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Two extensions for fitting discrete time term structure models with normally distributed factors

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This paper provides extensions to procedures for the implementation of two well-known term structure models. In the first part, a misleading implication given in two textbooks concerning the ability to fit a Ho–Lee type term structure tree through trial and error is corrected, and it is shown that the tree can be fitted precisely with a simple and easily programmable formula. In the second part, a previously published result that obtains the drift for a single-factor discrete time Heath–Jarrow–Morton model is extended to a multi-factor world. In both cases numerical examples are provided.

Keywords: term structure, Ho–Lee model, Heath–Jarrow–Morton model

1. Introduction

The family of term structure models in which the underlying factors are normally distributed includes the spot rate models of Vasicek (1977) and Ho and Lee (1986) and the forward rate model of Heath, Jarrow and Morton (1992). Judging from citations in the literature and the popularity of these models in the financial community, they have achieved great success. Each has its own limitations, however, and considerable research continues to this day on improvements.

Implementations of the models are often done by fitting binomial and multinomial trees. Trees are excellent means of applying these term structure models. They are easy to understand and computationally accurate, provided they are implemented in an efficient manner and a sufficiently large number of time steps is used.

This paper addresses two specific problems in the implementation of two of the models. First, it examines the problem of fitting a normally distributed spot rate to a binomial tree, in what amounts essentially to the Ho–Lee model. Two popular textbooks suggest that this process requires trial and error: it is shown here that it does not. Second, it provides an extension to an already-published technique for fitting the Heath–Jarrow–Morton (HJM) model to a tree. The extension generalizes the

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previously published result, which was performed only for the case of a single factor, to a multi-factor world. As is well known, a single-factor HJM model can be fit to a binomial tree. A multi-factor model requires a multinomial tree. The results presented here show how the tree can be fitted in a relatively simple manner.

2. Fitting a term structure model with one normally distributed factor to a binomial tree

This part of the paper derives a simple technique for fitting a one-factor normally distributed spot rate to an arbitrage-free binomial term structure tree. As noted, published work on this topic has suggested that such models can be fitted only by trial and error. This paper shows that there is a simple solution and provides an easily programmable algorithm for fitting such a tree.

2.1. *Previous research*

The Ho–Lee (1986) model was a path-breaking development in the history of term structure models. It showed that the existing term structure could be incorporated into an arbitrage-free binomial tree. The original version of the model, however, assumes constant volatility across the term structure and across time.

In recent books on derivative securities, both Jarrow and Turnbull (2000) and Arditti (1996) illustrate the fitting of a binomial tree to a normally distributed spot rate model, while permitting the spot rate volatility to differ deterministically across time. Technically such a model is not identical to the original Ho–Lee model, but the difference lies only in the fact that the original model assumed constant volatility across time. Hence, it can be reasonably assumed that the Jarrow–Turnbull and Arditti applications are essentially Ho–Lee models with the flexibility to accommodate different deterministic volatilities across time.

In the Jarrow–Turnbull and Arditti specifications, the user can input an initial term structure of spot zero coupon bond prices as well as the volatilities of the spot rate for various maturities. The model then produces an arbitrage-free binomial tree for a one-factor normally distributed process. After specifying the initial one-period spot rate r, Jarrow and Turnbull proceed to obtain the next two possible spot rates, $r(1)_U$ and $r(1)_D$ for the second period. Their procedure is summarized in the following sentence from their book:

> We must pick these values to be consistent with the initial term structure as shown in Table 15.1; we do this by trial and error. (p. 457)

The authors then go on to state that, after checking to see if their guesses are consistent with the no-arbitrage condition and volatility constraint,

> Had we been wrong, we would have revised $r(1)_U$ and $r(1)_D$ by iteration until these two conditions were satisfied. (p. 459)

Arditti (1996) suggests that a direct solution, in the form of a drift term, can be obtained but does not actually show that a general solution exists. Indeed one might surmise from Arditti's approach that for the *n*th time step, one must solve a system of *n*-1 equations, with each solution potentially of a different form.

Evidently the closest anyone appears to have come to a general solution is in an unpublished working paper by Grant and Vora (2003). They show that the tree can be fit using a drift adjustment term.[1] This adjustment is based on the volatility structure of the spot rate across time. They show that simple arithmetic calculations involving rows and columns of the matrix of volatilities provides the drift adjustment.

Our focus is on providing a simple algorithm that obtains this solution directly in an easily programmable form. For mathematical convenience, this paper follows the structure of the problem as specified by Jarrow and Turnbull, but some notational adaptations are required. In doing so, it is shown that the Jarrow–Turnbull and Arditti implication that the solution must be obtained by trial and error is not correct.

2.2. *The structure of the problem*

One begins by assuming that for a trading interval $[0, T]$ the term structure of continuously compounded interest rates is specified at time 0 as $r(0, 1), r(0, 2), \ldots, r(0, T)$. Accordingly, the price of a zero coupon bond of maturity t, $t = 1, 2, \ldots, T$, is given as[2]

$$B(0, t) = \exp(-r(0, t)t)$$

The volatility of the spot rate at any point in time is specified by a term structure of volatilities where σ_t is the volatility of the spot rate at time t.[3] These two pieces of information are sufficient to derive the martingale evolution of the entire tree of interest rates for all t up to and including T. As is common in many term structure models, the convenient martingale probability of 0.5 is assigned for each up move.[4]

We begin with the common assumption that there exists a spot rate of interest, which is the rate on the shortest maturity bond. A hypothetical instrument is specified, called a money market account, the value of which is the accumulation of $1 invested at time 0. The money market account value reflects the accrual of interest at the one-period spot rate of interest over the sequence of time steps.

It is well known that the elimination of arbitrage in a term structure is consistent with the local expectations hypothesis (LEH) of Cox, Ingersoll and Ross (1981). In simple terms, the LEH states that in the absence of arbitrage opportunities the current value of any security, portfolio, or trading strategy

[1] The Grant–Vora approach specifies that the drift has two components, a term reflecting the evolution of the term structure in the absence of uncertainty, and another term that reflects the adjustment required to prevent arbitrage. Technically both of these terms could be combined into a single term.

[2] Technically, it is not necessary to specify the actual term structure of interest rates. One must know only the zero coupon bond prices. While this seems like a minor point, it is actually quite convenient. An interest rate is really only a non-linear transformation of a bond price. Numerous transformations, making different compounding assumptions, are quite possible. Continuous compounding is used here, which leads to a tree of continuously compounded rates, but any other form of compounding could easily be used. Ultimately, however, the closed-form solutions cannot be obtained without working with continuous rates, but the results can then be translated into the analogous discretely compounded rates. In short, the volatility structure must apply to the continuously compounded rates.

[3] This notation refers to the volatility at time zero for the spot rate at time t.

[4] Ho and Lee do not assume martingale probabilities of 0.5. They derive their probabilities to be consistent with the absence of arbitrage. Term structure models can usually assume probabilities of 0.5, while allowing other model parameters to make up for the loss of this degree of freedom.

is the expected value of that strategy one period later where expectations are taken using the martingale probabilities.[5] An equivalent implication is that the current value of any asset, derivative, portfolio, or dynamic trading strategy normalized by the value of a money market account is its expected future value normalized by the money market account where expectations are taken using the martingale probability measure.[6] For zero coupon bonds, the formal statement is

$$\frac{B(0,2)}{A(0)} = E_0^Q\left[\frac{B(1,2)}{A(1)}\right]$$
$$\frac{B(0,3)}{A(0)} = E_0^Q\left[\frac{B(2,3)}{A(2)}\right]$$
$$\cdots$$
$$\frac{B(0,T)}{A(0)} = E_0^Q\left[\frac{B(T-1,T)}{A(T-1)}\right]$$

where the E_0^Q refers to expectations taken at time 0 under the martingale probability measure Q, and $A(t)$ is the value of the money market account at time t, $t=0, 1, 2, \ldots, T-1$. If this condition is met, then no arbitrage opportunities exist, and likewise, if no arbitrage opportunities exist, this condition is met.[7]

The current one-period spot rate is denoted as $r_n{}^i$ where n is the time step and i is the number of times the rate has increased from the beginning of the tree. Thus, at time zero, $r_0{}^0$ is the spot rate. The next period the rate can go up to $r_1{}^1$, the rate at time 1 if the rate has gone up once since the beginning, or down to $r_1{}^0$, the rate at time 1 if the rate has gone up zero times since the beginning. Figure 1 illustrates the tree we shall fit out to three time periods. This formulation will be sufficient to identify the pattern. Let us now move forward and identify the rates at time 1.

2.3. *The time 1 interest rates, $r_1{}^1$ and $r_1{}^0$*

At time 0, the martingale condition is

$$\frac{B(0,2)}{A(0)} = E_0^Q\left[\frac{B(1,2)}{A(1)}\right]$$

In other words, the expected normalized value of a two-period bond at time 1 is its current normalized value. Without loss of generality, we can initialize the money market account value at $A(0)=\$1$. Then $A(1)=\exp(r_0{}^0)$ where $r_0{}^0$ is the starting spot rate. Note that $A(1)$ is known and could technically be moved from inside the expectations brackets. The expectation of $B(1, 2)$ can be found as

[5] The reference to 'one period' later technically refers to the shortest possible holding period in the model. In a discrete time model, the shortest possible holding period is one time step. In a continuous time model, the shortest possible holding period is one instant in time.

[6] The money market account is referred to as the numeraire, but it is not the only numeraire that could be used. The price of a zero coupon bond is a common alternative.

[7] The local expectations hypothesis is based on the work of Harrison and Pliska (1981) or Harrison and Kreps (1979) but, in the term structure framework, primarily attributable to Cox, Ingersoll, and Ross (1981).

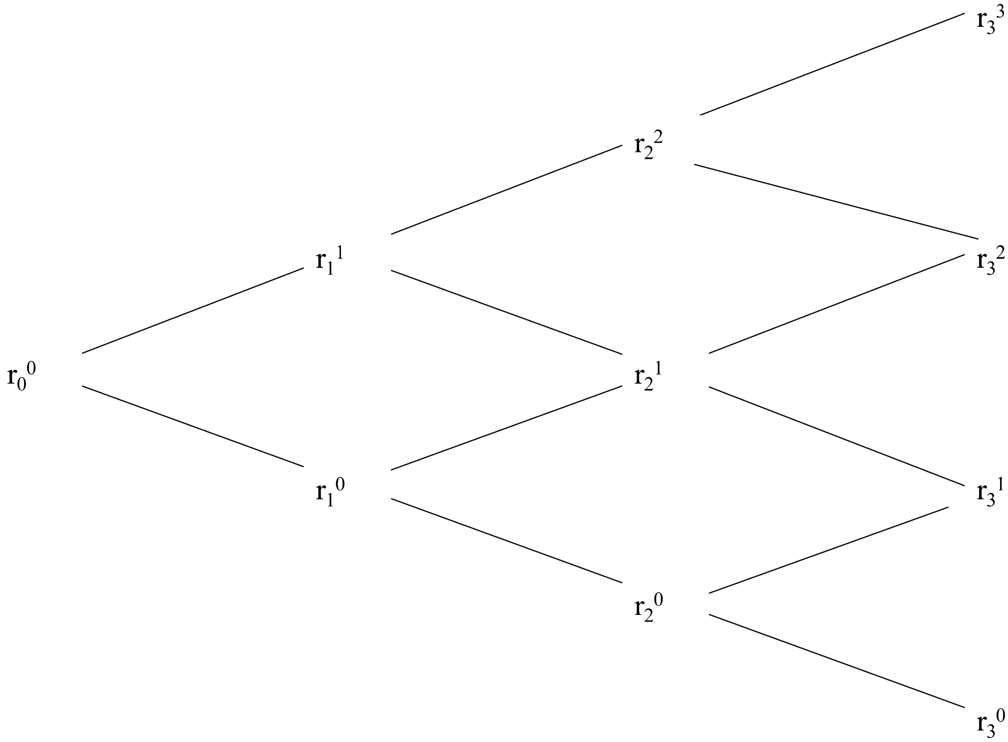

Fig. 1. One-period spot rate tree.

$0.5\exp(-r_1{}^1) + 0.5\exp(-r_1{}^0)$. Thus, we have

$$\begin{aligned} \frac{B(0,2)}{A(0)} &= \frac{B(0,2)}{1} = B(0,2) \\ &= E_0^Q\left[\frac{B(1,2)}{A(1)}\right] = \frac{E_0^Q[B(1,2)]}{A(1)} \\ &= \frac{0.5\exp\left(-r_1{}^1\right) + 0.5\exp\left(-r_1{}^0\right)}{\exp\left(r_0{}^0\right)} \end{aligned}$$

The volatility of the spot rate at time 1 is specified as σ_1 and is a known input. For the rates $r_1{}^1$ and $r_1{}^0$ to be consistent with a volatility of σ_1, we require that

$$\sigma_1 = \frac{\left(r_1{}^1 - r_1{}^0\right)}{2}$$

This formulation is just the standard definition of volatility in a binomial distribution with probability 0.5.[8]

These two specifications comprise a system of two equations in two unknowns. Even though Jarrow and Turnbull (2000) and Arditti (1996) do not provide a solution, the values of ${r_1}^1$ and ${r_1}^0$ are easily found as

$$
\begin{aligned}
{r_1}^1 &= -\ln\left(\frac{2\exp\left({r_0}^0\right)B(0,2)}{1+\exp(2\sigma_1)}\right)\\
{r_1}^0 &= {r_1}^1 - 2\sigma_1
\end{aligned}
$$

2.4. *The 'time 2' interest rates, ${r_2}^2$, ${r_2}^1$, and ${r_2}^0$*

To extend the tree out to the next time step, we require the bond price $B(0, 3)$. Under the local expectations hypothesis, the following condition must hold:

$$\frac{B(0,3)}{A(0)} = E_0^Q\left[\frac{B(2,3)}{A(2)}\right]$$

Note that the time 2 value of the three-period bond is required as well as the time 2 value of the money market account. In the first step we were able to remove the value $A(1)$ from outside of the expectations operator, because it was known to be $\exp({r_0}^0)$. Now we cannot remove $A(2)$ because it is not known at time 0. A simple way to obtain the result is as follows. Given the above equation, by the law of iterated expectations:

$$E_0^Q\left[\frac{B(2,3)}{A(2)}\right] = E_0^Q\left[E_1^Q\left\{\frac{B(2,3)}{A(2)}\right\}\right]$$

The term in brackets can be evaluated by positioning ourselves at time 1.

$$E_1^Q\left[\frac{B(2,3)}{A(2)}\right] = \frac{B(1,3)}{A(1)}$$

Because $A(2)$ is known at time 1, one can move it outside the expectations operator. Thus,

$$
\begin{aligned}
\frac{B(1,3)}{A(1)} &= E_1^Q[B(2,3)]\frac{1}{A(2)}\\
B(1,3) &= E_1^Q[B(2,3)]\frac{A(1)}{A(2)}
\end{aligned}
$$

Multiplying by $A(1)/A(2)$ is the same as discounting the expectation at the current one-period rate. One then needs to evaluate the expectation of $B(1, 3)/A(1)$ at time 0. The expectations at time 1 of $B(2, 3)$

[8] In other words, the variance of the rate is by definition the expected value of the square of the rate minus the square of the expected value of the rate.

$$\sigma_1 = 0.5\left({r_1}^1\right)^2 + 0.5\left({r_1}^0\right)^2 - \left(\frac{{r_1}^1 + {r_1}^0}{2}\right)^2$$

This value can be shown to reduce to the specification shown above in the text.

for the up and down states are easily found as

$$B(1,3)^{+} = \frac{0.5B(2,3)^{++} + 0.5B(2,3)^{+-}}{\exp(r_1{}^1)} = \frac{0.5\exp\left(-r_2{}^2\right) + 0.5\exp\left(-r_2{}^1\right)}{\exp(r_1{}^1)}$$

$$B(1,3)^{-} = \frac{0.5B(2,3)^{-+} + 0.5B(2,3)^{--}}{\exp(r_1{}^0)} = \frac{0.5\exp\left(-r_2{}^1\right) + 0.5\exp\left(-r_2{}^0\right)}{\exp(r_1{}^0)}$$

where the $+$ and $-$ symbols indicate the sequence of interest rate moves associated with the given zero coupon bond price. Thus, one takes the expectation of these bond prices and discount back one period:

$$E_0^Q\left[\frac{B(1,3)}{A(1)}\right] = \frac{0.5\left(\frac{0.5\exp\left(-r_2{}^2\right)+0.5\exp\left(-r_2{}^1\right)}{\exp(-r_1{}^1)}\right) + 0.5\left(\frac{0.5\exp\left(-r_2{}^1\right)+0.5\exp\left(-r_2{}^0\right)}{\exp(-r_1{}^0)}\right)}{\exp(r_0{}^0)} = B(0,3)$$

This expression can be easily calculated and will equal B(0, 3)/A(0) = B(0, 3). For time 1, there are two volatility constraints:

$$\sigma_2 = \frac{\left(r_2{}^2 - r_2{}^1\right)}{2}$$

$$\sigma_2 = \frac{\left(r_2{}^1 - r_2{}^0\right)}{2}$$

where σ_2 is the volatility of the spot rate at time 2. There is now a system of three equations and three unknowns. The solution is

$$r_2{}^2 = -\ln\left(\frac{4\exp\left(r_0{}^0\right)B(0,3)}{\exp(-r_1{}^1)(1+\exp(2\sigma_2)) + \exp(-r_1{}^0)(\exp(2\sigma_2)+\exp(4\sigma_2))}\right)$$

$$r_2{}^1 = r_2{}^2 - 2\sigma_2$$

$$r_2{}^0 = r_2{}^1 - 2\sigma_2$$

2.5. *The general solution*

One could continue in this manner, solving a separate system of equations for the spot rate at any time step, but a general solution is available. The solutions one has seen so far and those one could have obtained carrying the tree out further constitute an identifiable pattern. The general solution for $n > 0$ is

$$r_n{}^n = \ln\left(\frac{Q_n}{2^n B(0, n+1)}\right)$$

where

$$Q_n = \exp\left(-\sum_{j=0}^{n-1} r_j{}^j\right)\prod_{j=1}^{n}\left(1 + \exp\left(2\sum_{k=0}^{j-1}\sigma_{n-k}\right)\right)$$

with

$$r_n{}^j = r_n{}^n - 2(n-j)\sigma_n \text{ for } j = 0, 1, \ldots, n-1$$

This formula is easily programmable. In the next section, this is illustrated using the example from Jarrow and Turnbull.

2.6. *An illustrative example*

Table 1 presents the numerical data used by Jarrow and Turnbull. First the rates, $r_1{}^1$ and $r_1{}^0$ are found at time step 1. So, starting with $n=1$, we have

$$Q_1 = \exp\left(-\sum_{j=0}^{0} r_j{}^j\right)\prod_{j=1}^{1}\left(1+\exp\left(2\sum_{k=0}^{j-1}\sigma_{1-k}\right)\right)$$

$$= \exp\left(-r_0{}^0\right)(1+\exp(2\sigma_1)) = \exp(-.061982)(1+\exp(2(.017))) = 1.91230567.$$

The interest rates are

$$r_1{}^1 = \ln\left(\frac{Q_n}{2^n B(0, 2)}\right) = \ln\left(\frac{1.91230567}{2^1(0.8798)}\right) = 0.083223$$

$$r_n{}^j = r_n{}^n - 2(n-j)\sigma_n$$

$$r_1{}^0 = r_1{}^1 - 2\sigma_1 = 0.083223 - 2(0.017) = 0.049223$$

These are the correct interest rates as given by Jarrow and Turnbull.

Now let us move forward to the next period, where we must solve for the rates $r_2{}^2$, $r_2{}^1$, and $r_2{}^0$ at time step 2. Thus, $n=2$ and we have

$$Q_n = \exp\left(-\sum_{j=0}^{n-1} r_j{}^j\right)\prod_{j=1}^{n}\left(1+\exp\left(2\sum_{k=0}^{j-1}\sigma_{n-k}\right)\right)$$

$$Q_2 = \exp\left(-\sum_{j=0}^{1} r_j{}^j\right)\prod_{j=1}^{2}\left(1+\exp\left(2\sum_{k=0}^{j-1}\sigma_{2-k}\right)\right)$$

$$= \exp\left(-\left(r_0{}^0 + r_1{}^1\right)\right)(1+\exp(2\sigma_2))(1+\exp(2\sigma_2+2\sigma_1))$$

$$= \exp(-(0.061982+0.083223))(1+\exp(2(0.015)))(1+\exp(2(0.015)+2(0.017)))$$

$$= 3.628117$$

$$r_2{}^2 = \ln\left(\frac{Q_2}{2^2 B(0, 3)}\right) = \ln\left(\frac{3.628117}{4(0.8137)}\right) = 0.108583$$

$$r_2{}^1 = 0.108583 - 2(0.015) = 0.078583$$

$$r_2{}^0 = 0.108583 - 4(0.015) = 0.048583$$

These answers agree with those given by Jarrow and Turnbull.

Table 1. Interest rate data from Jarrow and Turnbull (2000, p. 456).

Maturity in years (t)	Bond price $B(0, t)$	Yield (%)	Volatility
1	0.9399	6.1982	0.017
2	0.8798	6.4030	0.015
3	0.8137	6.8721	0.011
4	0.7552	7.0193	

Finally, let us move one time step further where $n=3$. We have

$$Q_n = \exp\left(-\sum_{j=0}^{n-1} r_j{}^j\right)\prod_{j=1}^{n}\left(1+\exp\left(2\sum_{k=0}^{j-1}\sigma_{n-k}\right)\right)$$

$$Q_3 = \exp\left(-\sum_{j=0}^{2} r_j{}^j\right)\prod_{j=1}^{3}\left(1+\exp\left(2\sum_{k=0}^{j-1}\sigma_{3-k}\right)\right)$$

$$= \exp\left(-\left(r_0{}^0+r_1{}^1+r_2{}^2\right)\right)(1+\exp(2\sigma_3))(1+\exp(2\sigma_3+2\sigma_2))(1+\exp(2\sigma_3+2\sigma_2+2\sigma_1))$$

$$= \exp(-(0.061982+0.083223+0.108583))(1+\exp(2(0.011)))(1$$

$$+\exp(2(0.011)+2(0.015)))(1+\exp(2(0.011)+2(0.015)+2(0.017)))$$

$$=6.73269977$$

$$r_3{}^3 = \ln\left(\frac{Q_3}{2^3 B(0,4)}\right) = \ln\left(\frac{6.73269977}{8(0.7552)}\right) = 0.108307$$

$$r_3{}^2 = 0.108307 - 2(0.011) = 0.086307$$

$$r_3{}^1 = 0.108307 - 4(0.011) = 0.064307$$

$$r_3{}^0 = 0.108307 - 6(0.011) = 0.042307$$

Again, these are the values obtained by Jarrow and Turnbull.

2.7. *The generality of the formula*

The formula derived in Section 2 is quite general. Although the formula is obtained working with a one-factor Ho–Lee/Heath–Jarrow–Morton model with varying deterministic volatility, one could just as easily have derived this formula with any other one-factor spot rate model. One could have assumed mean reversion. No particular stochastic process is required. All one would need is the spot rate, zero coupon bond prices, and the volatilities.

An interesting question is whether a similar formula could be obtained for the case of a multi-factor model and whether such an approach could lead to a recombining tree. Those questions are left for future research and this paper now takes a look at a method of easily fitting a multinomial tree to a multi-factor Heath–Jarrow–Morton model.

3. Fitting a multi-factor Heath–Jarrow–Morton model to a multinomial tree

In the HJM model the term structure is specified by the set of forward rates $f(t, T)$. Each $f(t, T)$ is the forward rate observed at time t for the instantaneous holding period beginning at time T. As the term structure changes through time, a given forward rate $f(t, T)$ changes to $f(t+dt, T)$, $f(t+2dt, T)$, … ultimately to $f(T, T)$, the instantaneous spot rate. The HJM model assumes that a given forward rate evolves according to the following stochastic differential equation

$$df(t, T) = \mu(t, T)dt + \sum_{i=1}^{n} \sigma(t, T)dW_i(t)$$

where $\mu(t, T)$ is the drift term at time t for the instantaneous forward rate of time T, $\sigma_i(t, T)$ is the volatility at time t of the ith factor driving the instantaneous forward rate of time T, and $W_i(t)$ is a standard Wiener process at time t applicable to the ith factor. In other words, there are n factors driving any given forward rate. These factors are mutually independent as well as independent across time. HJM have shown that to prevent arbitrage, the drift is restricted to the following specification under the equivalent martingale probability measure:

$$\mu(t, T) = \sum_{i=1}^{n} \sigma_i(t, T) \int_t^T \sigma_i(t, v)dv$$

Hence, the drift is a function of the volatility structure.

3.1. *Discretizing the HJM model*

Fitting the HJM model is primarily an exercise in fitting a tree with rates that are normally distributed in the limit, have the correct volatilities, and prohibit arbitrage. For a one-factor model, a binomial tree can be specified by starting with the stochastic process:

$$\Delta f(t, T) = f(t+\Delta t, T) - f(t, T) = \mu(t, T)\Delta t + \sigma(t, T)\Delta W(t, T)$$

where $\Delta W(t, T)$ is a binomial random walk, which takes on a value of $+\sqrt{\Delta t}$ or $-\sqrt{\Delta t}$ with equal probability. Without loss of generality, one can take the value of Δt to equal one. Fitting a binomial tree involves identifying a division of the time interval $[t, T]$ into an equal number of units of time and finding the drift term $\mu(t, T)$ for each rate $\mu(t, t+1)$, $\mu(t, t+2)$, …, $\mu(t, T-1)$ with $t = 0, 1, 2, …, T-1$. That is, one must find the drift term for each forward rate at each point in time during the period spanned by the tree.

Although some efforts to discretize the specification of the drift under continuous time as stated above have led to approximations, Heath, Jarrow and Morton (1990b, pp. 60–2, 75), Ritchken (1996, pp. 578–80) and Grant and Vora (1999, pp. 95–6) show that this approach is not correct and permits small arbitrage profits. The correct approach is to solve for the drift specification completely within a discrete time framework. Their results show that the drift specification can be obtained easily through a correlation matrix of the volatilities of the forward rates in the model.[9] In other words, the volatility structure can be used to obtain the drift directly and exactly.[10]

3.2. *Extending the Grant and Vora drift specification to the case of n factors*

Grant and Vora (1999) develop a formula for the drift of the HJM model in a single-factor world and show how the formula can be obtained from the matrix of volatilities. The contribution of this part of the paper is to extend the Grant–Vora specification to the case of n factors. We begin with the general specification of the evolution of the forward rates in discrete time:

$$f(t+1,j)=f(t,j)+\mu(t,j)+\sum_{i=1}^{n}\sigma_i(t,j)\Delta W_i(t)$$

The price of any \$1 zero coupon bond can be obtained by discounting the sequence of forward rates from today until the bond's maturity:

$$P(t+1,\, T)=\exp\left(-\sum_{j=t+1}^{T-1}f(t+1,j)\right)$$

Substituting the first equation into the second, we obtain

$$\begin{aligned}P(t+1,\, T)&=\exp\left(-\sum_{j=t+1}^{T-1}\left(f(t,j)+\mu(t,j)+\sum_{i=1}^{n}\sigma_i(t,j)\Delta W_i(t)\right)\right)\\&=\exp\left(-\sum_{j=t+1}^{T-1}f(t,j)-\sum_{j=t+1}^{T-1}\mu(t,j)-\sum_{i=1}^{n}\sum_{j=t+1}^{T-1}\sigma_i(t,j)\Delta W_i(t)\right)\end{aligned}$$

By definition the price of any asset is the discounted expectation of its future value. Thus,

$$P(t,\, T)=E_t^Q[P(t+1,\, T)]P(t,\, t+1)$$

where expectations are taken with the martingale probability measure, which assures the absence of arbitrage opportunities. Inserting the formulation above for $P(t+1,\, T)$ into the equation above for

[9] In our previous discussion of the Grant–Vora matrix with respect to the Ho–Lee model, we referred to it as just a matrix of volatilities. The off-diagonal elements are not covariances, but with respect to the HJM model, the off-diagonal elements are indeed covariances. In the Ho–Lee model, the volatilities refer to the volatility of the change in the spot rate across time. Obviously changes in the spot rate are uncorrelated across time. In the HJM model, the volatilities refer to the volatilities of forward rates at any point in time. These forward rates are all driven by a single factor and, hence, are perfectly correlated. Thus, the off-diagonal elements are the products of volatilities and an implicit correlation of 1.

[10] Technically the Grant–Vora technique is still an approximation because it assumes that the binomial distribution is equivalent to the normal distribution. This statement is true only in the limit, but most actual applications of the binomial model would be implemented with a sufficiently large number of time steps.

$P(t, T)$ and noting that $P(t, t+1)=\exp(-f(t, t))$ gives

$$P(t, T)=\exp(-f(t, t))E_t^Q\left[\exp\left(-\sum_{j=t+1}^{T-1} f(t,j)-\sum_{j=t+1}^{T-1}\mu(t,j)-\sum_{i=1}^{n}\sum_{j=t+1}^{T-1}\sigma_i(t,j)\Delta W_i(t)\right)\right]$$

We can combine the terms

$$\exp(-f(t, t)) \text{ and } \exp\left(-\sum_{j=t+1}^{T-1} f(t,j)\right)$$

into simply

$$\exp\left(-\sum_{j=t}^{T-1} f(t,j)\right)$$

and move this expression outside of the expectations operator, since this is just the set of forward rates in the current term structure. Thus,

$$P(t, T)=\exp\left(-\sum_{j=t}^{T-1} f(t,j)\right)E_t^Q\left[-\sum_{j=t+1}^{T-1}\mu(t,j)-\sum_{i=1}^{n}\sum_{j=t+1}^{T-1}\sigma_i(t,j)\Delta W_i(t)\right]$$

Since the Wiener processes, $\Delta W_i(t)$ are independent and normally distributed, we need not concern ourselves with any correlations among the Wiener processes for different factors and across time. Thus, the negative of the term in brackets is normally distributed with expected value

$$\sum_{j=t+1}^{T-1}\mu(t,j)$$

and variance,

$$\sum_{i=1}^{n}\sum_{j=t+1}^{T-1}\sum_{k=t+1}^{T-1}\sigma_i(t,j)\sigma_i(t,k)$$

In other words, we are working with a multivariate normal distribution, and thus, each factor is itself normally distributed, but more importantly, each factor is independent of all other factors so there is no correlation structure to consider.

From the moment generating function of a normally distributed random variable, the expectation of $\exp(-x)$ where x is normally distributed with expectation μ_x and variance $\sigma_x{}^2$ is $\exp(-\mu_x+(1/2)\sigma_x{}^2)$. Using this result, we obtain

$$P(t, T)=\exp\left(-\sum_{j=t}^{T-1} f(t,j)\right)\exp\left(-\sum_{j=t+1}^{T-1}\mu(t,j)+(1/2)\sum_{i=1}^{n}\sum_{j=t+1}^{T-1}\sum_{k=t+1}^{T-1}\sigma_i(t,j)\sigma_i(t,k)\right)$$

For any maturity j, $P(t, T)=\exp\left(-\sum_{j=t}^{T-1} f(t,j)\right)$. Using this result we now have

$$P(t, T)=P(t, T)\exp\left(-\sum_{j=t+1}^{T-1}\mu(t,j)+(1/2)\sum_{i=1}^{n}\sum_{j=t+1}^{T-1}\sum_{k=t+1}^{T-1}\sigma_i(t,j)\sigma_i(t,k)\right)$$

which means that

$$1 = \exp\left(-\sum_{j=t+1}^{T-1} \mu(t,j) + (1/2)\sum_{i=1}^{n}\sum_{j=t+1}^{T-1}\sum_{k=t+1}^{T-1} \sigma_i(t,j)\sigma_i(t,k) \right)$$

Taking logs, we obtain the solution for the sum of all $\mu(t, j)$:

$$\sum_{j=t+1}^{T-1} \mu(t,j) = (1/2)\sum_{i=1}^{n}\sum_{j=t+1}^{T-1}\sum_{k=t+1}^{T-1} \sigma_i(t,j)\sigma_i(t,k)$$

Specific values for each $\mu(t, j)$ can be obtained by examining individual cases. For $t=0$ and $T=2$,

$$\mu(0,1) = (1/2)\sum_{i=1}^{n} \sigma_i(0,1)^2$$

For $t=0$ and $T=3$,

$$\mu(0,1) + \mu(0,2) = (1/2)\sum_{i=1}^{n} \left(\sigma_i(0,1)^2 + \sigma_i(0,2)^2 + 2\sigma_i(0,1)\sigma_i(0,2) \right)$$

Using the results for $\mu(0, 1)$, we have

$$\mu(0,2) = (1/2)\sum_{i=1}^{n} \left(\sigma_i(0,2)^2 + 2\sigma_i(0,1)\sigma_i(0,2) \right)$$

Continuing in this manner, we obtain the general result that

$$\mu(t,T) = (1/2)\sum_{i=1}^{n} \sigma_i(t,T)^2 + \sum_{i=1}^{n} \sigma_i(t,T)\sum_{j=1}^{T-1} \sigma_i(t,j)$$

For the special case of one factor, this formula reduces to the Grant and Vora result.

Grant and Vora illustrate how their drift adjustment term can be derived from the matrix of volatilities of the forward rates. It is now illustrated how this can be done in the multi-factor model. Consider the following matrix of volatilities.

$$S_{t,T}{}^{(i)} = \begin{pmatrix} \sigma_i(t,1)^2 & \sigma_i(t,2)\sigma_i(t,1) & \dots & \sigma_i(t,T)\sigma_i(t,1) \\ \sigma_i(t,1)\sigma_i(t,2) & \sigma_i(t,2)^2 & \dots & \sigma_i(t,T)\sigma_i(t,2) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_i(t,1)\sigma_i(t,T) & \sigma_i(t,2)\sigma_i(t,T) & \dots & \sigma_i(t,T)^2 \end{pmatrix}$$

This $T \times T$ matrix contains the volatilities of factor i in the diagonal elements and products of the volatilities as the off-diagonal elements. For the one-factor case, there is a single matrix of this sort. Grant and Vora show that the drift term $\mu(j, T)$, where j is any value from t to T, can be obtained as either (1) one-half the sum of all of the terms in the jth row and jth column or (2) one-half the sum all the elements of the $T \times T$ matrix minus the sum of all elements of a sub-matrix with $T-1$ rows and $T-1$ columns.

Let us now introduce a $T\times 1$ column vector of ones:

$$\mathrm{I_T}=\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}.$$

By matrix multiplication and addition, we obtain the scalar value $\mu(t,\ T)$:

$$\mu(t,\ T)=(1/2)\sum_{i=1}^{n}\left(\mathrm{I}_T{}'S_{t,T}{}^{(i)}I_T-\mathrm{I}_{T-1}{}'S_{t,T-1}{}^{(i)}I_{T-1}\right).$$

where $\mathbf{I_T}'$ is the transpose of the $T\times 1$ column vector of ones. The expression $\mathbf{I_T}'\mathbf{S_{t,\ T}}^{(\mathbf{i})}\ \mathbf{I_T}$ sums the elements of the $T\times T$ matrix of volatilities and products of volatilities. From this the sum of the elements of the $T-1$ submatrix of volatilities and products of volatilities is subtracted. These results are then added for each factor.

3.3. *An illustrative example*

Now illustrated is the implementation of a two-factor HJM model with a term structure over two time periods. First note that when implementing a binomial tree with one factor, the random process ΔW takes a value of $+1$ or -1 with equal probability. The forward rate evolution is, thus, specified as

$$f(t+1,\ T)=f(t,\ T)+\mu(t,\ T)+\sigma(t,\ T) \text{ or}$$
$$f(t+1,\ T)=f(t,\ T)+\mu(t,\ T)-\sigma(t,\ T)$$

with equal martingale probabilities. This specification results in the forward rate having the appropriate drift and volatility.

Now we have a second factor, however, and must consider four possible moves in each forward rate.[11] Thus,

$$f(t+1,\ T)=f(t,\ T)+\mu(t,\ T)+\sigma_1(t,\ T)-\sigma_2(\mathrm{t,\ T}) \text{ or}$$
$$f(t+1,\ T)=f(t,\ T)+\mu(t,\ T)+\sigma_1(t,\ T)+\sigma_2(\mathrm{t,\ T}) \text{ or}$$
$$f(t+1,\ T)=f(t,\ T)+\mu(t,\ T)-\sigma_1(t,\ T)-\sigma_2(\mathrm{t,\ T}) \text{ or}$$
$$f(t+1,\ T)=f(t,\ T)+\mu(t,\ T)-\sigma_1(t,\ T)+\sigma_2(\mathrm{t,\ T}).$$

If the probabilities are set at ¼ each, the expected change in any forward rate will be $\mu(t,\ T)$, and its variance will be $\sigma_1(t,\ T)^2+\sigma_2(t,\ T)^2$, which are the desired values. Since there will be four outcomes in a one-period model, there will be 16 in a two-period model. Although there are some cases and

[11] Jarrow (2002, pp. 293–6) presents a trinomial formulation of the two-factor case. He does this by altering the stochastic process such that the desired expected value and variance are obtained, but the stochastic process is quite different from the original stochastic process governing the evolution of the forward rates. For example, one of the volatility terms has a coefficient of the square root of 2. It is not clear whether this modification is an improvement, though it would reduce the calculations.

assumptions under which the tree can be made to recombine, a non-recombining tree is used in this example.[12]

We start with the following data:

$$f(0,0)=0.075, f(0,1)=0.08, f(0,2)=0.09$$

$$\sigma_1(0,1)=0.02,\ \sigma_1(0,2)=0.0225$$

$$\sigma_2(0,1)=0.01,\ \sigma_2(0,2)=0.015$$

$$\sigma_1(1,2)=0.01,\ \sigma_2(1,2)=0.005$$

Based on these forward rates, the prices of zero coupon bonds will be

$$P(0,1)=\exp(-0.075)=0.927743$$

$$P(0,2)=\exp(-(0.075+0.08))=0.856415$$

$$P(0,3)=\exp(-(0.09+0.08+0.075))=0.782705$$

We shall need to calculate the drifts, $\mu(0, 1)$, $\mu(0, 2)$, and $\mu(1, 2)$. Using the formula developed in this paper, we obtain

$$\mu(0,1)=(1/2)\sum_{i=1}^{n}\sigma_i(0,1)^2=(1/2)\left((.02)^2+(.01)^2\right)=0.00025$$

$$\mu(0,2)=(1/2)\sum_{i=1}^{n}\left(\sigma_i(0,2)^2+2\sigma_i(0,1)\sigma_i(0,2)\right)$$

$$=(1/2)\left((.0225)^2+2(.02)(.0225)+(.015)^2+2(.01)(.015)\right)=0.000966$$

$$\mu(1,2)=(1/2)\sum_{i=1}^{n}\sigma_i(1,2)^2=(1/2)\left((.01)^2+(.005)^2\right)=0.0000625.$$

To identify the four separate outcomes at time 1, let us use the letters A, B, C, and D. At time 2, the outcomes will be identified as AA, AB, AC, AD, …, DA, DB, DC, DD, with the first letter indicating the preceding outcome and the second referring to the current outcome. We shall calculate the following rates:

$$f(1,1)_{\mathrm{A}}=f(0,1)+\mu(0,1)+\sigma_1(0,1)-\sigma_2(0,1)$$

$$f(1,2)_{\mathrm{A}}=f(0,2)+\mu(0,2)+\sigma_1(0,2)-\sigma_2(0,1)$$

$$f(1,1)_{\mathrm{B}}=f(0,1)+\mu(0,1)+\sigma_1(0,1)+\sigma_2(0,1)$$

$$f(1,2)_{\mathrm{B}}=f(0,2)+\mu(0,2)+\sigma_1(0,2)+\sigma_2(0,2)$$

$$f(1,1)_{\mathrm{C}}=f(0,1)+\mu(0,1)-\sigma_1(0,1)-\sigma_2(0,1)$$

$$f(1,2)_{\mathrm{C}}=f(0,2)+\mu(0,2)-\sigma_1(0,2)-\sigma_2(0,2)$$

$$f(1,1)_{\mathrm{D}}=f(0,1)+\mu(0,1)-\sigma_1(0,1)+\sigma_2(0,1)$$

$$f(1,2)_{\mathrm{D}}=f(0,2)+\mu(0,2)-\sigma_1(0,2)+\sigma_2(0,2)$$

[12] For example, if we assume $\sigma_i(j, t)=\sigma_i(k, \mathrm{t})$ for all t, then the tree recombines.

Inserting the values into these formulas gives

$$f(1,1)_{\mathrm{A}}=0.090250, f(1,2)_{\mathrm{A}}=0.098466, f(1,1)_{\mathrm{B}}=0.110250, f(1,2)_{\mathrm{B}}=0.128466$$

$$f(1,1)_{\mathrm{C}}=0.050250, f(1,2)_{\mathrm{C}}=0.053466, f(1,1)_{\mathrm{D}}=0.070250, f(1,2)_{\mathrm{D}}=0.083466$$

For time step 2, we require the values

$$f(2,2)_{\mathrm{AA}}=f(1,2)_{\mathrm{A}}+\mu(1,2)+\sigma_1(1,2)-\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{AB}}=f(1,2)_{\mathrm{A}}+\mu(1,2)+\sigma_1(1,2)+\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{AC}}=f(1,2)_{\mathrm{A}}+\mu(1,2)-\sigma_1(1,2)-\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{AD}}=f(1,2)_{\mathrm{A}}+\mu(1,2)-\sigma_1(1,2)+\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{BA}}=f(1,2)_{\mathrm{B}}+\mu(1,2)+\sigma_1(1,2)-\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{BB}}=f(1,2)_{\mathrm{B}}+\mu(1,2)+\sigma_1(1,2)+\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{BC}}=f(1,2)_{\mathrm{B}}+\mu(1,2)-\sigma_1(1,2)-\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{BD}}=f(1,2)_{\mathrm{B}}+\mu(1,2)-\sigma_1(1,2)+\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{CA}}=f(1,2)_{\mathrm{C}}+\mu(1,2)+\sigma_1(1,2)-\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{CB}}=f(1,2)_{\mathrm{C}}+\mu(1,2)+\sigma_1(1,2)+\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{CC}}=f(1,2)_{\mathrm{C}}+\mu(1,2)-\sigma_1(1,2)-\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{CD}}=f(1,2)_{\mathrm{C}}+\mu(1,2)-\sigma_1(1,2)+\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{DA}}=f(1,2)_{\mathrm{D}}+\mu(1,2)+\sigma_1(1,2)-\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{DB}}=f(1,2)_{\mathrm{D}}+\mu(1,2)+\sigma_1(1,2)+\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{DC}}=f(1,2)_{\mathrm{D}}+\mu(1,2)-\sigma_1(1,2)-\sigma_2(1,2)$$

$$f(2,2)_{\mathrm{DD}}=f(1,2)_{\mathrm{D}}+\mu(1,2)-\sigma_1(1,2)+\sigma_2(1,2)$$

Plugging in the values into the above formulas gives

$$f(2,2)_{\mathrm{AA}}=0.103528, f(2,2)_{\mathrm{AB}}=0.113528, f(2,2)_{\mathrm{AC}}=0.083528, f(2,2)_{\mathrm{AD}}=0.093528$$

$$f(2,2)_{\mathrm{BA}}=0.133528, f(2,2)_{\mathrm{BB}}=0.143528. f(2,2)_{\mathrm{BC}}=0.113528, f(2,2)_{\mathrm{BD}}=0.123528$$

$$f(2,2)_{\mathrm{CA}}=0.058528, f(2,2)_{\mathrm{CB}}=0.068528, f(2,2)_{\mathrm{CC}}=0.038528, f(2,2)_{\mathrm{CD}}=0.048528$$

$$f(2,2)_{\mathrm{DA}}=0.088528, f(2,2)_{\mathrm{DB}}=0.098528, f(2,2)_{\mathrm{DC}}=0.068528, f(2,2)_{\mathrm{DD}}=0.078528$$

From these forward rates, the zero coupon bond prices can be easily computed, and the tree of zero coupon bond prices is shown in Fig. 2.

Finally one should check to ensure that these results are correct. They should conform to the equivalent martingale condition that the expected value of the price of any bond one period later can be discounted by the one-period rate to obtain the current price, with expectations taken using the martingale probabilities. At time 0, one can check whether the following two conditions hold:

$$P(0,2)=E_0^Q[P(1,2)]P(0,1)$$

$$P(0,3)=E_0^Q[P(1,3)]P(0,1)$$

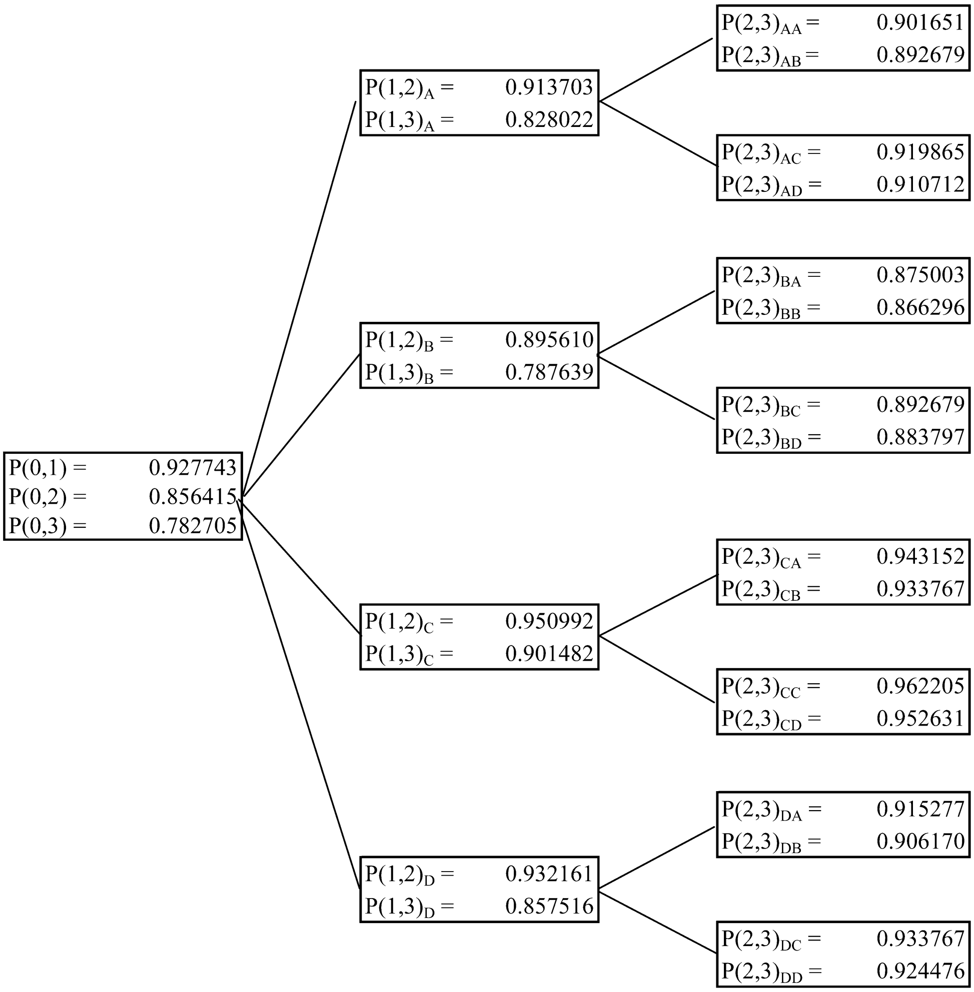

Fig. 2. Tree of zero coupon bond prices for two-factor two-period Heath–Jarrow–Morton model.

Checking these results, we have

$$E[P(1,2)]P(0,1) = (1/4)(0.913702 + 0.895610 + 0.950992 + 0.932161)0.927743$$
$$= 0.856415, \text{ which is } P(0,2)$$
$$E[P(1,3)]P(0,1) = (1.4)(0.828022) + 0.787639 + 0.901482 + 0.857516)0.927743$$
$$= 0.782704, \text{ which is } P(0,3) \text{ subject to a small round} - \text{off error}$$

At time 1, we can check whether the condition holds in each of four states:

$$E[P(2,3)]P(1,2) = P(1,3) \text{ for each state A, B, C, and D}$$

These results are obtained as follows:

$$E[P(2,3)]P(1,2)_{\mathrm{A}} = (1/4)(0.901651 + 0.892679 + 0.919865 + 0.910712)0.913703$$
$$= 0.828022, \text{ which is } P(1,3)_{\mathrm{A}}$$
$$E[P(2,3)]P(1,2)_{\mathrm{B}} = (1/4)(0.875003 + 0.866296 + 0.892679 + 0.883797)0.895610$$
$$= 0.787639, \text{ which is } P(1,3)_{\mathrm{B}}$$
$$E[P(2,3)]P(1,2)_{\mathrm{C}} = (1/4)(0.943152 + 0.933767 + 0.962205 + 0.952631)0.950992$$
$$= 0.901482, \text{ which is } P(1,3)_{\mathrm{C}}$$
$$E[P(2,3)]P(1,2)_{\mathrm{D}} = (1/4)(0.915277 + 0.906170 + 0.933767 + 0.924476)0.932161$$
$$= 0.857516, \text{ which is } P(1,3)_{\mathrm{D}}$$

4. Conclusions

The results of this paper provide two extensions that should facilitate the implementation of term structure models in discrete time frameworks. First it is shown that it is not necessary to guess the values of the next two possible interest rates in a one-factor arbitrage-free term structure with a normally distributed spot rate. The solutions can be obtained directly using a simple and easily programmable algorithm. Next we generalize the Grant–Vora result, which obtains the drift function for a discrete time binomial tree in a one-period HJM setting, to a multi-factor case. In both cases, it is illustrated how to implement these results.

Given the complexity and time consumption of interest rate tree calculations, steps to improve, simplify and expedite the process are valuable contributions to our understanding and implementation of the models. In particular, they show the generality of some of the results we know and use on a routine basis. In addition, they expedite the calibration of these models to the term structure and the volatility structure for caps and swaptions.

References

Arditti, F.D. (1996) *Derivatives: A Comprehensive Resource for Options, Futures, Interest Rate Swaps, and Mortgage Securities*, Harvard Business School Publishing, Boston, MA.

Cox, J.C., Ingersoll, J.E. Jr. and Ross, S.A. (1981) A re-examination of traditional hypotheses about the term structure of interest rates, *The Journal of Finance,* **36,** 769–99.

Grant, D. and Vora, G. (1999) Implementing no-arbitrage term structure of interest rate models in discrete time when interest rates are normally distributed, *The Journal of Fixed Income,* **8,** 85–98.

Grant, D. and Vora, G. (2003) Analytical implementation of the Ho and Lee model of the evolution of the spot interest rate, *Global Finance Journal,* **14,** 19–47.

Harrison, J.M. and Pliska, S.R. (1981) Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Processes and their Applications,* **11,** 215–60.

Harrison, J.M. and Kreps, D.M. (1979) Martingales and arbitrage in multiperiod securities markets, *Journal of Economic Theory,* **20,** 381–408.

Heath, D., Jarrow, R. and Morton, A. (1990a) Bond pricing and the term structure of interest rates: a discrete time approximation, *Journal of Financial and Quantitative Analysis,* **25,** 419–40.

Heath, D., Jarrow, R. and Morton, A. (1990b) Contingent claim valuation with a random evolution of interest rates, *Review of Futures Markets,* **9,** 54–76.

Heath, D., Jarrow, R. and Morton, A. (1992) Bond pricing and the term structure of interest rates: a new methodology, *Econometrica,* **60,** 77–106.

Ho, T.S.Y. and Lee, S.-B. (1986) Term structure movements and pricing interest rate contingent claims, *The Journal of Finance,* **41,** 1011–30.

Jarrow, R. (2002) *Modeling Fixed-Income Securities and Interest Rate Options*, 2nd. edn, Stanford University Press, Stanford, CA.

Jarrow, R. and Turnbull, S. (2000) *Derivative Securities*, 2nd edn, South-Western College Publishing, Cincinnati.

Ritchken, P. (1996) *Derivative Markets: Theory, Strategy, and Applications*, HarperCollins, New York.

Vasicek, O. (1977) An equilibrium characterization of the term structure, *Journal of Financial Economics,* **5,** 177–88.