Rank and directional entropy

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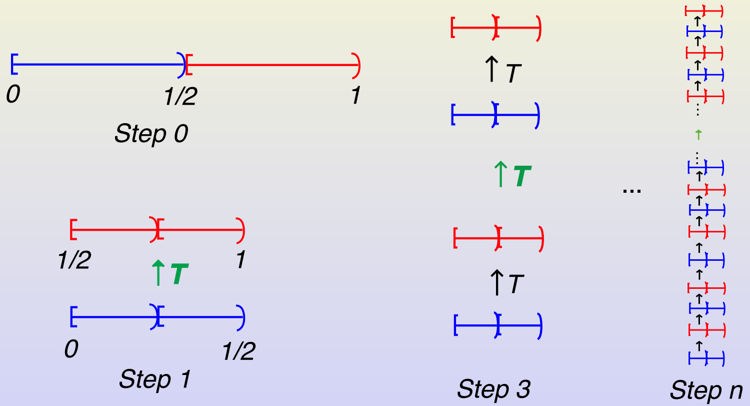
Cutting and Stacking

- Elementary method to construct examples in ergodic theory.
- Classical version: invertible Lebesgue measure preserving transformation $T : [0,1) \to [0,1)$.
- Equivalently, a measure preserving $\mathbb{Z}$ action (MP$\mathbb{Z}$A).
- Easily generalizes to $\mathbb{Z}^d$ or $\mathbb{R}^d$ to produce MP$\mathbb{Z}^d$A or MP$\mathbb{R}^d$A.
- More general than substitutions.

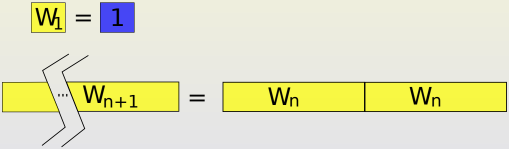
Entropy

- Kolmogorov-Sinai, 1959: entropy $h(T)$ of a measure preserving transformation T. Average "information" per time step.
- Straightforward generalization to d-dimensional entropy $h(T)$ of MP$\mathbb{Z}^d$A T.
- Adler-Konheim-McAndrew, 1965: Topological entropy $h_{\text{top}}(T)$ of continuous map (or $\mathbb{Z}^d$ action) T. Exponential growth in "complexity" $h(T) \leq h_{\text{top}}(T)$.
- Milnor, 1986: directional entropy $h_n(V, T)$ of MP$\mathbb{Z}^d$A, T. Here $V \subseteq \mathbb{R}^d$ subspace, $\dim(V) = n$.

Von Neuman's "adding machine"



Illustrated as block concatination

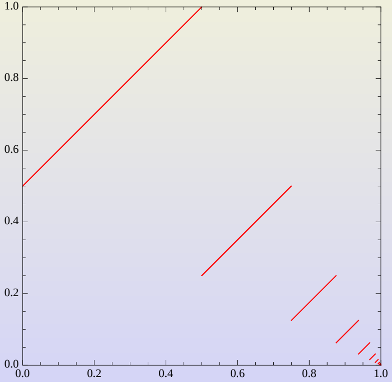


Picture shows base step and induction step, illustrating the combinatorial data needed for the construction:

$$W_1 = 0, \quad W_{n+1} = W_n W_n.$$

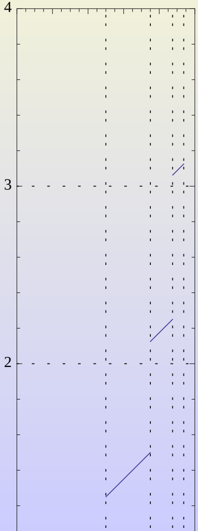
The tower is turned on its side, with individual levels blurred.

As $T : [0,1) \to [0,1)$

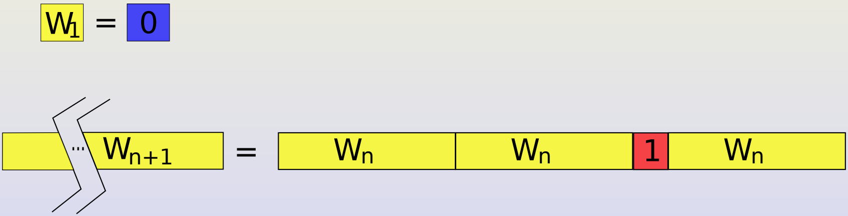


As Toeplitz sequence

Action together with partition equals process.

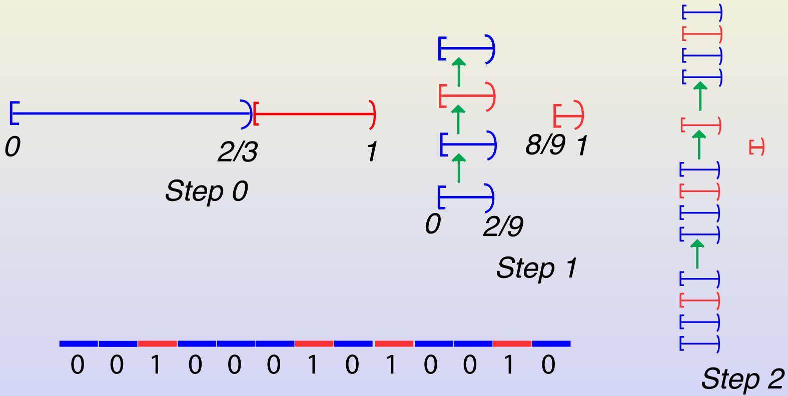


CHACON'S TRANSFORMATION



Here the combinatorial data is $W_1 = 0$ and $W_{n+1} = W_n W_n 1 W_n$.

Chacon's transformation



Rank 1

Definition. T is rank 1 if it can be constructed by cutting and stacking with one large tower in each step.

- Left over interval called a spacer.

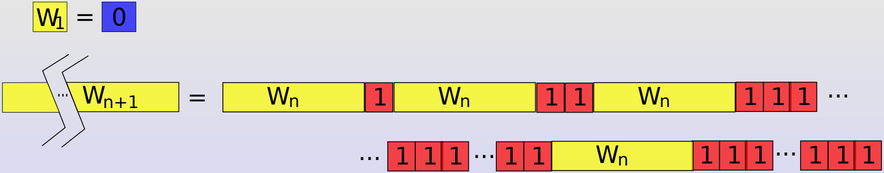
Theorem

Rank 1 implies (uniquely) ergodic. (Also minimal if number of adjacent spacers is bounded.)

- Adding machine has discrete spectrum. Chacon's transformation has continuous spectrum (i.e., is weakly mixing.)
- Any ergodic T with discrete spectrum is rank 1 (e.g., irrational rotation transformation).

Rank 1 mixing

- (Smorodinski)-Adams (1998) version (see also Ornstein (1968)).



Recurrence relation: $W_1 = 0$, $W_{n+1} = W_n 1 W_n 1^2 \dots W_n 1^{q_n}$.
Mixing provided $q_n \nearrow \infty$ sufficiently fast.

The Morse dynamical system

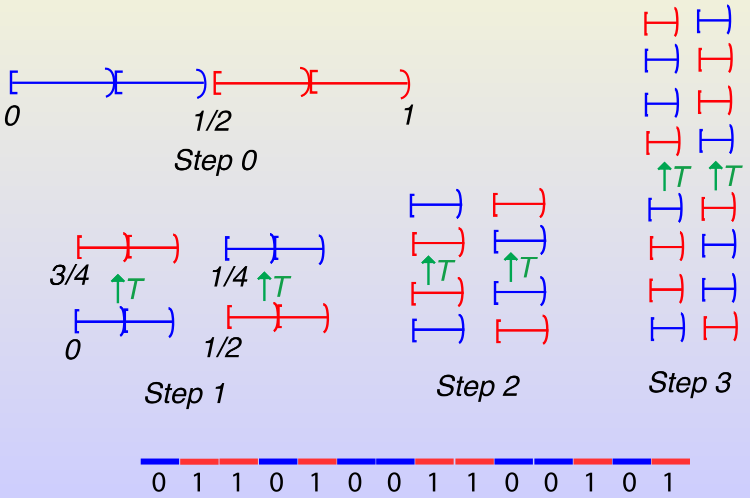
$W_1^0 = 0$

$W_1^1 = 1$

$W_{n+1}^0 = W_n^0 \, W_n^1$

$W_{n+1}^1 = W_n^1 \, W_n^0$

Morse sequences



Finite Rank

In this example, there are 2 towers at each step. We say T has rank ≤ 2.

- A. del Junco showed this T is not rank 1. Thus T is rank 2.
- The spectrum of T is simple, and mixed (both discrete and continuous).
- Can similarly definerank $\leq$ r, rank r, and finite rank.

Theorem (see Queffelec, (1987/2010))

A substitution on r letters is rank $\leq r$.

Rank, Spectrum and Entropy

Theorem (Baxter, 1971)

Finite rank implies $h(T) = 0$.

Proof.

- Rank n implies spectral multiplicity $M_T \leq n$ (Chacon, 1970).
- Positive entropy ($h(T) > 0$) implies $M_T = +\infty$ (Bernoulli factor) (Sinai's Theorem).

□

Rohlin Towers

- Let $T : X \to X$ be a MP$\mathbb{Z}$A on a probability space $(X, \mathcal{B}, \mu)$.
- If $B, TB, T^2B, \ldots, T^{h-1}B$ are pairwise disjoint, we call it Rohlin tower with height h and base B.
- The error is $E = \left(\cup_{k=0}^{h-1} T^k B\right)^c$.
- Call $\xi = \{B, TB, \ldots, T^{h-1}B, E\}$ a Rohlin partition.

Theorem (Rohlin's Lemma)

If T is ergodic, then for any $h \in \mathbb{N}$ and $\epsilon > 0$, there is a height h Rohlin tower with $\mu(E) < \epsilon$.

Rank 1

- Let ξ_n be a sequence of partitions. Say ξ_n separates ($\xi_n \to \varepsilon$) if for any $A \in \mathcal{B}$ there is $A_n \leq \xi_n$ so that $\mu(A \Delta A_n) \to 0$.

Definition

T is rank 1 if there is a sequence ξ_n of Rohlin towers so that $\xi_n \to \varepsilon$.

Cutting and stacking definition of Rank 1 implies this one: $\xi_n \to \varepsilon$ follows from $\mathrm{diam}(B_n) \to 0$.

Theorem (Baxter,1971)

ξ_n may be chosen so that $\xi_n \leq \xi_{n+1}$ and $B_{n+1} \subseteq B_n$.

Thus all these T may be obtained by cutting and stacking.

"Funny" Rank 1

- Call a finite $R \subseteq \mathbb{Z}$ a shape.
- Suppose $\mu(B) > 0$ and $T^k B \cap T^\ell B = \emptyset$ for all $k, \ell \in R$, $k \neq \ell$.
- Call $\xi = \{E, T^k B : k \in R\}$ a funny Rohlin tower.
    - In rank 1, $R = \{0, 1, \ldots, h-1\}$.
- Define funny rank 1 analogously.

Shape matters! Rank 1 implies "loosely Bernoulli" (Katok, 1977, Ornstein-Rudolph-Weiss 1982), but funny rank 1 does not (Ferenczi, 1985).

Actions of $\mathbb{Z}^d$

- Let $(X, \mathcal{B}, \mu)$ be a probability space.
- Let $T_1, T_2 : X \to X$ be MP$\mathbb{Z}$As that commute: $T_1 T_2 = T_2 T_1$.
- For $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, define MP$\mathbb{Z}^2$A $T^{\mathbf{n}} = T_1^{n_1} T_2^{n_2}$.
- Similar definition for MP$\mathbb{Z}^d$A, (i.e., $T_1, T_2, \ldots, T_d$ commute).
- Call a finite $R \subseteq \mathbb{Z}^d$ a shape.

Definition. A shape-R Rohlin tower consists of disjoint sets $T^{\mathbf{n}}B, \mathbf{n} \in R$. The partition $\xi = \{E, T^{\mathbf{n}}B : \mathbf{n} \in R\}$ is a Rohlin partition.

$\mathbb{Z}^d$ rank 1

Definition

A MP$\mathbb{Z}^d$A T is rank 1 if there is a sequence ξ_n of shape R_n Rohlin partitions so that $\xi_n \to \varepsilon$.

Proposition (R-Sahin, 2010)

Rank 1 (any shape) implies ergodic and simple spectrum.

Corollary

Rank 1 (any shape) implies $h(T) = 0$.

$\mathbb{Z}^d$ RANK r

DEFINITION

Suppose T is a MP$\mathbb{Z}^d$A there are shapes R_n^j and positive measure sets B_n^j, for $j = 1, \ldots, r$ and $n \in \mathbb{N}$, so that

$$\xi_n = \{T^{\mathbf{n}} B_n^j : \mathbf{n} \in R_n^j, j = 1, \ldots, n\} \cup \{X \setminus \cup_{j=1}^n \cup_{\mathbf{n} \in R_n^j} T^{\mathbf{n}} B_n^j\}$$

is a partition, and $\xi_i \to \varepsilon$. We say T is rank $\leq r$ for shapes $\{R_n^1, R_n^2, \ldots, R_n^j\}$.

Rank r if rank $\leq r$ and not rank $\leq r - 1$.

PROPOSITION

Rank $\leq r$ implies $M_T \leq r$ and $h(T) = 0$, but not necessarily ergodic.

Følner sequences

A sequence $\mathcal{R} = \{R_k\}$ of shapes in $\mathbb{Z}^2$ is a *Følner sequence* (van Hove sequence) if for any $\mathbf{n} \in \mathbb{Z}^2$

$$\lim_{k \to \infty} \frac{|R_k \triangle (R_k + \mathbf{n})|}{|R_k|} = 0,$$

- A natural choice is rectangles

$$R_k = [0, \dots, w_k - 1] \times [0, \dots, h_k - 1],$$

where $w_k, h_k \to \infty$.

Types of rank 1

- Rank 1: no shape restriction.
- Følner rank 1: R_n a Følner sequence.

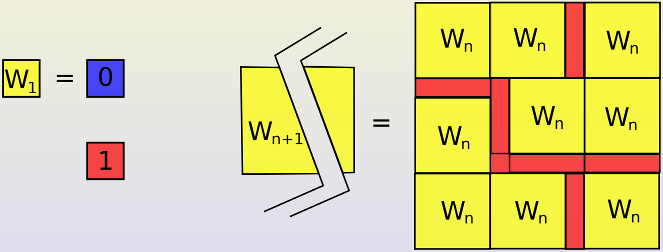
Proposition (R-Sahin, 2010)

If Følner, can get $\xi_n \leq \xi_{n+1}$ with the same $\mathcal{R} = \{R_n\}$.

  - Cutting and stacking works!

- Rectangular rank 1: rectangles
- Geometric restrictions (on rectangular Rank 1):
  - Bounded eccentricity: $1/K \leq w_k/h_k \leq K$.
  - Subexponential eccentricity: $\log(w_k)/h_k \to 0$ $(w_k \geq h_k)$.

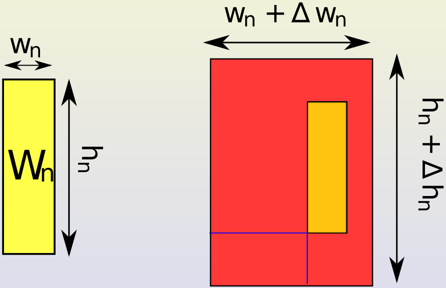
Chacon $\mathbb{Z}^2$ actions



Weak mixing, not strong mixing, & "MSJ" (R-Park, 1991).

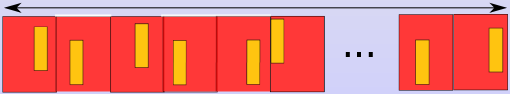
Note. $w_n/h_n = 1$: "bounded" eccentricity.

RUDOLPH'S EXAMPLE



One of $(\Delta w_n)(\Delta h_n)$ blocks.

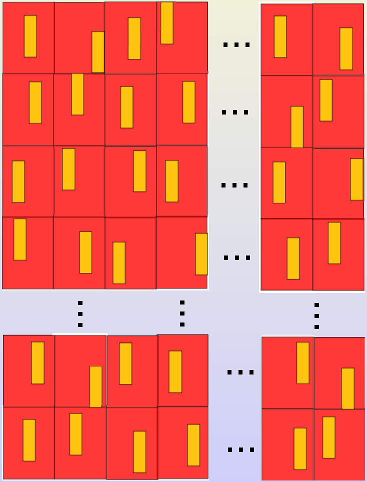
N_n of these blocks in a row.



Rudolph's example (continued)

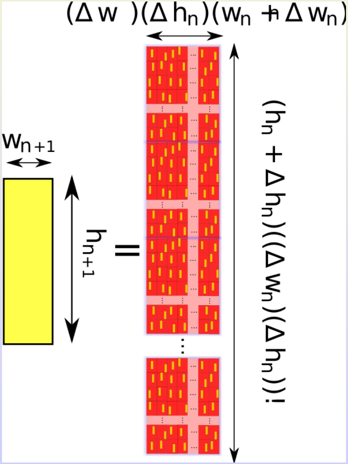
A block consisting of all possible $((\Delta w_n)(\Delta h_n))^{N_n}$ rows, in some particular order.

There are $\left(((\Delta w_n)(\Delta h_n))^{N_n}\right)!$ of these.



RUDOLPH'S EXAMPLE (CONTINUED)

- All $\left(((\Delta w_n)(\Delta h_n))^{N_n}\right)!$ blocks (every possible order) stacked.
- $w_{n+1} = ((\Delta w_n)(\Delta h_n))^{N_n} \times (w_n + \Delta w_n)$.
- $h_{n+1} = \left(((\Delta w_n)(\Delta h_n))^{N_n}\right)! \times ((\Delta w_n)(\Delta h_n))^{N_n} \times (h_n + \Delta h_n)$.



Properties of Rudolph's example

- Requires appropriate choice of $\Delta w_n \to \infty$, $\Delta h_n \to \infty$ and $N_n \to \infty$.
- Side lengths
  $w_{n+1} = ((\Delta w_n)(\Delta h_n))^{N_n} (w_n + \Delta w_n)$, and
  $h_{n+1} = \left(((\Delta w_n)(\Delta h_n))^{N_n}\right)! \, ((\Delta w_n)(\Delta h_n))^{N_n} (h_n + \Delta h_n)$.
- Sides satisfy $\log(h_n)/w_n \to \infty$. Super exponential eccentricity.

Theorem (Rudolph, 1978)

Horizontal T_1 is Bernoulli shift with arbitrary finite entropy $0 < h(T_1) < \infty$.

Review d-dimensional entropy

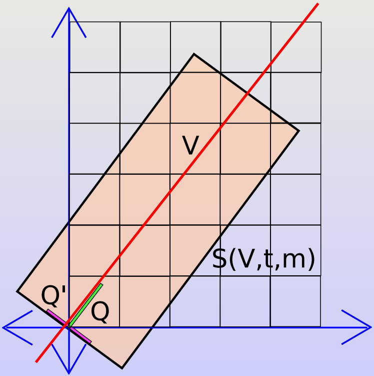
Before defining directional entropy, we briefly review the ordinary (d-dimensional) entropy of a MP$\mathbb{Z}^d$A T.

- Let ξ be a finite partition. The entropy of ξ is
$$H(\xi) = -\sum_{A\in\xi} \mu(A)\log\mu(A).$$
- Define $\xi_n = \bigvee_{\mathbf{n}\in[0,\dots,n)^d} T^{-\mathbf{n}}\xi$
- The ξ-entropy of T is
$$h(T,\xi) = \lim_{n\to\infty} \frac{1}{n^d} H(\xi^n).$$
- The entropy of T is given by
$$h(T) = \sup_\xi h(T,\xi).$$

This gives usual entropy of transformation T when $d = 1$.

Preliminaries for directional entropy

- Subspace $V \subseteq \mathbb{R}^d$, $n = \dim(V) < d$.
- $Q \subseteq V$, $Q' \subseteq V^{\perp}$ unit cubes, and
- $S(V, t, m) = (tQ + mQ')$ (we call it a window.)



DIRECTIONAL ENTROPY (MILNOR, 1986)

Let T be a MP$\mathbb{Z}^d$A, with ξ a finite partition, and $\dim(V) = n$.

- $\xi_{V,t,m} := \bigvee_{\mathbf{n} \in S(V,t,m)} T^{-\mathbf{n}} \xi.$
- $h_n(T, V, \xi, m) := \limsup_{t \to \infty} \frac{1}{t^n} H(\xi_{V,t,m}).$
- $h_n(T, V, \xi) := \sup_{m > 0} h_n(T, V, \xi, m)$

DEFINITION (MILNOR, 1986)

If $1 \leq n < d$, n-dimensional directional entropy in direction V is

$$h_n(T, V) = \sup_\xi h_n(T, \xi, V).$$

If $n = d$, then $h_d(T, V) = h(T)$, (where $V = \mathbb{R}^d$).

Directional entropy ($\mathbb{Z}^2$ case)

- $h_1(V,T) < \infty$ for some V, implies $h_2(T) = 0$.
  - Ledrappier's $\mathbb{Z}^2$ shift T has $h_1(T,V) > 0$ for all V.
  - K. Park (unpublished, c 1987) Chacon MP$\mathbb{Z}^2$A T has $h_1(T,V) = 0$ for all V.
- $h_1(T,V) = ||(p,q)||^{-1} h(T^{(q,p)})$, $V = (p,q)\mathbb{R}$, $p/q \in \mathbb{Q}$.
  - Rudolph rank 1 $\mathbb{Z}^2$ has $h_1(V,T) > 0$ where $V = \mathbf{e}_1\mathbb{R}$.
- (K. Park, 1999) If $V = \mathbf{v}\mathbb{R}$, $||\mathbf{v}|| = 1$, then $h_1(T,V) = h(F^{t\mathbf{v}})$ for the unit $\mathbb{R}^2$ suspension $F^{\mathbf{t}}$ of T.
- (K. Park, 1999) The function $h(\mathbf{v}) = h(T, \mathbf{v}\mathbb{R})$, $||\mathbf{v}|| = 1$, is upper semicontinuous, and $\{\mathbf{v} : h(\mathbf{v}) = 0\}$ is G_δ.

Theorems

The first result has no assumptions beyond rectangular rank 1.

Theorem 1. (R-Sahin, 2010)

Let T be a rectangular rank-1 MP$\mathbb{Z}^d$A. Then there is a 1-dimensional subspace $V \subseteq \mathbb{R}^d$ so that $h_1(T, V) = 0$.

With addition hypotheses on the eccentricity, we can say more.

Theorem 2. (R-Sahin, 2010)

Let T be a rectangular rank-1 MP$\mathbb{Z}^d$A with subexponential eccentricity. If $V \subseteq \mathbb{R}^d$ is an n-dimensional subspace, $1 \leq n \leq d$, then $h_n(T, V) = 0$.

Two lammas

Lemma (Milnor, 1988)

The formulas that define directional entropy simplify to

$$h_n(T,V,\xi,m) = \lim_{t\to\infty} \frac{1}{t^n} H(\xi_{V,t,m}), \text{ and}$$

$$h_n(T,V,\xi) = \lim_{m\to\infty} h_n(T,V,\xi,m).$$

Theorem (Boyle-Lind, 1997)

If $\xi_k \le \xi_{k+1}$ *and* $\xi_k \to \epsilon$ *then*

$$h_n(T,V) = \lim_{k\to\infty} h_n(T,V,\xi_k).$$

Zero-entropy lemma

Lemma

Suppose $\xi_k \leq \xi_{k+1}$ *and* $\xi_k \to \varepsilon$. *If* $t_j \to \infty$, *and*

$$\lim_{j\to\infty} \frac{1}{(t_j)^n} H((\xi_k)_{V,t_j,m}) = 0,$$

for all k *and all* $m > 0$, *then* $h_n(T,V) = 0$.

We will use this lemma in the proofs of both theorems.

Proofs (set-up)

We do the case $d = 2$.

Let $V \subseteq \mathbb{R}^2$ be a 1-dimensional subspace (to be specified later for Theorem 1), and let ξ_k be a sequence of shape-R_k Rohlin towers for T.

Assume WOLOG:

- $\xi_k \leq \xi_{k+1}$ (Baxter's Theorem),
- R_k is $w_k \times h_k$ where $h_k \leq w_k$ for all k.

Note. There are no eccentricity assumptions in Theorem 1.

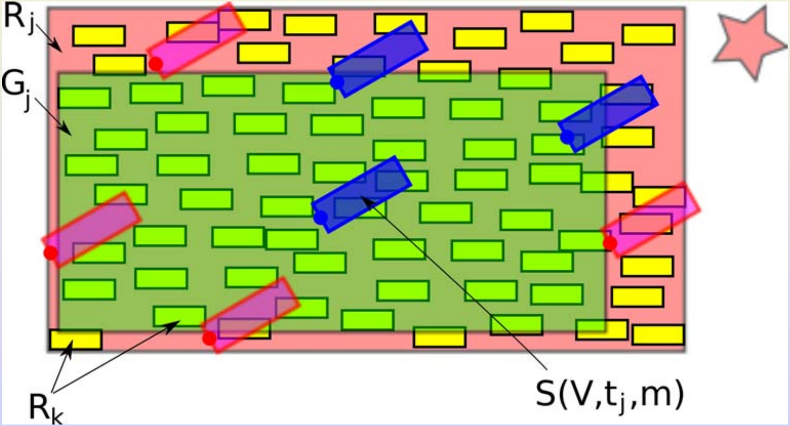
Let $t_j \to \infty$ be a slowly increasing sequence, to be specified later.

Ultimate Goal. For fixed m, k, show that $H((\xi_k)_{V,t_j,m})/t_j \to 0$.

Proofs (labels)

- Let $j > k$.
- Call a level $T^{\mathbf{n}}B_j$ in ξ_j **good** if $S(V, t_j, m) \subseteq R_j - \mathbf{n}$.
- Let $G_j \subseteq \mathbb{Z}^2$ be the set of good levels.
- Let $F_j = (\cup_{\mathbf{n} \in G_j} T^{\mathbf{n}}B_j)^c$.
- And, recall $E_j = (\cup_{\mathbf{n} \in R_j} T^{\mathbf{n}}B_j)^c$.

Proofs (Good Levels)



Proofs (New partitions)

- $\xi_j^* := \{T^{\mathbf{n}} B_j : \mathbf{n} \in G_j\} \cup \{F_j\}$.
- $\eta_j := (\xi_k)_{T,t_j,m} \vee \xi_j^*$.
- Note that $(\xi_k)_{T,t_j,m} \leq \eta_j$.
- Thus $H((\xi_k)_{T,t_j,m}) \leq H(\eta_j)$.
- So it suffices to show $H(\eta_j)/t_j \to 0$.
- (This will achieve our **Ultimate Goal.**)

Proofs (Relations among partitions)

Key observation: Each of the sets $T^{\mathbf{n}}B_j$ for $\mathbf{n} \in G_j$ belong to the partition η_j.

"Goodness" insures the partition $(\xi_k)_{V,t_j,m}$ is "constant" on levels $T^{\mathbf{n}}B_j$, for $\mathbf{n} \in G_j$. In other words, each $T^{\mathbf{n}}B_j$ is a subset of some $A \in (\xi_k)_{V,t_j,m}$.

$$
\begin{aligned}
H(\eta_j)/t_j &= -\frac{1}{t_j} \sum_{A \in \eta_j} \mu(A) \log \mu(A) \\
&= -\frac{1}{t_j} \left(\sum_{\mathbf{n} \in G_j} \mu(T^{\mathbf{n}}B_j) \log \mu(T^{\mathbf{n}}B_j) + \sum_{A \in \eta'_j} \mu(A) \log \mu(A) \right) \\
&= -\frac{1}{t_j} \left(|G_j| \mu(B_j) \log \mu(B_j) - \sum_{A \in \eta'_j} \mu(A) \log \mu(A) \right).
\end{aligned}
$$

Proofs (Left term Goal)

$$\begin{aligned} -\tfrac{1}{t_j}|G_j|\mu(B_j)\log\mu(B_j) &\leq -\tfrac{1}{t_j}|R_j|\mu(B_j)\log\mu(B_j) \\ &= -\left(\frac{w_j h_j}{t_j}\right)\left(\frac{1-\epsilon_j}{w_j h_j}\right)\log\left(\frac{1-\epsilon_j}{w_j h_j}\right) \\ &\leq \frac{\log(w_j h_j) - \log(1-\epsilon_j)}{t_j}, \end{aligned}$$

where $\epsilon_j = \mu(E_j)$.

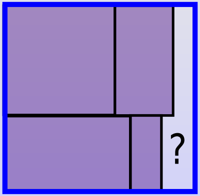
Left Term Goal. Show $\log(w_j h_j)/t_j \to 0$. (Insubstantial entropy from (uniformly covered) good set)

Local entropy lemma

Theorem (Shields, 1996)

Suppose ξ is a partition, $\xi' \subseteq \xi$ and $\beta = \mu(\cup_{A\in\xi'} A)$. Then

$$-\sum_{A\in\xi'} \mu(A)\log\mu(A) \leq \beta\log|\xi'| - \beta\log\beta.$$



Proofs (Right Term)

- $|\xi'_j| \leq (|R_k| + 1)^{|S(V,t_j,m)|}$.
- $\log |\xi'_j| = |S(V,t_j,m)| \log(|R_k| + 1) \leq 2|S(V,t_j,m)| \log |R_k|$.
- $|S(V,t_j,m)| \leq 2t_j m$.
- $\log |R_k| = K$.

Thus

$$\log |\xi'_j| \leq 2K t_j m.$$

Proofs (Right Term Goal)

Also

$$\beta = \mu(F_j) = |B_j \backslash G_j| \mu(B_j) + \mu(E_j) \leq \frac{|B_j \backslash G_j|}{w_j h_j} + \epsilon_j.$$

So by the local entropy lemma

$$-\frac{1}{t_j} \sum_{A \in \xi'} \mu(A) \log \mu(A) \leq 2Km \left(\frac{|B_j \backslash G_j|}{w_j h_j} + \epsilon_j \right) - \frac{\beta \log \beta}{t_j}.$$

(t_j/t_j cancels in the first term). Since $\beta < 1$, $(\beta \log \beta)/t_j \to 0$.

Right Term Goal. $\frac{|B_j \backslash G_j|}{w_j h_j} \to 0$. (This is essentially that measure of bad part, $\beta \to 0$.)

Proof of Theorem 1 (Left Term Goal)

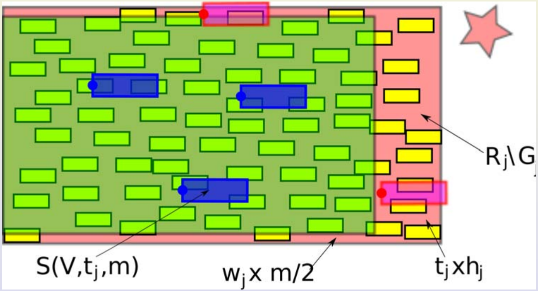
- Assume $w_j \geq h_j$ for all j.
- Take $V = \mathbf{e}_1\mathbb{R}$.
- We want $t_j \to \infty$ so that $\frac{\log(w_j)}{t_j} \to 0$ and $\frac{t_j}{w_j} \to 0$.

Define $t_j = \sqrt{w_j \log w_j}$.

$\frac{\log(w_j h_j)}{t_j} \leq \frac{2\log(w_j)}{t_j} \to 0$. **Left Term Goal Achieved.**

Proof of Theorem 1 (Right Term Goal)

We have, $|R_j \backslash G_j| \leq h_j t_j + m w_j$.



$$\frac{|R_j \backslash G_j|}{w_j h_j} = \frac{t_j}{w_j} + \frac{m}{h_j} \to 0,$$

since $\frac{t_j}{w_j} = \frac{\sqrt{w_j \log w_j}}{w_j} = \sqrt{\frac{\log w_j}{w_j}} \to 0$. **Right Term Goal Achieved.**

Proof of Theorem 2 (Left Term Goal)

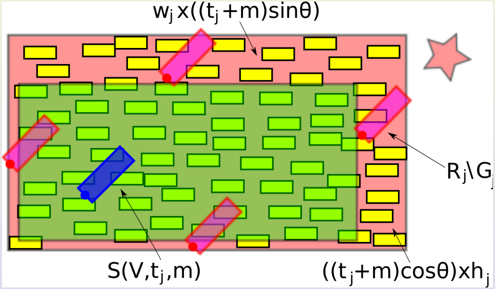
- Take $V \subseteq \mathbb{R}^2$, $\dim(V) = 1$.
- Assume $w_j \geq h_j$ and **define** $t_j = \sqrt{h_j \log w_j}$.
  - $\frac{\log w_j}{t_j} = \frac{\log w_j}{\sqrt{w_j \log(w_j)}} = \sqrt{\frac{\log w_j}{w_j}} \to 0$
  - $\frac{t_j}{h_j} = \frac{\sqrt{h_j \log w_j}}{h_j} = \sqrt{\frac{\log w_j}{h_j}} \to 0$
    (by subexponential eccentricity).

$$\frac{\log(w_j h_j)}{t_j} \leq \frac{2\log(w_j)}{t_j} \to 0.$$

Left Term Goal achieved.

Proof of Theorem 2 (Right Term Goal)

We have, $|R_j \backslash G_j| \leq h_j(t_j + m)\cos\theta + w_j(t_j + m)\sin\theta$.



$$\frac{|R_j \backslash G_j|}{w_j h_j} = \frac{t_j + m}{w_j}\cos\theta + \frac{t_j + m}{h_j}\sin\theta \to 0,$$

since $\frac{t_j}{h_j} \to 0$, (and $\frac{t_j}{w_j}, \frac{m}{h_j}, \frac{m}{w_j} \to 0$.) **Right Term Goal achieved.**

Rank r

Here is what we can prove in rank r. For simplicity, we discuss only the case T is an ergodic rectangular rank ≤ 2 MP$\mathbb{Z}^2$A. Let R_n^1 be $w_n^1 \times h_n^1$ and R_n^2 be $w_n^2 \times h_n^2$.

Theorem A. If $w_n^1 \geq h_n^1$ and $w_n^2 \geq h_n^2$ for infinitely many n then there exists V so that $h_1(T, V) = 0$ (i.e., $h(T_1) = 0$).

Theorem B. Under the same hypotheses as above, if $\log(w_n^1)/h_n^1 \to 0$, and $\log(w_n^2)/h_n^2 \to 0$, then $h_1(T, V) = 0$ for all 1-dimensional V.

Theorem C. If $w_n^1 \geq h_n^1$ and $w_n^2 \leq h_n^2$ for all n, and $\log(w_n^1)/h_n^1 \to 0$, and $\log(h_n^2)/w_n^2 \to 0$, then $h_1(T, V) = 0$ for all 1-dimensional V.

Examples from aperiodic order

- As mentioned before, a substitution on r letters has rank $\leq r$. This is also true for a substitution tiling with r distinct prototiles. The eccentricity is bounded. This implies a substitution tiling system has all directional entropies zero.
- Another way to prove this is to note that the complexity of a substitution tiling satisfies $c(n) \leq Kn^e$ (where $e = d$ in the self similar case).
- A. Julien (2009) proved $c(n) \leq Kn^e$ for a cut and project tiling where the acceptance domain is polyhedral and "almost canonical". This implies all directional entropies zero.
- More generally a model set with a topologically and measure theoretically regular acceptance domain has discrete spectrum, so is rank 1. This implies all directional entropies zero.

Other examples

- Ledrappier's shift has $c(n) = Ke^{2n}$ (exponential complexity in smaller dimension). It has positive directional entropy in every direction.
- Radin showed that any uniquely ergodic $\mathbb{Z}^2$ SFT has $c(n) \leq Ke^{\ell n}$. Can it have positive directional entropy.
- Not for the examples that come from substitutions and model sets!

Loosely Bernoulli

Say MP$\mathbb{Z}^d$A T with $h_d(T) = 0$ is entropy zero loosely Bernoulli (LB) if a suspension of T (to a MP$\mathbb{R}^d$A) can be time changed to a suspension of some R discrete spectrum (action by rotations on a compact group).

Theorem (Johnson-Sahin, 1998)

A rectangular rank 1 MP$\mathbb{Z}^2$A T with bounded eccentricity is loosely Bernoulli.

- This T can be chosen to have T_1 be non LB.
- Johnson-Sahin (1998) prove that the same result holds for rank $r > 1$ provided towers have uniformly bounded eccentricity.

Loosely Bernoulli

Theorem (R-Sahin 2011?)

If T is a loosely Bernoulli MP$\mathbb{Z}^d$A with $h_d(T) = 0$ then $h_n(T, V) = 0$ for all V.

Implications:

- Ledrappier's shift is not loosely LB (a "folk theorem").
- Rudolph's rank 1 is not LB.

$\mathbb{Z}^d$ Rohlin lemma

- Say the Rohlin lemma holds for a shape $R \subseteq \mathbb{Z}^d$ if for any ergodic $\mathbb{Z}^d$ action T, and $\epsilon > 0$, there exists $B \in \mathcal{B}$ so that X is partitioned by $\xi = \{E, T^{\mathbf{n}}B : \mathbf{n} \in R\}$ and $\mu\left(\cup_{\mathbf{n}\in R} T^{\mathbf{n}}B\right) > 1 - \epsilon$.
- A shape R tiles $\mathbb{Z}^d$ if there exists $C \subseteq \mathbb{Z}^d$ so that $\{T^{\mathbf{n}}R : \mathbf{n} \in C\}$ is a partition of $\mathbb{Z}^n$.

Theorem (Ornstein-Weiss, 1980)

A Rohlin lemma holds for a shape R if and only if R tiles $\mathbb{Z}^d$.