive. Later we will need to use the well-known fact that the primitive elements of such a Hopf algebra are precisely the linear homogeneous polynomials.

4. Restriction Invariants

As above, let $\mathcal{P}$ be a set of types of Whitney systems which is closed under sums and restrictions. We now introduce another bialgebra structure on the monoid algebra of $\mathcal{P}$. Let $M(\mathcal{P})$ denote the monoid algebra of $\mathcal{P}$ over K, together with linear maps $\delta : M(\mathcal{P}) \to M(\mathcal{P}) \otimes M(\mathcal{P})$ and $\alpha : M(\mathcal{P}) \to K$ given by

$$\delta[H] = [H] \otimes [H]$$

and

$$\alpha[H] = 1,$$

for all $[H] \in \mathcal{P}$. $M(\mathcal{P})$ is thus a bialgebra, with coproduct δ and counit α.

The full dual algebra $M(\mathcal{P})^*$ is called the **algebra of (K-valued) invariants** on $\mathcal{P}$. It can be identified in the obvious manner with the algebra of all functions from $\mathcal{P}$ to K, under pointwise sum, product, and scalar multiplication. For any $[H] \in \mathcal{P}$, the invariant $n_H \in M(\mathcal{P})^*$ is defined by be letting $n_H[G]$ be the cardinality of the set $\{U \subseteq S(G) \, : \, [G|U] = [H]\}$, for all $[G] \in \mathcal{P}$. The subalgebra of $M(\mathcal{P})^*$ generated by the set $\{n_H : [H] \in \mathcal{P}\}$ is denoted by $M(\mathcal{P})'$ and called the algebra of **restriction invariants** on $\mathcal{P}$.

Proposition 2. *The linear map $J : M(\mathcal{P}) \to C(\mathcal{P})$ defined by*

$$J[H] = \sum_{U \subseteq S(H)} [H|U],$$

for all $[H] \in \mathcal{P}$, is a bialgebra isomorphism.

Proof. For all $[H] \in \mathcal{P}$, we have by definition,

$$\begin{aligned} \Delta \circ J[H] &= \sum_{U \subseteq S(H)} \sum_{U_1 \cup U_2 = U} [H|U_1] \otimes [H|U_2] \\ &= \sum_{U_1, U_2 \subseteq S(H)} [H|U_1] \otimes [H|U_2] \\ &= J[H] \otimes J[H] \\ &= (J \otimes J) \circ \delta[H]. \end{aligned}$$

Also,

$$\begin{aligned} \epsilon \circ J[H] &= \sum_{U \subseteq S(H)} \epsilon[H|U] \\ &= 1 \\ &= \alpha[H]. \end{aligned}$$

Hence J is a coalgebra map. J is also an algebra map, because

$$\begin{aligned} J([H_1][H_2]) &= \sum_{U \subseteq S(H_1 + H_2)} [(H_1 + H_2)|U] \\ &= \sum_{U_1 \subseteq S(H_1)} \sum_{U_2 \subseteq S(H_2)} [H_1|U_1][H_2|U_2] \\ &= J[H_1]J[H_2], \end{aligned}$$

for all $[H_1], [H_2] \in \mathcal{P}$. Finally, note that J has inverse given by

$$J^{-1}[H] = \sum_{U \subseteq S(H)} (-1)^{|S(H) \setminus U|} [H|U].$$

Therefore J is an isomorphism. $\square$

The composition $M(\mathcal{P}) \xrightarrow{J} C(\mathcal{P}) \hookrightarrow \widehat{C(\mathcal{P})}$ is an injective bialgebra map. Let J' denote the restriction to $C(\mathcal{P})'$ of the transpose of this map.

Corollary 1. *$J' : C(\mathcal{P})' \to M(\mathcal{P})'$ is an algebra isomorphism, which maps δ_H to n_H, for all $[H]$ in $\mathcal{P}$.*

Proof. We know J' is an algebra map because J is a coalgebra map, and for all $[G], [H] \in \mathcal{P}$, we have

$$\begin{aligned} \langle J'(\delta_H)\, ,\, [G] \rangle &= \langle \delta_H\, ,\, J[G] \rangle \\ &= \sum_{U \subseteq S(G)} \langle \delta_H\, ,\, [G|U] \rangle \\ &= \langle n_H\, ,\, [G] \rangle. \end{aligned}$$

$\square$

We now have the following structure theorem for $M(\mathcal{P})'$.

Theorem 2. *The linear map from $M(\mathcal{P})'$ onto the polynomial algebra $K[\mathcal{P}_\circ]$ defined by $n_B \to [B]$, for all $[B] \in \mathcal{P}_\circ$, is an algebra isomorphism.*

Proof. The result is immediate from theorem 1 and corollary 1. □

By corollary 1 and equation 5 one obtains the following formula for the invariant n_H as a polynomial in the functions n_B, for B connected.

$$n_H = \sum_{[G]\leq[H]} c^{-1}(G,H) \prod_{B\in\beta(G)} n_B. \tag{7}$$

By equation 6, we have

$$n_H = \frac{1}{c(H,H)} \left[\prod_{B\in\beta(H)} n_B - \sum_{[G]<[H]} c(G,H) n_G \right].$$

5. Hopf Algebras and Additive Invariants

For all $n \geq 1$, let $\hat{I}_n$ denote the image in $M(\mathcal{P})$ of the ideal $I_n \subseteq C(\mathcal{P})$, under the inverse isomorphism J^{-1}. The set of ideals $\{\hat{I}_n \ : \ n \geq 1\}$ forms a local base at 0 for a topology on $M(\mathcal{P})$, with respect to which the mapping J is a homeomorphism. Let $\widehat{M(\mathcal{P})}$ denote the completion of $M(\mathcal{P})$ (thus $\widehat{M(\mathcal{P})} \simeq K[[\hat{\mathcal{P}}_\circ]]$ as algebras, where $\hat{\mathcal{P}}_\circ = \{J^{-1}[B] \ : \ [B] \in \mathcal{P}_\circ\}$). The composition $M(\mathcal{P}) \xrightarrow{J} C(\mathcal{P}) \hookrightarrow \widehat{C(\mathcal{P})}$ extends uniquely to a continuous Hopf algebra isomorphism $\hat{J} : \widehat{M(\mathcal{P})} \to \widehat{C(\mathcal{P})}$.

The continuous dual of $\widehat{M(\mathcal{P})}$ can be identified with the algebra of restriction invariants $M(\mathcal{P})'$. Therefore $M(\mathcal{P})'$ is a Hopf algebra, and the restricted transpose map $J' : C(\mathcal{P})' \to M(\mathcal{P})'$ is a Hopf algebra isomorphism. It follows that the coproduct ψ of $M(\mathcal{P})'$ is given by

$$\psi(n_H) = \sum_{[H_1][H_2]=[H]} n_{H_1} \otimes n_{H_2}, \tag{8}$$

for all $[H] \in \mathcal{P}$.

Let $\hat{I}$ denote the ideal $\hat{I}_1 \cdot \widehat{M(\mathcal{P})} = \ker \alpha$ of $\widehat{M(\mathcal{P})}$. The function $\log : 1 + \hat{I} \to \hat{I}$ defined by

$$\log(x) = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} (x-1)^n$$

is a bijection with inverse $\exp : \hat{I} \to 1 + \hat{I}$ given by

$$\exp(x) = \sum_{n\geq 0} \frac{x^n}{n!}.$$

For $[H] \in \mathcal{P}$, define $f_H \in M(\mathcal{P})'$ by $\langle f_H \, , \, 0\rangle = 0$ and $\langle f_H \, , \, [G]\rangle = \langle n_H \, , \, \log[G]\rangle$, for all $[G] \in \mathcal{P}$. Note that $\log[G]$ converges, because $[G] - 1$ is in the kernel of α, and thus $[G] \in 1 + \hat{I}$.

Proposition 3. *For all* $[H] \in \mathcal{P}$,

$$f_H = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{\substack{[H_1]\cdots[H_n]=[H] \\ [H_i]\neq 1}} \prod_{i=1}^{n} n_{H_i}.$$

Proof. For any $[G] \in \mathcal{P}$, we have

$$\begin{aligned} \langle f_H \ , \ [G]\rangle &= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \langle n_H \ , \ ([G]-1)^n\rangle \\ &= \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{[H_1]\cdots[H_n]=[H]} \prod_{i=1}^{n} \langle n_{H_i} \ , \ [G]-1\rangle, \end{aligned}$$

from which the result follows. □

In particular, $f_B = n_B$, for all $[B] \in \mathcal{P}_\circ$. The inverse correspondence is given by

$$n_H = \sum_{n\geq 0} \frac{1}{n!} \sum_{[H_1]\cdots[H_n]=[H]} \prod_{i=1}^{n} f_{H_i}, \tag{9}$$

which is a finite sum, because $f_1 = 0$.

Proposition 4. $\psi(f_H) = f_H \otimes \alpha + \alpha \otimes f_H$, *for all* $[H] \in \mathcal{P}$.

Proof. By definition, for all $[G_1], [G_2] \in \mathcal{P}$,

$$\begin{aligned} \langle \psi(f_H) \ , \ [G_1] \otimes [G_2]\rangle &= \langle f_H \ , \ [G_1][G_2]\rangle \\ &= \langle n_H \ , \ \log([G_1][G_2])\rangle \\ &= \langle n_H \ , \ \log[G_1] + \log[G_2]\rangle \\ &= \langle f_H \ , \ [G_1]\rangle + \langle f_H \ , \ [G_2]\rangle \\ &= \langle f_H \otimes \alpha + \alpha \otimes f_H \ , \ [G_1] \otimes [G_2]\rangle. \end{aligned}$$

□

Thus f_H is an **additive** invariant, for all $[H] \in \mathcal{P}$, that is, $\langle f_H \ , \ [G_1 + G_2]\rangle = \langle f_H \ , \ [G_1]\rangle + \langle f_H \ , \ [G_2]\rangle$, for all $[G_1], [G_2] \in \mathcal{P}$.

The next result shows that the invariant f_H is equal to the sum of the linear terms in the expression for n_H as a polynomial in the n_B's.

Proposition 5. *For all* $[H] \in \mathcal{P}$,

$$f_H = \sum_{[B]\in\mathcal{P}_\circ} c^{-1}(B, H) n_B.$$

Proof. We know that

$$n_H = \sum_{[G]\leq[H]} c^{-1}(G, H) \prod_{B\in\beta(G)} n_B,$$

for all $[H] \in \mathcal{P}$. Thus by proposition 3, we have

$$f_H = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{\substack{[H_1]\cdots[H_n]=[H]\\ [H_i]\neq 1}} \prod_{i=1}^{n} \left[\sum_{[G]\leq[H_i]} c^{-1}(G, H_i) \prod_{B\in\beta(G)} n_B \right].$$

It follows from proposition 4 and the fact that the primitive elements of $M(\mathcal{P})'$ are precisely the linear homogeneous polynomials in the n_B's , that all non-linear terms in the above expression cancel. All linear terms occur when $n = 1$, thus f_H is equal to the sum of the linear terms of

$$\sum_{[G]\leq[H]} c^{-1}(G, H) \prod_{B\in\beta(G)} n_B,$$

from which the result follows. ☐

Corollary 2. *For all $[G], [H] \in \mathcal{P}$,*

$$\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{\substack{[H_1]\cdots[H_n]=[H]\\ [H_i]\neq 1}} \binom{G}{H_1, \ldots, H_n} = \begin{cases} c^{-1}(G, H) & \textit{if } G \in \mathcal{P}_\circ \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. By proposition 3, we have

$$f_H = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{\substack{[H_1]\cdots[H_n]=[H]\\ [H_i]\neq 1}} \prod_{i=1}^{n} n_{H_i}.$$

Expanding the product yields

$$f_H = \sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{\substack{[H_1]\cdots[H_n]=[H]\\ [H_i]\neq 1}} \sum_{[G]\in\mathcal{P}} \binom{G}{H_1, \ldots, H_n} n_G,$$

which is equal to

$$\sum_{[G]\in\mathcal{P}} \left[\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} \sum_{\substack{[H_1]\cdots[H_n]=[H]\\ [H_i]\neq 1}} \binom{G}{H_1, \ldots, H_n} \right] n_G.$$

The result follows by comparing the above with the expression for f_H given in proposition 5. ☐

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6. An Application

Suppose G is a graph with vertex set V and edge set E. Let $p(G)$ denote the number of connected components of G. The **coboundary rank** $\rho(G)$ and **cycle rank** $r(G)$ are given by $\rho(G) = |V| - p(G)$ and $r(G) = |E| - |V| + p(G)$, respectively. Graphs $G = G(V, E)$ and $H = H(W, F)$ have the same **cycle type** whenever there is a bijection $f : E \to F$ such that $C \subseteq E$ is a cycle of G if and only if $f(C)$ is a cycle of H. Hence the cycle type of a graph G is determined by the cycle matroid of G. Note that if G and H have the same cycle type, then $\rho(G) = \rho(H)$ and $r(G) = r(H)$, while $p(G)$ and $p(H)$ may differ.

Let $\mathcal{G}$ be the set of all types of Whitney systems of the form G_d, where G is a graph (see example 1 of section 2). Since $[G_d] = [H_d]$ if and only if G and H have the same cycle type, it follows that the set $\mathcal{G}$ is in one-to-one correspondence with the set of all cycle types of graphs, and the set $\mathcal{G}_\circ$ corresponds to the set of cycle types of doubly connected graphs.

$M(\mathcal{G})'$ is therefore the algebra of (cycle type) invariants of graphs. For all $i, j \geq 0$, the invariant $m_{ij} \in M(\mathcal{G})'$ is defined by

$$m_{ij} = \sum n_H,$$

where the sum is over all types $[H]$ having coboundary rank i and cycle rank j. Whitney showed in [12] that the chromatic polynomial of a graph $G = (V, E)$ is given by

$$\chi(G, \lambda) = \sum_{i,j \geq 0} (-1)^{i+j} m_{ij}(G) \lambda^{|V|-i}.$$

The main results in Whitney's paper [13] state that the invariants m_{ij} can be expressed as polynomials with rational coefficients in the invariants n_B, for B doubly connected, and that the invariants n_B are algebraically independent over $\mathbb{Q}$. These results are implied by theorem 2, in the special case that $\mathcal{P} = \mathcal{G}$.

The "multiplicative" property of the m_{ij}

$$m_{ij}[G + H] = \sum_{\substack{i_1+i_2=i \\ j_1+j_2=j}} m_{i_1 j_1}[G] m_{i_2 j_2}[H]$$

follows immediately from formula 8 for the coproduct of $M(\mathcal{G})'$.

The additive invariants f_{ij} studied in the second half of Whitney's paper [12] can be expressed in terms of the functions f_H of the previous section as $f_{ij} = \sum f_H$, where the sum is over all block-types $[H]$ having coboundary rank i and cycle rank j. The additive property, $f_{ij}[G + H] = f_{ij}[G] + f_{ij}[H]$, follows from proposition 4, and proposition 3 implies that

$$f_{ij} = \sum_{k \geq 1} \sum_{\substack{i_1+\cdots+i_k=i \\ j_1+\cdots+j_k=j \\ i_r, j_r \neq 0}} \prod_{r=1}^{k} m_{i_r j_r},$$

which is the formula for the f_{ij} given by Whitney. His characterization of f_{ij} as the linear terms of the polynomial expansion of m_{ij} in terms of the n_B's follows from proposition 5.

7. An Example

In this section, we consider the simplest non-trivial example of a set of types of Whitney systems closed under sums and restrictions. In this case the only connected type, denoted by x, is that of a one-point Whitney system (either a loop or non-loop). The set of types $\mathcal{P}$ therefore consists of all non-negative integral powers of x, and is linearly ordered by degree. The Hopf algebra $\widehat{C(\mathcal{P})}$ is isomorphic to the power series algebra $K[[x]]$, with coproduct given by $\Delta(x) = x \otimes 1 + x \otimes x + 1 \otimes x$. The antipode of $\widehat{C(\mathcal{P})}$ is determined, according to equation 1, by

$$S(x) = \sum_{n \geq 1} (-x)^n = \frac{-x}{1+x}.$$

The reader familiar with formal groups will recognize $\widehat{C(\mathcal{P})}$ as the contravariant bialgebra of the multiplicative formal group law (see [6]).

For $k \geq 0$, let $n_k = n_H$, where H is a Whitney system of type x^k. Theorem 1 states that the algebra of restriction invariants $M(\mathcal{P})'$ is the polynomial algebra $K[n_1]$. The expression for n_k as a polynomial in n_1 is easy to obtain; since $\langle n_1 \, , \, x^n \rangle = n$ and, in general, $\langle n_k \, , \, x^n \rangle = \binom{n}{k}$, it follows that $n_k = \binom{n_1}{k}$, for all $k \geq 0$.

For all n and k, the number of coverings $c(x^k, x^n)$ is equal to the number of surjections from an n-element set onto a k-element set, which is given by $k!S(n,k)$, where the $S(n,k)$ denote Stirling numbers of the second kind. It follows that $c^{-1}(x^k, x^n) = s(n,k)/k!$, where the $s(n,k)$ denote Stirling numbers of the first kind. In the present example, the general formula [7] states that

$$n_k = \sum_{r \leq k} c^{-1}(x^r, x^k) n_1^r, \tag{10}$$

which is equivalent to the classical expression for falling factorials in terms of powers.

Proposition 5 implies that $f_k = c^{-1}(x_1, x_k) n_1$, which is equal to $(-1)^{k-1} n_1 / k$, for all $k \geq 1$. Therefore, using equation 9, we obtain

$$\begin{aligned} n_k &= \sum_{r \geq 0} \frac{1}{r!} \sum_{\substack{k_1 + \cdots + k_r = k \\ k_i \neq 0}} \prod_{i=1}^{r} \frac{(-1)^{k_i - 1}}{k_i} n_1 \\ &= \sum_{r \geq 0} \left(\frac{(-1)^{k-r}}{r!} \sum_{\substack{k_1 + \cdots + k_r = k \\ k_i \neq 0}} \prod_{i=1}^{r} \frac{1}{k_i} \right) n_1^r. \end{aligned}$$

Comparing this with the expression for n_k given in equation 10, we obtain the following identity for the Stirling numbers of the first kind.

$$s(k,r) = \frac{(-1)^{k-r} k!}{r!} \sum_{\substack{k_1 + \cdots + k_r = k \\ k_i \neq 0}} \prod_{i=1}^{r} \frac{1}{k_i}.$$

References

[1] N. Biggs, *On the Algebra of Graph Types*, Graph Theory and Related Topics (Academic Press, New York, 1979), 81-89.

[2] N. Biggs, *On Cluster Expansions in Graph Theory and Physics*, Quarterly Journal of Mathematics, Oxford (2), 29 (1978), 159-173.

[3] N. Biggs, *Interaction Models*, London Mathematical Society Lecture Note Series 30, Cambridge University Press, Cambridge, 1977.

[4] G. D. Birkhoff, *A Determinant Formula for the Number of Ways of Coloring a Map*, Annals of Mathematics 14 (1912), 42-46.

[5] P. Erdös, L. Lovász, and J. Spencer, *Strong Independence of Graphcopy Functions*, Graph Theory and Related Topics (Academic Press, New York, 1979), 165-172.

[6] M. Hazewinkel, *Formal Groups and Applications*, Academic Press, New York, 1978.

[7] G. C. Rota, *On the Foundations of Combinatorial Theory I, Theory of Mobius Functions*, Z. Wahrscheinlichkeitstheorie 2 (1964), 340-368.

[8] W. Schmitt, *Antipodes and Incidence Coalgebras*, Journal of Combinatorial Theory A 46 (1987), 264-290.

[9] M. Sweedler, *Hopf Algebras* (Benjamin, New York, 1969).

[10] W. T. Tutte, *On Dichromatic Polynomials*, Journal of Combinatorial Theory 2 (1967) 301-320.

[11] W. T. Tutte, *All the Kings Horses (A Guide to Reconstruction)*, Graph Theory and Related Topics (Academic Press, New York, 1979), 15-33.

[12] H. Whitney, *A Logical Expansion in Mathematics*, Bulletin of the American Mathematical Society 38 (1932), 572-579.

[13] H. Whitney, *The Coloring of Graphs*, Annals of Mathematics 33 (1932), 688-718.