THE PROOF OF A FAMOUS THEOREM

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1. Getting Started

Here is the result I want to prove.

Theorem 1.

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{1}$$

I will need the following lemma concerning the conversion from rectangular to polar coordinates for double integrals, which I shall not prove.

Lemma 2. *$dx\,dy = r\,dr\,d\theta$.*

Proof of Theorem 1. Let

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{2}$$

Then

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy. \end{aligned}$$

Converting to polar coordiantes, $x^2 + y^2 = r^2$, and applying Lemma 2, we get

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2}\,r\,dr\,d\theta. \tag{3}$$

The inside integral (in the variable r) is evaluated using basic calculus. We use the *substitution* $u = r^2$, so that $du = 2r\,dr$. Then

$$\int_0^{\infty} e^{-r^2}\,dr = \frac{1}{2}\int_0^{\infty} e^{-u}\,du = \frac{1}{2}. \tag{4}$$

Thus by (3) we have

$$I^2 = \frac{1}{2}\int_0^{2\pi} d\theta = \pi, \tag{5}$$

so that $I = \sqrt{\pi}$. □ □

2. How to do it!

For more ideas see Grätzer [1].

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References

[1] George Grätzer, *First Steps in LaTeX*, Birkhäuser Springer-Verlag, 1999.