MATH 6000: HW # 1

ROBBIE ROBINSON

Problem 1. *Prove that*

$$\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}. \tag{1}$$

Solution to Problem 1. We will assume the following lemma:

Lemma 1. $dxdy = r\,drd\theta$

Now let

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx. \tag{2}$$

Then

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy \\ &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,drd\theta, \end{aligned}$$

using $x^2 + y^2 = r^2$ and Lemma 1. The inside integral evaluates

$$\int_0^{\infty} e^{-r^2} r\,dr = \frac{1}{2}\int_0^{\infty} e^{-u} du = \frac{1}{2},$$

using the substitution $u = r^2$ and $du = 2r\,dr$. So

$$I^2 = \frac{1}{2}\int_0^{2\pi} d\theta = \pi. \tag{3}$$

The result (1) now follows from (2) and (3).

Problem 2. *Show that the product of any two functions of bounded variation is a function of bounded variation.*

Solution. We first note that this does not hold for monotonic functions since if $f(x) = x$ on $[-1, 1]$, $f^2(x) = x^2$ is not monotonic.

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