

The Proof of a Famous Theorem

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1 Getting Started

Here is the result I want to prove.

Theorem 1

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \quad (1)$$

I will need the following lemma concerning the conversion from rectangular to polar coordinates for double integrals, which I shall not prove.

Lemma 2 $dx dy = r dr d\theta$.

Proof of Theorem 1: Let

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}. \quad (2)$$

Then

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy. \end{aligned}$$

Converting to polar coordinates, $x^2 + y^2 = r^2$, and applying Lemma 2, we get

$$I^2 = \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta. \quad (3)$$

The inside integral (in the variable r) is evaluated using basic calculus. We use the *substitution* $u = r^2$, so that $du = 2r dr$. Then

$$\int_0^{\infty} e^{-r^2} r dr = \frac{1}{2} \int_0^{\infty} e^{-u} du = \frac{1}{2}. \quad (4)$$

Thus by (3) we have

$$I^2 = \frac{1}{2} \int_0^{2\pi} d\theta = \pi, \quad (5)$$

so that $I = \sqrt{\pi}$. \square

2 How to do it!

For more ideas see Grätzer [1].

References

- [1] George Grätzer, *First Steps in L^AT_EX*, BIRKHÄUSER SPRINGER-VERLAG, 1999.