

# The George Washington University Combinatorics Seminar

Wednesday, October 8, 2008, 4:45 - 5:45 p.m.

Monroe Hall, Room 267  
2115 G Street, N.W., Washington, D.C.

## Questions (and a few answers) in additive number theory and combinatorial geometry

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If you are like me, you have been wondering how to best approximate the entire nine-dimensional unit sphere by just twenty-five points. It turns out that a good starting point is to find five elements in the cyclic group of order twenty-five which are least dependent on one another. Let me explain.

A finite set  $X$  of points on the sphere  $S^d$  is a *spherical  $t$ -design*, if for every polynomial  $f$  of total degree  $t$  or less, the average value of  $f$  over  $S^d$  is equal to the arithmetic average of its values on  $X$ .

A finite subset  $A$  of an abelian group  $G$  is  *$t$ -independent* in  $G$ , if no multi-subset of  $A$  with at most  $t$  elements can be divided into two disjoint parts so that the two parts have the same (multi-subset) sum.

Spherical 2-designs and 2-independent sets are not difficult to find; for example, the vertices of a regular simplex form a 2-design of size  $d + 2$  on  $S^d$ , and taking exactly one of each element or its negative (when different) yields a 2-independent set of size  $\lfloor (n - 1)/2 \rfloor$  in  $\mathbb{Z}_n$ . (These cardinalities are minimal and maximal, respectively.)

Much less is known for  $t \geq 3$ , but we can answer the questions in our first paragraph, as follows. Consider  $A = \{1, 4, 6, 9, 11\}$  in  $G = \mathbb{Z}_{25}$ . We find that  $A$  is 3-independent in  $G$  (of maximum size; also, there is no set of size five which is 4-independent in  $G$ ). Now take

$$X = \left\{ \frac{1}{\sqrt{5}} (z^j(1), z^j(4), z^j(6), z^j(9), z^j(11)) : 1 \leq j \leq 25 \right\}$$

on  $S^9$  where

$$z(a) = \cos\left(\frac{2\pi}{25}a\right) + i \cdot \sin\left(\frac{2\pi}{25}a\right).$$

Then  $X$  is a 3-design on  $S^9$  (and there is no set of size twenty-five which is a 4-design on  $S^9$ ).

In this talk we discuss these and other concepts in additive number theory and geometric combinatorics from a new, unified viewpoint, and feature some of the famous and less well known conjectures along the way.