**A tale of two theorems :**

**the Bolyai-Gerwein Theorem and the Banach-Tarski Paradox**

*Problem: Given two regions in the Euclidean plane or Euclidean three space, under what conditions is it possible to subdivide one into a finite number of pieces and to rearrange to form the other?*

As usual, both the meaning of the question and the answer to it depend critically on the meaning of the terms in the statement – in this case, the words *region, subdivide, piece* and *rearrange*.

**Version 1:** In the plane, let a *region* be a closed bounded set with straight line boundary pieces (call it polygonal), *subdivide* mean to cut with straight line cuts, *pieces* to be subregions of the same type and *rearrange* to mean reassembly up to Euclidean congruence - where two regions meet, if at all, in points or boundary straight line segments. Two regions related in this way will be called *congruent-by-dissection*.

A very satisfying theorem (and not that hard to prove) shows that the obvious necessary condition is also sufficient.

***THEOREM:*** *(Bolyai-Gerwein) Two polygonal regions in the plane are congruent-by-dissection if and only if they have the same area.*

**Version 2:** In three space, let a *region* be a set bounded by planar pieces, *subdivide* mean to cut with planar cuts, *pieces* mean subregions of the same type and *rearrange* mean to reassembly up to Euclidean congruence (where two regions meet, at all, in points or boundary straight line segments or boundary planar pieces). Two regions related in this way will be called *congruent-by-dissection*.

**Example:** The cube of unit volume and the regular tetrahedron of unit volume are not congruent by dissection. (This example, due to Dehn in 1900, answered *Hilbert’s Third Problem* in the negative and showed, in essence, that no theory of volumes in dimension three that applied to all polyhedra could avoid limit procedures.)

This suggests asking whether there is a less restrictive notion of *piece* whereby the cube and tetrahedron of the same volume are equivalent; going whole hog, we have the following.

**Version 3:** In Euclidean three space, let a *region* be a bounded set with interior, *subdivide* mean set theoretic union, *pieces* mean subsets and *rearrange* mean reassembly up to Euclidean congruence (i.e., disjoint union of congruent subpieces). Two regions related in this way will be called *equidecomposible*.

***THEOREM:*** *(Banach-Tarski)* ***Any*** *two regions in Euclidean three space are equidecomposible.*

***Corollary:*** *One can subdide the unit ball into a finite number of pieces and reassemble them to form two (or four or eight or …) unit balls.*

This apparent paradox (this is usually called the *Banach-Tarski Paradox)* is a consequence of the Axiom of Choice. The proof involves a detour through the theory of non-abelian free groups and is fundamentally a statement about the group of isometries of Euclidean three space. This particular paradox does not occur in two dimensional Euclidean geometry but does in two dimensional hyperbolic geometry.