Simulation Based Designs for Accelerated Life Tests

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Abstract

In this paper we present a Bayesian decision theoretic approach to the design of accelerated life tests (ALT). We discuss computational issues regarding the evaluation of expectation and optimization steps in the solution of the decision problem. We illustrate how Monte Carlo methods can be used in preposterior analysis to find optimal designs and how the required computational effort can be avoided by using curve fitting techniques. In so doing, we adopt the recent Monte Carlo-based approaches of Muller and Parmigiani (1995) and Muller (1999) to develop optimal Bayesian designs. These approaches facilitate the preposterior analysis by replacing it with a sequence of scatter plot smoothing/regression techniques and optimization of the corresponding fitted surfaces. We present our development by considering single and multiple-point fixed, as well as, sequential design problems when the underlying life model is exponential, and illustrate the implementation of our approach with some examples.

Key Words: Accelerated life tests, Bayesian designs, Gibbs sampling, Preposterior analysis.

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1 Introduction and Overview

In accelerated life tests (ALT's), components/systems are subjected to an environment that is more severe than the use environment in order to induce early failures. The accelerated environment is created by increasing the levels of one or more of the stress variables that constitute the environment. The data collected in the accelerated environment are then used for inference about the failure characteristics of the items tested at the use environment. An important assumption which facilitates the inference is the form of the time transformation function which describes the relationship between the failure characteristic of interest and the applied stress level. As a result of using time transformation functions, Bayesian inferences from ALT models become intractable. One way to alleviate this problem is to use the Markov Chain Monte Carlo methods as discussed in Erkanli and Soyer (1997).

Most of the Bayesian literature in ALT's have focused on developing inference methods; see for example Mazzuchi and Soyer (1992), van Dorp et al. (1996), and Mazzuchi, Soyer and Vopatek (1997). The majority of the work published on design of ALT's relied on sample theoretic methods, see for example, Nelson (1990). Exceptions to these are the recent Bayesian approaches by Chaloner and Larntz (1992), Verdinelli, Polson and Singpurwalla (1993) and Soyer and Vopatek (1995). Most of these Bayesian approaches are based on the theory of optimal Bayesian designs for linear models, see for example, Chaloner (1984). Thus, the results are applicable to ALT designs when the life model is normal or lognormal. When the underlying life models are exponential or Weibull, nonlinearities arise and approximate designs can be obtained by use of special techniques such as linear Bayesian methods, as dicussed, for example, in Soyer and Vopatek (1995).

The design problem in accelerated life testing is concerned with specification of the number and magnitude of the accelerated stress levels, and the number of items to be tested at these levels. From a Bayesian perspective the design of an ALT is a decision problem. In what follows, we will present the decision theoretic setup for the design problem and discuss its solution. In

Section 2 we present the optimal design of an ALT as a decision problem by considering utility (loss) functions that reflect concerns on quality of inference from the ALT. We discuss the general setup and the solution of the design problem. Extensions to other utility (loss) functions, such as those reflecting economic concerns, are straigthforward. In Section 3, we consider a specific ALT model with an exponential life distribution and a power law time transformation function and present inference using Markov Chain Monte Carlo methods. In Section 4, we illustrate how the Markov Chain Monte Carlo methods can be used for evaluating the posterior expected loss and discuss the potential computational burden involved in finding the optimal design. To alleviate such computational burden, we adopt the curve fitting approach of Muller and Parmigiani (1995) to design of ALT experiments. This approach facilitates the preposterior analysis by replacing it with a sequence of scatter plot smoothing/regression techniques and optimization of the corresponding fitted curves/surfaces. Extension of our approach to sequential ALT designs is discussed in Section 5. In Section 6, we present examples to illustrate the implementation of our approach.

2 Decision Theoretic Setup for the Design Problem

The Bayesian approach to the optimal design problem requires specification of three components:

- a loss (utility) function that reflects the consequences of selecting a specific design;
- a statistical model;
- a prior distribution reflecting designer's a priori beliefs about all unknown quantities.

To introduce some notation, let d denote a specific design (the decision variable), λ_u represent the failure characteristic of interest (such as the use environment failure rate), and a denote an action based on data (for example, a prediction for λ_u). We denote the loss function as $L(\lambda_u, d, a)$, and the statistical model as $p(D|\lambda_u, d)$, respectively, where D is the data observed from experiment d. Finally, we describe prior beliefs about λ_u by the prior distribution $p(\lambda_u)$. It is possible that λ_u is a vector or a function of other unknown parameters.

Once the design d is specified and the ALT is performed, data D is observed and uncertainty about λ_u is revised according to the laws of probability, that is,

$$p(\lambda_u|D,d) \propto p(D|\lambda_u,d)p(\lambda_u) \tag{1}$$

Then, the optimal action a is chosen based on the data. This process can be represented by the decision tree in Figure 1 where the squares denote the decision nodes and the circles represent the chance (random) nodes. In Figure 1, the decision node D_1 represents the choice of the design (for example, the accelerated stress environment). Random node R_1 represents the outcome of the experiment, that is, the observed data. The selection of the Bayes rule a given data D is represented by decision node D_2. Finally, the random node R_2 represents the unknown failure characteristic λ_u and $L(\lambda_u,d,a)$ is the associated loss that reflects the quality of inference about λ_u. For example, when $L(\lambda_u,d,a) = (\lambda_u - a)^2$, the optimal Bayes rule a^* is the posterior mean and the last two nodes can be replaced by $E_{\lambda_u|D,d}L(\lambda_u,d,a^*) = V(\lambda_u|D,d)$, the posterior variance of λ_u.

Put Figure 1 here

The solution of the design problem is obtained in the conventional manner by "folding back" the decision tree through taking expectations at random nodes and minimizing the expected loss at the decision nodes of Figure 1. The following steps describe this procedure.

At R_2:

Evaluate $E_{\lambda_u|D,d}[L(\lambda_u,d,a)|D,d]$.

At D_2:

Compute $a^* = \arg\min\{E_{\lambda_u|D,d}[L(\lambda_u,d,a)|D,d]\}$ and evaluate $L^*(D,d) = E_{\lambda_u|D,d}[L(\lambda_u,d,a^*)|D,d]$.

At R_1:

Evaluate $L^*(d) = E_{D|d}[L^*(D,d)]$. We note that the expectation is taken with respect to the (prior) predictive distribution $p(D|d)$, where

$$p(D|d) = \int p(D|\lambda_u, d)p(\lambda_u)d\lambda_u. \tag{2}$$

At D_1:

Compute $d^* = \arg\min\{L^*(d)\}$, the optimal d.

In other words, the optimal design is obtained as

$$d^* = \arg\min_d\{E_{D|d}E_{\lambda_u|D,d}[L(\lambda_u, d, a^*)|D, d]\}, \tag{3}$$

where $a^* = \arg\min_a\{E_{\lambda_u|D,d}[L(\lambda_u, d, a)|D, d]\}$.

The above setup can be adopted for any form of the loss function $L(\lambda_u, d, a)$ (See, for example, Verdinelli et al., 1993). Assume that the decision at D_2 is to choose a point estimator for λ_u after observing the data. Furthermore, assume that quality of the inference about λ_u is the only concern and specify a quadratic utility function $L(\lambda_u, d, a) = (\lambda_u - a)^2$. Then at R_2 we have $E_{\lambda_u|D,d}[(\lambda_u - a)^2]$, and at D_2 the optimal decision is $a^* = E[\lambda_u|D, d]$. Now the solution to the design problem is given by the design that minimizes the preposterior variance. Of course, it is possible to include a cost component, $C(d)$ in $L(\lambda_u, d, a)$ to reflect concerns such as cost of testing at a given environment, but this will not be considered in our development.

Except in few special cases, the optimal design d^* can not be obtained analytically. In ALT models where the underlying failure distribution is exponential or Weibull, posterior and/or predictive inferences are not available in closed analytical form (Erkanli and Soyer, 1997). Thus, evaluation of the design in (3) requires either use of approximation techniques, as in Soyer and Vopatek (1995), or Monte Carlo methods. The linear Bayesian designs of Soyer and Vopatek (1995), however, are limited to few models and the straight Monte Carlo approach is not computationally efficient. An alternate approach is to use Monte Carlo simulations and smoothing techniques of Muller and Parmigiani (1995) where the preposterior analysis is replaced by a curve fitting approach. This approach has been used by Erkanli, Soyer, and Angold (1998) in developing two-phase designs for prevalence estimation and will be dicussed in detail in Sections 4 and 5 in the context of ALT designs. Other simulation based designs and computational strategies are presented in Muller (1999).

3 The Exponential Life Model and ALT Designs

In this section, we consider single and multiple-point fixed design problems when the life model is exponential, and discuss how the decision theoretic approach can be used in determining the optimal allocation of test items into different stress levels. The proposed setup can be easily adopted to other lifetime distributions. To establish notation, let λ_i denote the failure rate at the accelerated stress environment S_i. We assume that life length X_i is exponential with failure rate λ_i, denoted as $[X_i|\lambda_i] \sim \text{Exp}(\lambda_i)$, and the time transformation function is given by the power law model

$$\lambda_i = \alpha S_i^{\beta}, \tag{4}$$

implying that $[X_i|\alpha, \beta, S_i] \sim \text{Exp}(\alpha S_i^{\beta})$. Let m denote the number of distinct stress levels used for the ALT and n_i denote the number of items tested at the stress level S_i and $n = \sum_{i=1}^{m} n_i$ is the predetermined number of items to be used in the ALT. We are interested in making inference about $\lambda_u = \alpha S_u^{\beta}$, the failure rate at the use stress environment. Furthermore, we assume that there is no censoring in the ALT, however, censoring can be easily accomodated using appropriate modifications of the likelihood function in (6) for the kinds of censoring involved. For fixed value of n, the design problem is to select

- $m \leq n$, the number of distinct stress levels,
- accelerated stress levels $S_i,\ i = 1, \ldots, m$,

- number of items tested at each stress level, $n_i,\ i = 1, \ldots, m$,

in such a way that expected loss is minimized. Here, a specific design is given by $d = \{m,\ S_i,\ n_i,\ i = 1, \ldots, m\}$. We assume that the loss function is given by $L(\lambda_u, d, a) = (\lambda_u - a)^2$ and thus the optimal design is given by

$$d^* = \arg\min_d \{E_{D|d}[V(\lambda_u|D, d)]. \tag{5}$$

We specify independent priors on α and β as $\alpha \sim \text{Gamma}(a, b)$ and $\beta \sim p(\beta)$, respectively. We note that the evaluation of the optimal design requires the computation of posterior variance $V(\lambda_u|D, d)$, where $D = \{T_1, \ldots, T_m\}$ and $T_i = \sum_{j=1}^{n_i} x_j$ is the total time on test at stress environment S_i.

Given failure data D, the likelihood function of (α, β) is given by

$$\text{lik}(\alpha, \beta) = \prod_{i=1}^{m} (\alpha S_i^{\beta})^{n_i} \exp(-\alpha S_i^{\beta} T_i) = (\alpha^n \prod_{i=1}^{m} S_i^{\beta n_i}) \exp(-\alpha \sum_{i=1}^{m} S_i^{\beta} T_i), \tag{6}$$

and the joint posterior density of (α, β) is

$$p(\alpha, \beta|D) \propto [\alpha^{a+n-1} \exp\{-\alpha(b + \sum_{i=1}^{m} S_i^{\beta} T_i)\}](\prod_{i=1}^{m} S_i^{\beta n_i}) p(\beta). \tag{7}$$

The joint posterior density in (7) can not be obtained in closed form for any choice of the prior $p(\beta)$. Consequently, the posterior variance $V(\lambda_u|D, d)$ can not be computed analytically. One way to compute the posterior variance is to use a Markov Chain Monte Carlo method such as the Gibbs sampler. The implementation of the Gibbs sampler requires the full conditional distributions $p(\alpha|\beta, D)$ and $p(\beta|\alpha, D)$. The Gibbs sampler creates a sample from the joint posterior distribution $p(\alpha, \beta|D)$ by successive drawings from the full conditional distributions, see Gelfand and Smith (1990) for a general discussion. It can be easily seen that, with the gamma prior on α, the full conditional posterior distribution of α is obtained from (7) in closed form as

$$[\alpha|\beta, D] \sim \text{Gamma}(a + n, b + \sum_{i=1}^{m} S_i^{\beta} T_i). \tag{8}$$

Using prior independence of α and β, the full conditional posterior density of β is obtained as

$$p(\beta|\alpha, D) \propto (\prod_{i=1}^{m} S_i^{\beta n_i}) \exp(-\alpha \sum_{i=1}^{m} S_i^{\beta} T_i) p(\beta), \tag{9}$$

which is not of any familiar form. Note that, in the special case of a single stress, i.e., $m = 1$, we can use a rejection sampling method within the Gibbs sampler by analytically evaluating the mode of conditional likelihood β given α and D, which is obtained as $\widehat{\beta} = \{\log(n) - \log(\alpha T_i)\}/\log(S_i)$. Unfortunately, in the case of multiple stresses $\{S_1, \ldots, S_m\}$, evaluation of the mode requires a numerical optimization routine at each Gibbs iteration, which is not computationally efficient. However, an attractive feature of the conditional posterior in (9) is that it is logconcave if the prior $p(\beta)$ is log-concave. Thus, the adaptive rejection sampling method of Gilks and Wild (1992) can be implemented to sample from (9) at each iteration of the Gibbs sampler. If log-concavity is not satisfied, we can use a Metropolis step instead.

Once a sample is obtained from the joint posterior $p(\alpha, \beta|D)$, the posterior distribution $p(\lambda_u|D, d)$ and the variance $V(\lambda_u|D, d)$ can be approximated by Monte Carlo sample averages. In finding the optimal design we need to obtain the preposterior variance

$$E_D[V(\lambda_u|D, d)] = \int V(\lambda_u|D, d) p(D|d) dD, \tag{10}$$

which can be written as

$$E_D[V(\lambda_u|D, d)] = \int \int V(\lambda_u|D, d) p(D|\lambda_u, d) p(\lambda_u) d\lambda_u dD, \tag{11}$$

where $V(\lambda_u|D, d)$ is evaluated via the Gibbs sampler for a given design $d = \{m, S_i, n_i, i = 1, \ldots, m\}$. Thus, one way to find the optimal design is to use a Monte Carlo method to evaluate (11) for a given design d and then minimize it with respect to d.

Before proceeding any further, we would like to mention a special case that arises when $m = 1$ and the parameter β is known. In this case, the marginal posterior distribution of α is obtained (see Equation 8) as a gamma distribution with parameters $a + n$, and $b + T_i d^{\beta}$, where

$d = \{S_i\}$ is the single stress level, and $D = T_i$. It can be easily shown that the posterior variance of λ_u is obtained as

$$V(\lambda_u|D,d) = V(\alpha S_u^\beta|D,d) = \frac{(a+n)(S_u/d)^{2\beta}}{(T_i + \frac{b}{d^\beta})^2},$$

from which the preposterior variance is computed, by integrating out the above with respect to the (prior) predictive distribution of T_i, as

$$V(\lambda_u|d) = \frac{a(a+1)\, S_u^{2\beta}}{(a+n+1)\, b^2},$$

which does not depend on the stress $d = S_i$. Thus, when β is known in the power law model, it does not matter at what stress level the items are tested. This fact was also shown by Vopatek (1992) for the special case of the power law model with $\beta = 1$ and pointed out by Verdinelli et al. (1993) for the case of the lognormal life model.

4 Monte Carlo Approaches for Fixed Design

The standard Monte Carlo evaluation of (11) consists of the following steps:

Step 1: Choose $d = \{m,\, S_i,\, n_i,\ i = 1, \ldots, m\}$.

Step 2: Generate (α_r, β_r) from the prior $p(\alpha, \beta)$, $r = 1, \ldots, R$.

Step 3: For $i = 1, \ldots, m$, independently generate T_{ir} from $p(T_i|\lambda_i)$ using the power law $\lambda_i = \alpha S_i^\beta$. Note that $p(T_i|\lambda_i)$ is a gamma density in this case.

Step 4: For each $D_r = \{T_{1r}, T_{2r}, \ldots, T_{mr}\}$ evaluate $V[\lambda_u|D_r, d]$ using Gibbs sampler.

Step 5: Compute preposterior variance for d using Monte Carlo average

$$\frac{1}{R}\sum_{r=1}^{R} V[\lambda_u|D_r, d]. \tag{12}$$

Step 6: Go to Step 1 and repeat the steps 2-5 for a different design d.

The optimal design is then selected as the d with the minimum value of (12). However, the implementation of the Monte Carlo approach just described is not computationally efficient since it requires, for each design d, R draws from the statistical model and evaluation of the posterior variance using the Gibbs sampler. Typically the Gibbs sampler will require large number of iterations for convergence at each step. Thus, the approximation of the integral may require a large scale computational effort. As pointed out by Mueller (1999), such a large scale simulation for each design d fails to exploit the continuity of the expected loss surface.

To avoid the potential computational burden, we will adopt the curve fitting approach proposed by Muller and Parmigiani (1995) for finding optimal designs. This approach facilitates preposterior analysis by replacing the expectation in (11) with a smoothing step. The posterior variance is evaluated as $l_r = V(\lambda_{ur}|D_r, d_r)$ for each simulated experiment $\{d_r, \alpha_r, \beta_r, D_r\}$. A smooth surface $L(d)$ is then fitted to the points (d_r, l_r), and the optimal design is found by minimizing the fitted surface $L(d)$. In the proposed approach, the expectation step in the solution of the decision problem is replaced by a scatter plot smoother based on Monte Carlo samples of $\{\alpha_r, \beta_r, D_r\}$. The minimization step is replaced by the minimization of the fitted smoother. The proposed approach can be implemented to the ALT design problem with $d = \{m,\ S_i,\ n_i,\ i = 1, \ldots, m\}$ and $\theta = (\alpha, \beta)$ as follows:

Step 1: Select designs $d_r,\ r = 1, \ldots, R$.

Step 2: First draw R points D_r, θ_r from the joint density $p(D, \theta|d_r)$. This is done by independently generating $(T_{1r}, T_{2r}, \ldots, T_{mr})$ from $p(T_i|\lambda_i, d_r)$ using the power law $\lambda_i = \alpha S_i^{\beta}$, for $i = 1, \ldots, m$. Next, use the Gibbs sampler, evaluate $l_r = V(\lambda_{ur}|D_r, d_r)$, and record the Monte Carlo sample points (d_r, D_r, l_r).

Step 3: Fit a surface $L(d)$ to the points (d_r, l_r).

Step 4: Find the minimum over d of $L(d)$.

In Step 1, we choose the design sequence $\{d_r\}$ deterministically from the set of feasible candidates. We assume that total number of n items will be tested in an ALT. This can be considered as a budget constraint in the design problem. It is important to note that d consists of both discrete and continuous components. Given n, the discrete components $\{m, n_i, i = 1, \ldots, m\}$ of d are constrained as $n = \sum_{i=1}^{m} n_i$ and $m \leq n$. Previous studies of designs in ALT's suggested that $m = 2$ point designs are optimal for relationships such as power law model with two parameters, see, for example, Chaloner and Larntz (1992). Thus, our development will focus on two-point designs which will be compared with the one-point designs.

Given the values m and n, all combinations of n_1 and $n_2 = n - n_1$ will constitute the discrete part of the design variables. If n is large one can define continuous decision variables $f_1 = n_1/n$ and $f_2 = 1 - f_1$ and work with the approximate design $d = (f_1, f_2, S_1, S_2)$. In this case the fitted surface of Step 3 and the optimization of Step 4 will involve three dimensions. The illustrations given in Section 6 involve small values of n and therefore all combinations of n_1 and n_2 are considered. This involves surface fitting and optimization for each combination of n_1 and n_2, where the fitted regression surface $L(d)$ is a scatter plot smoothing approximation to the corresponding conditional expectation surface. In the illustrations presented in Section 6, we use the local regression model Loess of Chambers and Hastie (1992) in Step 3.

For the case of two-point designs, an alternate design strategy is to consider a two-phase sequential design. The proposed approach described above can be easily extended for this case. An implementation of the approach in sequential design problems is considered in Erkanli et al. (1998) for prevalence estimation.

5 Monte Carlo Approach for Sequential Designs

In the two-phase ALT design n_1 items are tested at S_1 at the first phase and the remaining $n - n_1$ items are tested at S_2 at the second phase. The design variables at the first phase are $d_1 = (n_1, S_1)$ and given d_1, total time on test, T_1, is observed at the first phase. The design variable $d_2 = S_2$ at the second phase is determined after T_1 is observed. In what follows, we assume that the loss function is $L(\lambda_u, d, a) = (\lambda_u - a)^2$ and thus the optimal design is chosen by minimizing the preposterior variance. That is, $d^* = \arg\min\{E_{D|d}[V(\lambda_u|D, d)]\}$, where $d = (d_1, d_2)$ and $D = (T_1, T_2)$. The decision tree for the sequential design problem is shown in Figure 2.

Put Figure 2 Here

The solution of the design problem is obtained by folding the tree back as before. The solution procedure is as follows:

At random node R_2:

Evaluate $V^*(d_1, d_2, T_1) = E_{T_2|d,T_1}[V(\lambda_u|D,\ d)]$.

At decision node D_2:

Compute $d_2^* = \arg\min\{V^*(d_1, d_2, T_1)\}$ and evaluate $V^*(d_1, T_1) = V^*(d_1,\ d_2^*, T_1)$.

At random node R_1:

Evaluate $V^*(d_1) = E_{T_1|d_1}[V^*(d_1, T_1)]$.

At decision node D_1:

Compute $d_1^* = \arg\min\{V^*(d_1)\}$.

The optimal solution to the sequential design problem is not trivial as it involves implicit computation of expectations and minimization at each step. It requires computing posterior expected loss and preposterior loss. None of these quantities can be obtained analytically. An evaluation strategy based on Monte Carlo simulations and smoothing techniques will be presented next. As in Section 4, this approach replaces expectations by smoothing steps and the decision points are replaced by minimization of the smoothed surfaces. The algorithm is as follows:

Step 1: Select designs $d_r = (d_{r1}, d_{r2})$, $r = 1, \ldots, R$.

Step 2: Draw R points $(T_{1r}, T_{2r}, \theta_r)$ from the joint density $p(T_1, T_2, \theta | d_r)$, where $\theta = (\alpha, \beta)$. Use the Gibbs sampler, evaluate $l_{r2} = V(\lambda_{ur} | T_{1r}, T_{2r}, d_r)$ and record the Monte Carlo sample points (d_r, T_{1r}, T_{2r}), and l_{r2}.

Step 3: Fit a surface $L_2(d_1, d_2, T_1)$ to the points $(d_r, T_{1r}, T_{2r}, l_{r2})$. This step is analogous to the expectation performed at the node R_2 of the decision tree.

Step 4: Find the minimum over d_2 of $L_2(d_1, d_2, T_1)$. Replace the Monte Carlo points by the pairs (d_{r1}, l_{r1}), where $l_{r1} = L_2(d_{r1}, d^*_{r2}, T_{1r})$ and d^*_{r2} is the minima of $L_2(d_1, d_2, T_1)$. Here the minimal value of $d^*_2 = S_2$ is the solution to the second stage of the design problem and it depends on $d_1 = (n_1, S_1)$ and T_1. This step is analogous to the minimization performed at the node D_2.

Step 5: Fit a surface $L_1(d_1)$ to the points (d_{r1}, l_{r1}). This step is analogous to the expectation performed at the random node R_1 of the decision tree.

Step 6: Find the minimum over $d_1 = (n_1, S_1)$ of $L_1(d_1)$. Here the minimal value of d_1 is the solution to the first stage of the design problem. This step is analogous to the minimization performed at the decision node D_1.

In Step 3, a nonparametric surface $L_2(d_1, d_2, T_1)$ can be fitted to the points $(d_r, T_{1r}, T_{2r}, l_{r2})$. We can think about the points $(d_r, T_{1r}, T_{2r}, l_{r2})$ as the data of a regression problem with l_{r2} as the dependent variable and (d_1, d_2, T_1) as the predictors, or the independent variables. In Step 4, we minimize the fitted surface L_2 with respect to $d_2 = S_2$ which is a one-dimensional optimization. This usually requires numerical techniques since most nonparametric regression surface estimators are not available in closed forms. In Step 5, surface $L_1(d_1)$ is fitted to the points (d_{r1}, l_{r1}), and, in Step 6, $L_1(d_1)$ is minimized with respect to d_1 which is a two-dimensional optimization. The optimal solution at the decision node D_1 of the tree gives us the optimal stress level and the number of items to be tested at the first phase of the ALT. It is important to note that the optimal decision at D_2 represents the optimal stress level at the second phase and this decision depends on T_1, the outcome of the first phase.

In our implementation in Section 6, following Section 4, we will focus on cases with small values of n and consider all combinations of n_1 and n_2 in our designs. This will result in one-dimensional optimization phases 1 and 2 of the design problem, i.e., in steps 4 and 6 above.

6 Illustrations

6.1 Fixed Point Designs

In this section we will consider fixed point designs and illustrate the implementation of Monte Carlo based algorithms of Section 4. In so doing, we focus on one and two-point designs. For illustrative purposes we assume that total number of items to be tested in the ALT is $n = 4$ and $(S_1, S_2) \in [1.05, 5.4] \times [1.05, 5.4]$, where $S_u = 1.05$ is the use-stress. We note that the design space is given by:

$$\{(n_1, n_2) \in \{(4,0), (2,2), (1,3)\}, (S_1, S_2) \in [1.05, 5.4] \times [1.05, 5.4]\} \tag{13}$$

where $(n_1 = 4, n_2 = 0)$ represents a one-point design, that is, $S_1 = S_2 = S$. We assume independent gamma priors for α and β: $\alpha \sim \text{Gamma}(20, 1000)$ and $\beta \sim \text{Gamma}(3, 1)$. As in Section 4, our focus is on ALT designs that minimize the preposterior variance $E_D[V(\lambda_u|D, d)]$ where $D = (T_1, T_2)$ and $d = (n_1, S_1, n_2, S_2)$. The optimal levels (S_1, S_2) under each (n_1, n_2) combination are evaluated and the optimal design is chosen from these. In implementing the surface fitting algorithm, $R = 2000$ simulated designs are considered and the Gibbs sampler is used to evaluate $V(\lambda_u|D, d)$ for each simulated design. We generate the designs for a given combination of (n_1, n_2) and apply the algorithm. In Figure 3, we present the case of one-point design $(n_1 = 4, n_2 = 0)$. Since $S_1 = S_2 = S$, this case involves only curve fitting in the last step of the algorithm.

Put Figure 3 Here: One-point design $(n_1 = 4, n_2 = 0)$.

The fitted curve in Figure 3 is very flat, but it can be shown that the optimal stress level is $S^* = 1.05$, so that it is optimal to put all 4 items at the use-stress.

In Figure 4 (the pre-posterior variance is multipled by 1000 for a better resolution of the fitted surface), we illustrate the case of $(n_1 = 2, n_2 = 2)$. In this two-point design the optimal stress levels are $S_1^* = 1.05$ and $S_2^* = 5.40$. But it is interesting to note that another design which is close to the optimal is given by $S_1^* = S_2^* = 1.05$. In other words, the two-point design $(n_1 = 2, n_2 = 2)$ performs slightly better than the one-point design $(n_1 = 4, n_2 = 0)$.

Put Figure 4 Here: Two-point design $(n_1 = 2, n_2 = 2)$.

In Figure 5 (the pre-posterior variance is multipled by 1000 for a better resolution of the fitted surface), we illustrate the final case, the two-point design $(n_1 = 1, n_2 = 3)$. In this case, the optimal stress levels are $S_1^* = 5.40$ and $S_2^* = 1.05$. In looking at these three cases, we see that as more items can be allocated to a stress level, the optimal level moves towards the use-stress. This was also observed by Soyer and Vopatek (1995).

Put Table 1 Here.

We can compare the three cases with respect to the optimal values of the preposterior variance. Table 1 shows that the minimum value of $E_D[V(\lambda_u|D, d)]$ is obtained by allocating two items to the highest stress environment and two items to the use-stress environment. Furthermore, the table shows that the two-point design is better than the one-point design in this case.

6.2 Sequential Designs

In this section we consider a two-phase sequential design and illustrate the implementation of the approach presented in Section 5. As in the fixed point designs discussed in Section 6.1 above, we assume that total number of items to be tested is $n = 4$ and $(S_1, S_2) \in [1.05, 5.4] \times [1.05, 5.4]$, where $S_u = 1.05$ is the use-stress. But now the design space is given by

$$\{n_1 \in (1, 2, 3),\ (S_1, S_2) \in [1.05, 5.4] \times [1.05, 5.4]\},$$

where n_1 is the number of items to be tested at the first phase, i.e., at stress environment S_1, and S_2 represents the stress environment at the second phase of the sequential ALT. Note that once n_1 is determined, the remaining $n_2 = n - n_1$ items will be tested at the second phase. As in the case of fixed point design, we will assume independent gamma priors for α and β of the power law model and consider $R = 2000$ simulated designs. The Gibbs sampler is used to evaluate $V(\lambda_u|D, d)$ for each design and the surface fitting algorithm of Section 5 is applied.

It is important to note that the optimal decision obatined at the second phase, i.e., the choice of S_2, depends on both the decision and the observed failure data from the first phase. In other words, for given values of $d_1 = (n_1, S_1)$ and T_1, a one-dimensional optimization is performed at the second phase and the optimal strees environment S_2^* is determined. Examples of optimal values of S_2^* for velected values of S_1 and T_1 are displayed in Table 2 for the case of $n_1 = 2$. For example, if $S_1 = 2.04$ is applied at the first phase at which the total test time of T_1 is observed, then the optimal stress level at the second phase is $S_2^* = 4.03$. In general, for the case of $n_1 = n_2 = 2$, if the S_1 is chosen closer to the use-stress and the observed total test time, T_1, is small (i.e., failures can be generated in a short period of time), then it is optimal to test the remaining items also closer to use-stress at the second phase.

Put Table 2 Here.

Our analysis show that it was optimal to test at the use-stress level and allocate two items at the first phase, i.e., $S_1^* = 1.05$ and $n_1^* = n_2^* = 2$. In Figure 6, we present preposterior variance of λ_u as a function of the stress level S_1. As pointed out above, once two items are tested at stress level $S_1^* = 1.05$ and T_1 is observed at the first phase, the optimal strees level for the second phase is obtained using a one-dimensional optimization and developing a tabular form similar to Table 2. We note that the proposed approach can easily be extended to multi-phase testing for small to moderate number of items to be tested, since at each stage only one-dimensional optimization is required. Extension of our approach for general n using continuous decision variables, $f_i = n_i/n$, representing the fraction of items to be tested at phase i, is currently under investigation and will be presented elsewhere.

Put Figure 5 Here: The $(n_1 = 1, n_2 = 3)$ case.

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Table 1: Comparison of Designs.

Designs	S_1^*	S_2^*	$E_D[V(\lambda_u \mid D, d^*)]$
$(n_1 = 4, n_2 = 0)$	1.05	1.05	0.0000241
$(n_1 = 2, n_2 = 2)$	1.05	5.40	0.0000229
$(n_1 = 1, n_2 = 3)$	5.40	1.05	0.0046428

Table 2: Selected values of S_2^* as a function of S_1 and T_1: $(n_1 = 2,\ n_2 = 2)$.

S_1	T_1	$S_2^*(S_1, T_1)$
1.05	456.08	1.05
1.05	85.53	5.46
2.04	103.95	1.05
2.04	7.43	4.03
3.12	46.83	1.05
3.12	0.08	1.05
5.40	34.61	1.05
5.40	4.22	1.05

Figure 1: Decision Tree for the Fixed Design Problem

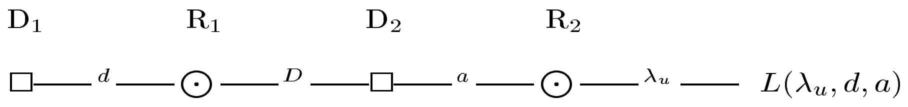

Figure 2: Decision Tree for the Sequential Design Problem

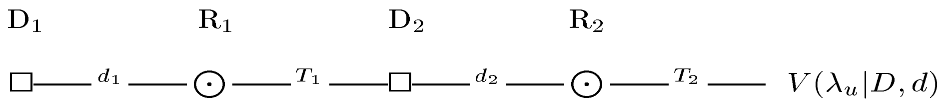

Figure 3: One-point Design $(n_1 = 4,\ n_2 = 0)$.

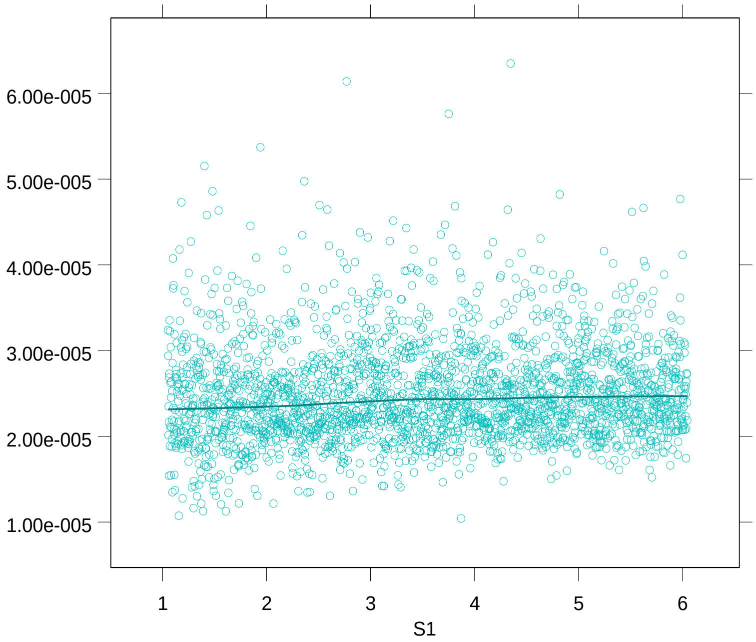

Figure 4: Two-point Design ($n_1 = 2,\ n_2 = 2$).

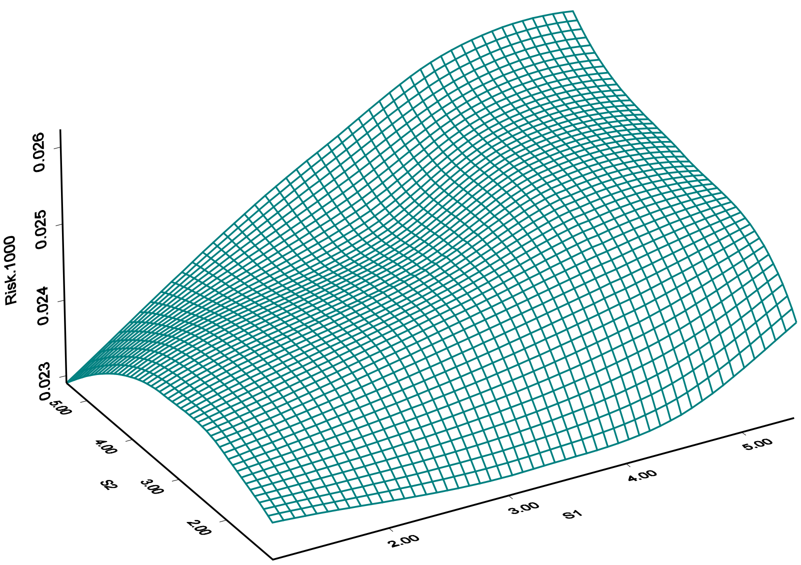

Figure 5: Two-point Design ($n_1 = 1,\ n_2 = 3$).

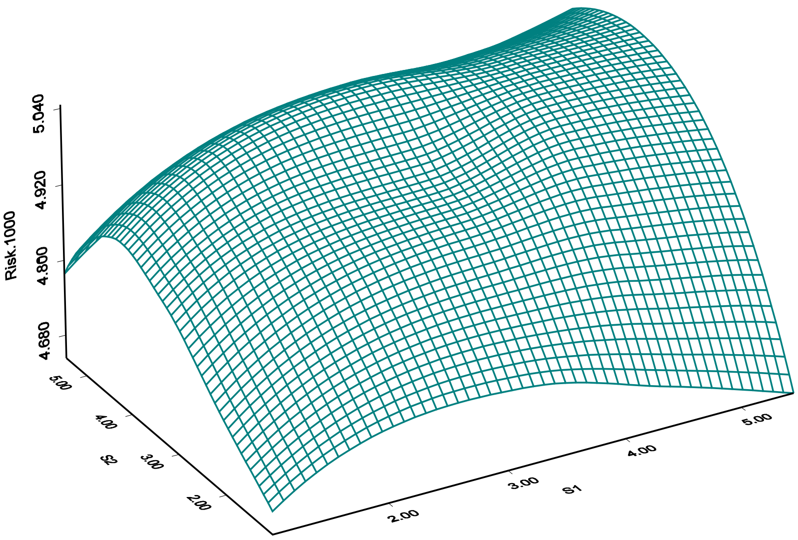

Figure 6: Pre-posterior variance as a function of S_1: $(n_1 = 2,\ n_2 = 2)$.