Ergod. Th. & Dynam. Sys. (2011), **31**, 285–299
doi:10.1017/S0143385709000911

Rank-one $\mathbb{Z}^d$ actions and directional entropy

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(*Received* 9 *September* 2008 *and accepted in revised form* 14 *July* 2009)

Abstract. We study the dynamic properties of rank-one $\mathbb{Z}^d$ actions as a function of the geometry of the shapes of the towers generating the action. Some basic properties require only minimal restrictions on the geometry of the towers. Our main results concern the directional entropy of rank-one $\mathbb{Z}^d$ actions with rectangular tower shapes, where we show that the geometry of the rectangles plays a significant role. We show that for each $n \leq d$ there is an n-dimensional direction with entropy zero. We also show that if the growth in eccentricity of the rectangular towers is sub-exponential, then all directional entropies are zero. An example of D. Rudolph shows that, without a restriction on eccentricity, a positive entropy direction is possible.

1. *Introduction*

Rank-one transformations play a central role in the theory of ergodic measure-preserving transformations. Having first been identified as a distinct class by Chacon in [**3**], their properties have been studied extensively (see for example [**1, 6, 7, 11**]). Rank-one transformations have also served as an important tool for exploring the range of possible behavior of measure-preserving transformations (see for example [**3, 15, 22**]). The idea of rank-one can be easily generalized to measure-preserving actions of $\mathbb{Z}^d$. Informally, we think of a rank-one action of $\mathbb{Z}^d$ on a Lebesgue probability space $(X, \mathcal{A}, \mu)$ as a limit of actions defined on a sequence of Rohlin towers whose levels generate $\mathcal{A}$. In the classical case, i.e. the case $d = 1$, the most natural shapes for these towers are intervals. Even in that case, however, it is possible to define rank-one more generally by allowing the towers to have more exotic shapes. Thouvenot, for example, gives a definition of a class of transformations called *funny* rank-one, in which the tower shapes are arbitrary Følner sets [**5**]. Even in the case $d = 1$ the tower shapes do matter, because while all rank-one transformations with interval tower shapes are loosely Bernoulli [**16**], Ferenczi showed that there exists a funny rank-one transformation that is not [**5**].

Many of the most basic properties of rank-one transformations extend to the $\mathbb{Z}^d$ case, $d > 1$, with essentially no restrictions on the tower shapes. These include ergodicity, entropy zero, and simple spectrum (see §3). While for $d > 1$ it might appear that rectangular tower shapes are the obvious analogues of the interval tower shapes of the case $d = 1$, even within the class of rank-one $\mathbb{Z}^d$ actions with rectangular towers there remains some choice in the shapes of the towers. In particular, the dimensions of the rectangles can grow to infinity at different rates. Because the choice of natural tower shapes is less obvious for $d > 1$, we drop the terminology *funny* rank-one, and replace it with adjectives describing the choices made in the tower shapes. If the sequence of towers all are rectangular, we call the action *rectangular rank-one*.

If T is a rank-one $\mathbb{Z}^d$ action with the property that the sequence of towers are rectangles of uniform bounded eccentricity then T is loosely Bernoulli [**8**]. While it is not known if there is a rectangular rank-one $\mathbb{Z}^d$ action that is not loosely Bernoulli, the fact that the proof in [**8**] does not readily extend to more general sequences of rectangles, together with the one-dimensional result of Ferenczi, suggests that even within the class of rectangular rank-one $\mathbb{Z}^d$ actions, we have the possibility of different dynamical behavior.

In this paper we study the directional entropy of rectangular rank-one $\mathbb{Z}^d$ actions with various growth conditions on the tower shapes. Directional entropy was introduced by Milnor in [**12, 13**], and has been studied extensively (see [**2, 9, 17, 19, 24**]). Here we show that the shape of the towers plays a role in the possible directional entropies that can occur. We first show that any rectangular rank-one $\mathbb{Z}^d$ action has at least one zero-entropy direction. On the other hand, Rudolph shows in [**21**] that given any $h > 0$, there exists a rectangular rank-one $\mathbb{Z}^2$ action whose horizontal sub-action is Bernoulli with entropy h. Our main result shows that when the rectangles satisfy a condition we call *sub-exponential eccentricity*, the resulting action has directional entropy zero in every direction. More generally, our result shows that for any sub-exponentially eccentric rank-one $\mathbb{Z}^d$ action, the ℓ-dimensional entropy is zero for all $\ell < d$, for all ℓ-dimensional hyperplanes in $\mathbb{Z}^d$. We note that in [**4**] the authors construct a family of examples of rank-one $\mathbb{Z}^d$ actions which, in our terminology, are bounded eccentricity rank-one actions. There, they compute the entropy of every transformation in the action to be zero. In the terminology of directional entropy, they show that their examples have directional entropy zero in every rational direction.

In order to prove our main theorem we also establish a higher-dimensional generalization of a result of Baxter [**1**], which may be of independent interest. We show that given a rank-one $\mathbb{Z}^d$ action, with a natural restriction on the tower shapes, there is an isomorphic rank-one $\mathbb{Z}^d$ action such that the towers have the same shapes as the original action and the sequence of towers form a refining sequence of generators.

The organization of the paper is as follows. In §2 we establish the notation we use in the paper. In §2.4 we introduce the idea of directional entropy and summarize some results from the theory that we will use. In §3 we describe a hierarchy of types of rank-one actions based on progressively more restrictive conditions on the sequence of towers. We also establish properties of rank-one actions, highlighting their relationship to this hierarchy.

In this section we also state the generalization of the result in [**1**] about refining sequences of partitions, leaving the proof to §5. Finally in §4 we prove our main results about the directional entropy of rank-one $\mathbb{Z}^d$ actions.

2. *Basic definitions and a review of directional entropy*

2.1. *Shapes.* A *shape* R is a finite subset of $\mathbb{Z}^d$. Let $|R|$ denote the cardinality of R. For another shape S the inner S-boundary of R is defined by

$$\partial_S(R) = \bigcup_{R^c \cap (S+\vec{v}) \neq \emptyset} R \cap (S + \vec{v}).$$

A sequence $\mathcal{R} = \{R_k\}$ of shapes is a *Følner sequence* (see for example [**25**]) if for any $\vec{v} \in \mathbb{Z}^d$

$$\lim_{k\to\infty} \frac{|R_k \triangle (R_k + \vec{v})|}{|R_k|} = 0,$$

or equivalently if for any shape S,

$$\lim_{k\to\infty} \frac{|\partial_S(R_k)|}{|R_k|} = 0.$$

This follows from the fact that $\partial_S(R_k) \subseteq \bigcup_{\vec{v}\in S-S} R_k \triangle (R_k + \vec{v})$.

2.2. *Partitions.* Let $(X, \mathcal{M}, \mu)$ be a Lebesgue probability space and let L be a finite set. A *partition* $\mathcal{P}$ with *alphabet* L is a measurable function $\mathcal{P} : X \to L$. Equivalently, we think of $\mathcal{P} = \{P_a = \mathcal{P}^{-1}(a) \mid a \in L\}$ as a finite labeled collection of pairwise disjoint measurable sets (called *atoms*) so that $\bigcup_{a\in L} P_a = X$. We write $P(x)$ for the unique atom $P_a \in \mathcal{P}$ so that $\mathcal{P}(x) = a$ (i.e., $P(x) := \mathcal{P}^{-1}(\mathcal{P}(x))$). For two partitions with alphabet L we define

$$d(\mathcal{P}, \mathcal{Q}) = \sum_{a\in L} \mu(P_a \triangle Q_a).$$

It is well known that d is a complete metric on the space of all partitions with a fixed alphabet (i.e., this space is essentially a closed subset of $L^1(X, \mu)$).

For a sequence $\mathcal{P}_k$ of partitions, where L_k is the alphabet of $\mathcal{P}_k$, we say $\mathcal{P}_k \to \epsilon$ if for any $A \in \mathcal{M}$, there exists $I_k \subseteq L_k$ such that

$$\lim_{k\to\infty} \mu\left(\left(\bigcup_{a\in I_k} P_a\right) \triangle A\right) = 0.$$

Let $\mathcal{P}$ and $\mathcal{Q}$ be partitions with alphabets L and M. We say $\mathcal{P} \leq \mathcal{Q}$ if each $P_a \in \mathcal{P}$ is a union of elements of $\mathcal{Q}$: $P_a = \bigcup_{b\in I} Q_b$ for some $I \subseteq M$. We define $\mathcal{P} \vee \mathcal{Q}$ to be the partition with atoms $P \cap Q$, where $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$. The alphabet for $\mathcal{P} \vee \mathcal{Q}$ is $L \times M$.

2.3. *Towers.* Let T be a free measure-preserving $\mathbb{Z}^d$ action on a Lebesgue probability space $(X, \mathcal{A}, \mu)$. Let $R \subseteq \mathbb{Z}^d$ be a shape and let $B \in \mathcal{A}$, $\mu(B) > 0$, satisfy

$$T^{\vec{v}_1} B \cap T^{\vec{v}_2} B = \emptyset \quad \text{for all } \vec{v}_1, \vec{v}_2 \in R \text{ with } \vec{v}_1 \neq \vec{v}_2.$$

Let $E = (\bigcup_{\vec{v} \in R} T^{\vec{v}} B)^c$. We call the partition $\mathcal{P} = \{T^{\vec{v}} A \mid \vec{v} \in R\} \cup \{E\}$ a *Rohlin tower*, or more specifically, a *T-tower* with *shape-R* and *base* B. The sets $T^{\vec{v}} B$, $\vec{v} \in R$ are called the *levels* of the tower and E is called the *error set*. The tower partition $\mathcal{P}$ is defined by labeling the level $T^{\vec{v}} B$ with $\vec{v}$, and labeling the set E with $\vec{\epsilon} = \frac{1}{2}(1, 1, \ldots, 1)$. Thus the alphabet of $\mathcal{P}$ is $L = R \cup \{\vec{\epsilon}\}$.

2.4. *Directional entropy.* In this section we state the definitions and results about the entropy of partitions and directional entropy that are necessary for our work. We refer the reader to [**2, 9, 12, 13, 19**] for a detailed development of the theory of directional entropy.

We define the *entropy* of a partition $\mathcal{P}$ with alphabet L by

$$H(\mathcal{P}) = \sum_{a \in L} -\mu(P_a) \log(\mu(P_a)),$$

where we define $0 \log(0) = 0$.

LEMMA 2.1. [**23**, Lemma I.6.8] *Suppose $\mathcal{P}$ is a partition with alphabet L. Let $M \subseteq L$ and let $\beta = \mu(\bigcup_{b \in M} P_b)$. Then*

$$-\sum_{b \in M} \mu(P_b) \log \mu(P_b) \leq \beta \log |M| - \beta \log \beta.$$

Let V be an n-dimensional subspace of $\mathbb{R}^d$, with $1 \leq n < d$ and let $V^{\perp}$ be its orthogonal complement. Choose an orthonormal basis for V and $V^{\perp}$. Let Q be the corresponding unit cube in V with the orthonormal basis vectors as sides and let Q' be the corresponding unit cube in $V^{\perp}$, centered at $\vec{0}$. Let

$$S(V, t, m) = (tQ + mQ') \cap \mathbb{Z}^d, \tag{1}$$

and for a partition $\mathcal{P}$ let

$$\mathcal{P}_{V,t,m} = \bigvee_{\vec{w} \in S(V,t,m)} T^{-\vec{w}} \mathcal{P}.$$

The following definitions are essentially due to Milnor [**13**], but closely match the definitions given in [**2**].

First define

$$h_n(T, V, \mathcal{P}, m) = \limsup_{t \to \infty} \frac{1}{t^n} H(\mathcal{P}_{V,t,m}).$$

Then put

$$h_n(T, V, \mathcal{P}) = \sup_{m > 0} h_n(T, V, \mathcal{P}, m), \tag{2}$$

and

$$h_n(T, V) = \sup_{\mathcal{P} \text{ finite}} h_n(T, \mathcal{P}, V).$$

This is the n-dimensional *directional entropy* in the direction V.

Remark 2.2. Although the unit cubes Q and Q' in (1) require a choice of basis in V and $V^{\perp}$, the entropy does not depend on this choice. Milnor [**13**] showed that the supremum over $m > 0$ in (2) can be replaced by the supremum over all compact compact sets $M \subseteq V^{\perp}$. Now let vol_V denote normalized Lebesgue measure on V. Milnor [**13**] also showed that if $S(V, t, m)$ in (1) is replaced with $(tK + mQ') \cap \mathbb{Z}^d$, where $K \subseteq V$ is any compact set with $\mathrm{vol}_V(\partial_V K) = 0$, and if we define

$$h_n(K) = \sup_{\mathcal{P}} \sup_{m>0} \limsup_{t\to\infty} \frac{1}{t^n} H(\mathcal{P}_{V,t,m}),$$

then $h_n(K) = \mathrm{vol}_V(K) \cdot h_n(T, V)$. Boyle and Lind [**2**] define an n-frame to be a set $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_n$ of n linearly independent (not necessarily orthonormal) vectors in $\mathbb{R}^d$. Let V be the n-dimensional subspace spanned by the frame, and let $K \subseteq V$ be the parallelepiped generated by the frame. Milnor [**12, 13**] and Boyle and Lind [**2**] study $h_n(\vec{w}_1 \wedge \vec{w}_2 \wedge \cdots \wedge \vec{w}_n)$, which is defined to be $h_n(K)$ for $S(V, t, m) = (tK + mQ') \cap \mathbb{Z}^d$. Similarly, Park [**19**], and Kamiński and Park, [**9**] study $h_1(\vec{w})$.

The next two results are useful tools to compute directional entropy.

LEMMA 2.3. (Milnor [**13**])

$$h_n(T, V, \mathcal{P}, m) = \lim_{t\to\infty} \frac{1}{t^n} H(\mathcal{P}_{V,t,m})$$

and

$$h_n(T, V, \mathcal{P}) = \lim_{m\to\infty} h_n(T, V, \mathcal{P}, m).$$

LEMMA 2.4. [**2**, Proposition 6.15] *Let* $1 \le n \le d$. *If* $\mathcal{P}_k \le \mathcal{P}_{k+1}$ *and* $\mathcal{P}_k \to \epsilon$ *then*

$$h_n(T, V) = \lim_{k\to\infty} h_n(T, V, \mathcal{P}_k).$$

Comment. The hypothesis 'expansive' is included in [**2**] but is not used in the proof.

We note that it is well-known that if $V_1 \subseteq V_2$ are subspaces of $\mathbb{R}^d$ with $n_1 = \dim(V_1) < n_2 = \dim(V_2) \le d$, then $h(T, V_2) > 0$ implies $h(T, V_1) = \infty$. The following straightforward observation is key to the structure of our later arguments.

LEMMA 2.5. *Let* $\mathcal{P}_k$ *be a sequence of partitions with* $\mathcal{P}_k \le \mathcal{P}_{k+1}$ *and* $\mathcal{P}_k \to \epsilon$. *Let* V *be an* n*-dimensional subspace of* $\mathbb{R}^d$. *Suppose there exists a sequence* $t_j \to \infty$ *so that for all* k, *and for all* $m > 0$,

$$\lim_{j\to\infty} \frac{1}{(t_j)^n} H((\mathcal{P}_k)_{V,t_j,m}) = 0. \tag{3}$$

Then $h_n(T, V) = 0$.

Proof. This follows immediately from Lemmas 2.3 and 2.4. □

3. *Rank one*

The most general definition of a rank-one action that we will consider is the following.

Definition 3.1. Let T be a free measure-preserving $\mathbb{Z}^d$ action on a Lebesgue probability space $(X, \mathcal{A}, \mu)$. We say T is $\mathcal{R}$ *rank-one* if $\mathcal{R} = \{R_k\}$ is a sequence of shapes, and there is a sequence $\mathcal{P}_k$ of T-towers of shape R_k such that $\mathcal{P}_k \to \epsilon$. We say T is *rank-one* if it is $\mathcal{R}$ rank-one for some $\mathcal{R}$.

This definition places no restrictions on the shapes of the towers, except that implicitly $\mathcal{P}_k \to \epsilon$ implies $|R_k| \to \infty$. One classical result about rank-one transformations that does not depend on the geometry of the tower shapes R_k is ergodicity.

PROPOSITION 3.2. *If T is a rank-one action of $\mathbb{Z}^d$, then T is ergodic.*

Proof. This is essentially the same as the well known proof for rank-one $\mathbb{Z}$ actions (see for example [**7**]). Let A be an invariant set of positive measure. Since $\mathcal{P}_k \to \epsilon$ we can, for any $\epsilon > 0$, find a tower $\mathcal{P}_k$ such that one of the levels of the tower is more than $(1-\epsilon)$-covered by A. The invariance of A guarantees that this property holds for the entire tower. Since ϵ is chosen to be arbitrary, we conclude that A has arbitrarily large measure. □

Given the generality of our definition it is natural to ask how strange the tower shapes can actually be. While we do not address this question in detail in this paper, we will give a simple example of a rank-one $\mathbb{Z}^2$ action with a one-dimensional tower. It turns out, however, that this particular example is also rank-one with non-degenerate rectangular tower shapes [**10, 20**].

Example 3.3. Consider the irrational rotation R_α. Since R_α is rank-one as a $\mathbb{Z}$ action, there exists a sequence of towers $\mathcal{Q}_k = \{R_\alpha^n B_k \mid n = 0, 1, \ldots, \ell_k - 1\}$ where $\mathcal{Q}_k \to \epsilon$. Now take R_β where β, $(\beta/\alpha) \notin \mathbb{Q}$ and define a free $\mathbb{Z}^2$ action on the circle by $T^{(n,m)} = R_\alpha^n R_\beta^m$. Let $\mathcal{F} = \{[0, 1, \ldots, \ell_k] \times \{0\}\}$. Then T is $\mathcal{F}$ rank-one.

In Definition 3.1 the towers have no *a priori* relationship to one another. In constructing examples, though, the towers are usually obtained by 'cutting and stacking' procedures which yield a refining sequence of tower partitions. In particular, each tower partition is measurable with respect to all subsequent tower partitions. The following definition incorporates this structure into the tower partitions.

Definition 3.4. We say T is *stacking $\mathcal{R}$ rank-one*, $\mathcal{R} = \{R_k\}$, if T is $\mathcal{R}$ rank-one for a sequence $\mathcal{P}_k$ of T-towers with shape R_k that also satisfies $\mathcal{P}_{k+1} \geq \mathcal{P}_k$.

Suppose T is a stacking $\mathcal{R}$ rank-one action with $\mathcal{R} = \{R_k\}$ such that $\cup(R_k - R_k) = \mathbb{Z}^d$. Then it is easy to see that T is isomorphic to an action T_1 constructed by a 'cutting and stacking' construction using the shapes R_k (see for example [**18**] for a formal definition of such a construction in the case where $d = 2$).

Two more classical results about rank-one transformations can be extended to the case of rank-one $\mathbb{Z}^d$ actions, $d > 1$, by adding only the stacking hypothesis.

THEOREM 3.5. *If T is a stacking rank-one $\mathbb{Z}^d$ action then T has simple spectrum.*

Proof. The argument in Baxter [**1**] for the case $d = 1$ remains valid in the case $d > 1$. For $f \in L^2(X, \mu)$, let $U_T^{\vec{v}} f(x) = f(T^{\vec{v}} x)$. The cyclic subspace generated by f, denoted $\mathcal{H}(f)$, is the closure of the span of $\{U_T^{\vec{v}} f \mid \vec{v} \in \mathbb{Z}^d\}$. Simple spectrum means that there exists f so that $\mathcal{H}(f) = L^2(X, \mu)$. Since T is stacking rank-one, there exist an infinite sequence $F_k = T^{\vec{v}_k} B_k$ of pairwise disjoint T-tower levels. Let $f = \sum \chi_{F_k}$. Baxter's argument shows that this function f satisfies $\mathcal{H}(f) = L^2(X, \mu)$. The argument does not depend on the dimension of the acting group. □

An immediate corollary is the following.

COROLLARY 3.6. *If T is a stacking rank-one $\mathbb{Z}^d$ action then $h_d(T) = 0$.*

Proof. Again, Baxter's argument from [**1**] holds with no changes. If $h_d(T) > 0$, then by Sinai's theorem for $\mathbb{Z}^d$ actions (see [**14**]) it follows that T has a Bernoulli factor T'. Any Bernoulli $\mathbb{Z}^d$ action T' has countable Lebesgue spectrum, and in particular, non-simple spectrum. This would imply that T has non-simple spectrum. □

In the case $d = 1$, Baxter shows that rank-one transformations with interval tower shapes are stacking rank-one transformations, also with interval tower shapes. The fairly degenerate towers in Example 3.3 can also easily be chosen to be stacking. In order to generalize Baxter's proof to $\mathbb{Z}^d$, $d > 1$, however, we need to impose one extra condition on the sequence $\mathcal{R}$ of shapes. The class of rank-one actions that we consider is the following.

Definition 3.7. We say a $\mathbb{Z}^d$ action T is *Følner rank-one* if it is $\mathcal{R}$ rank-one for some Følner sequence $\mathcal{R}$ of shapes.

The following result generalizes Baxter's $d = 1$ result to the case of Følner rank-one $\mathbb{Z}^d$ actions, $d > 1$, adding one additional conclusion: the tower shapes can be exactly preserved.

THEOREM 3.8. *Let $\mathcal{R} = \{R_k\}$ be a Følner sequence in $\mathbb{Z}^d$ with $\vec{0} \in R_k$ for all k. Let T be an $\mathcal{R} = \{R_k\}$-rank-one $\mathbb{Z}^d$ action. Then there exists a sequence of T-towers $\mathcal{Q}_k$ with shape R_k so that $\mathcal{Q}_k \leq \mathcal{Q}_{k+1}$ for all k and $\mathcal{Q}_k \to \epsilon$. In particular, T is stacking $\mathcal{R}$ rank-one.*

The proof of Theorem 3.8 appears in §5.

4. *Growth conditions and directional entropy*

In this section we consider rectangular rank-one $\mathbb{Z}^d$ actions. Let $\vec{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. We say $\vec{n} \geq \vec{m}$ if $n_i \geq m_i$ for all $i = 1, \ldots, d$. In this case we define a shape, called a *rectangle*, by $[\vec{m}, \vec{n}] = \{\vec{v} \in \mathbb{Z}^d \mid \vec{m} \leq \vec{v} \leq \vec{n}\}$. This definition of a rectangle is sufficiently general to make the degenerate tower shapes in Example 3.3 rectangles. For this reason, the definition (below) of rectangular rank-one actions includes an additional condition that guarantees the rectangles will have the same dimension as the acting group. This condition also provides a structure that is both natural from the point of ergodic theory and necessary for some of our arguments.

Definition 4.1. We say T is *rectangular rank-one* if it is $\mathcal{R}$ rank-one for a Følner sequence $\mathcal{R}$ of rectangles.

Note that sequence $\mathcal{R} = \{R_1, R_2, \ldots\}$ of rectangles $R_j = [\vec{m}_j, \vec{n}_j]$ in $\mathbb{Z}^d$ is a Følner sequence if and only if for any $\vec{w} > \vec{0}$ one has $\vec{w}_j = (w_1^j, w_2^j, \ldots, w_d^j) := \vec{n}_j - \vec{m}_j > \vec{w}$ for all k sufficiently large (we say $\vec{w}_j \to +\infty$). The following theorem is our first result on directional entropy. Unlike the results to follow it, it requires no restrictions on the geometry of the rectangles.

THEOREM 4.2. *Let T be a rectangular rank-one $\mathbb{Z}^d$ action. Then there is a one-dimensional subspace V of $\mathbb{R}^d$ so that $h_1(T, V) = 0$.*

On the other hand, given Rudolph's example [**21**], we know that rectangular rank-one actions with at least one positive-entropy direction do exist. In Rudolph's construction, the long sides of the rectangles grow super-exponentially as a function of the short sides. Our next result shows that this is necessary: one cannot have directional entropy in the absence of exponential growth of the longest side relative to the shortest side.

Given a sequence $\mathcal{R}$ of rectangles let

$$s_j = \min_{i=1,\ldots,d} w_i^j \quad \text{and} \quad \ell_j = \max_{i=1,\ldots,d} w_i^j.$$

Definition 4.3. We say that a Følner sequence $\mathcal{R}$ of rectangles has *sub-exponential eccentricity* if

$$\limsup_{j\to\infty} \frac{\log(\ell_j)}{s_j} = 0. \tag{4}$$

The following is our main result.

THEOREM 4.4. *Let $\mathcal{R}$ be a Følner sequence of rectangles with sub-exponential eccentricity. If T is an $\mathcal{R}$ rank-one $\mathbb{Z}^d$ action, then $h_n(T, V) = 0$ for each n-dimensional subspace V, for all $1 \le n \le d$.*

The same geometric idea underlies the proofs of Theorems 4.2 and 4.4. In the next section we describe the general setup and prove some key lemmas used in both proofs. This will help make the role of sub-exponential growth in Theorem 4.4 clearer.

4.1. *Geometric preliminaries.* For any rectangular rank-one $\mathbb{Z}^d$ action T, we can find a Følner sequence $\mathcal{R} = \{R_k\}$ of rectangles, and towers $\mathcal{P}_k$ of shape R_k, such that $\mathcal{P}_k \to \epsilon$. By Theorem 3.8 we may assume that $\mathcal{P}_k \le \mathcal{P}_{k+1}$.

Let V be an n-dimensional subspace of $\mathbb{R}^d$. We will prove Theorems 4.2 and 4.4 by establishing Lemma 2.5 for the sequence $\mathcal{P}_k$. The details of the sequence $t_j \to \infty$ will be specified separately in each proof, but will always depend on the geometry of the rectangles in $\mathcal{R}$.

The alphabet of $(\mathcal{P}_k)_{V,t_j,m}$ is $(R_k \cup \{\vec{e}\})^{S(V,t_j,m)}$. For $j > k$ let us call a level $T^{\vec{v}} B_j$ of $\mathcal{P}_j$ *good* if $\vec{v} \notin \partial_{S(V,t_j,m)}(R_j)$. The good levels have the property that $\vec{v} + S(V, t_j, m) \subseteq R_j$. Define $Y_j \subseteq X$ to be the complement of the union of the good levels of $\mathcal{P}_j$, together with the error set E_j. Let

$$\mathcal{P}_j^*(x) = \begin{cases} \mathcal{P}_j(x) & \text{if } x \in Y_j^c, \text{ and} \\ * & \text{if } x \in Y_j, \end{cases}$$

and let $\mathcal{Q}_j = \mathcal{P}_j^* \vee (\mathcal{P}_k)_{V,t_j,m}$. Clearly $H((\mathcal{P}_k)_{V,t_j,m}) \le H(\mathcal{Q}_j)$, so it will suffice to prove

$$\lim_{j\to\infty} \frac{1}{(t_j)^n} H(\mathcal{Q}_j) = 0. \tag{5}$$

Let G_j be the set of atoms of $\mathcal{Q}_j$ that do not contain the symbol $*$. Then

$$H(\mathcal{Q}_j) = -\sum_{P\in G_j} \mu(P) \log \mu(P) \tag{6}$$

$$-\sum_{P\in G_j^c} \mu(P) \log \mu(P). \tag{7}$$

We estimate (6) and (7) separately in the next two lemmas.

LEMMA 4.5.

$$-\sum_{P\in G_j}\mu(P)\log\mu(P)\le -\log(1-\mu(E_j))+\sum\log w_i^j. \tag{8}$$

Proof. We first observe that it will suffice to show

$$G_j\subseteq\{P_a\in\mathcal{P}_j\mid a\in R_j\backslash\partial_{S(V,t_j,m)}(R_j)\}. \tag{9}$$

For this implies

$$|G_j|\le|R_j|-|\partial_{S(V,t_j,m)}(R_j)|\le|R_j|,$$

and (8) follows because for $P\in G_j$,

$$\mu(P)=\frac{1-\mu(E_j)}{|R_j|}\le\frac{1}{|R_j|}\quad\text{and}\quad |R_j|=\prod_{i=1}^{d}w_i^j.$$

To prove (9), let $P\in G_j$. Since $G_j\subseteq\mathcal{Q}_j$, $P=P_1\cap P_2$ where $P_1\in\mathcal{P}_j^*$ and $P_2\in(\mathcal{P}_k)_{V,t_j,m}$. It suffices to show $P_1\subseteq P_2$ since this implies $P=P_1$. Now $P\in G_j$ means the label of P does not contain the symbol $*$. Thus $P\subseteq Y_j^c$, and it follows that P is a good level of $\mathcal{P}_j$. But the name $(\mathcal{P}_k)_{V,t_j,m}(x)=b$ is constant on every good level. Since $P_2=(\mathcal{P}_k)^{-1}_{V,t_j,m}(b)$ is an atom, it follows that $P_1\subseteq P_2$. □

LEMMA 4.6.

$$-\sum_{P\in G_j^c}\mu(P)\log\mu(P)\le 2\mu(Y_j)|S(V,t,m)|\log|R_k|-\mu(Y_j)\log\mu(Y_j). \tag{10}$$

Proof. Lemma 2.1 gives that

$$-\sum_{P\in G_j^c}\mu(P)\log\mu(P)\le\mu(Y_j)\log|G_j^c|-\mu(Y_j)\log\mu(Y_j).$$

Since the alphabet of $\mathcal{Q}_j$ restricted to G_j^c is $\{*\}\times(R_k\cup\{\vec{e}\})^{S(V,t,m)}$, we have

$$|G_j^c|\le(|R_k|+1)^{|S(V,t,m)|}$$

and the result follows. □

We will also need the following geometric estimate.

LEMMA 4.7. *Let V be an n-dimensional subspace of $\mathbb{R}^d$, and let Q and Q' be fixed cubes as described above. Then for $t, m>0$,*

$$|S(V,t,m)|\le(t+2d^{1/2})^n(m+2d^{1/2})^{d-n}. \tag{11}$$

Proof. Let $C_{\vec{v}}$ be the cube in $\mathbb{R}^d$ centered at $\vec{v}\in\mathbb{Z}^d$ and let

$$C(V,t,m)=\bigcup_{\vec{v}\in S(V,t,m)}C_{\vec{v}}.$$

Then $|S(V,t,m)|=\mathrm{vol}_{\mathbb{R}^d}(C(V,t,m))$. Now let Q_1 and Q_1' be the cubes in V and $V^\perp$ obtained by attaching a thickness $d^{1/2}$ frame all around Q and Q'. Clearly $C(V,t,m)\subseteq tQ_1+mQ_1'$. Now, since $\mathrm{vol}_V(tQ_1)=(t+2d^{1/2})^n$ and $\mathrm{vol}_{V^\perp}(mQ_1')=(m+2d^{1/2})^{d-n}$, it follows that $\mathrm{vol}_{\mathbb{R}^d}(tQ_1+mQ_1')=(t+2d^{1/2})^n(m+2d^{1/2})^{d-n}$. Thus

$$|S(V,t,m)|\le(t+2d^{1/2})^n(m+2d^{1/2})^{d-n}. \qquad \square$$

4.2. *Proof of Theorem 4.2.* Suppose, without loss of generality, that for all $j \geq 1$, the maximal dimension of R_j is in the direction $\vec{e}_1$, i.e. $\vec{w}_1^j = \ell_j$. Let V be the $n = 1$ dimensional subspace of $\mathbb{R}^d$ spanned by $\vec{e}_1$. We set

$$t_j = \sqrt{\ell_j \log(\ell_j)}, \tag{12}$$

which implies

$$\lim_{j\to\infty} \frac{t_j}{\ell_j} = 0, \tag{13}$$

and

$$\lim_{j\to\infty} \frac{\log(\ell_j)}{t_j} = 0. \tag{14}$$

We will prove (5).

By (11) and Lemma 4.6

$$\begin{aligned} &-\sum_{P\in G_j^c} \mu(P)\log\mu(P) \\ &\quad \leq 2\mu(Y_j)(t_j + 2d^{1/2})(m + 2d^{1/2})^{d-1}\log|R_k| - \mu(Y_j)\log\mu(Y_j). \end{aligned}$$

Combining this with (6), (7) and Lemma 4.5 yields

$$\begin{aligned} \frac{1}{t_j}H(\mathcal{Q}_j) \leq &-\frac{\log(1-\mu(E_j))}{t_j} + \sum_{i=1}^{d}\frac{\log(w_i^j)}{t_j} + 2\mu(Y_j)(m+2d^{1/2})^{d-1}\left(\frac{t_j + 2d^{1/2}}{t_j}\right) \\ &\times \log|R_k| - \frac{\mu(Y_j)\log\mu(Y_j)}{t_j}. \end{aligned} \tag{15}$$

The first term in (15) goes to zero since $\mathcal{P}_j \to \epsilon$. The second term is bounded above by $d\log(\ell_j)/t_j$, which goes to zero by (14). In the last term, $\mu(Y_j)\log\mu(Y_j)$ is bounded and $t_j \to \infty$. It remains to analyze the third term. First note that the third term is clearly bounded above. Since, in this case, $S(V, t_j, m)$ is a parallelepiped with sides parallel to the sides of R_j we have the following estimate on the size of the boundary of a rectangle:

$$|\partial_{S(V,t_j,m)}(R_j)| \leq 2t_j\prod_{k=2}^{d} w_j^k + 2m\sum_{i=2}^{d}\prod_{k\neq i} w_j^k. \tag{16}$$

Combining this with the fact that $\ell_j = w_j^1$, we have

$$\begin{aligned} \mu(Y_j) &\leq \left[2t_j\prod_{k=2}^{d} w_j^k + 2m\sum_{i=2}^{d}\prod_{k\neq i} w_j^k\right]\frac{1}{|R_j|} + \mu(E_j) \\ &= \frac{2t_j}{\ell_j} + \sum_{i=2}^{d}\frac{2m}{w_j^i} + \mu(E_j). \end{aligned} \tag{17}$$

It follows that $\lim_{j\to\infty}\mu(Y_j) = 0$, using (13) for the first term of (17), the fact that $\mathcal{R} = \{R_j\}$ is a Følner sequence for the second, and the fact that $\mathcal{P}_j \to \epsilon$ for the third. □

4.3. *Proof of Theorem 4.4.* Fix $1 \le n < d$, and let V be an n-dimensional subspace of $\mathbb{R}^d$. Recall that ℓ_j denotes the largest dimension, and s_j the smallest dimension of R_j. It follows from (4), the assumption of sub-exponential eccentricity, that by passing to a subsequence we can assume

$$\lim_{j\to\infty} \frac{\log(\ell_j)}{s_j} = 0. \tag{18}$$

To prove the theorem using the same idea as in the proof of Theorem 4.2, we choose a sequence of times $t_j \to \infty$ that simultaneously satisfy

$$\lim_{j\to\infty} \frac{t_j}{s_j} = 0, \tag{19}$$

which replaces (13), and

$$\lim_{j\to\infty} \frac{\log(\ell_j)}{t_j} = 0, \tag{20}$$

which is (14). In particular, these two limits require t_j to grow faster than $\log(\ell_j)$ but slower than s_j, and it follows from (18) that $t_j = \sqrt{s_j \log(\ell_j)}$ will do. Again, we will prove (5).

By Lemmas 4.6 and 4.7, we have

$$\begin{aligned} &-\sum_{P\in G_j^c} \mu(P)\log\mu(P) \\ &\quad \le 2\mu(Y_j)(t+2d^{1/2})^n(m+2d^{1/2})^{d-n}\log|R_k| - \mu(Y_j)\log\mu(Y_j). \end{aligned}$$

This, together with (6), (7) and (8), implies

$$\begin{aligned} \frac{1}{(t_j)^n} H(Q_j) \le -\frac{\log(1-\mu(E_j))}{(t_j)^n} + \sum_{i=1}^{d} \frac{\log(w_i^j)}{(t_j)^n} + 2\mu(Y_j)(m+2d^{1/2})^{d-n} \\ \times \left(\frac{(t_j+2d^{1/2})^n}{(t_j)^n}\right)\log|R_k| - \frac{\mu(Y_j)\log\mu(Y_j)}{(t_j)^n}. \end{aligned} \tag{21}$$

As in the proof of Theorem 4.2, we estimate each of the four terms of (21) separately. The arguments for the first and last terms are the same as before. Similarly, (20) is sufficient to guarantee that the second term goes to zero. The key to estimating the third term is the following estimate of the size of the boundary, which plays the same role that (16) played in the last proof.

LEMMA 4.8. *For j sufficiently large,*

$$|\partial_{S(V,t_j,m)}(R_j)| \le 4t_j \sum_{i=1}^{d} \prod_{k\ne i} w_j^k. \tag{22}$$

Proof. It follows from the proof of Lemma 4.7 that for large enough t_j the diameter δ of $S(V, t_j, m)$ satisfies $\delta \le t_j + 2d^{1/2}$ and for t_j large enough, $\delta < 2t_j$. Let $R_j = [\vec{n}^j, \vec{m}^j]$, and let $R_j' = [\vec{n}^j + \vec{\delta}, \vec{m}^j - \vec{\delta}]$, where $\vec{\delta} = (\delta, \delta, \ldots, \delta)$. Then

$$\partial_{S(V,t_j,m)}(R_j) \subseteq R_j \backslash R_j' \subseteq \bigcup_{i=1}^{d} (L_i^j \cup H_i^j),$$

where

$$L_i^j = [n_i^j, n_i^j + \delta] \times \prod_{k \neq i} [n_k^j, m_k^j] \quad \text{and} \quad H_i^j = [m_i^j - \delta, m_i^j] \times \prod_{k \neq i} [n_k^j, m_k^j].$$

The lemma follows since $|L_i^j| = |H_i^j| = \delta \prod_{k \neq j} w_j^k$. □

Using Lemma 4.8, we have

$$\mu(Y_j) \leq |\partial_{S(V,t_j,m)}(R_j)| \frac{1}{|R_j|} + \mu(E_j)$$
$$= \sum_{i=1}^{d} \frac{4t_j}{w_i^j} + \mu(E_j). \tag{23}$$

The first term goes to zero by (19) and the second since $\mathcal{P}_j \to \epsilon$. Since $\mu(Y_j) \to 0$, we have (5), and this concludes the proof of Theorem 4.4. □

5. *The proof of Theorem 3.8*

This proof follows the ideas in Baxter [**1**], together with an improvement in Lemma 5.3 needed for the $\mathbb{Z}^d$ case. The proof applies more or less verbatim to Følner rank-one actions of any amenable group G.

We begin with some definitions that will simplify our arguments. Given shapes R and J in $\mathbb{Z}^d$ we say that J is *R-separated* if

$$(R + \vec{v}_1) \cap (R + \vec{v}_2) = \emptyset \quad \text{for all } \vec{v}_1, \vec{v}_2 \in J \text{ with } \vec{v}_1 \neq \vec{v}_2,$$

in which case $R + J := \bigcup_{\vec{v} \in J} R + \vec{v}$ is a disjoint union. We say that a shape S is a *stacking* of a shape R if there exists an R-separated set J so that $R + J \subseteq S$. We call J an *R-stacking set for S*.

The purpose of a stacking is that it tells us how a tower of shape S can be put together out of a tower of shape R. In particular, suppose R and S are shapes, with $\vec{0} \in R \subset S$, and suppose that J is an R-stacking set for S. Given a tower $\mathcal{Q}$ with shape S and base B, let $A = \bigcup_{\vec{v} \in J} T^{\vec{v}} B$. Then A is the base of tower $\mathcal{Q}_J$ of shape R, defined by $\mathcal{Q}_J = \{T^{\vec{v}} A \mid \vec{v} \in R\}$, such that $\mathcal{Q}_J \leq \mathcal{Q}$.

LEMMA 5.1. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be towers of shape S. Let R be a shape such that J is an R stacking set for S. Then*

$$d(\mathcal{Q}_J, \mathcal{Q}'_J) \leq d(\mathcal{Q}, \mathcal{Q}').$$

Proof. Let B and B' denote the bases of $\mathcal{Q}$ and $\mathcal{Q}'$. By definition we have that $A = \sum_{\vec{w} \in J} T^{\vec{w}} B$ and $A' = \sum_{\vec{w} \in J} T^{\vec{w}} B'$ are the bases of $\mathcal{Q}_J$ and $\mathcal{Q}'_J$. Then

$$\begin{aligned} d(\mathcal{Q}_J, \mathcal{Q}'_J) &= \sum_{\vec{v} \in R} \mu(T^{\vec{v}} A \triangle T^{\vec{v}} A') + \mu(E \triangle E') \\ &= |R| \mu(A \triangle A') + \mu(E \triangle E') \\ &= |R| \mu\bigg(\bigcup_{\vec{w} \in J} T^{\vec{w}} B \triangle \bigcup_{\vec{w} \in J} T^{\vec{w}} B'\bigg) + \mu(E \triangle E') \\ &\leq |J||R| \mu(B \triangle B') + \mu(E \triangle E') \leq |S| \mu(B \triangle B') + \mu(E \triangle E') \\ &= d(\mathcal{Q}, \mathcal{Q}'). \end{aligned}$$

□

Let $\mathcal{P}$ be a tower of shape R. For $A \in \mathcal{A}$ let I be the set of $a \in R$ so that

$$\mu(A \cap P_a) > \tfrac{1}{2}\mu(P_a). \tag{24}$$

Define $A(\mathcal{P}) = \bigcup_{a \in I} P_a$.

LEMMA 5.2. *Suppose $\mathcal{P}$ and $\mathcal{Q}$ are towers of shapes R and S satisfying $\vec{0} \in R \subset S$, such that $\mathcal{P} \le \mathcal{Q}$. Let A and B be the base sets of $\mathcal{P}$ and $\mathcal{Q}$ and suppose $A(\mathcal{Q}) \ne \emptyset$. If J is the maximal set of indices in S that satisfy*

$$\bigcup_{\vec{v} \in J} T^{\vec{v}} B \subset A(\mathcal{Q}) \quad \textit{and} \quad J \cap \partial_R(S) = \emptyset,$$

then $\mathcal{Q}_J$ is a tower of shape R with base $A' = \bigcup_{\vec{v} \in J} T^{\vec{v}} B$.

Proof. Since $\mathcal{Q}$ is a tower, it follows from (24) that the sets $\{T^{\vec{v}} A' \mid \vec{v} \in R\}$ are pairwise disjoint. Thus J is R-separated and an R-stacking set for S. □

Under the hypotheses of Lemma 5.2, we define $\mathcal{P}(\mathcal{Q}) = \mathcal{Q}_J$. Note that $\mathcal{P}(\mathcal{Q})$ and $\mathcal{P}$ have the same shape R, and that $\mathcal{P}(\mathcal{Q}) \le \mathcal{Q}$. The next lemma shows that if we know that the levels in $\mathcal{Q}$ approximate the levels in $\mathcal{P}$ well (or equivalently, the levels in $\mathcal{Q}$ approximate the base A of $\mathcal{P}$ well), then we also know that $\mathcal{P}(\mathcal{Q})$ is a good approximation of $\mathcal{P}$.

LEMMA 5.3.

$$d(\mathcal{P}(\mathcal{Q}), \mathcal{P}) \le |R| \cdot \mu(A(\mathcal{Q}) \triangle A) + |\partial_R(S)| \cdot \mu(B) + |R|\mu(E_{\mathcal{Q}}),$$

where $E_{\mathcal{Q}}$ denotes the error set of $\mathcal{Q}$.

Proof. Suppose J is the index set in S so that $\mathcal{P}(\mathcal{Q}) = \mathcal{Q}_J$. Then we have

$$\begin{aligned} d(\mathcal{P}(\mathcal{Q}), \mathcal{P}) &= |R|\mu(A(\mathcal{Q}) \triangle A) \\ &\le |R|\mu\Bigg((A(\mathcal{Q}) \triangle A) \cup \bigcup_{\vec{w} \in J \cap \partial_R(S)} T^{\vec{w}} B \cup E_{\mathcal{Q}}\Bigg) \\ &= |R|\mu(A(\mathcal{Q}) \triangle A) + |R|\mu\Bigg(\bigcup_{\vec{w} \in J \cap \partial_R(S)} T^{\vec{w}} B \cup E_{\mathcal{Q}}\Bigg) \\ &\le |R|\mu(A(\mathcal{Q}) \triangle A) + |R||J \cap \partial_R(S)|\mu(B) + |B|\mu(E_{\mathcal{Q}}), \end{aligned}$$

where we used the identity $(A \backslash C) \triangle B \subseteq (A \triangle B) \cup C$. The result follows since $|\partial_R(S)| \le |R|\,|J \cap \partial_R(S)|$, which is true since J is R-separated. □

The next result shows that given a sequence of towers, we can construct a new sequence, maintaining the shapes, such that each tower is well approximated by subsequent towers far enough in the sequence.

LEMMA 5.4. *Let $\mathcal{R} = \{R_k\}$ be a Følner sequence, and $\mathcal{P}_k$ be a sequence of towers of shapes R_k, such that $0 \in R_k$ and $\mathcal{P}_k \to \epsilon$. Then for any $k \ge 1$ and $\delta > 0$ we have for all sufficiently large $\ell > k$ that $R_k \subseteq R_\ell$ and $d(\mathcal{P}_k(\mathcal{P}_\ell), \mathcal{P}_\ell) < \delta$.*

Proof. Fix $k \geq 1$ and use the Følner property to choose ℓ large enough that $R_k \subseteq R_\ell$. Let A_k be the base of $\mathcal{P}_k$, and E_k the error set. In addition, since $\mathcal{P}_k \to \epsilon$, for all sufficiently large ℓ we have

$$\mu(A_k(\mathcal{P}_\ell) \triangle A_k) < \frac{\delta}{3|R_k|} \quad \text{and} \quad \mu(E_\ell) < \frac{\delta}{3|R_k|}.$$

For such an ℓ the towers $\mathcal{P} = \mathcal{P}_k$ and $\mathcal{Q} = \mathcal{P}_\ell$ satisfy the hypotheses of Lemma 5.2. Thus we can apply Lemma 5.3 to conclude

$$d(\mathcal{P}_k(\mathcal{P}_\ell), \mathcal{P}_\ell) \leq |R_k|\mu(A_k(\mathcal{P}_\ell) \triangle A) + |\partial_{R_k}(R_\ell)|\mu(A_\ell) + |R_k|\mu(E_\ell). \tag{25}$$

Since $\mathcal{P}_\ell$ is a tower, for all ℓ we have $\mu(A_\ell) \leq 1/|R_\ell|$. Hence we have

$$|\partial_{R_k}(R_\ell)|\mu(A_\ell) \leq \frac{|\partial_{R_k}(R_\ell)|}{|R_\ell|} \tag{26}$$

and since $\mathcal{R}$ is a Følner sequence this can be made less than $(\delta/3)$ for all ℓ sufficiently large. Using these estimates in (25) we have

$$d(\mathcal{P}_k(\mathcal{P}_\ell), \mathcal{P}_\ell) < \delta. \qquad \square$$

We are now ready for the proof of Theorem 3.8.

Proof of Theorem 3.8. Let $\delta_k > 0$ satisfy $\sum \delta_k < \infty$. Using Lemma 5.4, we can then assume, by passing to a subsequence, that the sequence $\mathcal{P}_k$ of T-towers satisfies $\rho(\mathcal{P}_k(\mathcal{P}_{k+1}), \mathcal{P}_{k+1}) < \delta_k$ for all k.

We will now define a doubly infinite sequence of T-towers $\mathcal{P}_{k,\ell}$, $k \geq 1$, $\ell \geq 0$. We start by putting, for each k, $\mathcal{P}_{k,0} := \mathcal{P}_k$ and

$$\mathcal{P}_{k,1} := \mathcal{P}_{k,0}(\mathcal{P}_{k+1,0}) = \mathcal{P}_k(\mathcal{P}_{k+1}).$$

Note that $\mathcal{P}_{k,1}$ has shape R_k and satisfies $\mathcal{P}_{k,1} \leq \mathcal{P}_{k+1,0}$. By Lemma 5.2 there exists a stacking set $I_k \subseteq R_{k+1} \backslash \partial_{R_k}(R_{k+1})$, which, in particular, is R_k-separated.

Now suppose we have defined $\mathcal{P}_{k,m}$ for all k and for all $0 \leq m < \ell$. Then for $k = 1, 2, \ldots,$ we define

$$\mathcal{P}_{k,\ell} = (\mathcal{P}_{k+1,\ell-1})_{I_k}. \tag{27}$$

By induction, $\mathcal{P}_{k+1,\ell-1}$ is a tower of shape R_{k+1}, so, as in our previous discussion, (27) is well defined. The result is a tower $\mathcal{P}_{k,\ell}$ of shape R_k satisfying $\mathcal{P}_{k,\ell} \leq \mathcal{P}_{k+1,\ell-1}$.

Repeatedly using Lemma 5.1 and equation (27), we have by induction that

$$\rho(\mathcal{P}_{k,\ell}, \mathcal{P}_{k,\ell+m}) \leq \sum_{j=\ell}^{\ell+m-1} \delta_j.$$

This shows that $\mathcal{P}_{k,\ell}$ is a Cauchy sequence in ℓ for each k, and we define $\mathcal{Q}_k = \lim_\ell \mathcal{P}_{k,\ell}$. It follows that $\mathcal{Q}_k$ is a partition of shape R_k. Also $\mathcal{Q}_k = (\mathcal{Q}_{k+1})_{I_k}$, so that $\mathcal{Q}_k \prec \mathcal{Q}_{k+1}$. Moreover, $d(\mathcal{Q}_k, \mathcal{P}_k) < \sum_{j \geq k} \delta_k \to 0$. This implies $\mathcal{Q}_k \to \epsilon$, since

$$\begin{aligned} \mu(A(\mathcal{Q}_k) \triangle A) &\leq \mu(A(\mathcal{Q}_k) \triangle A(\mathcal{P}_k)) + \mu(A(\mathcal{P}_k) \triangle A) \\ &\leq d(\mathcal{Q}_k, \mathcal{P}_k) + \mu(A(\mathcal{P}_k) \triangle A). \end{aligned} \qquad \square$$

Acknowledgements. The research of the second author was supported in part by a DePaul University Research Council Paid Leave. The second author also wishes to thank the Department of Mathematics of the George Washington University for their hospitality during her sabbatical when most of this research was conducted.

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