A HALMOS-VON NEUMANN THEOREM FOR MODEL SETS, AND ALMOST AUTOMORPHIC DYNAMICAL SYSTEMS

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Dedicated to Anatole Katok in celebration of his 60th birthday.

ABSTRACT. A subset $\Sigma \subseteq \mathbb{R}^d$ is the spectrum of a model set $\Lambda \subseteq \mathbb{R}^d$ if and only if it is a countable subgroup. The same result holds for $\Sigma \subseteq \widehat{G}$ for a large class of locally compact abelian groups G.

1. Introduction

A *model set* Λ is a special kind of uniformly discrete and relatively dense subset of a locally compact abelian group G. Model sets were first studied systematically in 1972 by Yves Meyer [15], who considered them in the context of Diophantine problems in harmonic analysis. More recently, model sets have played a prominent role in the theory of quasicrystals, beginning with N. G. de Bruijn's 1981 discovery [4] that the vertices of a Penrose tiling are a model set. Much of the interest in model sets is due to the fact that although they are aperiodic, model sets have enough "almost periodicity" to give them a discrete Fourier transform. This corresponds to spots, or Bragg peaks, in the X-ray diffraction pattern of a quasicrystal.

In this paper, we study model sets from the point of view of ergodic theory and topological dynamics. A translation invariant collection X of model sets in G has a natural topology, and with respect to this topology the translation action of G on X is continuous. We think of this action as a *model set dynamical system*. Model set dynamical systems are closely related to tiling dynamical systems (see [30], [24]), and the first examples that were worked out, Penrose tilings [25], "generalized Penrose tilings" [25], [10], and chair tilings [26], [1], were tilings with model sets as vertices. In each of these examples, the corresponding dynamical system can be shown to be an almost 1:1 extension of a *Kronecker dynamical system* (a rotation action on a compact abelian

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group). A general proof of this fact for model set dynamical systems was given by Schlottman [29].

Almost automorphic functions were defined by Bochner around 1955 as a generalization Bohr almost periodic functions. In 1965, Veech [31] defined an *almost automorphic dynamical system* to be the orbit closure of an almost automorphic function. This generalizes the fact that the orbit closure of an almost periodic function is an equicontinuous dynamical system (i.e., a dynamical system topologically conjugate to a Kronecker system). Veech showed that almost automorphic dynamical systems are always minimal. He also proved a structure theorem [31], which says that a dynamical system is almost automorphic if and only if it is topologically conjugate to an almost 1:1 extension of a Kronecker system (which is its unique maximal equicontinuous factor).

It follows from Veech's structure theorem, combined with Schlottman's result above, that all model set dynamical systems are almost automorphic. Some older well known examples also turn out to be almost automorphic: the *Sturmian sequences*, defined in 1955 by Gotschalk and Hedlund [7], and the *Topeplitz sequences*, introduced in 1952 by Oxtoby [20] (see [32]). Perhaps not surprisingly, these sequences can be interpreted as model sets[1] in $G = \mathbb{Z}$.

Later work on almost automorphic dynamical systems (see e.g., [21], [14], [32], [2]) brought to light the existence of a dichotomy. In one case, which we will regard in this paper as somewhat pathological, a lack of unique ergodicity can lead to positive topological entropy (see [14], [32]). On the other hand, in the uniquely ergodic case one has metric isomorphism to the Kronecker factor. This implies topological entropy zero, and also pure point spectrum. The latter suggests the Halmos-von Neumann Theorem.

Published in 1942, the *Halmos-von Neuman theorem* [8] is one of the oldest results of ergodic theory. Extended to actions of locally compact metric abelian groups G, it says that every ergodic measure preserving dynamical system with pure point spectrum $\Sigma \subseteq \widehat{G}$ is metrically isomorphic to a unique Kronecker system. This Σ is always a countable dense subgroup, of $\widehat{G}$, and (this part is sometimes called the *Halmos-von Neumann representation theorem*) any $\Sigma \subseteq \widehat{G}$ can arise as the spectrum of a Kronecker dynamical system. The main goal of this paper is to obtain a similar result for model set dynamical systems.

[1]As de Bruijn [4] observed, Sturmian sequences are closely related to Penrose tilings. Also, Toeplitz sequences are closely related to chair tilings (see [1], [26]).

Spectral problems about model sets naturally divide into two parts[2], which correspond to the two parts of the model set construction. The first part is algebraic, and is called a *cut and project scheme*. By itself, a cut and project scheme determines the Kronecker factor of a model set dynamical system. In Theorem 3.8 we obtain a Halmos-von Neumann representation theorem for cut and project schemes, showing (modulo a few algebraic assumptions) that any countable dense subgroup $\Sigma \subseteq \widehat{G}$ can be the spectrum. Equivalently, every Kronecker dynamical system can be realized in terms of a cut and project scheme. However, unlike the classical Halmos-von Neumann theorem, our result is not unique, and different cut and project schemes can have the same spectrum. In particular, all the cut and project schemes constructed via Theorem 3.8 have a compact group H as their *internal space*. Yet in some cases there exist cut and project schemes with the same spectrum, but where H is not compact. Ultimately, these lead to completely different model sets.

The second part of the model set construction is geometric, and involves a choice of (topologically nice) subset W, called a *window*, of the internal space H. Expressed in terms of dynamical systems theory, the choice of a window is equivalent to the construction of a particular almost 1:1 extension. It is well known in topological dynamics (see [21], [14]) that if the boundary of W has Haar measure zero, then the almost automorphic dynamical system is uniquely ergodic, and hence metrically isomorphic to its Kronecker factor (the same result is also known in context of model sets, see [17]). We show in Theorem 6.6 that if H is compact, then such a window W always exists. Combining this with Theorem 3.8, we obtain our main result, Theorem 6.7, which is a Halmos-von Neumann Theorem for model sets. In particular, it says that for any countable dense subgroup $\Sigma \subseteq \widehat{G}$, there exists a model set $\Lambda \subseteq G$ with Σ as its spectrum.

A second goal of this paper is to provide an exposition of the theory of model sets and model set dynamical systems. While this material is mostly not new (we follow [15], [29] and [17]) our emphasis is different because we concentrate on the connections to dynamical systems theory, and in particular, to the theory of almost automorphic dynamical systems. However, our exposition is not exhaustive, and there are several interesting topics we do not cover. These include the question of which model sets have a local matching rule (see [12]), and questions about the algebraic topology of model set spaces (see [5]). There is

[2]An important third part, pertaining to the diffraction spectrum, is briefly discussed in Section 7.

also an interesting new topology on collections of model sets, called the autocorrelation topology, that gives the Kronecker factor [18].

2. Cut and project schemes

2.1. **Locally compact abelian groups.** Most of the groups G discussed in this paper will be a Hausdorff, locally compact abelian groups (abbreviated LCA groups). We will always assume groups are metrizable and separable, which is equivalent to satisfying the second countability axiom. Since every σ-compact metric space is separable, and every separable, locally compact metric space is σ-compact, our groups are the same as the metrizable σ-compact locally compact abelian groups. Let $\widehat{G}$ denote the dual of G (see [19], [23]). Since metrizability is dual to σ-compactness (see [23]), it follows that $\widehat{G}$ is also a separable metric group. Metrizable compact (Hausdorff) abelian groups G (abbreviated CA groups) satisfy all these hypotheses. A discrete abelian group G is an LCA group in our sense if and only if it is countable. We abbreviate these as DA groups.

2.2. **Kronecker dynamical systems.** A *Kronecker dynamical system* is an action R of a non-compact LCA group G on a CA group Y by *rotations* (i.e., translations). We can write this as $R^g y = y + \varphi(g)$, where $\varphi : G \to Y$ is a continuous homomorphism. We are going to want R to be a *free action* (i.e., $R^g = I$ implies $g = 0$), which is equivalent to φ being an injection. We are also going to want R to be *minimal*[3], which is equivalent to φ having a dense range. We call a topological group homomorphism with dense range *topologically surjective*. For Kronecker systems, minimality is equivalent to unique ergodicity, and the combination of these two properties is called *strict ergodicity*.

Definition 2.1. A continuous LCA group homomorphism $\varphi : G \to Y$ is called a *compactification* of G if Y is CA, and φ is injective and topologically surjective.

Thus, R being a free, strictly ergodic Kronecker G-action on Y is equivalent to $\varphi : G \to Y$ being a compactification of G.

The dual $\widehat{Y}$ of a CA group Y is a DA group (see [23]). Since φ injective and topologically surjective, so is the dual homomorphism $\widehat{\varphi} : \widehat{Y} \to \widehat{G}$. This follows from the fact—which we will use repeatedly—that injectivity and topological surjectivity are dual properties (see [23]).

[3] A G action on Y compact metric is minimal if and only if every orbit is dense.

Define $\Sigma_R := \widehat{\varphi}(\widehat{Y})$, which is a countable dense subgroup of $\widehat{G}$. It is called the *point spectrum* of R. In particular, if $\chi := \widehat{\varphi}(f) \in \Sigma_R$ for some $f \in \widehat{Y}$, (i.e., $f : Y \to \mathbb{T} \subseteq \mathbb{C}$), then

$$\begin{aligned} f(R^g y) = \quad & f(y + \varphi(g)) = \quad f(\varphi(g))f(y) \\ = \quad & \widehat{\varphi}(f)(g)f(y) = \quad \chi(g)f(y), \end{aligned}$$

or more briefly

$$f \circ R = \chi f. \tag{2.1}$$

We call f as an *eigenfunction* for R, corresponding to the "*eigenvalue*" χ. A standard argument shows that this construction produces all L^2 functions (for Haar measure θ_Y on Y) that satisfy (2.1). Moreover, since $\mathrm{span}(\widehat{Y})$ is dense in L^2, it follows that R (i.e., any Kronecker system) has *pure point spectrum*.

Conversely, suppose Σ is a countable dense subgroup of $\widehat{G}$ (such a subgroup exists since $\widehat{G}$ is separable). Let $Y := \widehat{\Sigma}$ (which is compact) and let $\varphi := \widehat{i} : G \to Y$, where $i : \Sigma \to \widehat{G}$ is inclusion. Then φ is a compactification of G, and thus defines a free, strictly ergodic Kronecker system R with $\Sigma_R = \Sigma$. This is the *Halmos-von Neumann representation theorem* (see [8]).

Remark 2.2. A CA group Y that is a compactification of $\mathbb{Z}$ is called *monothetic*. A CA group Y that is a compactification of $\mathbb{R}$ is called *solenoidal*. We use the terms *d-monothetic*[4] and *d-solenoidal* for compactifications of $\mathbb{Z}^d$ and $\mathbb{R}^d$ respectively.

2.3. **Cut and project schemes.** A cut and project scheme is the algebraic part of the definition of a model set. A discrete closed subgroup of a LCA group G is called a *lattice* if it has a compact quotient.

Definition 2.3. A *cut and project scheme* is a triple $\mathcal{S} = (G, H, \Gamma)$ of LCA groups that satisfies:

(1) $\Gamma \subseteq G \times H$ is a lattice,
(2) $p_1|_\Gamma$ is injective and,
(3) $p_2(\Gamma)$ is dense in H.

Here, $p_1 : G \times H \to G$ and $p_2 : G \times H \to H$ denote the coordinate projections. The groups G and H are called (respectively) *physical space* and *internal space*. The subgroup $L := p_1(\Gamma) \subseteq G$ is called the *structure group*, and the mapping

$$g \mapsto g^* := p_2(p_1^{-1}(g)) : L \to H$$

[4]The definition of d-monothetic in [2], as a d-fold product of monothetic groups, is a special case of our definition.

is called the *star homomorphism*.

Definition 2.4. A cut and project scheme is called *aperiodic* if any of the following (equivalent) conditions hold

(1) $p_2|_\Gamma$ is injective,
(2) the *period group*, $J := \ker(*)$, is trivial, or
(3) $(\{0\} \times H) \cap \Gamma = \{0\}$.

We will usually assume a cut and project scheme is aperiodic.

Let $\mathcal{S} = (G, H, \Gamma)$ be a cut and project scheme. Let $Y := (G \times H)/\Gamma$, which by Definition 2.3 (1), is a CA group. Define

$$\varphi = \pi \circ i_1 : G \to Y, \tag{2.2}$$

where $i_1 : G \to G \times H$ is the coordinate injection and π is the canonical homomorphism.

Proposition 2.5. *The homomorphism φ in* (2.2) *is a compactification, so the G action R on Y, defined by*

$$R^g y := y + \varphi(g), \tag{2.3}$$

is a free, strictly ergodic Kronecker system. We call this the action the Kronecker system associated with cut and project scheme $\mathcal{S}$. *The spectrum of R is $\Sigma = \widehat{\varphi}(\widehat{Y})$.*

Proof. If $\varphi(g) = 0$, then $(g, 0) \in \Gamma$. Thus $g \in L$ and $g^* = p_2(g, 0) = 0$, so that $g = 0$ by the aperiodicity of $\mathcal{S}$. It follows that φ is injective.

To show φ is topologically surjective, it suffices to show $i_1(G) + \Gamma = (G \times \{0\}) + \Gamma$ is dense in $G \times H$. But

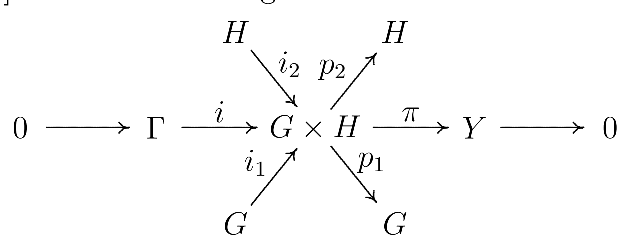
$$(G \times \{0\}) + \Gamma = (G \times \{0\}) \oplus (\{0\} \times p_2(\Gamma)) = G \times p_2(\Gamma),$$

which is dense by Definition 2.3, (3). Everything else follows. □

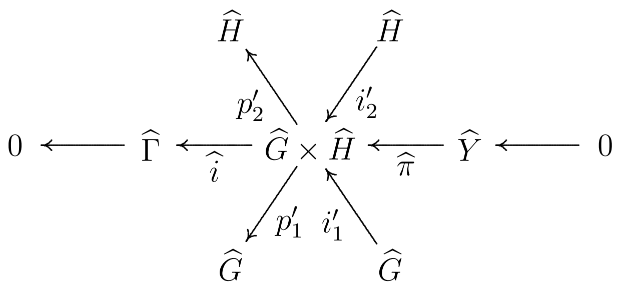
2.4. The spectrum of a cut and project schemes.

Definition 2.6. By the *spectrum* $\Sigma_{\mathcal{S}}$ of a cut and project scheme $\mathcal{S}$ we mean the spectrum $\Sigma_R = \widehat{\varphi}(\widehat{Y})$ of the corresponding Kronecker dynamical system R.

Now we will describe $\Sigma_{\mathcal{S}}$ in terms of $\mathcal{S}$ itself. To this end, we follow Y. Meyer [15]. Consider the diagram



where i denotes inclusion map. Note that the horizontal row is exact. The dual of this diagram is given by



where the coordinate injections i_k' satisfy $i_k' = \widehat{p_k}$, and the coordinate projections satisfy $p_k' = \widehat{i_k}$. Since the dual of an exact sequence is exact (see [19]) the horizontal row of this sequence is also exact.

Theorem 2.7 (Meyer, [15]). *Define* $\Delta := \widehat{\pi}(\widehat{Y}) \subseteq \widehat{G} \times \widehat{H}$. *Then* $\widehat{\mathcal{S}} = (\widehat{G}, \widehat{H}, \Delta)$ *is a cut and project scheme. We call it the* dual *of* $\mathcal{S}$.

Proof. We need to verify Definition 2.3. By duality theory, Δ is a lattice, so (1) holds.

By Proposition 2.5, $\varphi = \pi \circ i_1$ is a compactification, so it is topologically surjective. Thus its dual, $\widehat{\varphi} = p_1' \circ \widehat{\pi}$ is injective. This implies $p_1'|_\Delta$ is injective, proving (2).

We have $p_2'(\Delta) = p_2'(\widehat{\pi}(\widehat{Y}))$. To prove (3) we need to show this is dense, or equivalently that $p_2' \circ \widehat{\pi}$ is topologically surjective. This is equivalent to showing that

$$(p_2' \circ \widehat{\pi})\widehat{\ } = (\widehat{i_2} \circ \widehat{\pi})\widehat{\ } = \pi \circ i_2$$

is injective. If $(0,h) \in \ker(\pi)$ then $(0,h) \in \Gamma$, and also $p_1(0,h) = 0$. Since p_1 is injective, $h = 0$. □

Corollary 2.8. *The spectrum* $\Sigma_{\mathcal{S}}$ *of a cut and project scheme* $\mathcal{S}$ *is the structure group of the dual* $\widehat{\mathcal{S}}$ *of* $\mathcal{S}$.

Proof. $\Sigma_{\mathcal{S}} = \Sigma_R = \widehat{\varphi}(\widehat{Y}) = p_1'(\widehat{\pi}(\widehat{Y})) = p_1'(\Delta)$. □

Corollary 2.9.

(1) *A cut and project scheme* $\mathcal{S}$ *has a dense structure group* $L \subseteq G$ *if and only if its dual* $\widehat{\mathcal{S}}$ *is aperiodic.*
(2) *A cut and project scheme* $\mathcal{S}$ *is aperiodic if and only of its spectrum* Σ *is dense in* $\widehat{G}$.

Remark 2.10. The *flip* $\widetilde{\mathcal{S}} := (H, G, \Gamma)$ of a cut and project scheme $\mathcal{S} = (G, H, \Gamma)$, is a cut and project scheme if and only if $\mathcal{S}$ is aperiodic and has a dense structure group. In this case the flip of the dual $\widehat{\mathcal{S}}$ is also a cut and project scheme (the dual of the flip).

3. Realization for cut and project schemes

3.1. The realization property.

Definition 3.1. Let G be a non-compact LCA group. We say G satisfies the *realization property* if, given a countable dense subgroup of $\Sigma \subseteq \widehat{G}$, there is an aperiodic cut and project scheme $\mathcal{S} = (G, H, \Gamma)$ with $\Sigma_{\mathcal{S}} = \Sigma$.

It will be convenient to phrase realization in terms of the dual cut and project scheme.

Lemma 3.2. *Let G be a non-compact LCA group and let Σ be a countable dense subgroup of $\widehat{G}$. Suppose $\mathcal{S}' = (\widehat{G}, H', \Gamma')$ is a cut and project scheme with structure group Σ. Then the dual $\mathcal{S} := \widehat{\mathcal{S}'}$ is aperiodic and $\Sigma_{\mathcal{S}} = \Sigma$.*

Proof. By Theorem 2.7, $\mathcal{S}$ is a cut and project with $\widehat{\mathcal{S}} = \mathcal{S}'$, and by Corollary 2.8, $\mathcal{S}$ has spectrum Σ. Since Σ is dense it follows from (2) of Corollary 2.9 that $\mathcal{S}$ is aperiodic. □

3.2. The discrete case.

Theorem 3.3. *Every infinite DA group G satisfies the realization property.*

Proof. Note that $\widehat{G}$ is CA. Given a countable dense subgroup $\Sigma \subseteq \widehat{G}$, we put $H' = \Sigma_d$ (the subscript d indicates Σ given the discrete topology). Using the fact that $H' \subseteq \widehat{G}$ we define

$$\Gamma' = \{(k, k) : k \in \Sigma\} \subseteq \widehat{G} \times H'.$$

By Lemma 3.2 it suffices to show $\mathcal{S}' = (\widehat{G}, H', \Gamma')$ is a cut and project scheme with structure group Σ. We then have $\mathcal{S} = \widehat{\mathcal{S}'}$.

First, we have that Γ' is a lattice since H' is discrete, and $(\widehat{G} \times H')/\Gamma' = \widehat{G}$ is compact. This shows Definition 2.3 (1). For (2), we have $p_1'(k, k) = k$ which implies $k = 0$ for $(k, k) \in \ker(p_1')$. Clearly $p_1(\Gamma') = \Sigma$, so $\mathcal{S}'$ has structure group Σ. For (3), we note that p_2' is surjective. □

Remark 3.4.

(1) The examples constructed in Theorem 3.3 all have $H = \widehat{\Sigma_d}$, which is compact. We call this kind of cut and project scheme *internally compact.*

(2) The cut and project scheme $\mathcal{S}'$ in Theorem 3.3 has a *compact* physical space $\widehat{G}$. While such model sets do not lead to useful

quasicrystals (or to Kronecker systems), they do play an important role in the theory as duals to cut and project schemes with discrete physical spaces.

Example 3.5. Let $G = \mathbb{Z}$ and for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ let

$$\Sigma = \left\{e^{2\pi i n\alpha} : n \in \mathbb{Z}\right\} \subseteq \mathbb{T} = \widehat{\mathbb{Z}}.$$

Identify $\Sigma_d = \mathbb{Z}$ and let $H = \widehat{\Sigma_d} = \widehat{\mathbb{Z}} = \mathbb{T}$, so that $G \times H = \mathbb{Z} \times \mathbb{T}$. Then

$$\Gamma = \left\{(n, e^{2\pi i n\alpha}) : n \in \mathbb{Z}\right\} \subseteq G \times H$$

is a lattice, and $\mathcal{S} = (\mathbb{Z}, \mathbb{T}, \Gamma)$ is a cut and project scheme with spectrum Σ. Here, $R = R_\alpha$ is the *irrational rotation* on the circle $\mathbb{T}$.

Example 3.6. Again let $G = \mathbb{Z}$, but now let $\Sigma = e^{2\pi i \mathbb{Z}[\frac{1}{2}]}$, where $\mathbb{Z}[\frac{1}{2}] \subseteq \mathbb{Q}$ denotes the dyadic rationals. It is well known that $H := \widehat{\Sigma_d} = \mathbb{Z}_2$, the group of dyadic integers. Putting $\mathcal{S} = (\mathbb{Z}, \mathbb{Z}_2, \Gamma)$, where $\Gamma = \{(n, n) : n \in \mathbb{Z}\}$, we have that $\mathcal{S}$ is a cut and project scheme with spectrum Σ. Here R is the von Neumann adding machine, viewed as a rotation on $\mathbb{Z}_2$.

3.3. **Realization in a large class of groups.**

Definition 3.7. We say an LCA group G is *lattice dense* if every countable dense subgroup Σ of G has a subgroup $\Omega \subseteq \Sigma$ that is a lattice in G. A group *has lattice dense dual* if $\widehat{G}$ is lattice dense.

All CA groups and DA groups are both lattice dense and have lattice dense duals. Our main result in this section is the following theorem, which can be interpreted as a Halmos-von Neumann theorem for cut and project schemes.

Theorem 3.8. *Suppose G is a non-compact LCA group with a lattice dense dual. Then G satisfies the realization property. In particular, for any countable dense subgroup $\Sigma \subseteq \widehat{G}$, there is an aperiodic cut and project scheme $\mathcal{S} = (G, H, \Gamma)$ with spectrum $\Sigma_{\mathcal{S}} = \Sigma$.*

Before proving the theorem, we discuss some examples.

Proposition 3.9. *The group $\mathbb{R}^d$ is lattice dense and has a lattice dense dual.*

Proof. Since $\widehat{\mathbb{R}^d} \cong \mathbb{R}^d$ it is enough to show $\mathbb{R}^d$ is lattice dense.

Any set of d vectors in $\mathbb{R}^d$ that are linearly independent over $\mathbb{R}$ generates a lattice. Let L' be a lattice in $\mathbb{R}^d$ generated by $\mathbf{v}'_1, \ldots, \mathbf{v}'_d$. For ϵ sufficiently small, if $\mathbf{v}_1, \ldots, \mathbf{v}_d$ satisfy $||\mathbf{v}_j - \mathbf{v}'_j|| < \epsilon$ for all j, then $\mathbf{v}_1, \ldots, \mathbf{v}_d$ are also linear independent over $\mathbb{R}$, and thus also generate a lattice L. $\square$

Recall that a LCA group G is called *compactly generated* if there is a compact subset $K \subseteq G$ that generates G. The *Structure Theorem* (see [9]) says that a (not necessarily metrizable) LCA group is compactly generated if and only if it has the form $G = \mathbb{R}^{d_1} \times \mathbb{Z}^{d_2} \times Y$, where Y is compact. Such a G is metrizable if and only if Y is. If $Y = \mathbb{T}^{d_3} \times F$, where F is finite, G is called "elementary".

Corollary 3.10. *Any metrizable compactly generated LCA group G, and in particular, any elementary group, is lattice dense and has a lattice dense dual. In particular, all such groups satisfy the realization property.*

Proposition 3.9 and Corollary 3.10 show that all the physical spaces G likely to be of interest in the theory of quasicrystals satisfy the realization property. There are, however, important LCA groups that are not lattice dense (and do not have lattice dense duals).

Proposition 3.11. *The group (i.e., field) $\mathbb{Q}_p$ of p-adic numbers is not lattice dense and does not have a lattice dense dual.*

Proof. For $\alpha \in \mathbb{Q}_p$, $\alpha \neq 0$, the subgroup $\alpha\mathbb{Z}$ is infinite (see [9] for a definition of $\mathbb{Q}_p$). Moreover, $\overline{\alpha\mathbb{Z}} = \alpha\overline{\mathbb{Z}} = \alpha\mathbb{Z}_p$, where $\mathbb{Z}_p \subseteq \mathbb{Q}_p$ denotes the set of p-adic integers, a compact subgroup. It follows that there can be no lattice in $\mathbb{Q}_p$ since the closed subgroup generated by any $\alpha \neq 0$ is compact. Since $\widehat{\mathbb{Q}_p} \cong \mathbb{Q}_p$, it follows that $\mathbb{Q}_p$ does not have a lattice dense dual. □

The proof of Theorem 3.8 follows immediately from the next lemma.

Lemma 3.12. *Let G be a non-compact LCA group and let Σ be a countable dense subgroup of $\widehat{G}$. Suppose there is a lattice Ω in $\widehat{G}$ with $\Omega \subseteq \Sigma$. Then there is a cut an project scheme $\mathcal{S} = (G, H, \Gamma)$ with spectrum Σ.*

Proof. Let $\omega : \widehat{G} \to \widehat{G}/\Omega$ be the canonical homomorphism. Since Ω is a lattice, $\widehat{G}/\Omega$ is compact, and it contains a countable dense subgroup Σ/Ω. We put $H' := (\Sigma/\Omega)_d$ and define

$$\Gamma' := \{(g, h) \in \widehat{G} \times H' : \omega(g) \in H'\}. \tag{3.1}$$

Note that

$$p_1'(\Gamma') = \{g \in \widehat{G} : \omega(g) \in H'\} = \omega^{-1}(\Sigma/\Omega) = \Sigma. \tag{3.2}$$

By Lemma 3.2 it suffices to show that $\mathcal{S}' = (\widehat{G}, H', \Gamma')$ is a cut and project scheme, since (3.2) shows that $\mathcal{S}'$ has structure group Σ.

Clearly (1) Γ' is a lattice in $\widehat{G} \times H'$, (2) p_1' is injective (see (3.2)), and (3) p_2' is surjective (and thus topologically surjective) since $p_2'(\Gamma') = H_1$ by (3.1). $\square$

Example 3.13. Let $G = \mathbb{R}$ and for $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, let

$$\Sigma := \{n + m\alpha : (n,m) \in \mathbb{Z}^2\} \subseteq \widehat{G} = \widehat{\mathbb{R}} = \mathbb{R}.$$

For $\Omega = \mathbb{Z} \subseteq \Sigma \subseteq \mathbb{R}$ let

$$G_1 := \widehat{G}/\Omega = \mathbb{T} \cong \{z \in \mathbb{C} : |z| = 1\},$$

so that for $\Omega = \mathbb{Z} \subseteq \Sigma \subseteq \mathbb{R}$, we have

$$\Sigma_1 := \Sigma/\Omega \cong \left\{e^{2\pi i n\alpha} : n \in \mathbb{Z}\right\} \subseteq G_1,$$

is countable and dense. Letting $H' := (\Sigma_1)_d \cong \mathbb{Z}$ and $\Gamma_1 = (j \times j)(H')$, we note in passing that $\mathcal{S}_1 = (G_1, H', \Gamma_1)$ is the cut and project scheme dual to Example 3.5.

Following (3.1), we have

$$\Gamma' = \{(n + m\alpha, m) \in \mathbb{R} \times \mathbb{Z} : (n,m) \in \mathbb{Z}^2\},$$

so that $\mathcal{S}' = (\widehat{G}, H', \Gamma') = (\mathbb{R}, \mathbb{Z}, \Gamma')$ is a cut and project scheme with structure group Σ. It follows that $\mathcal{S} = \widehat{\mathcal{S}'} = (\mathbb{R}, \mathbb{T}, \Gamma)$, where $\Gamma = \{(n, e^{2\pi i n}) : n \in \mathbb{Z}\}$, is a cut and project scheme with spectrum Σ.

The structure group of $\mathcal{S}$ satisfies $L = \mathbb{Z}$. The associated Kronecker system R is the "irrational flow" on $\mathbb{T}^2$ with slope α. It is a suspension of the irrational rotation R_α from Example 3.5.

Remark 3.14. Many features of Example 3.13 are characteristic of the cut and project schemes $\mathcal{S}$ produced by Theorem 3.3:

1. The internal space H is always compact.
2. If G is not discrete then the structure group L is a lattice in G. In particular it is never dense.
3. The Kronecker system R is a suspension of the Kronecker system of another cut and project scheme, call it $\mathcal{S}_1$, which has physical space L, internal space H, and spectrum Σ/Ω.

Example 3.15. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}(2,\mathbb{R})$ and put $\Gamma = A\mathbb{Z}^2 \subseteq \mathbb{R}^2$. If $\frac{a}{c}, \frac{b}{d} \in \mathbb{R}\backslash\mathbb{Q}$, then $\mathcal{S}(A) = (\mathbb{R}, \mathbb{R}, \Gamma)$ is an aperiodic cut and project scheme with dense structure group $L = \{na + mc : (n,m) \in \mathbb{Z}^2\}$. Then $\mathcal{S}(A)\widehat{\ } = \mathcal{S}((A^{-1})^t)$ (the columns of $(A^{-1})^t$ are basis for the dual lattice for Γ).

Let us now specialize. Starting with the subgroup $\Sigma = \{n + m\alpha : (n,m) \in \mathbb{Z}^2\} \subseteq \widehat{\mathbb{R}}$ from Example 3.13, let $A = \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix}$. Then we get an aperiodic cut and project scheme $\mathcal{S} = (\mathbb{R}, \mathbb{R}, \Gamma') = \mathcal{S}((A^{-1})^t)$, with dense structure group, and spectrum Σ.

Even though this example has the same spectrum as Example 3.13, it looks quite different. Notably, H is not compact. As we will see, although the model sets obtained from these two cut and project schemes have the same spectrum, geometrically they are considerably different.

3.4. **Remarks on the general case.** Although the structure group L of a cut and project scheme may not be dense in G, it will always at least be relatively dense. Suppose G is a LCA group and let $L \subseteq G$ be a dense or relatively dense, countable subgroup. Here we consider the following question: Are there groups H and Γ so that $\mathcal{S} = (G, H, \Gamma)$ is a cut and project scheme with structure group L? So far, we know that the answer is yes if L contains a lattice (this follows from Lemma 3.12).

Let us now assume that L is dense in G, but not equal to G. Let d denote the translation invariant metric for the topology on G, and pull it back to L. In this topology (which is not discrete) L is not locally compact. Every metrizable LCA group is, in fact, completely metrizable (see [3] where it is shown that every locally compact topological group is completely uniformizable). It follows that G is the unique completion of L with respect to d. Define the "norm" $|g| := d(g, 0)$.

To see what we are after, let $\mathcal{S} = (G, H, \Gamma)$ be an aperiodic cut and project scheme with structure group L. Note that $* = p_2 \circ p_1^{-1} : L \to H$ is injective. Pulling back the invariant metric d' on H, we get a second metric d' on L, and we denote the corresponding norm $|g|' := d'(g, 0)$. Then H is the completion of L in d'. The fact that Γ is a lattice in $G \times H$ can be expressed in terms these two norms:

Lemma 3.16. *Let L be a countable abelian group, and suppose L has two norms $|\cdot|$ and $|\cdot|'$ (that come from two invariant topological group metrics). Let G and H (respectively) be the completions of L with respect to these norms (i.e., metrics). Define $\Gamma \subseteq G \times H$ by $\Gamma := (i \times i)(K)$. Then Γ is a lattice if and only if:*

1. *There exists $r > 0$ so that if $\ell \in L$ satisfies $|\ell| < r$, then $|\ell|' > r$.*
2. *There exists $R > 0$ so that for any $g, h \in L$ there exists $\ell \in L$ so that $|g - \ell| < R$ and $|h - \ell|' < R$.*

Proof. It is easy to see that (2) is equivalent to the condition that for any $(g, h) \in G \times H$ there exists $\ell \in L$ so that $|g - \ell| < R$ and $|h - \ell|' < R$. Combined with (1), this shows that Γ is a *Delone set* in $G \times H$ (see Definition 4.5 below). The lemma follows, since for a subgroup, the Delone property is equivalent to Γ being a lattice (see Lemma 4.6). $\square$

In effect, (1) and (2) say that the identity map needs to be wildly discontinuous.

Corollary 3.17. *Let L be a countable dense subgroup of a LCA group G and let d denote the metric induced on L by G. Then there is an aperiodic cut and project scheme $\mathcal{S} = (G, H, \Gamma)$ with structure group L if and only if* (i) *H is the completion of L with respect to another metric d',* (ii) *conditions* (1) *and* (2) *of* Lemma 3.16 *are satisfied, and* (iii) $\Gamma = (i \times i)(L)$.

Example 3.18. Consider $\Sigma = \mathbb{Z}[\frac{1}{p}] \subseteq \mathbb{Q}_p$. Then Σ inherits the p-adic valuation $|\cdot|_p$ from $\mathbb{Q}_p$. But $\mathbb{Z}[\frac{1}{p}]$ also has the usual absolute value $|\cdot|$, and with respect to that, its completion is $\mathbb{R}$. One can check that the conditions of Lemma 3.16 hold. Let $\Gamma' = (i \times i)(\mathbb{Z}[\frac{1}{2}]) \subseteq \mathbb{Q}_p \times \mathbb{R}$. Then $\mathcal{S}' = (\mathbb{Q}_p, \mathbb{R}, \Gamma')$ with is an aperiodic cut and project scheme with structure group Σ. From this it follows $\mathcal{S} = \widehat{\mathcal{S}'} = (\mathbb{Q}_p, \mathbb{R}, \Gamma)$ is a cut and project scheme with spectrum Σ.

4. Model sets

4.1. **Definition.** Let $\mathcal{S} = (G, H, \Gamma)$ be a cut and project scheme with G non-compact. A *precompact* subset $W \subseteq H$ is called the *window*. The *model set* corresponding to $\mathcal{S}$ and W is defined to be the subset of G (i.e., of L) given by the *selection* construction:

$$\Lambda(W) := p_1((G \times W) \cap \Gamma). \tag{4.1}$$

Selection is the geometric part of the definition of a model set.

More generally, any translation $\Lambda(W) - g$, $g \in G$ is also called a model set. We call

$$\mathcal{M}(W) := \{\Lambda(W + h) - g : (g, h) \in G \times H\} \tag{4.2}$$

the *model set family* associated with the window W.

Example 4.1. Let $\mathcal{S}$ be the cut and project scheme in Example 3.5 and let $W := [e^{2\pi i a}, e^{2\pi i b}]$ be a proper closed arc in $H = \mathbb{T}$. Then

$$\Lambda(W) = \{n \in \mathbb{Z} : e^{2\pi i n\alpha} \in W\} \subseteq L = \mathbb{Z}.$$

The *Sturmian case* is $a = \theta$ and $b = \theta + \alpha$. Assume $\theta \neq e^{2\pi i n\alpha}$ for any n (we call this *nonsingular* below). Then the "characteristic sequence" (see [13]) $x = (x_k) \in \{0, 1\}^{\mathbb{Z}}$, defined by $x_k := \chi_{\Lambda(W)}(k)$, is the classical *Sturmian sequence*.

More generally, nonsingularity means

$$e^{2\pi i n\alpha} \notin \{e^{2\pi i a}, e^{2\pi i b}\}. \tag{4.3}$$

Example 4.2. Take $\mathcal{S}$ as in Example 3.13, noting that $H = \mathbb{T}$ as in Example 3.5. Taking W from Example 4.1, we get the same $\Lambda(W)$, but this time as a subset of $\mathbb{R}$. Note that $L = \mathbb{Z} \subseteq \mathbb{R}$ is a lattice and is generated by $\Lambda(W)$.

Example 4.3. For $\mathcal{S}(A)$ in Example 3.15 let $W = [p, q] \subseteq \mathbb{R}$. Then $\Lambda(W) = \{na+mc : (n, m) \in \mathbb{Z}^2, na+mc \in [p, q]\} \subseteq \mathbb{R}$. In this example, "nonsingularity" means p and q are chosen so that $na + mc \notin \{p, q\}$ for any $(n, m) \in \mathbb{Z}^2$.

4.2. **Properties of model sets.**

Lemma 4.4.

(1) $\Lambda(W) - g \subseteq L$ *implies* $g \in L$.
(2) $g \in L$ *implies* $\Lambda(W) - g = \Lambda(W - g^*)$.
(3) $g \in J$ *implies* $\Lambda(W) - g = \Lambda(W)$.
(4) $\Lambda(W) - \Lambda(W) = \Lambda(W - W)$.
(5) *If* $\Lambda' \subseteq \Lambda(W)$ *then* Λ' *is a model set.*

Proof. For (1) $\Lambda(W) \subseteq L$ so $g = g_1 - g_2$ for $g_1, g_2 \in L$. For (2) we first note that $\Lambda(W) = \{g \in L : g^* \in W\}$. Thus $\Lambda(W - g^*) = \{g_1 \in G : (g_1+g)^* \in W\} = \Lambda(W)-g$. For (3), we have $\Lambda(W)-g = \Lambda(W-g^*) = \Lambda(W)$ where the first equality is by (2), and the second is by $g^* = 0$ since $g \in J$. The equation in the proof of (2) also proves (4). Since any subset of a precompact set is precompact, (5) follows. □

We are going to want model sets to have some additional properties. The most basic of these is the *Delone property.*

Definition 4.5. A subset $\Lambda \subseteq G$ is called *uniformly discrete* if there exists an open neighborhood U of 0 in G such that $(U + g) \cap \Lambda = \{g\}$ for any $g \in \Lambda$.

A subset $\Lambda \subseteq G$ is called *relatively dense* if there exists a compact $K \subseteq G$ such that $K + \Lambda = G$.

A *Delone set* is any set that is both uniformly discrete and relatively dense.

It will often be useful to express uniformly discrete and relatively dense in terms of the metric on G. Uniformly discrete means that there exists $r > 0$ such that $B_r(g) \cap \Lambda = \{g\}$ for all $g \in \Lambda$. Similarly, relatively dense means that there exists $R > 0$ such that $B_R(g) \cap \Lambda \neq \emptyset$ for all $g \in G$.

A set $\Lambda \subseteq G$ is called an (r, R)*-set* if it is a Delone set for the parameters $0 < r < R$. We denote the collection of all (r, R)-sets by $X_{r,R}$. We mention the following important fact without proof.

Lemma 4.6. *A subgroup Λ of a LCA group G is a Delone set if and only if it is a lattice.*

The next result shows that all model sets are uniformly discrete. With an additional hypotheses, they are also relatively dense. Moreover, both of these properties occur in a uniform way in a model set family.

Proposition 4.7. *Let $\mathcal{S} = (G, H, \Gamma)$ be a cut and project scheme and let $W \subseteq H$ be a window.*

(1) *There exists an open neighborhood V of 0 in G such that for any $\Lambda \in \mathcal{M}(W)$ and any $g \in \Lambda$, $(V + g) \cap \Lambda = \{g\}$.*
(2) *(Meyer [15]) If, in addition $W^\circ \neq \emptyset$, then there exists K compact so that for any $\Lambda \in \mathcal{M}(W)$, $K + \Lambda = G$.*

Before the proof we state a corollary and make some remarks.

Corollary 4.8. *If $W^\circ \neq \emptyset$ then there exist $R > r > 0$ so that $\mathcal{M}(W) \subseteq X_{r,R}$. In particular, every model set $\Lambda \in \mathcal{M}(W)$ is a Delone set.*

Remark 4.9.

(1) Nonempty interior is not necessary for a model set to be Delone. Consider a cut and project scheme $\mathcal{S} = (G, \{0\}, \Gamma)$, where $\Gamma \subseteq G$ is a lattice, and let $W = \{0\}$. Then one has $\Lambda(W) = \Gamma$.
(2) However, some additional hypotheses on a window W are necessary. In an aperiodic cut and project scheme $\mathcal{S}$, any finite window W leads to a finite (possibly empty) model set $\Lambda(W)$. This can not be relatively dense if G is not compact.

Proof of Proposition 4.7. For Part (1), we follow Moody [17]. By taking the closure of W if necessary, we may assume without loss of generality that W is compact. By the local compactness of G, there is a neighborhood U of 0 in G with $\overline{U}$ compact. Thus $\overline{U} \times (W - W)$ is compact, and since Λ is a lattice, $(U \times (W - W)) \cap \Gamma$ is finite. Since p_1 is injective, we can choose U so small that $\big(\overline{U} \times (W - W)\big) \cap \Gamma = \{(0, 0)\}$. Let V be a neighborhood of 0 in G with compact closure such that $V - V \subseteq U$.

Suppose for some $(g_0, h_0) \in G \times H$ there are $g_1, g_2 \in \Lambda(W + h_0) - g_0$ such that $(V + g_1) \cap (V + g_2) \neq \emptyset$. Then

$$p_1^{-1}(g_1 - g_2) \in ((V - V) \times (W - W)) \cap \Gamma \subseteq \{(0, 0\},$$

so $p_1^{-1}(g_1 - g_2) = (0, 0)$. Since p_1 is injective, $g_1 = g_2$. □

Part (2) of Proposition 4.7 follows from the next lemma.

Lemma 4.10 (Meyer [15], Lemma 10, p 50)**.** *Let $\mathcal{S} = (G, H, \Gamma)$ be a cut and project scheme. The for each $U \subseteq H$ open, with $\overline{H}$ compact, there exists a compact $K \subseteq G$ such that $G \times H = \Gamma + (K \times U)$.*

Proof. By Lemma 4.6, Γ is a lattice, so by Lemma 4.6 there exist $K_1 \subseteq G$ and $K_2 \subseteq H$ compact, such that $\Gamma + (K_1 \times K_2) = G \times H$. Since $p_2(\Gamma)$ is dense in H, there is a finite subset $F \subseteq \Gamma$ so that $K_2 \subseteq p_2(F) + U$. Let $K = K_1 - p_1(F)$. We have $G \times H = \Gamma + K_1 \times (p_2(F) + U) = \Gamma + (K_1 - p_1(F)) \times U = \Gamma + K \times U$. □

Usually, we will require a little more from a window W.

Definition 4.11. A window $W \subseteq H$ is *topologically regular* if it is the closure of its interior: $W = \overline{W^\circ}$. A model set $\Lambda(W)$ is called topologically regular if W is topologically regular.

In particular, topological regularity is more than enough to ensure $\Lambda(W)$ is Delone.

Lemma 4.12. *If W is topologically regular then $\overline{\Lambda(W)^*} = W$.*

Proof. $\Lambda(W)^* := \{g^* \in H : g \in \Lambda(W)\} = p_2(\Gamma) \cap W$. Since W is topologically regular $W^\circ \neq \emptyset$, and $\Lambda(W)^* \cap W^\circ = p_2(\Gamma) \cap W^\circ$, which is dense in W by Definition 2.3, (3). □

Remark 4.13. The windows in Examples 4.1 and 4.3 are topologically regular. Thus the model sets in the examples are Delone sets.

5. Model set dynamical systems

5.1. **The local topology.** Let G^∞ denote the one point compactification of G. For a Delone set $\Lambda \subseteq G$, let $\Lambda^\infty := \Lambda \cup \{\infty\} \subseteq G^\infty$, noting that Λ^∞ is closed in G^∞. We give $X_{r,R}$ the topology obtained by pulling back the Hausdorff topology (see [3]) on the closed subsets of G^∞. We have that $X_{r,R}$ is compact since it is closed.

Remark 5.1. This compactness result is essentially due to D. Rudolph [27], in the different looking but essentially equivalent context of tilings of $\mathbb{R}^d$. The terms *tiling topology* and *local topology* are often used in the literature for topologies this type. The definition given here, in terms of G^∞, is attributed in [5] to Johansen.

Define the translation action T of G on $X_{r,R}$ by

$$T^g \Lambda := \Lambda - g \tag{5.1}$$

It is easy to see that the G-action T is a continuous action in the local topology. We call T acting on $X_{r,R}$ a *pattern dynamical system.* More generally, any closed T-invariant subspace $X \subseteq X_{r,R}$ is called a pattern dynamical system.

Given a Delone set Λ, we define its *orbit closure* (or *dynamical hull*) to be the set

$$X(\Lambda) := \overline{\{T^g \Lambda : g \in G\}}. \tag{5.2}$$

Since $X(\Lambda) \subseteq X_{r,R}$ is clearly closed and T-invariant, $X(\Lambda)$ is a pattern dynamical system. If Λ is a model set, then $X(\Lambda)$ is called a *model set dynamical system.*

5.2. **Finiteness conditions.** A subset $\Lambda \subseteq G$ is called *locally finite* if $\Lambda \cap K$ is finite for any $K \subseteq G$ compact. Note that any uniformly discrete set (and thus any Delone set) is locally finite.

We say two subsets $A, B \subseteq G$ are *equivalent* if $B = A - g$.

Definition 5.2. A subset $\Lambda \subseteq G$ is said to have *finite local complexity* if for any $Q > 0$, the number of equivalence classes of subsets of the form $B_Q(g) \cap \Lambda$ for all $g \in G$, is finite.

Note that if $\Lambda \in X_{r,R}$ then it suffices to check only $Q = R$ in Definition 5.2.

Fix $0 < r < R$ and let $Q \geq R$. Given a finite collection $\mathcal{F}$ of finite subsets of $B_Q(0) \subseteq G$, we say $Z \subseteq X_{r,R}$ has $\mathcal{F}$-local complexity if every $F \in \{B_Q(\mathbf{s}) \cap \Lambda : g \in G, \Lambda \in X_{r,R}\}$ is equivalent to some $F' \in \mathcal{F}$. It is easy to see that $X(\mathcal{F})$, defined to be the collection of all $\Lambda \in X_{r,R}$ that have $\mathcal{F}$-local complexity, is closed. We say $X \subseteq X_{r,R}$ has *finite local complexity* if $X \subseteq X(\mathcal{F})$ for some $\mathcal{F}$.

Lagarias [11] showed that a set $\Lambda \subseteq G$ has finite local complexity if and only if $\Lambda - \Lambda$ is locally finite. A Delone set $\Lambda \subseteq G$ is called a *Meyer set* (see [16]) if $\Lambda - \Lambda$ is Delone. In particular, Meyer sets have finite local complexity.

Proposition 5.3. *If Λ is a topologically regular Model set, then $\Lambda - \Lambda$ is a topologically regular model set. In particular, every topologically regular model set Λ is a Meyer set. Thus a topologically regular model set has finite local complexity.*

This follows from (4) of Lemma 4.4.

Corollary 5.4. *If Λ is a topologically regular model set then the model set dynamical system $X(\Lambda)$ has finite local complexity.*

Remark 5.5. Lagarias used the term “finite type” for those $\Lambda \subseteq G$ that have the property we call “finite local complexity”. However, calling $X(\Lambda)$ a “finite type” dynamical system conflicts with the way “finite type” is used in symbolic dynamics, where one would probably call the

$X(\mathcal{F})$ a finite type dynamical system. It is only in a few cases that $X(\Lambda)$ is finite type in this more restricted sense (e.g., for Λ the vertices of a Penrose tiling, more generally see [12]).

5.3. **Repetitivity and minimality.** Suppose $\Lambda \subseteq X_{r,R}$ and $F \subseteq \Lambda$ is finite and nonempty. We denote the set of *occurrences* of F in Λ by

$$\Lambda[F] := \{g \in \Lambda : F + g \subseteq \Lambda\} = \bigcap_{f \in F} \Lambda - f. \tag{5.3}$$

Note that $0 \in \Lambda[F]$ and that $\Lambda[F]$ is uniformly discrete (since $\Lambda[F] \subseteq \Lambda$).

Definition 5.6. We say Λ is *repetitive* if every nonempty $\Lambda[F]$ is relatively dense.

In other words, Λ is repetitive if and only if every $\Lambda[F]$ is a Delone set.

Now suppose Λ has finite local complexity. A standard argument using *Gottschalk's Theorem* [6] (see e.g., [24]) shows that the dynamical system $X(\Lambda)$ is minimal if and only if Λ is repetitive. A topologically regular model set is not necessarily repetitive without a further assumption.

Definition 5.7. A topologically regular window $W \subseteq H$ (or the corresponding model set $\Lambda(W)$) is called *nonsingular* if $\Lambda(W) = \Lambda(W^\circ)$.

Lemma 5.8. *If W is nonsingular and $g \in L$, then $W - g^*$ is nonsingular.*

Proof. If $W - g^*$ is singular then $\Lambda(\partial(W - g^*)) \neq \emptyset$. But then $\Lambda(\partial(W) - g^*) = \Lambda(\partial(W)) - g \neq \emptyset$, so $\Lambda(\partial(W)) \neq \emptyset$ and W is singular. □

Proposition 5.9. *Let $\mathcal{S} = (G, H, \Gamma)$ be a cut and project scheme with $W \subseteq H$ a nonsingular topologically regular window. Suppose $F \subseteq \Lambda(W)$ is nonempty and finite. Then there there exists a nonsingular topologically regular window $V \subseteq W^\circ$ so that $\Lambda(W)[F] = \Lambda(V)$.*

Proof. Suppose $F \subseteq \Lambda(W) \subseteq L$. Then $\Lambda(W)[F] \subseteq L$ is nonempty and

$$\begin{aligned}
\Lambda(W)[F] &= \Lambda(W^\circ)[F] \\
&= \bigcap_{f \in F} \Lambda(W^\circ) - f \\
&= \bigcap_{f \in F} \Lambda(W^\circ - f^*) \\
&= \Lambda\left(\bigcap_{f \in F} W^\circ - f^*\right) = \Lambda(V_0),
\end{aligned}$$

where $V_0 = \bigcap_{f\in F} W^\circ - f^*$, which is open since F is finite. Put $V := \overline{V_0}$, which is topologically regular. Since $V = \cap_{f\in F}(W - f^*)$, it is nonsingular, since $W - f*$ is nonsingular by Lemma 5.8. It follows that V is nonsingular, and $\Lambda(V) = \Lambda(V^\circ) = \Lambda(W)[F]$. □

Corollary 5.10. *Every nonsingular topologically regular model set Λ is repetitive and has finite local complexity. Thus the corresponding model set dynamical system $X(\Lambda)$ is minimal.*

Remark 5.11. In Examples 4.1 and 4.3, conditions for nonsingularity were already given. These guarantee the minimality of $X(\Lambda)$.

5.4. **Aperiodicity.**

Definition 5.12. Let A be a subset of a LCA group G. We call $g \in G$ a *period* for A if $A - g = A$. A subset $A \subseteq G$ is called *aperiodic* if its only period is $g = 0$.

Roughly speaking, a "crystal" is a Delone set whose periods form a lattice in G. Here we really want our model sets to be "quasicrystals," so we insist that they be aperiodic. Aperiodicity has the following dynamical interpretation.

Lemma 5.13. *The orbit closure $X(\Lambda)$ of a Delone set Λ is free if and only if Λ is aperiodic.*

Here is the way that we ensure a model set is aperiodic.

Lemma 5.14. *If $\mathcal{S} = (G, H, \Gamma)$ is an aperiodic cut and project scheme and the window $W \subseteq H$ is topologically regular and aperiodic, then the model set $\Lambda(W) \subseteq G$ is aperiodic. Conversely, suppose that either $\mathcal{S}$ is not aperiodic or that W has a nontrivial period in $*(L)$. Then $\Lambda(W)$ is not aperiodic.*

Proof. Suppose $\Lambda(W) = \Lambda(W) - g$. Then $\Lambda(W) = \Lambda(W - g^*)$, and since W is regular, $W = W - g^*$ by Lemma 4.12. Since W is aperiodic $g^* = 0$, which implies $g = 0$ since $\mathcal{S}$ is aperiodic.

Conversely, if $g \in J\backslash\{0\} \subseteq L$ then $\Lambda(W) - g = \Lambda(W - g^*) = \Lambda(W)$ since $g^* = 0$. Similarly, if $W - g^* = W$ for some nonzero $g \in L$, then $\Lambda(W) - g = \Lambda(W - g^*) = \Lambda(W)$. □

Remark 5.15. In Example 4.1, $W = [e^{2\pi ia}, e^{2\pi ib}]$ is aperiodic if $b - a \in \mathbb{R}\backslash\mathbb{Q}$, a condition that always holds in the "Sturmian" case $a = \theta$, $b = \theta + \alpha$. (In fact, no $h \in *(L)$ can be a period of any arc $W \subseteq H$, so all $\Lambda(W)$ are aperiodic). In Example 4.3, aperiodicity is automatic.

5.5. **Parameterization.** Although the model set family $\mathcal{M}(W)$, defined in (4.2), is parameterized by $G \times H$, these parameters are redundant because $\Lambda(W+h)-g = \Lambda(W+h')-g'$ whenever $(g,h)-(g',h') \in \Gamma$. This suggests $Y = (G \times H)/\Gamma$ as a more natural parameter space.

Let us define $\alpha : Y \to M(W)$ by

$$\alpha(y) := \Lambda(W+h) - g$$

for any $(g,h) \in G \times H$ that satisfies $y = \pi(g,h)$. This mapping is clearly well defined and satisfies $\alpha(R^g y) = T^g \alpha(y)$ for $g \in G$, $y \in Y$, where R is the Kronecker dynamical system (2.3), and T is the translation action (5.1).

Lemma 5.16. *If $W \subseteq H$ is an aperiodic, topologically regular window, then α is injective (and thus a bijection). Conversely, whenever α is injective, W is aperiodic.*

Proof. Suppose $\Lambda(W+h') - g' = \Lambda(W+h) - g$. We may assume without loss of generality (by translating) that $(g',h') = 0$. Thus we have $\Lambda(W) = \Lambda(W+h) - g$. It follows that $g \in L$ and, by regularity that $W = W + h - g^*$. Since W is aperiodic, $h - g^* = 0$ or $h = g^*$. Written differently, $(g,h) = (g, g^*)$, which means $(g,h) \in \Gamma$.

We prove the converse by contradiction. Suppose $W + h = W$ for $h \neq 0$. Let $h' = \pi(0,h) \in Y$ and note that $h' \neq 0$. This is because $(0,h') \notin \Gamma$, which is because π_1 is injective. But $\alpha(h') = \alpha(0)$, so α is not injective. □

Unfortunately, $\mathcal{M}(W)$ turns out to have some undesirable properties in the local topology. These are associated with the singular model sets. However, non-singularity is common.

Lemma 5.17. *Let $\mathcal{S} = (G, H, \Gamma)$ be a cut and project scheme. If $W \subseteq H$ is a topologically regular window then there exists a dense G_δ subset $H_0 \subseteq H$ so that for all $h \in H_0$, $W + h$ is nonsingular.*

Basically this follows from the countability of Γ.

We define the *nonsingular parameters* $Y_0 := \pi(G \times H_0)$ and the *nonsingular model sets* $\mathcal{M}_0(W) = \alpha(Y_0)$. Since π is open, $Y_0 \subseteq Y$ is dense G_δ.

Proposition 5.18. *Let W be a topologically regular window. Then for any $\Lambda \in \mathcal{M}_0(W)$ the model set dynamical system $X(\Lambda) = \overline{\mathcal{M}_0(W)}$. In particular, all the non-singular model sets in the same family generate the same model set dynamical system.*

Proof. We will show that if $\Lambda, \Lambda' \in \mathcal{M}_0(W)$ then $\Lambda' \in X(\Lambda)$. We may assume without loss of generality that $\Lambda = \Lambda(W)$, where W is nonsingular and topologically regular, and that $\Lambda' = \Lambda(W + h)$ for some $h \in H_0$. By the definition of the local topology, it suffices to show that for any finite $F \subseteq \Lambda$, there exists $g \in G$ so that $F \subseteq \Lambda' - g$.

By Proposition 5.9, there exists a nonsingular, topologically regular window V so that $\Lambda(V) = \Lambda(W)[F] = \Lambda[F]$. We have

$$\Lambda(V + h) = \Lambda(W + h)[F] = \Lambda'[F].$$

But $V + h$ is topologically regular, so by Proposition 4.7, $\Lambda'[F]$ is relatively dense. By (5.3) we have that for any $g \in \Lambda'[F]$ that $F + g \subseteq \Lambda'$, which is the same as the condition above. $\square$

5.6. **The inverse of α.**

Theorem 5.19. *Let $\mathcal{S}$ be a cut and project scheme and let R be the corresponding Kronecker system. Let $W \subseteq H$ be an aperiodic, topologically regular window, and let $X(\Lambda)$ be the corresponding model set dynamical system. Then there is a continuous mapping $\beta : X(\Lambda) \to Y$ such that $\beta = \alpha^{-1}$ on $\mathcal{M}_0(W)$, and such that $\beta \circ T^g = R^g \circ \beta$ for all $g \in G$.*

Proof. For $\Lambda = \Lambda(W + h) - g \in \mathcal{M}(W)$ we define $\beta = \alpha^{-1}$ by $\beta(\Lambda) := \pi(g, h)$. Clearly $\beta = \alpha^{-1}$, and thus $\beta \circ T^g = R^g \circ \beta$. To prove the theorem it suffices to show β is uniformly continuous on $\mathcal{M}_0(W)$.

First assume that $\Lambda = \Lambda(W + h) \in \mathcal{M}_0(W)$. Then $\Lambda \subseteq L$, and since $W + h$ is topologically regular, $\overline{\Lambda^*} = W + h$ by Lemma 4.12. Note that $q \in \Lambda$ if and only if $q^* \in W + h$, or equivalently $-h \in W - q^*$. Thus $-h \in \bigcap_{q \in \Lambda} W - q^*$.

Suppose now that $k \in \cap_{q \in \Lambda} W - q^*$. Then $\overline{\Lambda^*} = W + k$, so that $W - h = W + k$. Since W is aperiodic, $k = -h$. It follows that $\cap_{q \in \Lambda} W - q^* = \{-h\}$. Putting

$$I_s = - \bigcap_{q \in B_s(0) \cap \Lambda} W - q^*, \tag{5.4}$$

we have that $\operatorname{diam}(I_s)$ is non-increasing, and that $\{h\} = \cap_{s=0}^{\infty} I_s$. Fix $\epsilon > 0$. Choose s so large that $\operatorname{diam}(I_s) < \epsilon/2$. Let

$$\delta < \min\left(\frac{1}{s + R}, \frac{\epsilon}{2}\right) \tag{5.5}$$

where $\mathcal{M}(W) \subseteq X_{r,R}$.

Suppose $\Lambda_1, \Lambda_2 \in \mathcal{M}_0(w)$ with $d(\Lambda_1, \Lambda_2) < \delta$. Let $\Lambda_1 := \Lambda(W + h_1) - g_1$ and $\Lambda_2 := \Lambda(W + h_2) - g_2$. By the definition of d, there exist g_1', g_2' with $|g_1'|, |g_2'| < \delta$ so that

$$(\Lambda_1 - g_1') \cap B_{\frac{1}{\delta}}(0) = (\Lambda_2 - g_2') \cap B_{\frac{1}{\delta}}(0). \tag{5.6}$$

Since $1/\delta > R$, there exists

$$g \in (\Lambda_1 - g_1') \cap (\Lambda_2 - g_2') \cap B_R(0) \neq \emptyset. \tag{5.7}$$

Thus

$$\begin{aligned} \Lambda_i - g_i' - g &= \Lambda(W + h_i) - (g_i + g_i' - g) \\ &= \Lambda\left(W + h_i - (g_i + g_i' + g)^*\right). \end{aligned} \tag{5.8}$$

Here we have used the fact that $g_i + g_i' + g \in L$, since by (5.7), $\Lambda_i - g_i' - g \subseteq L$. Now by (5.6)

$$(\Lambda_1 - g_1' - g) \cap B_s(0) = (\Lambda_2 - g_2' - g) \cap B_s(0),$$

so that by (5.4), $-(h_i - (g_i + g_i' + g)^*) \in I_s$, for $i = 1, 2$, which implies

$$\begin{aligned} |h_1 - (g_1 + g_1' + g)^* - (h_2 - (g_2 + g_2' + g)^*)| &\leq \operatorname{diam}(I_s) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

Note that for any $g \in L$, $\pi(g, g^*) = 0$, since $(g, g^*) \in \Gamma$. Because $g_i + g_i' + g \in L$, and π is a homomorphism, we have

$$\begin{aligned} \beta(\Lambda_i) &= \pi(g_i, h_i) \\ &= \pi(g_i, h_i) - \pi\big((g_i + g_i' + g), (g_i + g_i' + g)^*\big) \\ &= \pi\big(-(g_i' + g), h_i - (g_i + g_i' + g)^*)\big). \end{aligned}$$

It follows that

$$\begin{aligned} |\beta(\Lambda_1) - \beta(\Lambda_2)| &= |\pi(g_1' - g_2', h_1 - (g_1 + g_1' + g)^* - (h_2 - (g_2 + g_2' + g)^*))| \\ &\leq |g_1' - g_2', h_1 - (g_1 + g_1' + g)^* - (h_2 - (g_2 + g_2' + g)^*)| \\ &\leq |g_1' - g_2'| + |h_1 - (g_1 + g_1' + g)^* - (h_2 - (g_2 + g_2' + g)^*)| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

□

Corollary 5.20. *For any $\Lambda \in \mathcal{M}_0(W)$ the model set dynamical system $X(\Lambda)$ is an almost 1:1 extension of the Kronecker system R (from the cut and project scheme $\mathcal{S}$). In particular, $X(\Lambda)$ is almost automorphic.*

We now describe, without proof, the structure of the sets $\Lambda' \in X(\Lambda)\backslash\mathcal{M}_0(W)$. These are the sets Λ' that are limits of nonsingular model sets.

Theorem 5.21. *Suppose $\Lambda_n \to \Lambda'$ where $\Lambda_n \in \mathcal{M}_0(W)$ and $\Lambda' \in X(\Lambda)\backslash\mathcal{M}_0(W)$. Let $y_n = \beta(\Lambda_n) \in Y_0$ and let $y = \lim y_n \in Y\backslash Y_0$. Let $(g, h) \in G \times (H\backslash H_0)$ with $\pi(g, h) = y$ and define $\Lambda = \Lambda(W - h) + g \in X(\Lambda)$. Then*

(1) $\Lambda' \subseteq \Lambda$,

(2) *There exists* $W' \subseteq H$*, with* $W^\circ \subsetneqq W' \subsetneqq W$ *(with each inclusion proper) such that* $\Lambda' = \Lambda(W' - h) + g$*.*
(3) Λ' *is a model set that is Delone but (generally) not topologically regular.*

5.7. **Measurable properties of model sets.** We call a model set *measurably regular* if $\theta_H(\partial W) = 0$, where θ_H denotes Haar measure on H. It is easy to see that if W is measurably regular, then $\theta_Y(Y \backslash Y_0) = 0$. As a consequence we have the following.

Theorem 5.22. *Suppose* $\mathcal{S} = (G, H, \Gamma)$ *is an aperiodic cut and project scheme and* W *is a topologically and measurably regular, aperiodic window. Let* $\Lambda \in \mathcal{M}_0(W)$*. Then the model set dynamical system* $X(\Lambda)$ *is uniquely ergodic (and thus strictly ergodic).*

Corollary 5.23. *Under the conditions of* Theorem 5.22, $X(\Lambda)$ *and* R *(the Kronecker system for* $\mathcal{S}$*) are measurably isomorphic. In particular, they have the same point spectrum* $\Sigma_{\mathcal{S}}$*.*

Now we will discuss the geometric significance of unique ergodicity. A sequence $\mathcal{A} = \{A_n\}$ of compact subsets of G, with $\theta_G(A_n) > 0$, is called a *van Hove sequence* if for any $K \subseteq G$ compact,

$$\lim_{n\to\infty} \frac{\theta_G(\partial^K A_n)}{\theta_G(A_n)} = 0,$$

where

$$\partial^K A := (K + A)\backslash A^\circ \cup \Big((-K + \overline{G\backslash A} \cap A\Big),$$

(see [17]). We call a van Hove sequence *good* if for some $c > 0$,

$$\theta_G(A_n - A_n) \le c\,\theta_G(A_n). \tag{5.9}$$

The existence of a good van Hove sequence follows, for example, from the assumption that G is σ-compact (see [28], [29], [17]).

Suppose $\Lambda \subseteq G$ is *locally finite* (e.g., $\Lambda \in X_{r,R}$). Given a good van Hove sequence $\mathcal{A} = \{A_n\}$, we define the *metric density* of Λ by

$$\mathrm{dens}(\Lambda) = \lim_{n\to\infty} \frac{1}{\theta_G(A_n)} \mathrm{card}\,(\Lambda \cap A_n)\,.$$

If this limit exists, it is independent of the choice of van Hove sequence (see [17]).

Proposition 5.24 (Moody, [17])**.** *Let* $\mathcal{S} = (G, H, \Gamma)$ *be a cut and project scheme, let* $\mathcal{A} = \{A_n\}$ *be a good van Hove sequence for* G*, and let* $W \subseteq H$ *be a measurably regular window. Then*

$$\mathrm{dens}(\Lambda(W + h)) = \theta_H(W). \tag{5.10}$$

for θ_H a.e. $h \in H$. If, in addition, $\theta_H(\partial W) = 0$, then (5.10) *holds for all $h \in H$.*

The second part of this lemma says that if W is measurably regular, then the metric density of every model set $\Lambda \in \mathcal{M}(W)$ is $\theta_H(W)$. This comes close to proving the unique ergodicity of $X(\Lambda)$ (Theorem 5.22), but it is not quite equivalent. Unique ergodicity requires each nonempty $\Lambda'[F]$ for $\Lambda' \in X(\Lambda)$ to have positive density (see [28]). And, this will be the case if W is both topologically and measurably regular (Theorem 5.22).

6. Realization for model sets

6.1. **The existence of regular aperiodic windows.** In this section and the next, we study the question of when regular aperiodic windows exist. The next result shows that regularity comes almost for free[5].

Lemma 6.1. *Let H be a LCA group that is not discrete. Then there exists $r > 0$ so that $W = \overline{B_r(0)}$ is topologically and metrically regular.*

Proof. Let $S_r(0) := \{g : d(g, 0) = r\}$. Then $B_r(0) = \cup_{0 \le s < r} S_r(0)$, and since H is not discrete, $B_r(0)\backslash\{0\} \neq \emptyset$ for all $r > 0$. Thus W=$\overline{B_r(0)} = \{g : d(g, 0) \le r\}$ and $W^\circ = B_r(0)$. It follows that W is topologically regular.

By local compactness, we can choose R so that $\overline{B_R(0)}$ is compact. It follows that $\theta_R(\overline{B_R(0)}) < \infty$. If $\theta_H(S_r(0)) > 0$ for all $r \in (0, R)$, then there exists $N \in \mathbb{N}$ so that $I_N := \{r : \theta_H(S_r(0)) > 1/N\}$ is infinite. For otherwise, $(0, R) = \cup_N I_N$ would be countable. But the existence of such an N implies that $\theta_H(\overline{B_R(0)}) \ge \theta_H\left(\cup_{r \in I_N} S_r(0)\right) = \infty$. This contradicts the compactness of $\overline{B_R(0)}$. It follows that $\theta_H(S_r(0)) = 0$ for some (i.e., all but finitely many) $r \in (0, R)$. This implies that $\overline{B_r(0)}$ is topologically and metrically regular. □

Aperiodicity is a bit more challenging. In this section we consider the following case.

Definition 6.2. We say a LCA group H has *no compact subgroups* if the only compact subgroup of H is $\{0\}$.

Proposition 6.3. *Suppose H has no compact subgroups. Then every topologically regular window is aperiodic.*

Proof. $H_W := \{h : W + h = W\}$ is a closed subgroup of H. If $h \in W_H$ then $g_1, g_2 \in W$ so that $g_1 + h = g_2$. Equivalently, $h = g_2 - g_1$ so

[5]My thanks to Nelson Markley for telling me this proof.

that $h \in W - W$, and $H_W \subseteq W - W$. Since $W - W$ is compact $H_W = \{0\}$. □

Corollary 6.4. *If H has no compact subgroups then there is a topologically and measurably regular aperiodic window $W \subseteq H$.*

The following result from duality theory shows that the property no compact subgroups is, in fact, a relatively common.

Lemma 6.5. *A LCA group H has no compact subgroups if and only if $\widehat{H}$ is connected.*

For example, $H = \mathbb{R}^d$ has no compact subgroups (and neither does $\mathbb{Z}^d$). However, $\mathbb{Q}_p$ has $\mathbb{Z}_p$ as a compact subgroup.

6.2. **The compact case.** One large class of LCA groups which fail to have no compact subgroups are the CA groups. Nevertheless, the next theorem shows that CA groups always have good windows. Our approach follows the 1973 dissertation of Michael Paul [22], (see [21]), which concerns the symbolic dynamics of rotation actions of $\mathbb{Z}$ on monothtic groups (i.e., Kronecker $\mathbb{Z}$-actions).

Theorem 6.6. *If H is an infinite CA group then there exists an aperiodic, topologically and measurably regular window $W \subseteq H$.*

By combining Theorem 6.6 with Theorem 3.8 and Corollary 5.23, we obtain the main result of this paper. It is a Halmos von-Neumann theorem for model sets.

Theorem 6.7. *Let G be a LCA group with a lattice dense dual. Let $\Sigma \subseteq \widehat{G}$ be a countable dense subgroup. Then there exists a cut and project scheme $\mathcal{S} = (G, H, \Gamma)$ and a topologically and metrically regular window $W \subseteq H$, so that for any $\Lambda \in \mathcal{M}_0(W)$, the model set dynamical system $X(\Lambda)$ has spectrum Σ.*

Remark 6.8.

(1) $G = \mathbb{Z}^d$ and $G = \mathbb{R}^d$ satisfy the hypotheses of Theorem 6.7.
(2) The same result holds for any G satisfying the realization property.
(3) Since it is a corollary of Theorem 6.6, the internal spaces H constructed in Theorem 6.7 are always compact.

Proof of Theorem 6.6. The proof divides into two cases.

In **Case 1** we assume H is totally disconnected. Since H is infinite and metrizable, it is homeomorphic to the standard (middle thirds) Cantor set $C \subseteq [0, 1]$. From now on we identify C and H.

Let $b \in H$ be an element of C that is not an endpoint of any complementary interval and define $W := [0, b] \cap H$. Note that we have $\partial W = \{b\}$ so that $\theta_H(\partial W) = 0$. Thus W is topologically and measurably regular.

There exists a sequence $\{g_n\}$ in H with the following properties:

(1) $g_n \to b$,
(2) $g_n \in W$ for infinitely many n, and
(3) $g_n \in W^c$ for infinitely many n.
(4) $g_n \neq b$,

Note that, in fact, *any* convergent sequence satisfying (2) and (3) must also satisfy (1).

Now suppose $W + h = W$. Then on the one hand, $g_n + h \to b + h$. On the other hand, $g_n + h \in W + h = W$ if and only if $g_n \in W$. It follows that $g_n + h$ is a convergent sequence satisfying (2) and (3), so $g_n + h \to b$. Then $b + h = b$, so $h = 0$, and W is aperiodic.

Case 2. For any $\chi \in \widehat{H}$, $\chi(H)$ is a compact subgroup of $\mathbb{T}$. There are just two possibilities:

(1) $\chi(H) = \mathbb{T}$, or
(2) $\chi(H) = \mathbb{Z}/N = \{e^{2\pi ik/N} : k = 0, \dots, N-1\}$ for some $N \in \mathbb{N}$.

By our assumption, H is not totally disconnected. It follows from the Structure Theorem for CA groups (see [23]) that H has a connected subgroup. Some character, call it $\chi_1 \in \widehat{H}$, must be nontrivial on this subgroup. Since the continuous image of a connected set is connected, χ_1 is of the first type. Let $\chi_0 = 1$ denote the identity in $\widehat{H}$, and note that this is of the second type.

Because H is an infinite CA group, $\widehat{H}$ is countably infinite, and we will write $\widehat{H} = \{\chi_0, \chi_1, \chi_2, \dots\}$. Let $I_1 = \{1, i_2, i_3, \dots\}$ be a list of all characters of the first type, and let $I_2 = \{j_2, j_3, \dots\}$ be a list of all characters of the second type, excepting χ_0. Then $I_1 \cup I_2 = \mathbb{N}$, and at least one is infinite.

The *infinite dimensional torus* $\mathbb{T}^{\mathbb{N}}$ is a (non-metrizable) compact abelian group with dual $\mathbb{Z}^{\mathbb{N}}$. Define the mapping $\varepsilon : H \to \mathbb{T}^{\mathbb{N}}$ by

$$\varepsilon(h) = (\chi_1(h), \chi_2(h), \dots).$$

As a set of functions, $\widehat{H}$ separates points, so the mapping ε is an algebraic and topological isomorphism of H onto its image $\varepsilon(H)$ in $\mathbb{T}^{\mathbb{N}}$. We will identify H with its image. Let θ_H denote the Haar measure on H, viewed as a measure on $\mathbb{T}^{\mathbb{N}}$.

Identify $\mathbb{T}$ with $[0, 2\pi)$. For $i = 1, 2, \dots$ let $U_i = (a_i, b_i)$ be a collection of intervals in $(0, \pi)$ such that $a_{i+1} > b_i$. Let $U_i' = (c_i, d_i)$ be such that $d_i - c_i < \pi$, and such that

$$\chi_i(\chi_1^{-1}(U_i)) \cap U_i' \neq \emptyset, \tag{6.1}$$

Then (6.1) guarantees that

$$U_i' \cap \chi_i(H) \neq \emptyset. \tag{6.2}$$

Without loss of generality we may assume that if $i \in I_2$, then U_i' is small enough that $U_i' \cap \chi_i(H) = \{e_i\}$ is a single point.

Define $\mathbb{N}_i = \{i+1, i+2, \dots\}$ and put

$$\begin{aligned}
V_1 &= (U_1 \times \mathbb{T}^{\mathbb{N}_1}) \cap \varepsilon(H) \\
V_2 &= (U_2 \times U_2' \times \mathbb{T}^{\mathbb{N}_2}) \cap \varepsilon(H) \\
V_3 &= (U_3 \times \mathbb{T} \times U_3' \times \mathbb{T}^{\mathbb{N}_3}) \cap \varepsilon(H) \\
&\vdots \\
V_i &= (U_i \times \mathbb{T}^{i-2} \times U_i' \times \mathbb{T}^{\mathbb{N}_i}) \cap \varepsilon(H) \\
&\vdots
\end{aligned}$$

where (6.2) guarantees $V_i \neq \emptyset$ and the choice of U_i implies the sets V_i have disjoint closures. Define $W_0 := \cup_{i\in\mathbb{N}} V_i$, which is open, and define $W := \overline{W_0}$. It is clear that W is topologically regular.

By the construction we have $\partial(W) = \cup_i \partial(V_i)$ is a disjoint union. Thus, to show W is metrically regular, it suffices to show $\theta_H(\partial V_i) = 0$.

For $i = 1$, we have $\partial(V_i) = (\{a_1, b_1\} \times \mathbb{T}^{\mathbb{N}_1}) \cap \varepsilon(H)$. Let θ_1 be projection of θ_H to the first $\mathbb{T}$ factor of $\mathbb{T}^{\mathbb{N}}$, namely, $\theta_1(E) = \theta_H(E \times \mathbb{T}^{\mathbb{N}_1})$. It is easy to see that θ_1 is translation invariant and $\theta_1(\mathbb{T}) = 1$, so $\theta_1 = \lambda$, the normalized Lebesgue measure on $\mathbb{T}$. We can decompose

$$\theta_H = \int_{\mathbb{T}} \eta_t \, d\lambda(t),$$

where η_t is a finite measure on the t^{th} fiber $F_t := \{t\} \times \mathbb{T}^{\mathbb{N}_1}$. Then

$$\theta_H(\partial(V_1)) = \int_{\{a_1,b_1\}} \eta_t(F_t \cap \varepsilon(H)) \, d\lambda(t) = 0,$$

since λ has no atoms.

Now suppose $i > 1$, and without loss of generality, put $i = 2$. Let $\varepsilon_2(h) = (\chi_1(h), \chi_2(h))$ be the projection to the first $\mathbb{T}^2$ factor of $\mathbb{T}^{\mathbb{N}}$. Then $\varepsilon_2(H)$ is a closed subgroup of $\mathbb{T}^2$, and since $\chi_1(H) = \mathbb{T}$, $\varepsilon_2(H)$ is not discrete. As above, the projection θ_2 of θ_H to $\varepsilon_2(H)$, namely, $\theta_1(E) = \theta_H(E \times \mathbb{T}^{\mathbb{N}_2})$ for $E \subseteq \mathbb{T}^2$, is normalized Haar measure on $\varepsilon_2(H)$. Since $\varepsilon_2(H)$ is not discrete, θ_2 is nonatomic. Again we can decompose

$$\theta_H = \int_{\mathbb{T}^2} \eta_{\mathbf{t}} \, d\theta_2(\mathbf{t}),$$

where $\eta_{\mathbf{t}}$, $\mathbf{t} \in \mathbb{T}^2$, is the $\mathbf{t}^{\text{th}}$ finite measure on the fiber $F_{\mathbf{t}} := \{\mathbf{t}\} \times \mathbb{T}^{\mathbb{N}_2}$.

If $2 \in I_2$ then

$$\partial(V_2) = (\{(a_2, e_2), (b_2, e_2)\} \times \mathbb{T}^{\mathbb{N}_2}) \cap \varepsilon(H),$$

and we have

$$\theta_H(\partial(V_2)) = \int_{\{(a_2,e_2),(b_2,e_2)\}} \eta_t(F_{\mathbf{t}} \cap \varepsilon(H))\, d\theta_2(\mathbf{t}) = 0,$$

since θ_2 has no atoms.

Finally, suppose $2 \in I_1$. Then $\theta_2 = \lambda_2$, normalized Lebesgue measure on $\mathbb{T}^2$. Since $\partial(U_2 \times U_2')$ is a square in $\mathbb{T}^2$, it satisfies $\theta_2(\partial(U_2 \times U_2')) = 0$. Using the same argument as in the case $i = 1$, it follows that $\theta_H(\partial(V_2)) = 0$. Thus W is metrically regular.

It remains to show that W is aperiodic. Suppose to the contrary that $W + h = W$ for some $h \neq 0$. In our identification of H with $\varepsilon(H)$, let us write $h = (h_1, h_2, h_3, \dots)$ where $h_i = \chi_i(h)$. Since $h \neq 0$, there is a smallest i so that $h_i \neq 0$. If $i = 1$, then $U_1 + h_1 \neq U_1$ so $W + h \neq W$. Thus we assume $i > 1$.

If $i \in I_1$ then $\chi_i(H) = \mathbb{T}$ so $U_i' \cap \chi_i(H) = (c_i, d_i)$ and $(U_i' + h_i) \cap \chi_i(H) = (c_i + h_i, d_i + h_i)$. We have $(c_i + h_i, d_i + h_i) \backslash [c_i + d_i] \neq \emptyset$ since $d_i - c_i < \pi$ and $h_i \neq 0$. Choose $g \in H$ so that $\chi_i(g) \in (c_i + h_i, d_i + h_i) \backslash [c_i + d_i]$. In other words,

$$\exists\, g \in H \text{ such that } \chi_i(g) \in (U_i' + h_i) \backslash \overline{U_i'}. \tag{6.3}$$

Similarly, if $i \in I_2$ then $U_i' \cap \chi_i(H) = \{e_i\}$ and $(U_i' + h_i) \cap \chi_i(H) = \{e_i + h_i\}$. Note that $e_i + h_i \neq e_i$ and let $g \in H$ be such that $\chi_i(g) = e_i + h_i$. In this case we also have (6.3).

But (6.3) implies $g \in (W + h) \backslash W$, so $W + h \neq W$. Thus W is aperiodic.

□

7. The diffraction spectrum

In this final part of the paper, we describe how the dynamical spectrum of a model set Λ is related to its *diffraction spectrum*. To this end, given a LCA group G let us fix a van Hove sequence $\mathcal{A} = \{A_n\}$.

We first assume only that $\Lambda \subseteq X_{r,R}$ is locally finite and define a measure on G associated with Λ by

$$\Delta_\Lambda = \sum_{g \in \Lambda} \delta_g, \tag{7.1}$$

where δ_g denotes unit point measure at g.

For $K \subseteq G$ compact, we define a new measure on G by

$$\Delta^{(2)}_{\Lambda,K} = \frac{(\chi_K \Delta_\Lambda) * (\chi_K \Delta_{-\Lambda})}{\theta(K)},$$

where $*$ denotes convolution, and θ denotes Haar measure on G. An *autocorrelation* $\Delta^{(2)}_\Lambda$ of Δ_Λ is defined to be any weak-* topology limit point of the sequence $\{\Delta^{(2)}_{\Lambda,A_n}\}$. In a slight abuse of terminology we call $\Delta^{(2)}_\Lambda$ an autocorrelation of Λ.

It is easy to see that an autocorrelation is a discrete, positive measure on G that is supported on $\Lambda - \Lambda$. In particular, it has the form

$$\Delta^{(2)}_\Lambda = \sum_{g \in \Lambda - \Lambda} f(g)\delta_g,$$

for some $f(g) \geq 0$.

As one can also show, $\Delta^{(2)}_\Lambda$ is *positive definite* (see [23]) on G. Thus by Bochner's Theorem there is a unique positive measure γ on $\widehat{G}$ so that $\Delta^{(2)}_\Lambda = \widehat{\gamma}$. Here $\widehat{\gamma}$ denotes the *Fourier transform* of γ. We will write $\widehat{\Delta}_\Lambda := \gamma$ and call $\widehat{\Delta}_\Lambda$ a *diffraction spectrum* of Λ. The motivation for this definition is essentially physical.

In general, neither $\Delta^{(2)}_\Lambda$ nor $\widehat{\Delta}_\Lambda$ is unique. However, we say that Λ has a *unique autocorrelation* if there is just one autocorrelation measure $\Delta^{(2)}_\Lambda$. In this case there is also a unique diffraction spectrum $\widehat{\Delta}_\Lambda$ on G.

Proposition 7.1 (Schlottman, [29]). *Suppose Λ is a Delone set with finite local complexity such that $X(\Lambda)$ is uniquely ergodic. Then Λ has a unique autocorrelation and, in particular, a unique diffraction spectrum $\widehat{\Delta_\Lambda}$.*

Let $(\widehat{\Delta}_\Lambda)_d$ be the discrete part of $\widehat{\Delta}_\Lambda$ (in the sense of the Lebesgue Decomposition Theorem) and define $\widehat{\Lambda} := \text{supp}(\widehat{\Delta}_\Lambda)$. We call $\widehat{\Lambda}$ the *diffraction point spectrum* of Λ. We say Λ has *pure point diffraction spectrum* if $\widehat{\Delta}_\Lambda = (\widehat{\Delta}_\Lambda)_d$. A Delone set with pure point diffraction spectrum that is nor periodic is called a *quasicrystal.*

Proposition 7.2 (Schlottman, [29], Moody, [17]). *Suppose Λ is a Delone set with finite local complexity, such that the translation action T on $X(\Lambda)$ is uniquely ergodic. Let σ_T be the dynamical spectral measure of T. Then $\widehat{\Delta}_\Lambda << \sigma_T$ (in the Radon-Nikodym sense), so that in particular, $\widehat{\Lambda} \subseteq \Sigma_T$. If T has dynamical pure point spectrum, then Λ has pure point diffraction spectrum.*

It is easy to construct examples showing that, in general, $\widehat{\Lambda} \neq \Sigma_T$.

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