Mathematical Quasi-crystallography:

The First 30 Years

E. Arthur Robinson, Jr.
The George Washington University

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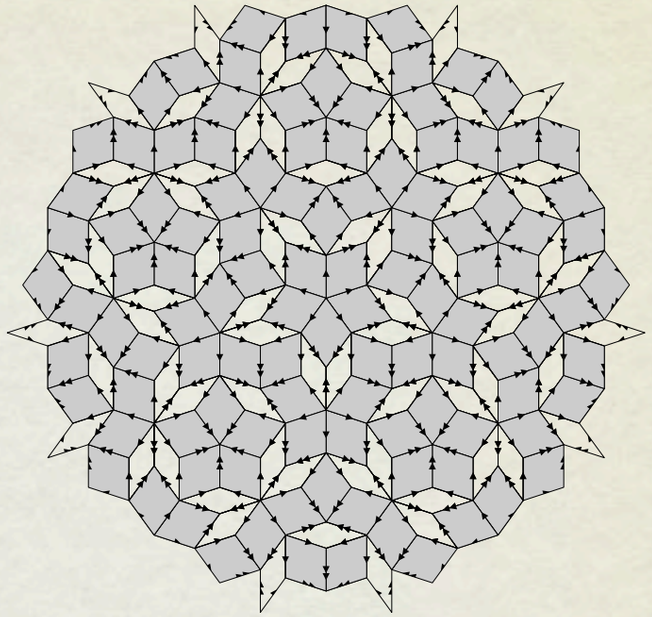
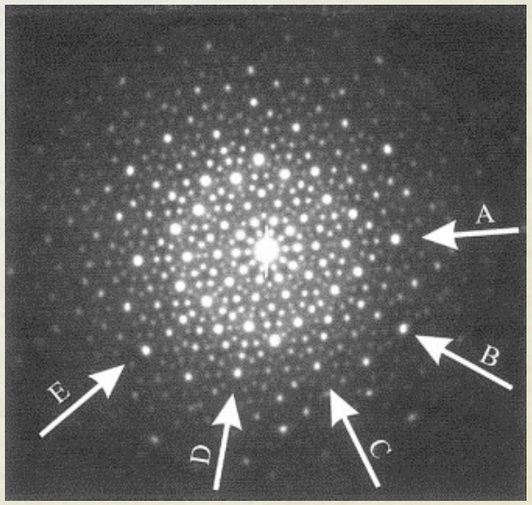
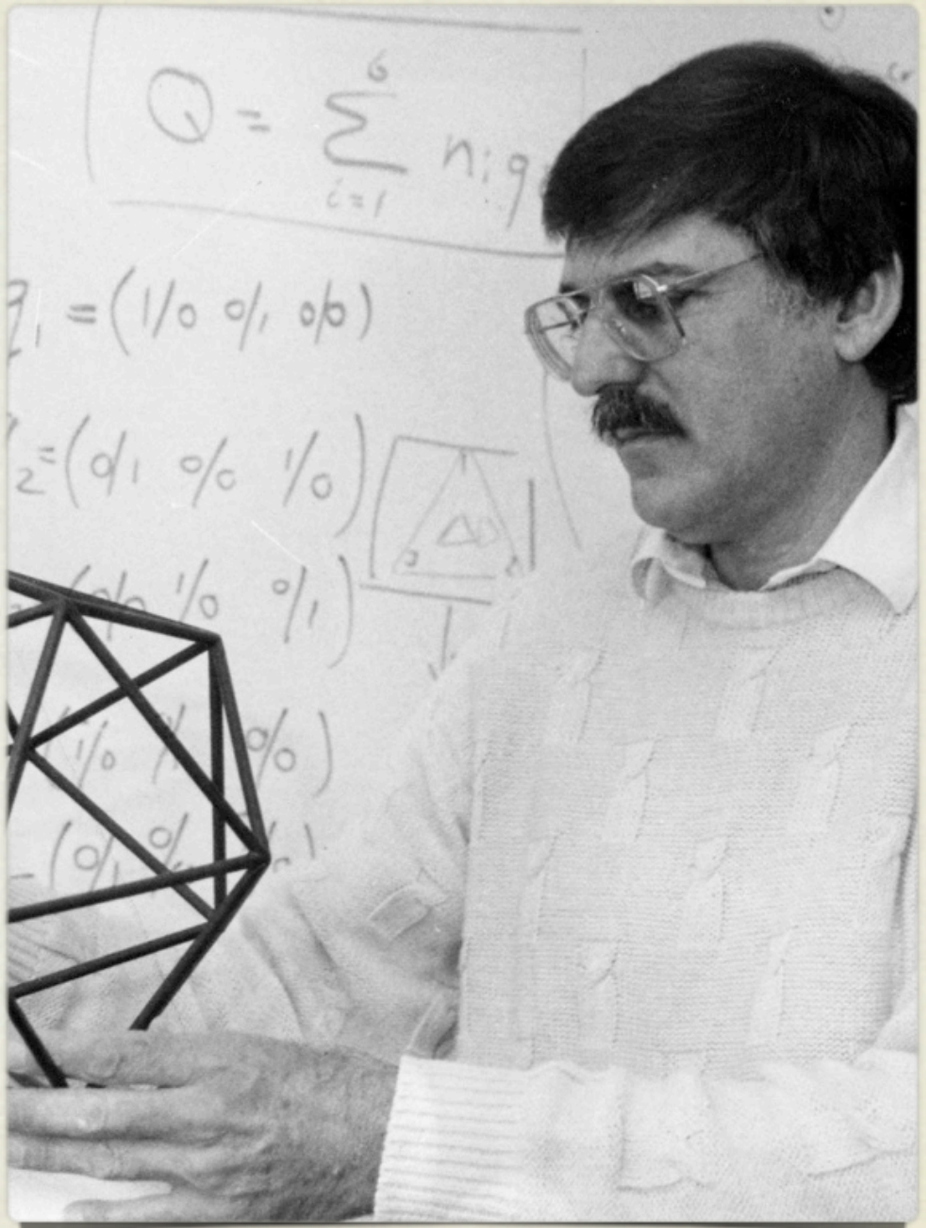
CONTENTS:





I. Quasicrystals discovered

- Dan Schectman, 1982.
- Materials Scientist visiting NIST (then called NBS) from Technion in Israel.
- Looking at X-ray diffraction patterns of Al-Mn alloys.
- Saw an “impossible” image.



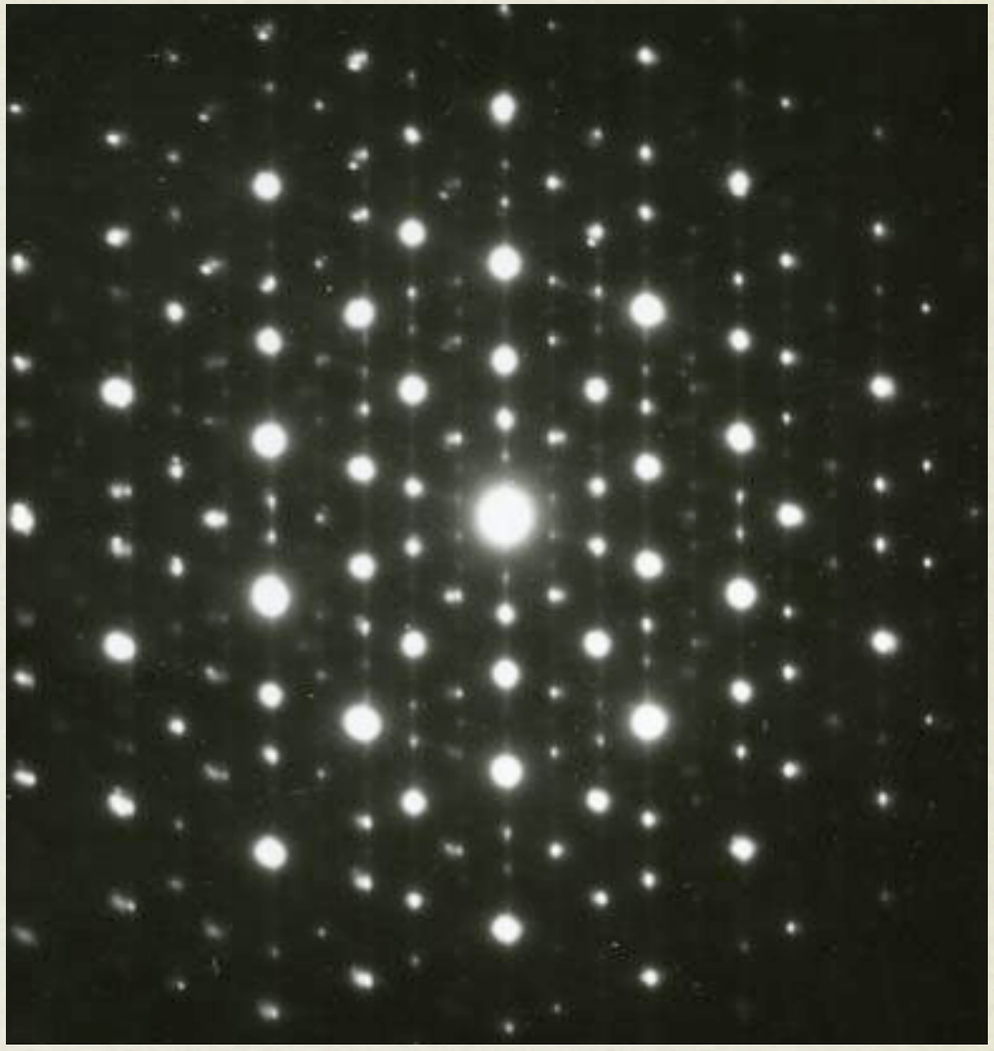
http://www.technioniit.com/2011/10/who-is-dan-shechtman.html

Impossibility

- Spots or "Bragg Peaks" indicate a ***crystal***: *atomic long range spatial order* (*periodic*).

- 5-fold (or 10-fold or icosahederal) symmetry, reflects the underlying symmetry of the atoms.

- **But!** "Crystallographic Restriction" forbids 5-fold symmetry for crystals.



Long Range "Aperiodic" Order

- **Shechtman, D.**; Blech, I.; Gratias, D.; Cahn, J., "*Metallic Phase with Long-Range Orientational Order and No Translational Symmetry*". Physical Review Letters **53**, 20, p. 1951, (1984).

- But long-range order must be strong enough to still result in Bragg peaks in diffraction pattern.

- Long-range order must be weak enough that 5-fold symmetry does not violate the Crystallographic Restriction.

Quasicrystals

- Levine, Dov; Steinhardt, Paul (1984). "*Quasicrystals: A New Class of Ordered Structures*". Physical Review Letters **53** (26): p. 2477.

  Coined the term "**quasicrystals**". Suggested a model based on **Penrose tilings** (i.e., a 3-dimensional version).
  The same model simultaneously proposed by several others. But many remained skeptical...

- Pauling, L. (1987). "*So-called icosahedral and decagonal quasicrystals are twins of an* 820-atom cubic crystal". Physical Review Letters **58** (4): 365–368.

  "Twinning" means two ordinary crystals fused together at angle.

Nobel Prize

- **Quasicrystals** now have widespread acceptance.
- *International Union of Crystallography* new 1992 definition of a crystal: a solid with a “discrete diffraction diagram”. (Formerly, this said “periodic atomic structure”).
- “Discrete diffraction” is meant to be taken as evidence of “long range order”.
- Dan Schectman won the 2011 Nobel Prize in Chemistry.

II. Crystallographic Restriction

A (classical) ***crystal*** is a "pattern" x in $\mathbf{R}^n$ that is ***periodic***: i.e., $T(x)=\{\mathbf{v} \in \mathbf{R}^n: x+\mathbf{v}=x\}$ is a full-rank ***lattice.***

Periodicity is "long range order".

The ***space group*** $S(x)$ is the group of rigid motions (i.e., isometries of $\mathbf{R}^n$) leaving x invariant. **(Symmetry type.)**

The ***point group***, defined $H(x) = S(x)/T(x)$, is isomorphic to a finite subgroup of $O(n)$.

THE CASE $n=2$

- When $n = 2$, there are 17 different space groups, sometimes called ***wallpaper groups.*** A "folk theorem": All types occur in Islamic art, and all were illustrated by M. C. Escher.

- $H(X)$ can either be $\mathbf{Z}_n$ or $\mathbf{D}_n$ for $n = 1, 2, 3, 4$, or 6. This is the crystallographic restriction for $n = 2$.

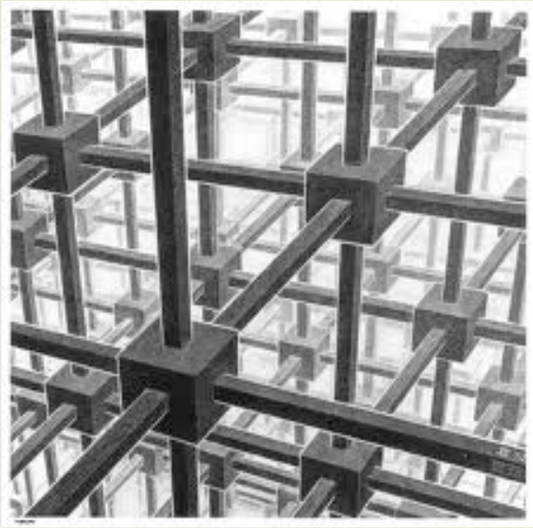
p3m1
(Owen Jones 1856)

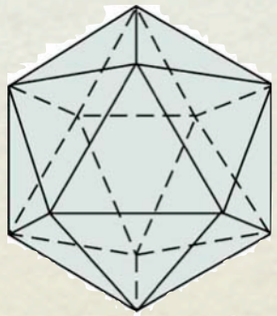
p4g (Escher)

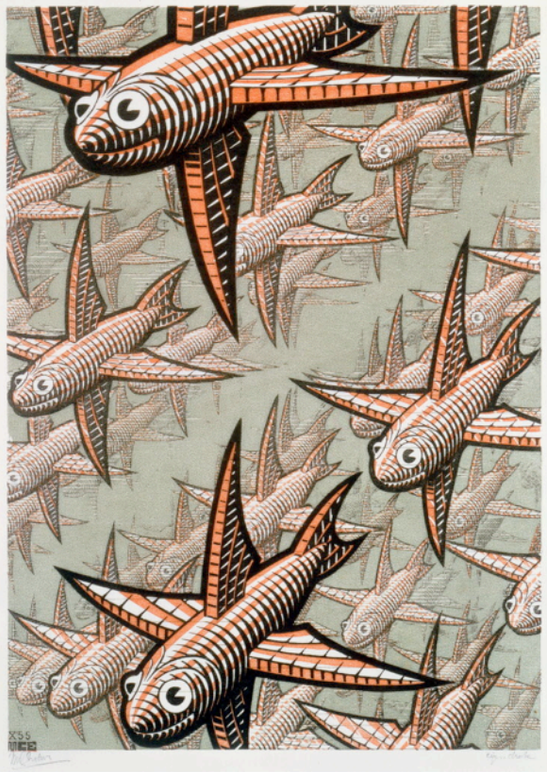
p6 (Alhambra)

The case $n=3$

- When $n = 3$, there are 230 different space groups. This result is due to Fyodorov and Schönflies (1892).
  - There are 14 possible space groups $T(X)$, called **Bravais lattices**.
  - There are 32 possible point groups $H(X)$. No point group is the icosahedral group.







Bieberbach's Theorem

- Hilbert's 18th problem from a list of 23, set out in 1900 by David Hilbert (Part 1): Is the number of n-dimensional space groups is finite for each n?

- **Theorem** (Bieberbach, 1911). The number of space groups in every dimension is finite. Every isomorphism is implemented "geometrically".

Parts 2 & 3 of Hilbert's 18th were to prove $S(x)$ is transitive on fundamental domains, and to prove Kepler's conjecture. Part 2 is False (Reinhardt, n=3, 1928 & Heesch, n=2, 1935. Part 3 was proved by Hales in 1998.

III. X-ray Diffraction

Let $x \subseteq \mathbf{R}^n$ be *discrete* and *relatively dense* (a ***Delone set***).

Let $\Delta_x = \sum_{\mathbf{v} \in x} \delta_{\mathbf{v}}$

Autocorrelate: $\Delta_x \circledast \Delta_x = \rho \Delta_{x-x}$ & positive definite.

So $\Delta_x \circledast \Delta_x = \widehat{\sigma}$ where σ is a measure on $\mathbf{R}^n$.

Lebesgue decomposition: $\sigma = \sigma_{\text{dis}} + \sigma_{\text{sc}} + \sigma_{\text{ac}}$.

Say x ***diffracts*** if $\sigma_{\text{dis}} \neq 0$.

Classical crystals diffract

- A (discrete) classical crystal x is a finite union of lattices (cosets). For simplicity, assume just one.
- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$, be a basis for x, and let x^* be dual lattice (generated by a dual basis: $\mathbf{v}^*_1, \mathbf{v}^*_2, \ldots, \mathbf{v}^*_n$ such that $\langle \mathbf{v}_i, \mathbf{v}^*_j \rangle = \delta_{i,j}$).
- For a lattice $x - x = x$ so $\Delta_x \circledast \Delta_x = \Delta_{x-x} = \Delta_x$.
- Then $(\Delta_x \circledast \Delta_x)\widehat{\ } = \Delta_x\widehat{\ } = \Delta_{x^*}$ by the Poisson summation formula: ***classical crystals diffract.***
- **What else diffracts?**

Penrose tilings

- ***Before*** Schectman's discovery, British crystallographer Alan L. Mackay published a curious paper:
  Mackay, A.L., "Crystallography and the Penrose Pattern", Physica **114** A, 609 (1982).

- The "Penrose Pattern" refers to the aperiodic tilings, discovered around 1976 by Roger Penrose, now called ***Penrose tilings.***

- Mackay essentially showed that (the vertices of) ***Penrose tilings* diffract.**

- He did this ***experimentally***, by making a tiny Penrose tiling and shining a laser through it!

IV. Penrose tilings

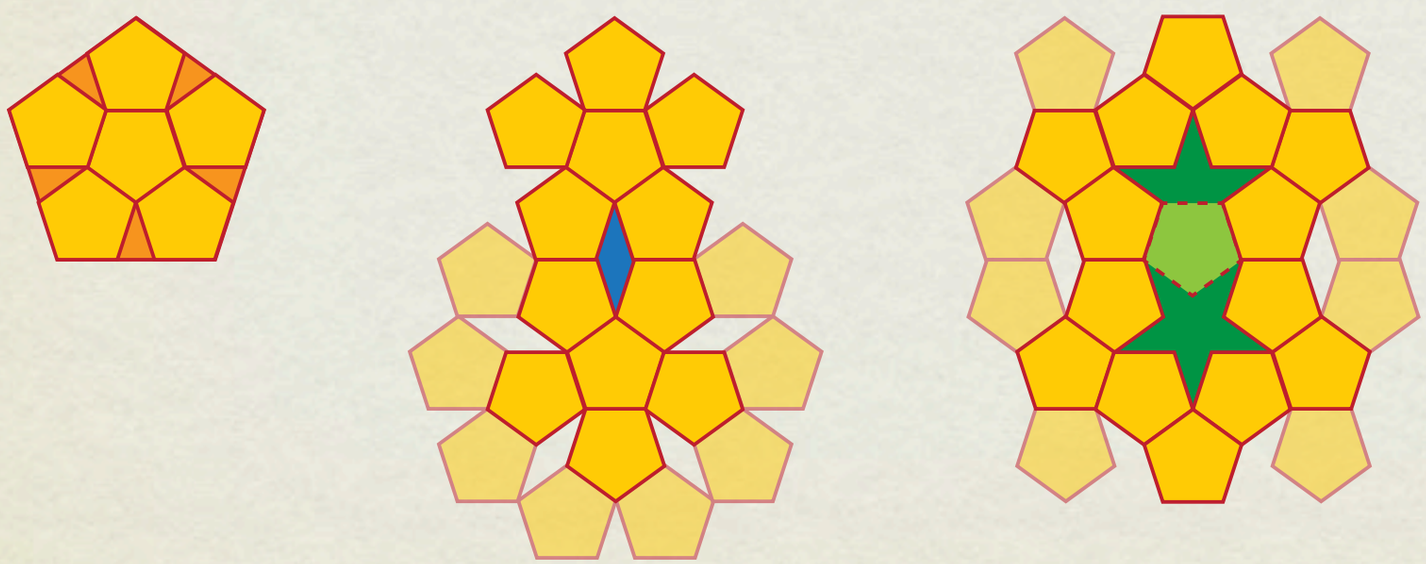
- **Roger Penrose** is a British mathematical physicist, mostly working on cosmology, but also interested in recreational math.

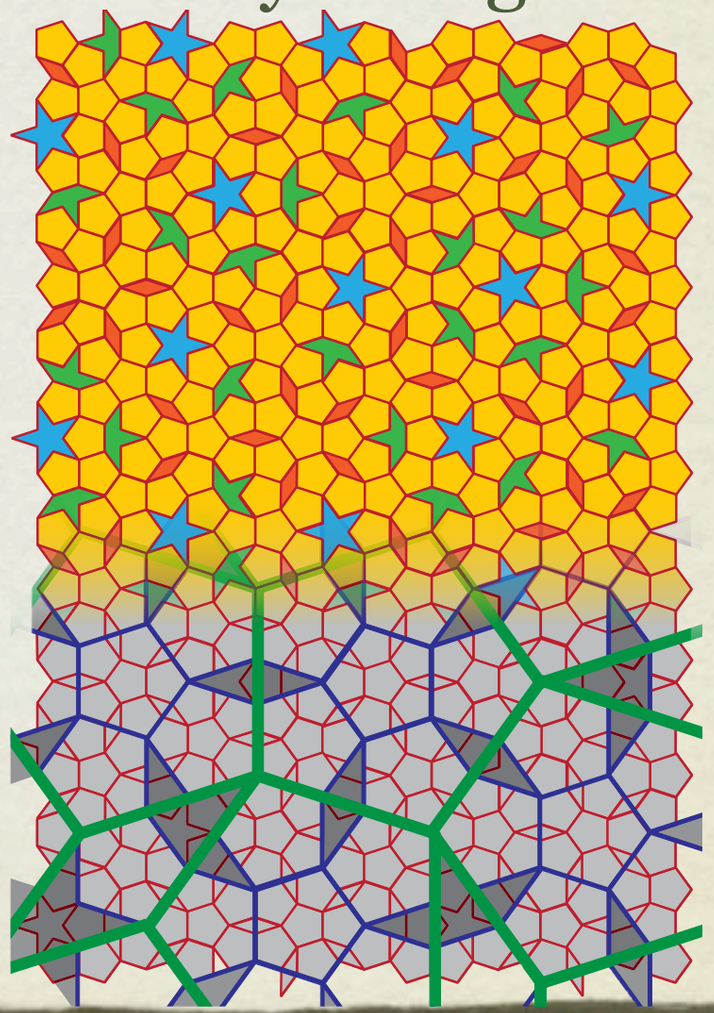
Shown at Mitchell Institute for Fundamental Physics and Astronomy, Texas A&M

http://en.wikipedia.org/wiki/Penrose_tiling

Origin in pentagonal symmetry

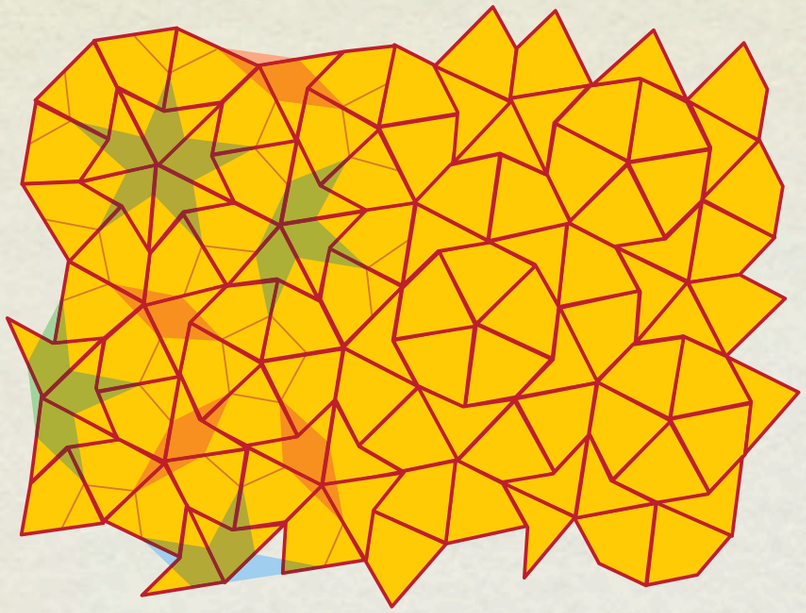
- Plane can be tiled by equilateral triangles, squares & (regular) hexagons, but not (regular) pentagons.
- **Penrose** (1976) was experimenting with efficiently filling the plane with pentagons.

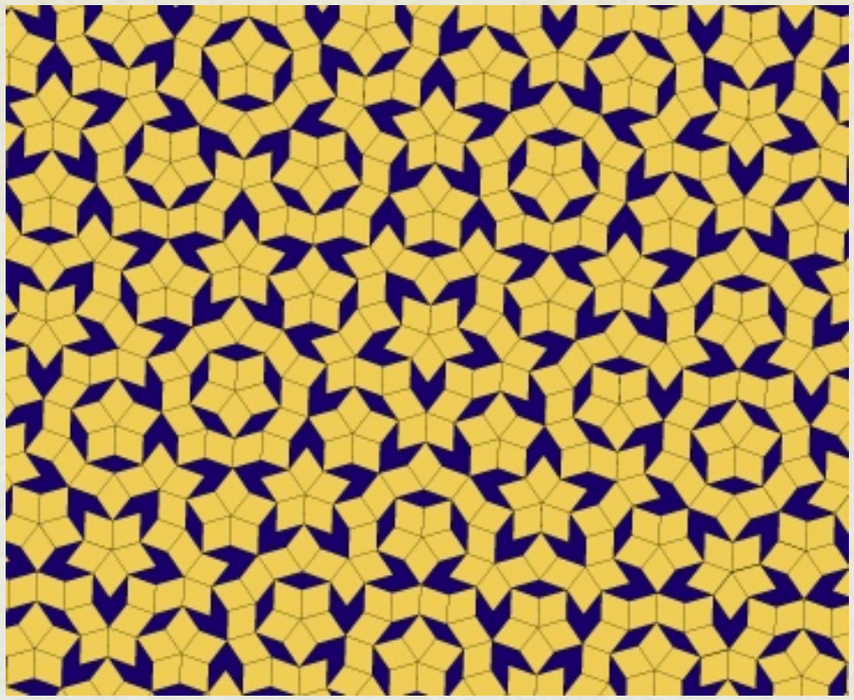




Various different Penrose tilings

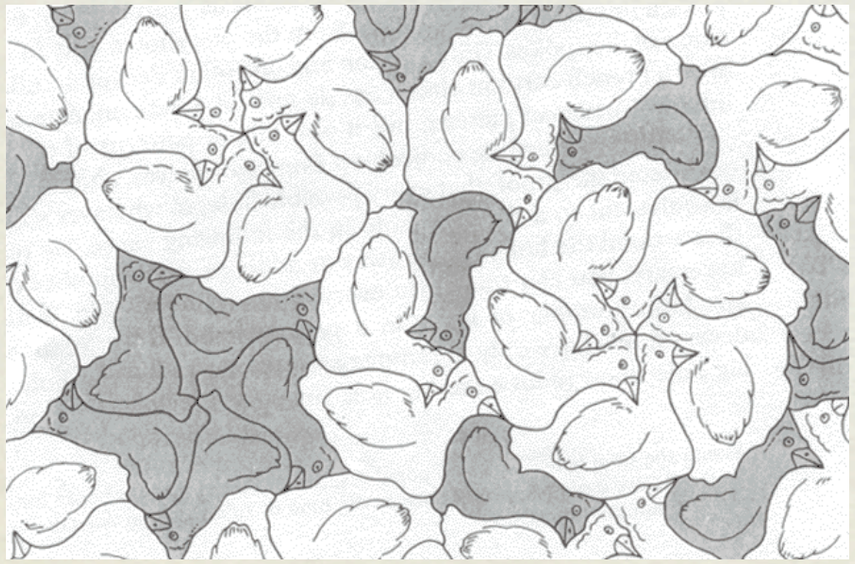
Kites & darts

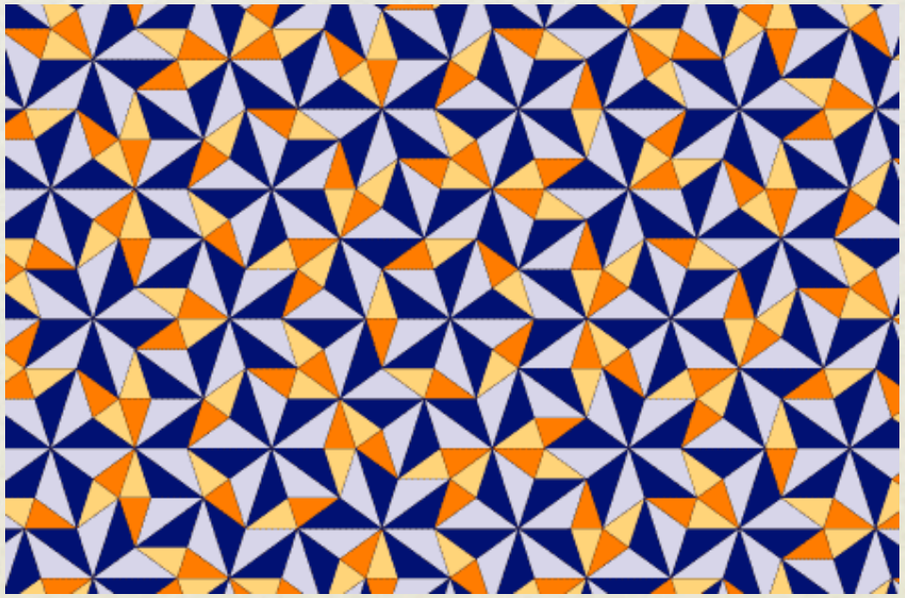




Rhombs

Penrose chickens

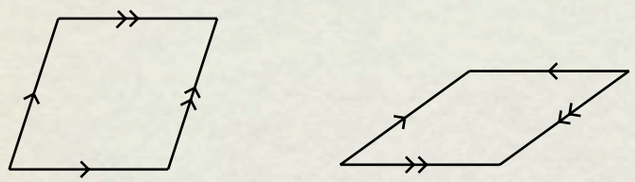


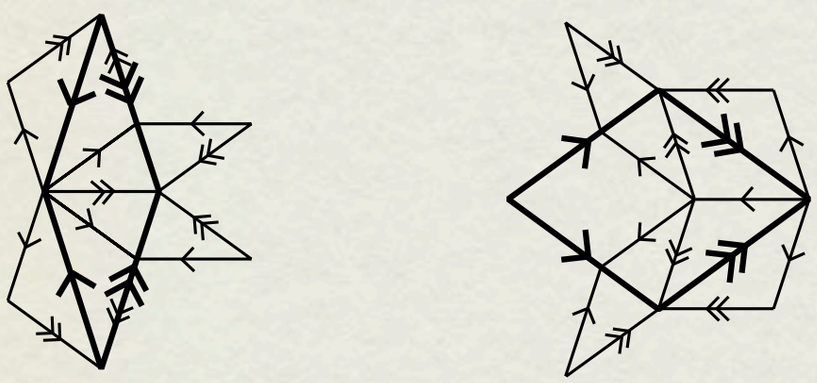


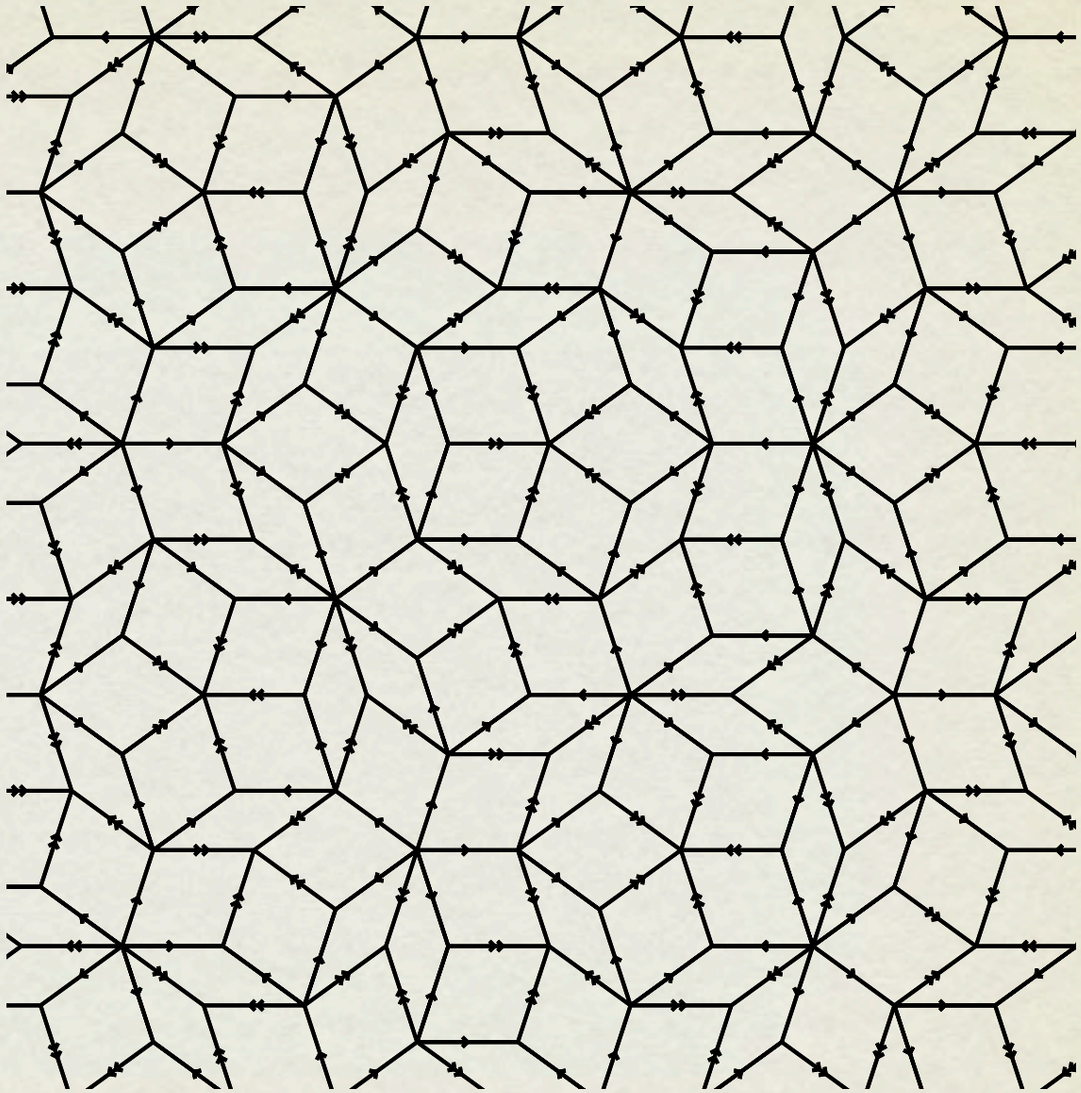
Robinson triangles

Basic properties

"Rhombs" version of Penrose tiling "matching rule" and by "inflation".





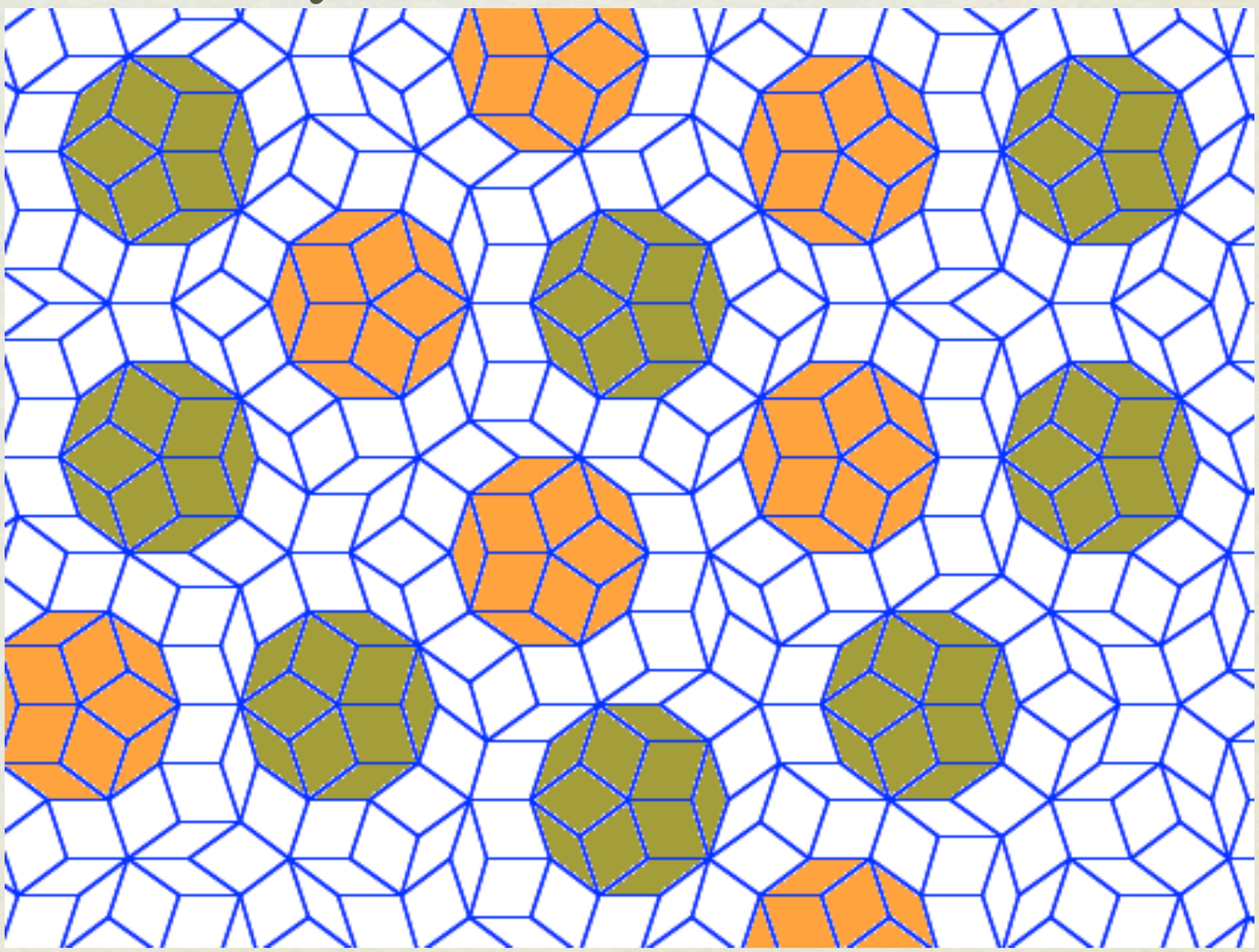


Aperiodic, not a crystal, but ***repetitive***, long-range order. Has 5-fold "***quasisymmetry***".

Repetitivity

A tiling set is **repetitive** if for every patch that occurs in any tiling, there exists L so that a copy of the patch occurs within L of any location. (Patch occurrences relatively dense.).

A tiling set is **uniformly repetitive** if patches occur with uniform positive frequency.



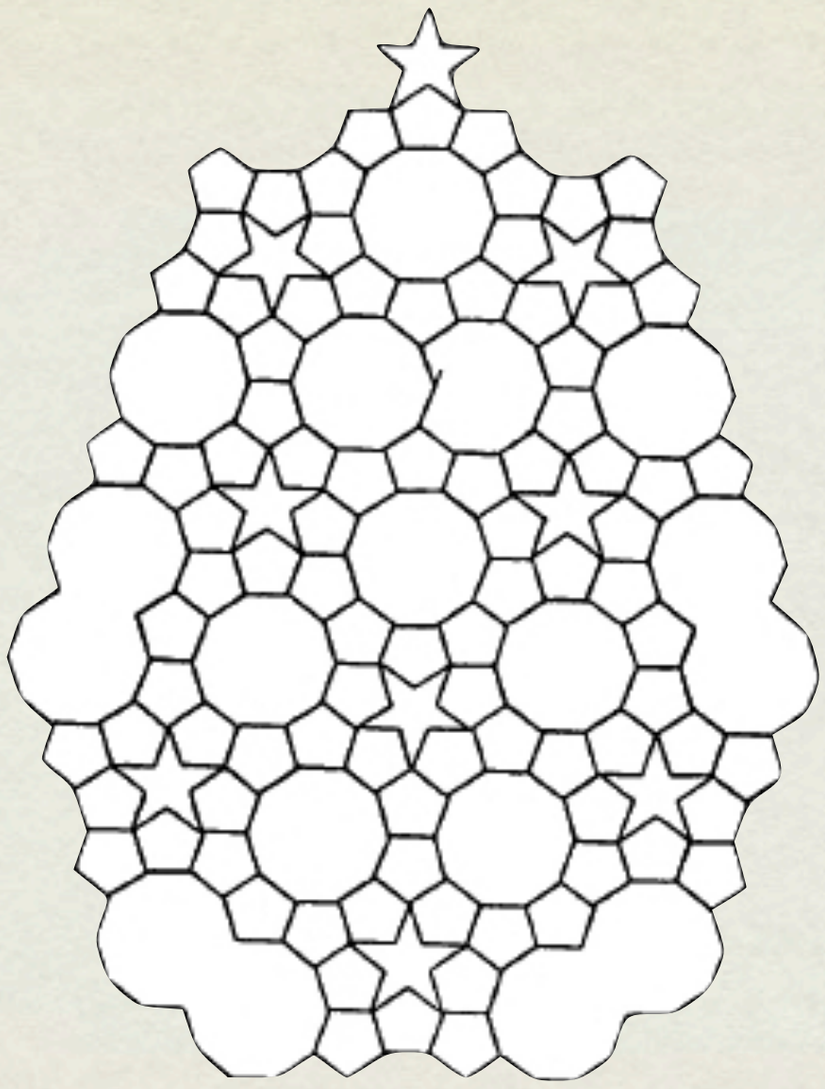
Repetitivity is type of "long range order".

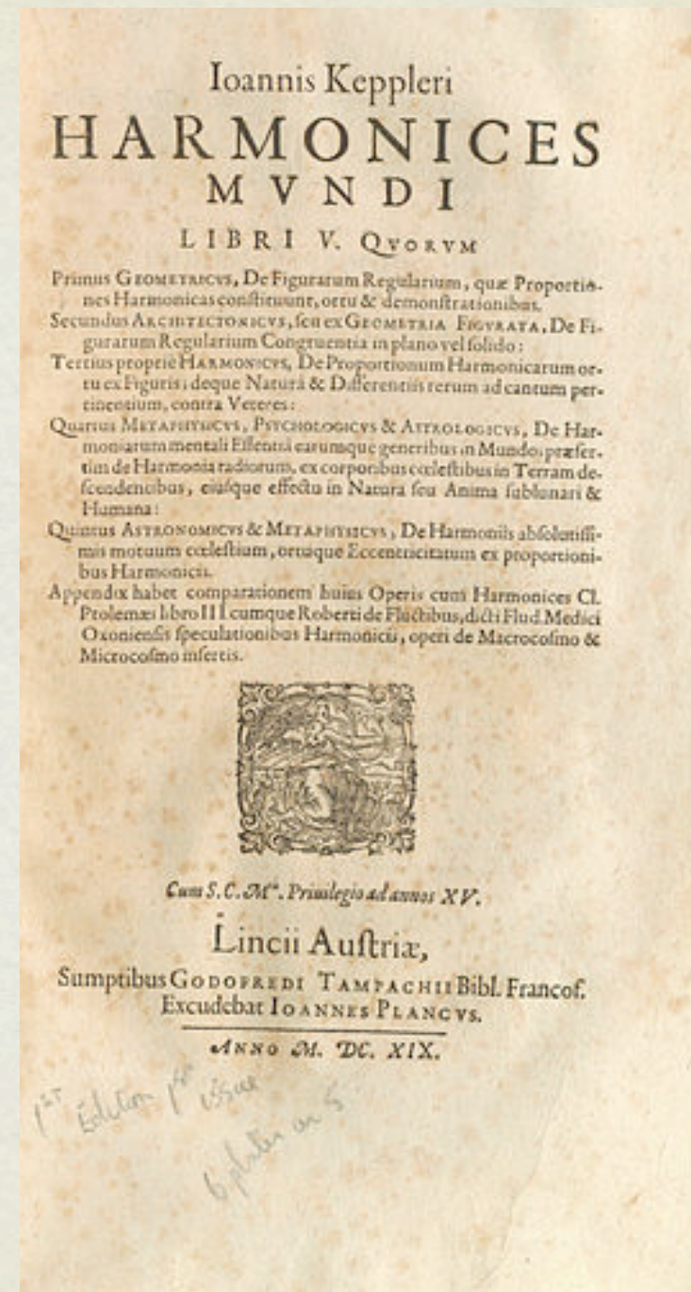
Penrose Tiles as Art

Tony Robbin 1994 “Coast”
Danish Technical University
(destroyed 2003).

GWU Math Department mural.
Saxe-Patterson, Taos, NM, 1998

Kepler's "Penrose" tilings





Ioannis Keppleri

HARMONICES MVNDI

LIBRI V. QVORVM

Primus GEOMETRICVS, De Figurarum Regularium, quæ Proportiones Harmonicas constituunt, ortu & demonstrationibus.

Secundus ARCHITECTONICVS, seu ex GEOMETRIA FIGVRATA, De Figurarum Regularium Congruentia in plano vel solido:

Tertius propriè HARMONICVS, De Proportionum Harmonicarum ortu ex Figuris; deque Naturâ & Differentiis rerum ad cantum pertinentium, contra Veteres:

Quartus METAPHYSICVS, PSYCHOLOGICVS & ASTROLOGICVS, De Harmoniarum mentali Essentiâ earumque generibus in Mundo; præsertim de Harmoniis radiorum, ex corporibus cœlestibus in Terram descendentibus, eisque effectu in Natura seu Anima sublunari & Humana:

Quintus ASTRONOMICVS & METAPHYSICVS, De Harmoniis absolutissimis motuum cœlestium, ortuque Eccentricitatum ex proportionibus Harmonicis.

Appendix habet comparationem huius Operis cum Harmonices Cl. Ptolemæi libro III cumque Roberti de Fluctibus, dicti Flud Medici Oxoniensis speculationibus Harmonicis, operi de Macrocosmo & Microcosmo insertis.

Cum S. C. M^tis. Priuilegio ad annos XV.

Lincii Austriæ,

Sumptibus GODOFREDI TAMPACHII Bibl. Francof.
Excudebat IOANNES PLANCVS.

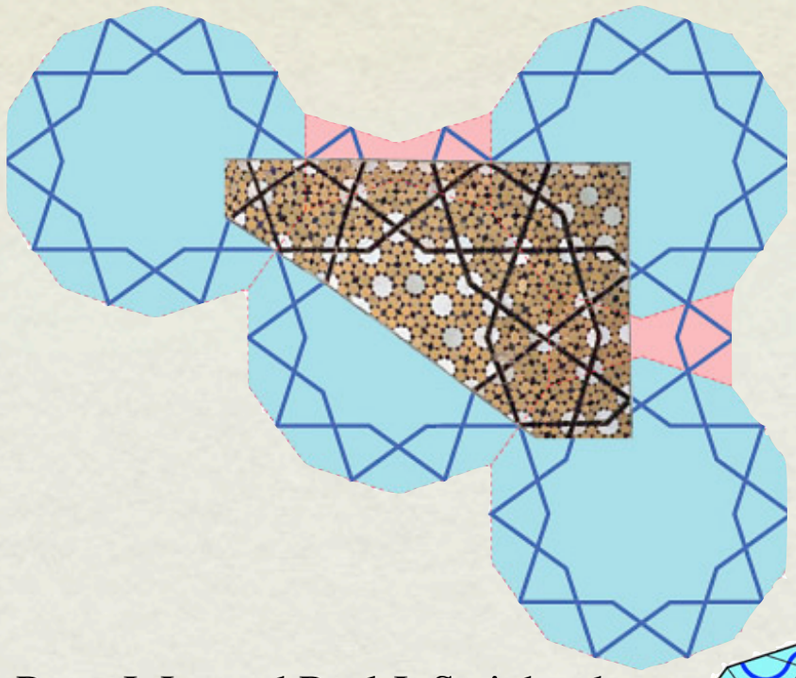
ANNO M. DC. XIX.

1st Edition 1st issue
6 plates on 5

From Kepler's *Harmonices Mundi*, 1619.

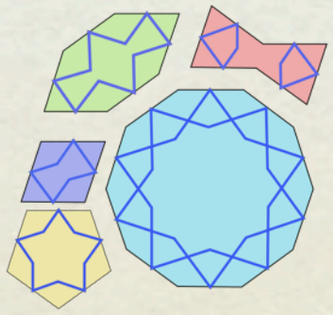
Called the fused decagons "monsters".

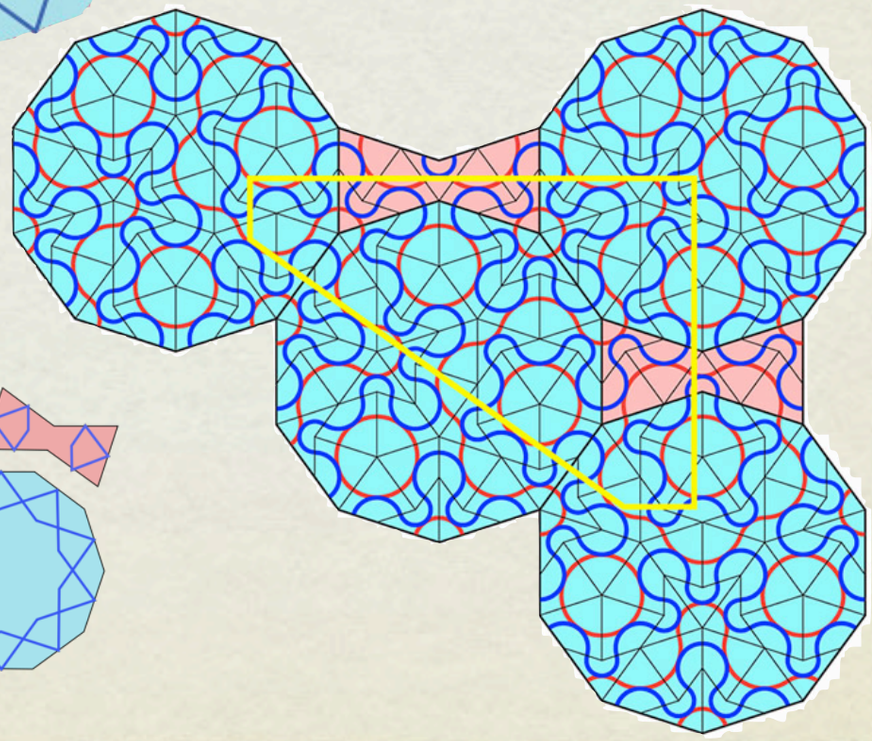
Islamic "Penrose" Tilings



Spandrel tiling with girih patterns (left), and with Kites & Darts Penrose tiling (below).

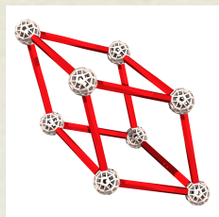
Peter J. Lu and Paul J. Steinhardt (2007). "Decagonal and Quasi-crystalline Tilings in Medieval Islamic Architecture". *Science* **315** (5815): 1106–1110.

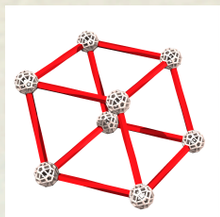




Peter J. Lu under tile spandrels, Darb-i Imam shrine, Isfahan, Iran, built 1453 (560 years).

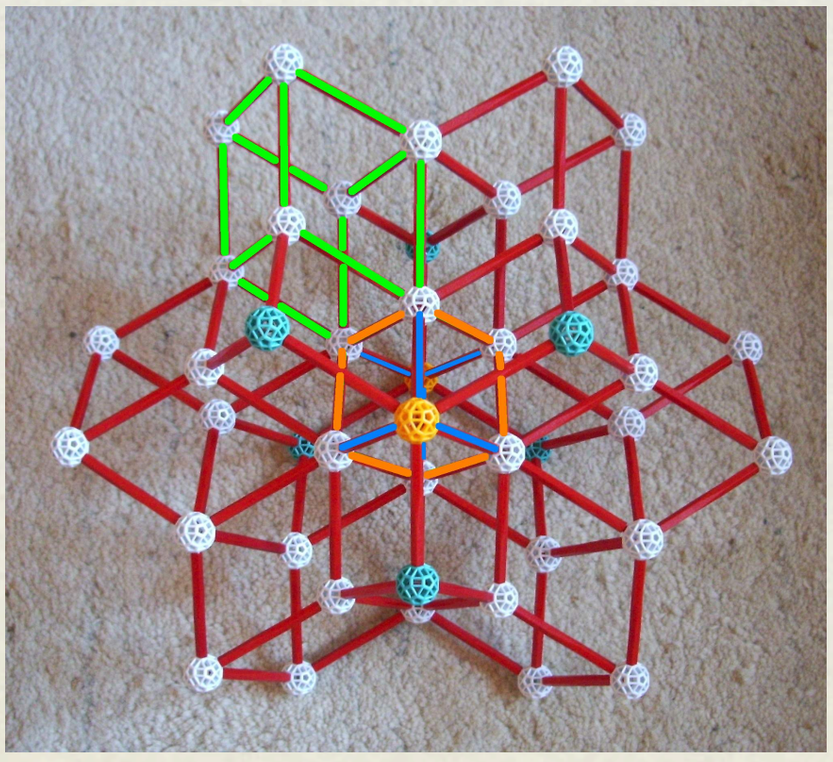
3-dimensional Penrose tiling

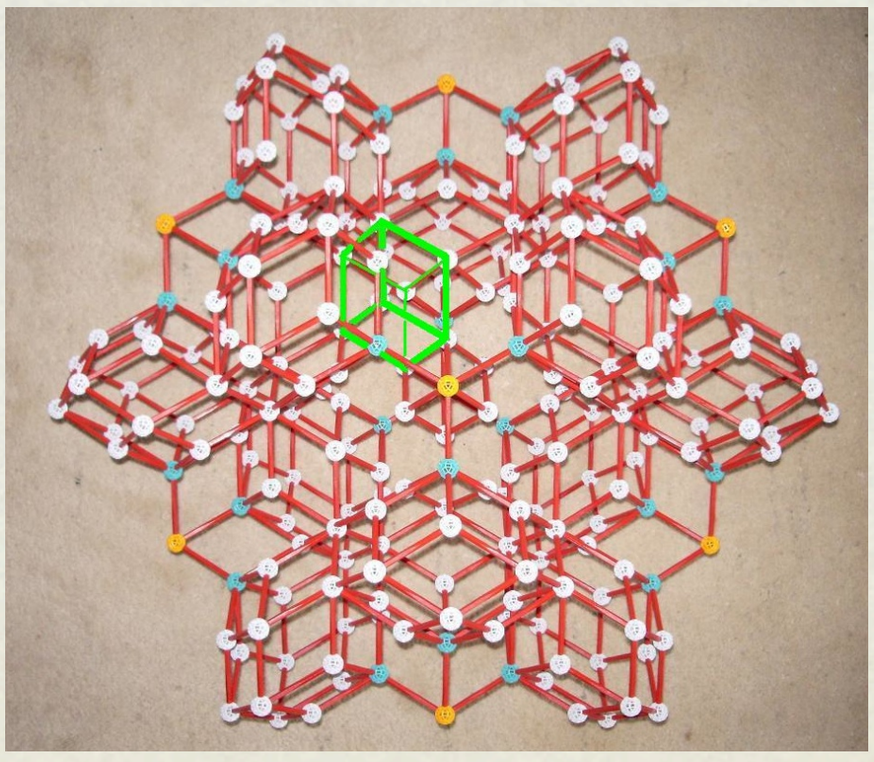




Deitl & Eschenberg,

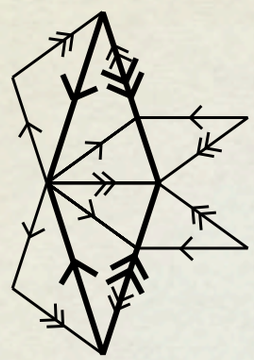
Made with “Zometool”

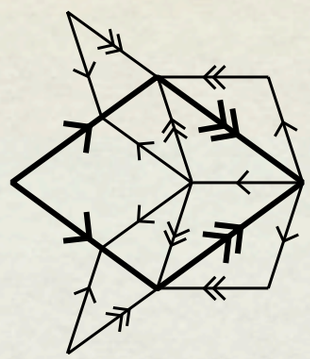


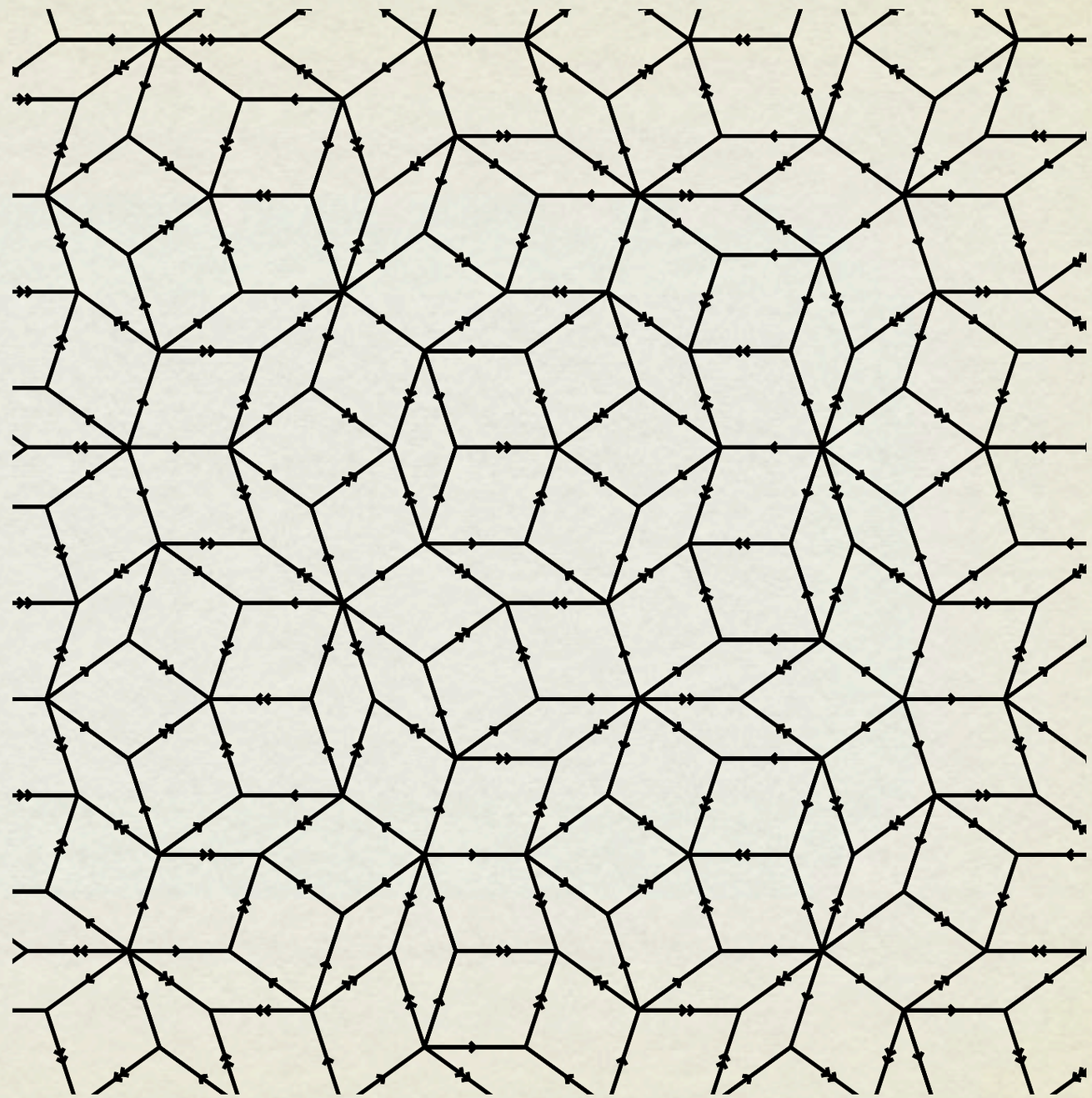


http://www.math.uni-augsburg.de/~eschenbu/

V. Three paradigms





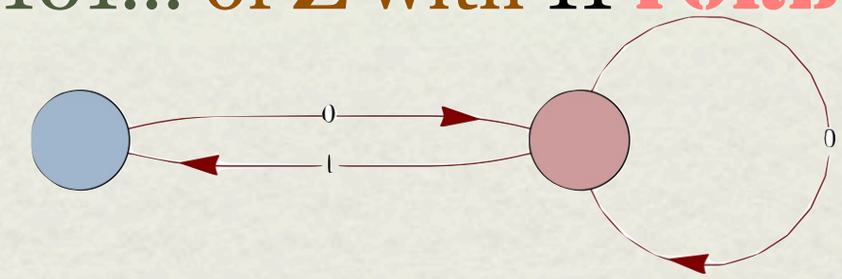


- ✤ **Matching rule.**
- ✤ **Inflation.**
- ✤ **Projection.**

✤Matching Rule

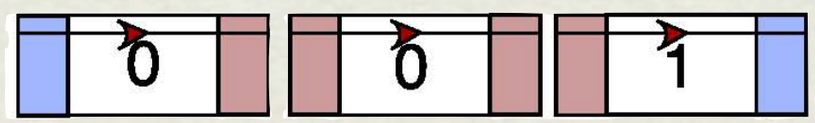
Subshift of finite type (SFT) $d = 1$:

- **Example:** Golden mean shift: $X \subseteq \{0,1\}^{\mathbf{Z}}$: tilings x =... 0100.101... of **Z** with **11** **FORBIDDEN**, T shift map.



- # w=01010..010 of length n is f_n. Entropy $h(T) = \log\left(\frac{1+\sqrt{5}}{2}\right)$
- *Has* ***periodic*** x=..0010.01001... and ***aperiodic*** points.

- 1-dim*Wang tiles:*



x = ...

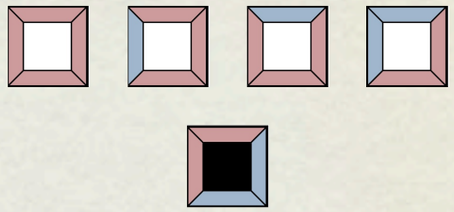
...

Wang Tiles

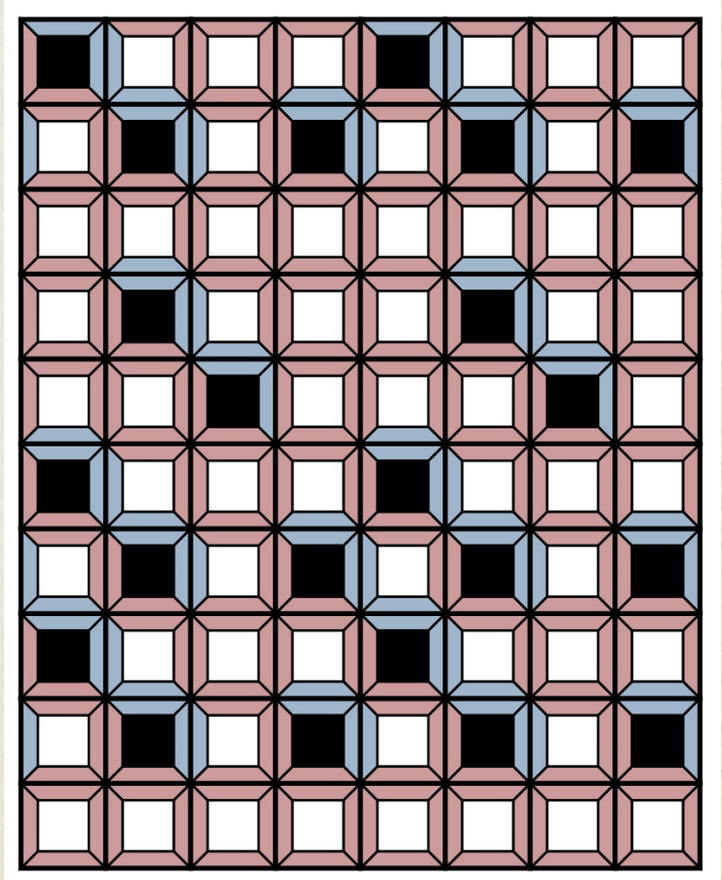
Wang tilings = 2-dim subshift of finite type:

- **Example:** Golden mean shift: $X \subseteq \{0,1\}^{\mathbb{Z}^2}$ with 11 & $\begin{smallmatrix}1\\1\end{smallmatrix}$ **FORBIDDEN**.

```
        1000101000
x=...   0001000010...
        1000011100
```



- Has both *periodic* and *aperiodic* points.
- Mixing, $h(T)>0$ but $h(T)$=? (open).



Hochman & Meyerovitch *A characterization of the entropies of multidimensional shifts of finite type*. Ann. of Math. **171** (2010):
$h(T)=\lim h_n$, where h_n recursive rational.

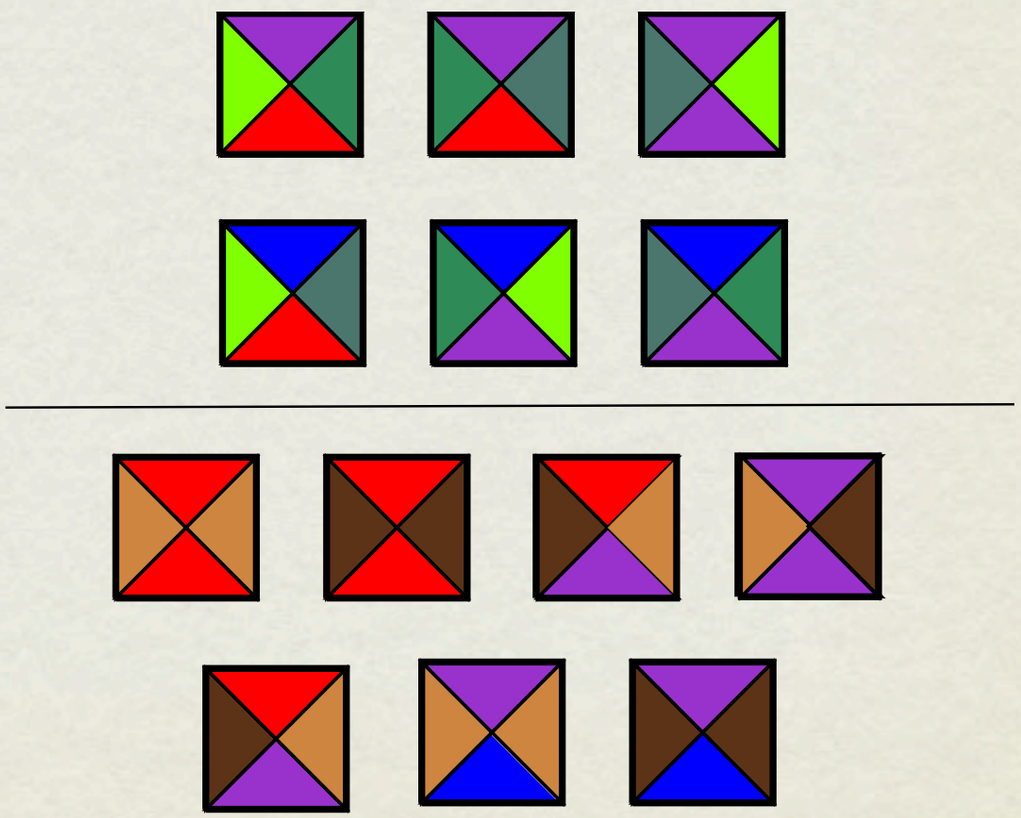
Wang's Conjecture

- **Conjecture** (Hao Wang, 1961) Every finite Wang tile set, d=2, that admits a *valid* tiling also admits a *valid periodic tiling*. (The tiling problem is decidable).

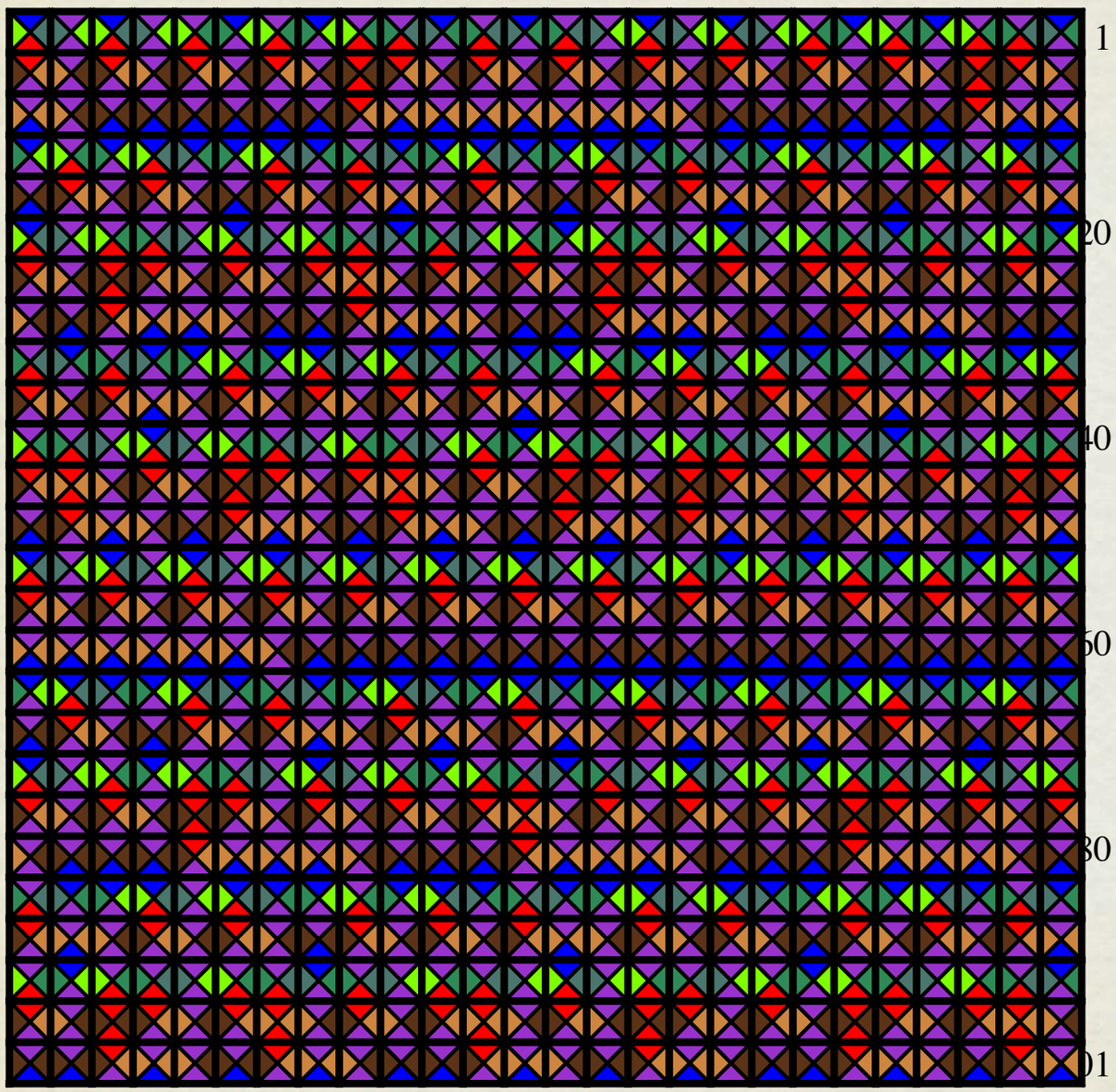
- True for $d = 1$ (1-dim. SFT). Often true $d = 2$, e.g. 2-dim Golden mean.

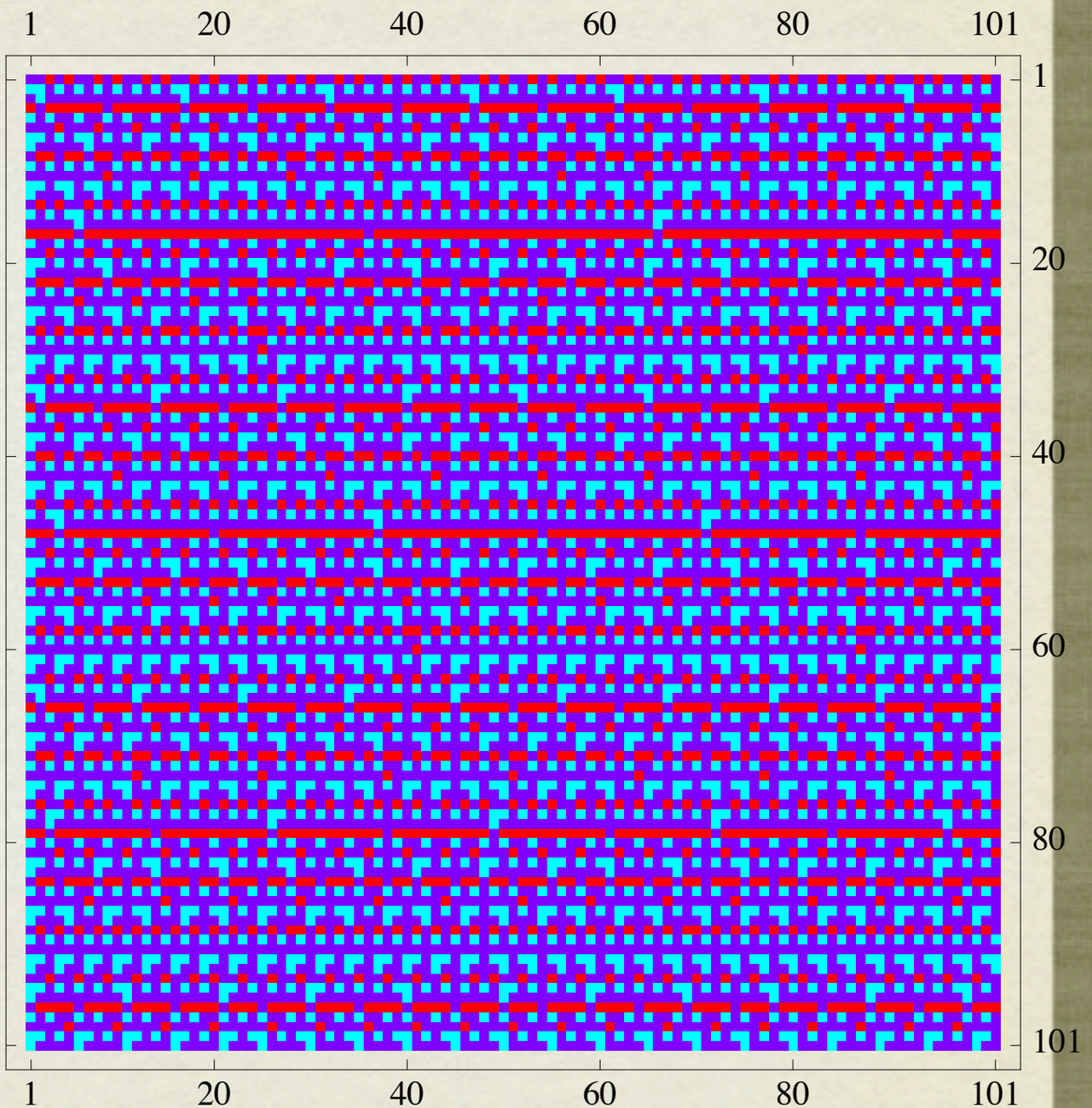
- Berger, 1967: False for $d = 2$. 20,000+ tile counterexample (*aperiodic* Wang tile set).

- Current record 13 tiles (K-C tiles), Kari-Culick, 1996.



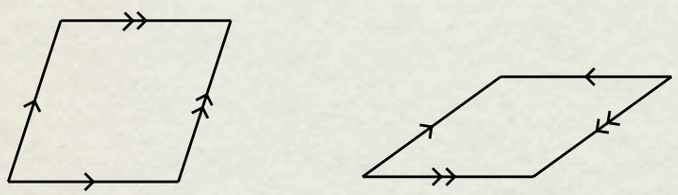
Tiling patches by KC-Tiles



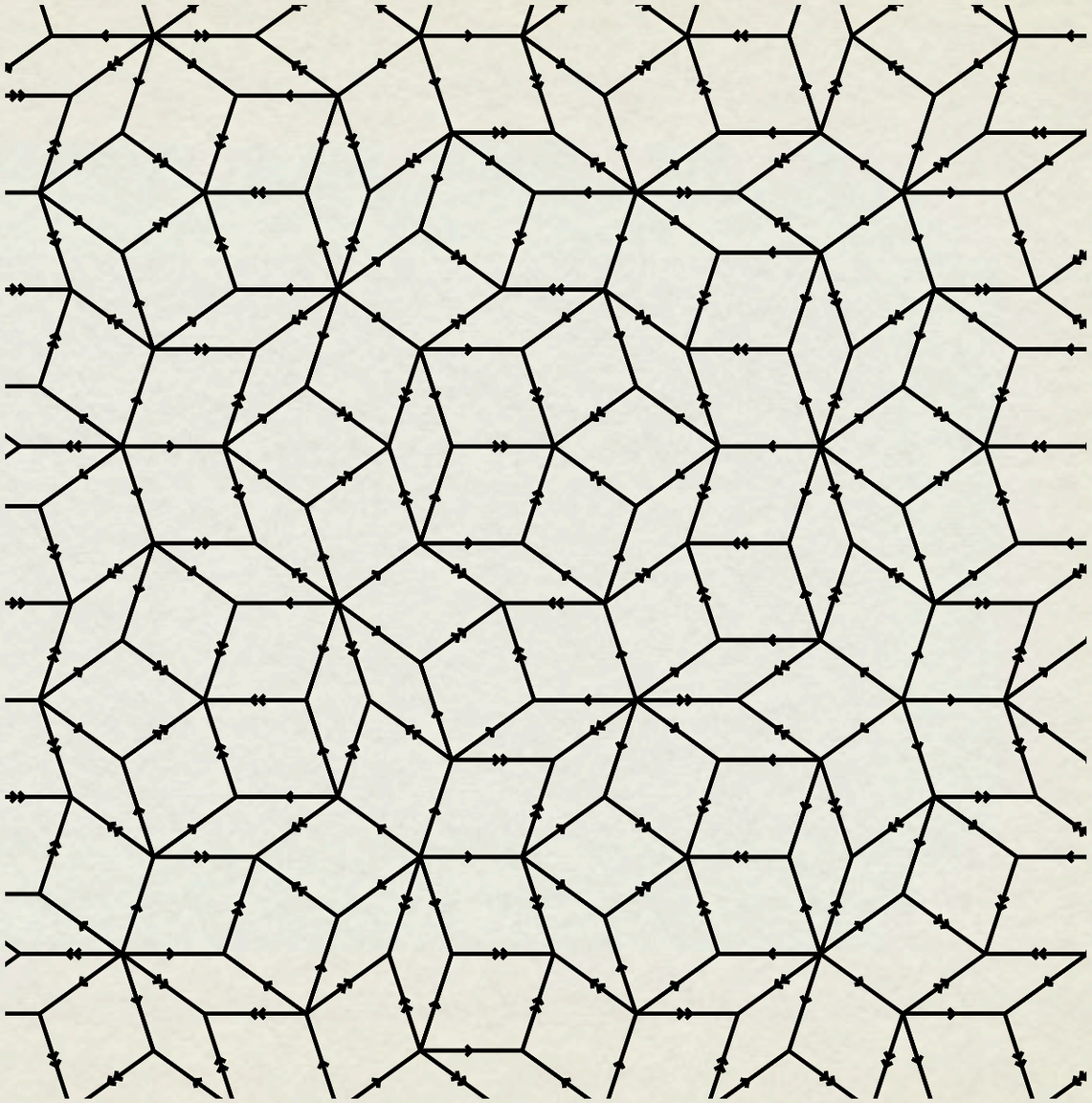


Penrose tiles as Wang tiles

Penrose tiles as rhombic aperiodic Wang tiles, i.e., $\mathbf{R}^2$ SFT.



Matching rules don't necessarily explain aperiodicity but might explain how quasicrystals grow. (Though probably not!)



✤ Inflation

Morse substitution (Morse sequence): M. Morse, 1921; A. Thue, 1906; E. Pruhet, 1851.

- *Substitution* σ: 0→01, 0→01
- *Iteration*: 0→01→0110→01101001→0110100110010110
- 1-sided limit (fixed point): $\sigma^\infty(0)$=1001011010010110...
- 2-sided fixed point:

  $x = \sigma^\infty(1)^R.\sigma^\infty(0)$=...10011001.0110100110...

Substitution dynamical system: shift T on $X=\mathrm{cl}\{T^n(x) : n\in\mathbf{Z}\}$.

- ***Properties***: aperiodic, hierarchical, repetitive (minimal), uniquely ergodic, *infinite* type, mixed spectrum, h=0.

Dynamical spectrum

- Transformation T on X (e.g., substitution shift):
- Eigenvalue-eigenfunction $f(Tx)=\lambda f(x)$
- Spectral measure $\hat{\gamma}_f(n) = \langle f(T^n x), f(x)\rangle$ (positive definite). Choose f so γ_f is maximal.
- Lebesgue decomposition: $\gamma_f = \delta_0 + \gamma_{\text{dis}} + \gamma_{\text{sc}} + \gamma_{\text{ac}}$.
- For an action of $\mathbf{R}^d$ one has $\hat{\gamma}_f(\mathbf{v}) = \langle f(T^{\mathbf{v}} x), f(x)\rangle$
- Can detect discrete spectrum, mixed spectrum, weak mixing, etc.

Other well known Substitutions

- **Morse**: 0→01, 1→10 "bijective", constant length.
- **Period doubling**: 0→01, 1→00 constant lengt.h

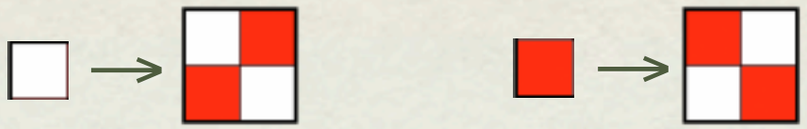
  0→01→01 00→0100 1010→...→0100010100010001...
- **Fibonacci**: 0→01, 1→0

  0 →0 1 →01 0 →010 01 →01001 010 →01001010 01001...
- **Dekking-Keane**: 0 → 0111, 1 → 0 (weakly mixing).

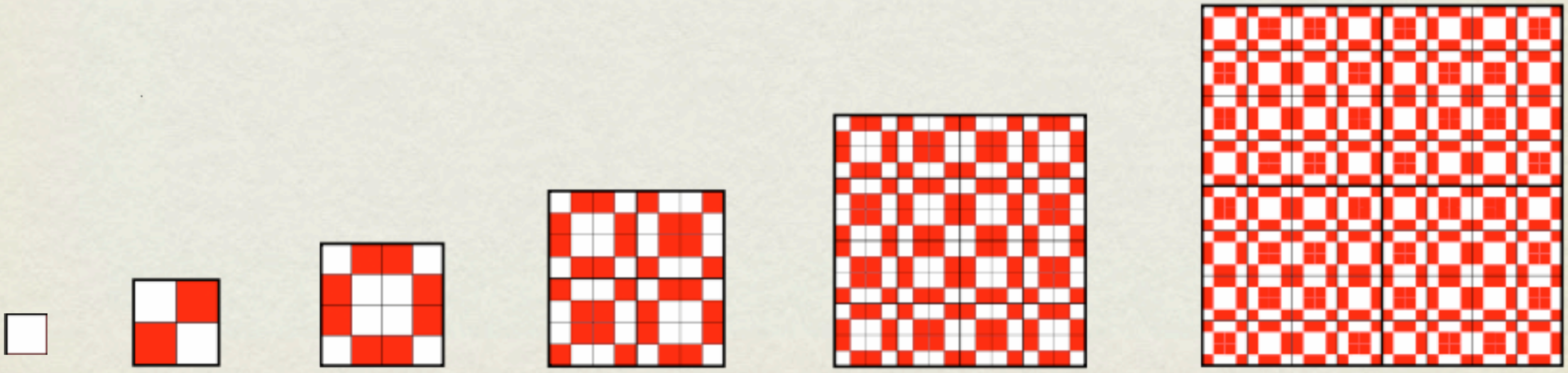
  0 → 0111 →0111 000 →... →0111000011101110111011100001...
- **Rudin-Shapiro:** 0 →01, 1 →02, 2 →31, 0 →32 (Lebesgue spectrum).
- **Cantor** 0 → 010, 1 → 111 not primitive.

  0 →010 →010 111 010 →101111010 111111111 010111010 ...

Morse tiling

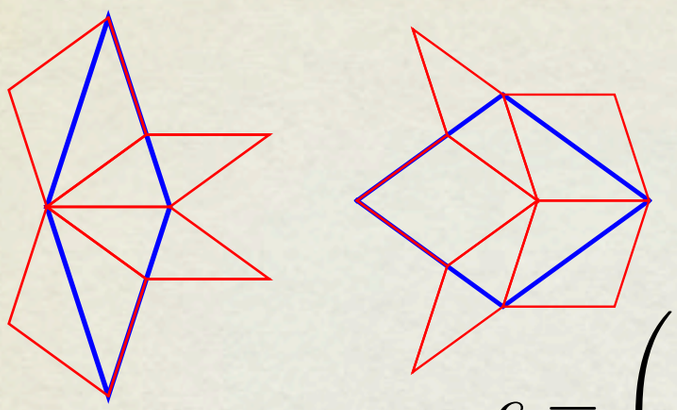


2-dimensional Morse substitution: ***inflation***.



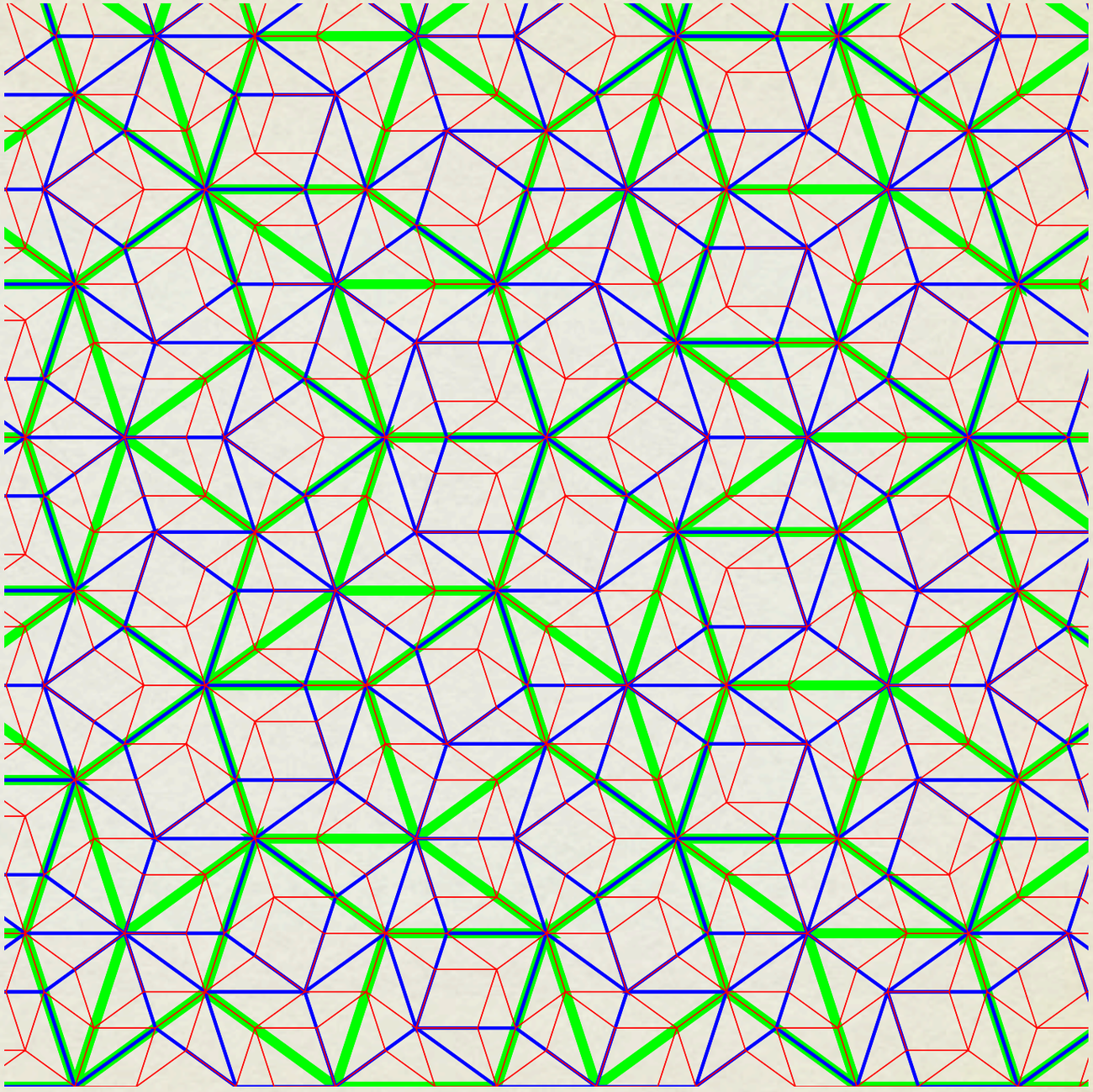
Penrose as inflation

Inflation on Penrose tiles



$$e = \left(\frac{1+\sqrt{5}}{2}\right)^2$$

Inflation explains aperiodicity, repetitivity, patch frequencies. Also explains 5-fold (quasi-)symmetry. Might explain diffraction.



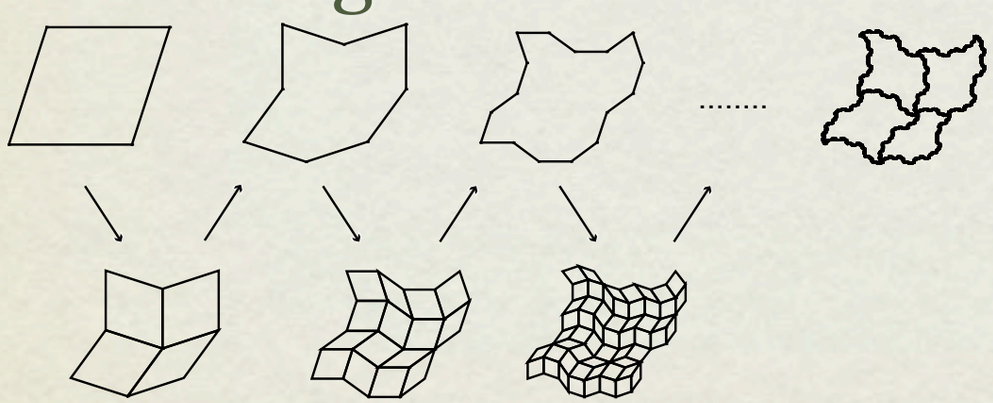
Hierarchical structure

Zoo of inflation tilings

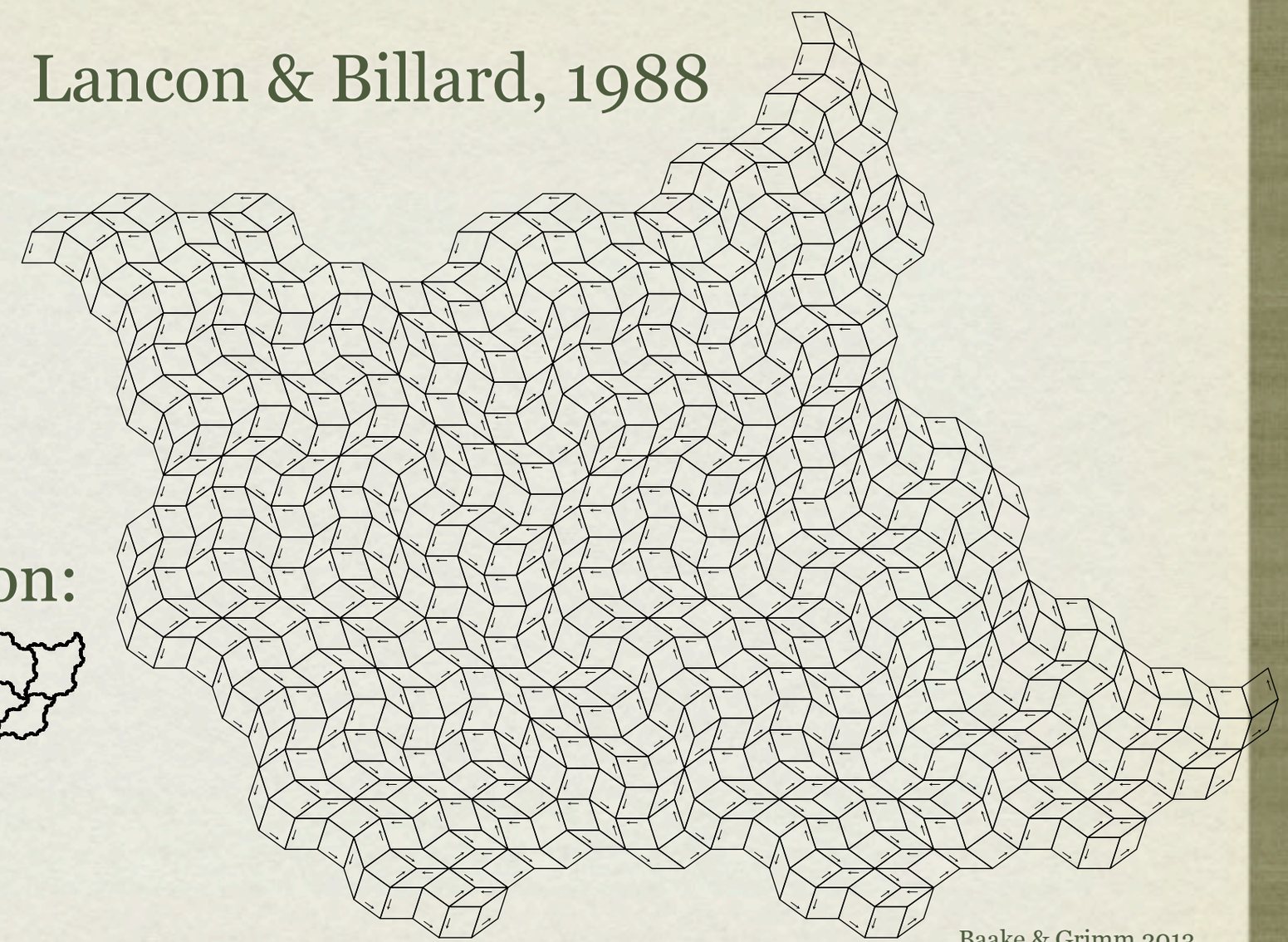
Binary tiling

$$e = \frac{5+\sqrt{5}}{2}$$

Perfecting the substitution:



Lancon & Billard, 1988

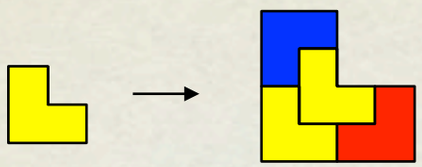


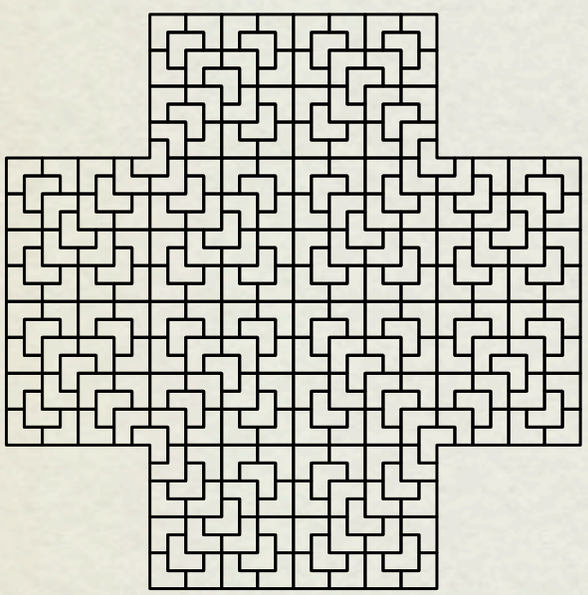
Baake & Grimm 2013

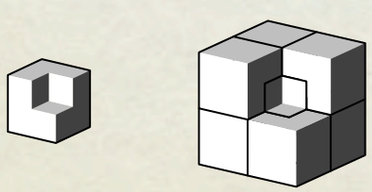
$$d_{\mathsf{H}} = \frac{\log(2)}{\log(2\cos\frac{\pi}{10})} = \frac{2\log(2)}{\log\left(\frac{5+\sqrt{5}}{2}\right)} \approx 1.078$$

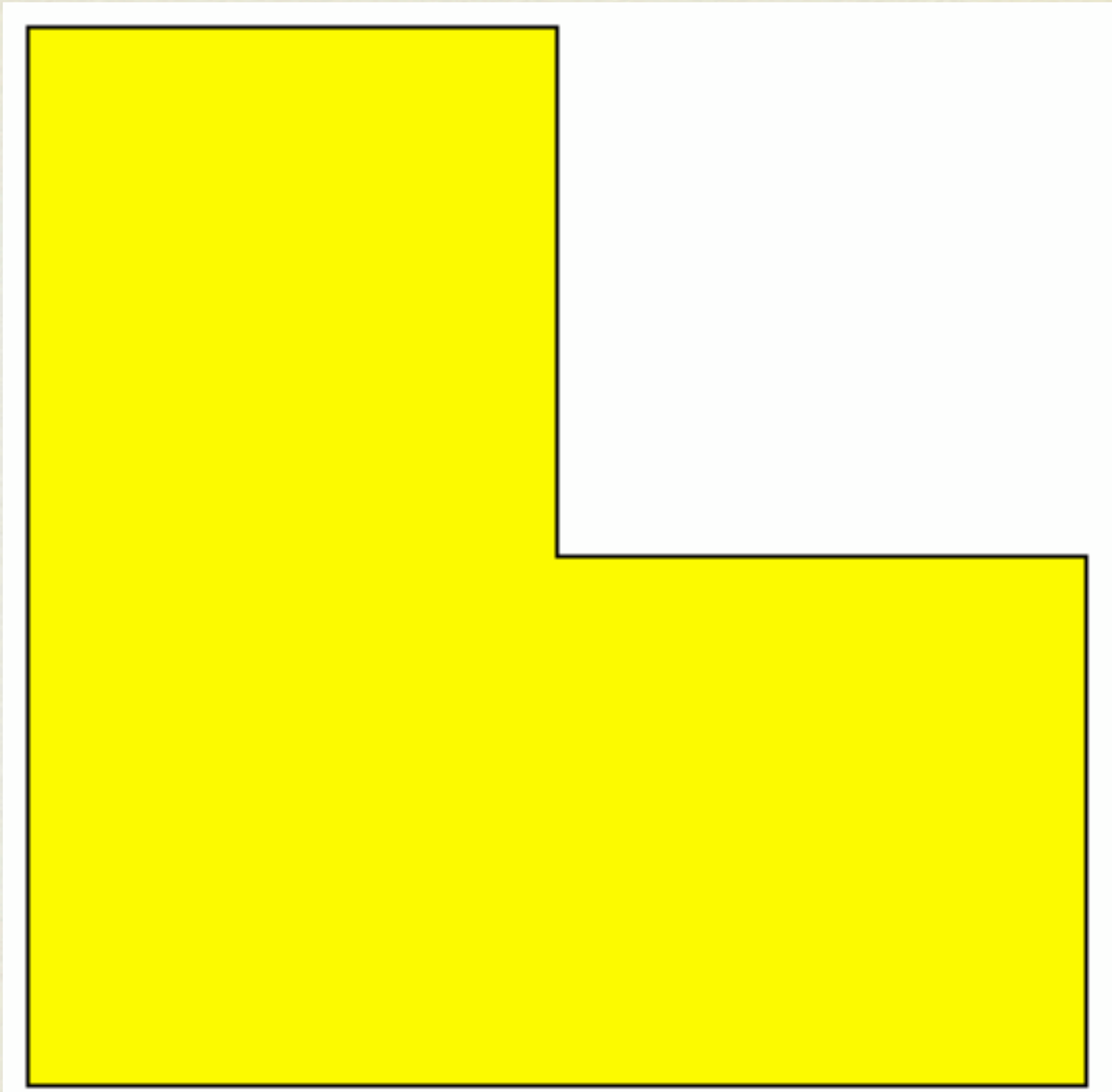
Zoo of inflation tilings

Chair tiling









Zoo of inflation tilings

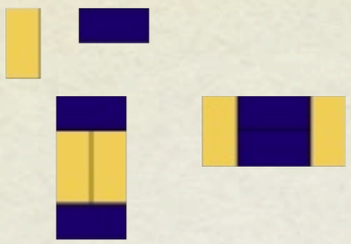
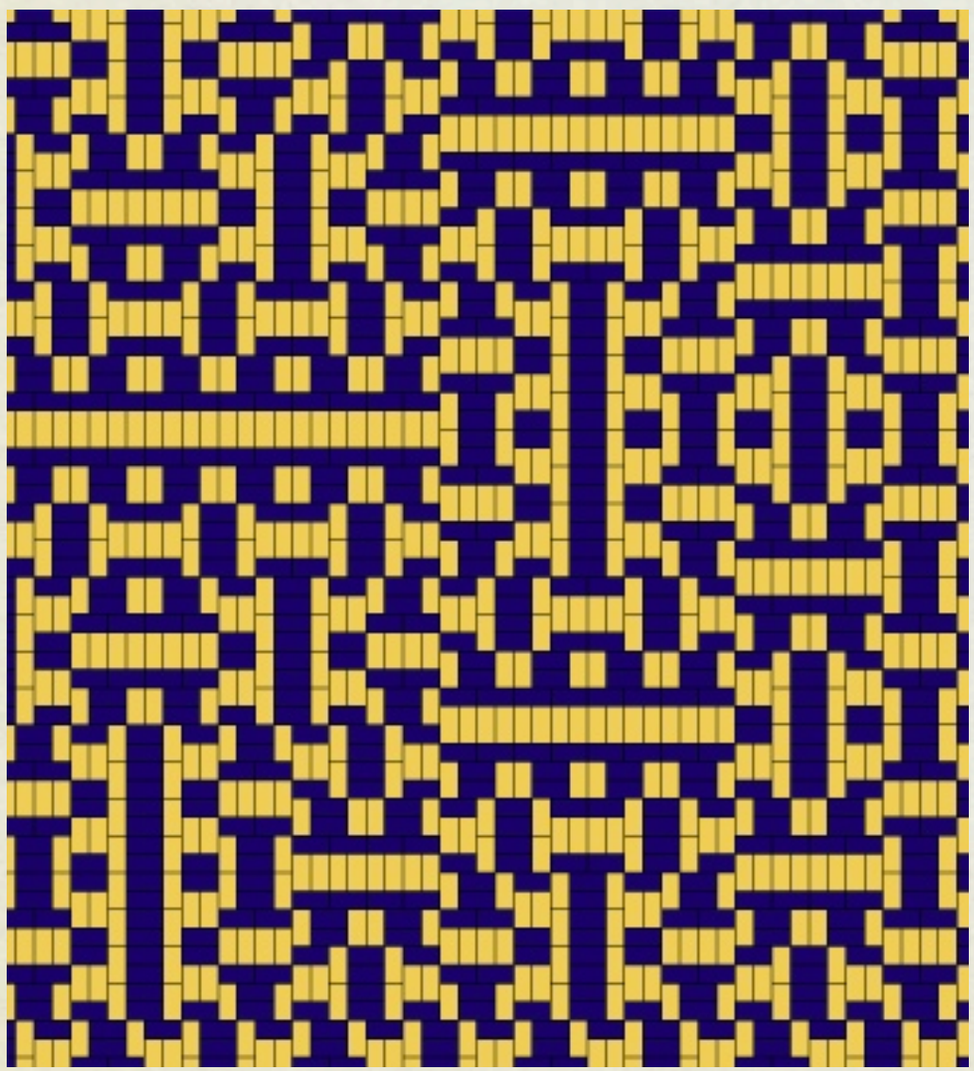
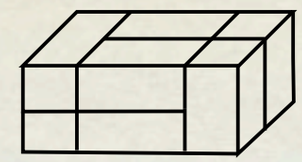
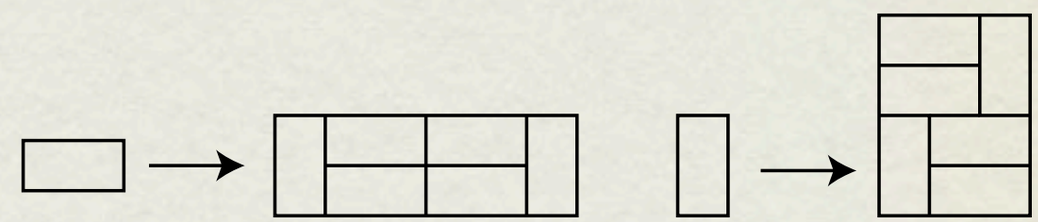


Table tiling

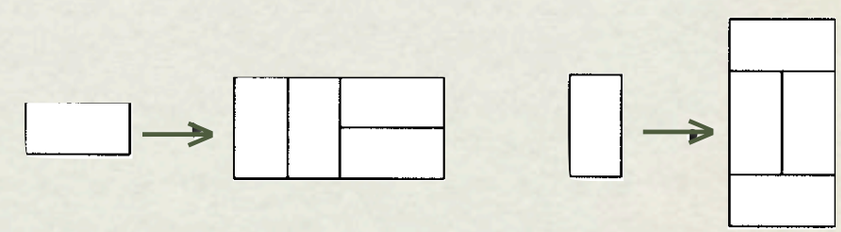




3-dimensional



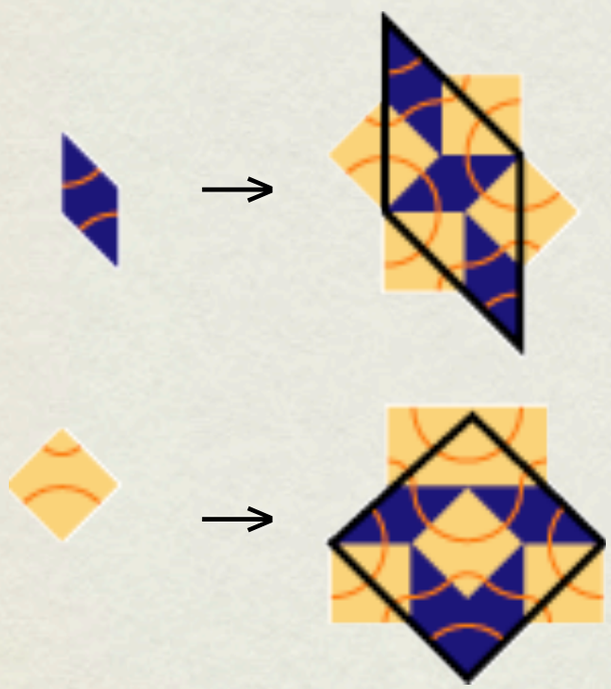
Self affine, not self similar.



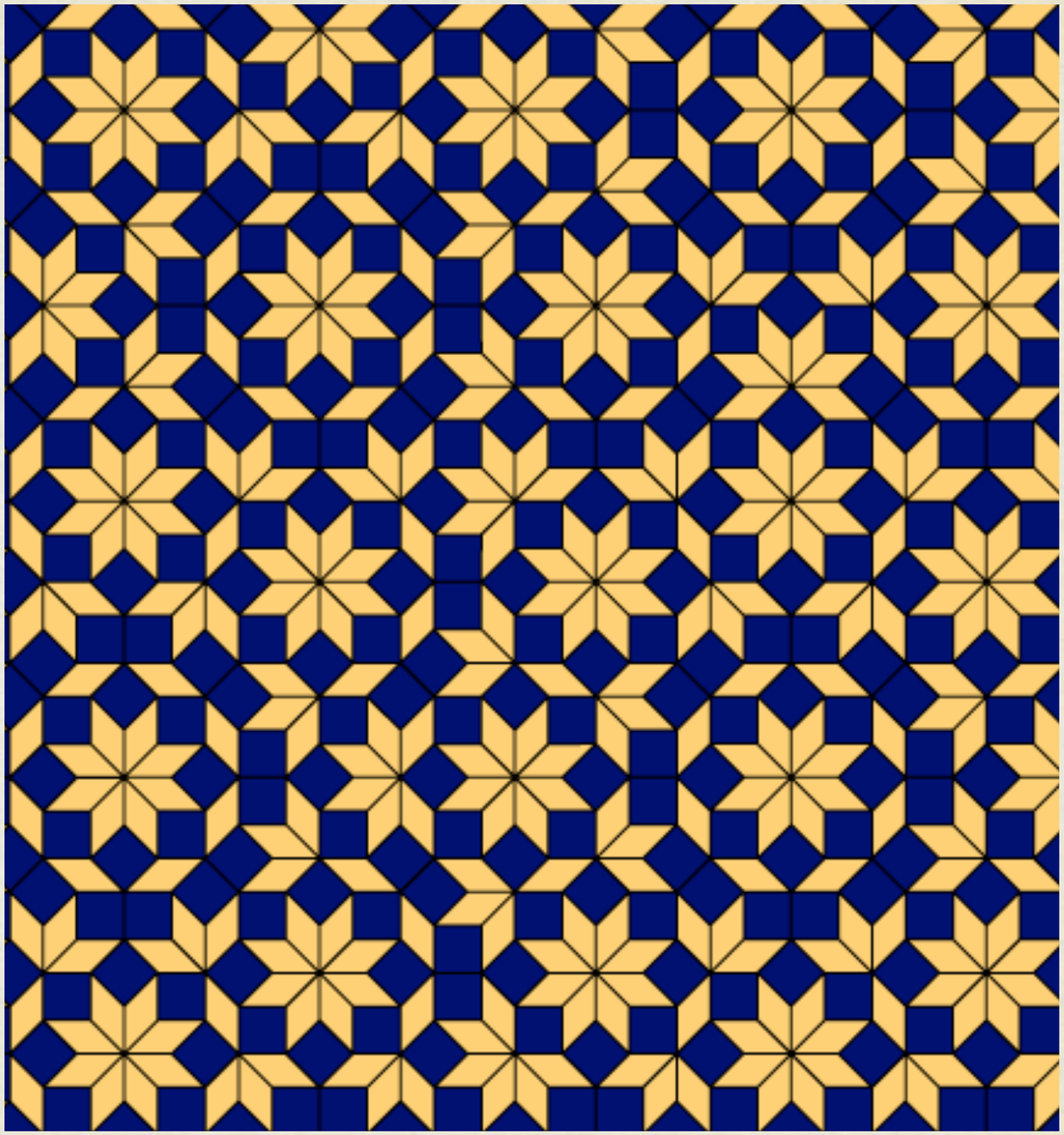
Alternate (with coincidence)

Zoo of inflation tilings

Ammann-Beenker
(Octagonal)

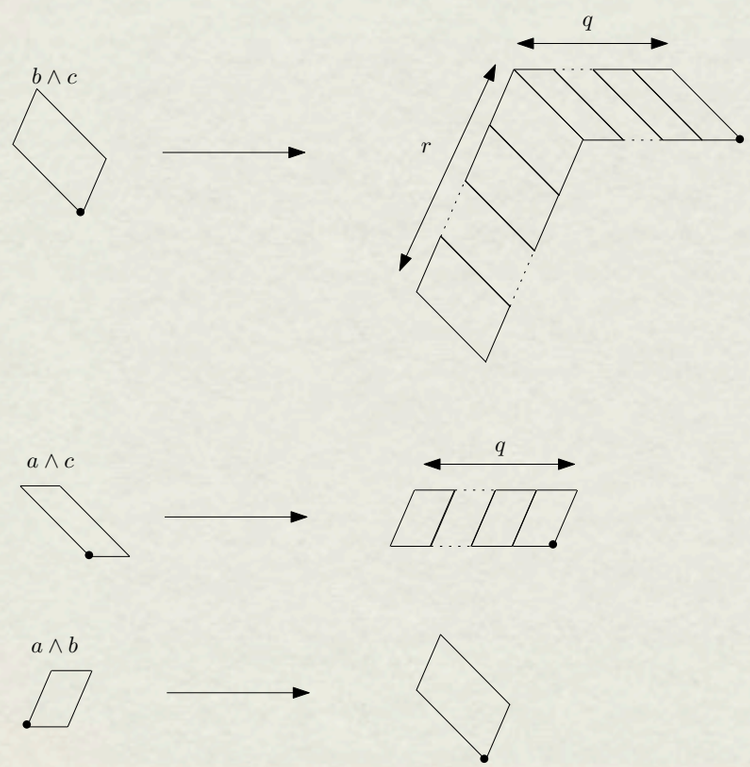


R. Ammann 1977
Beenker 1982

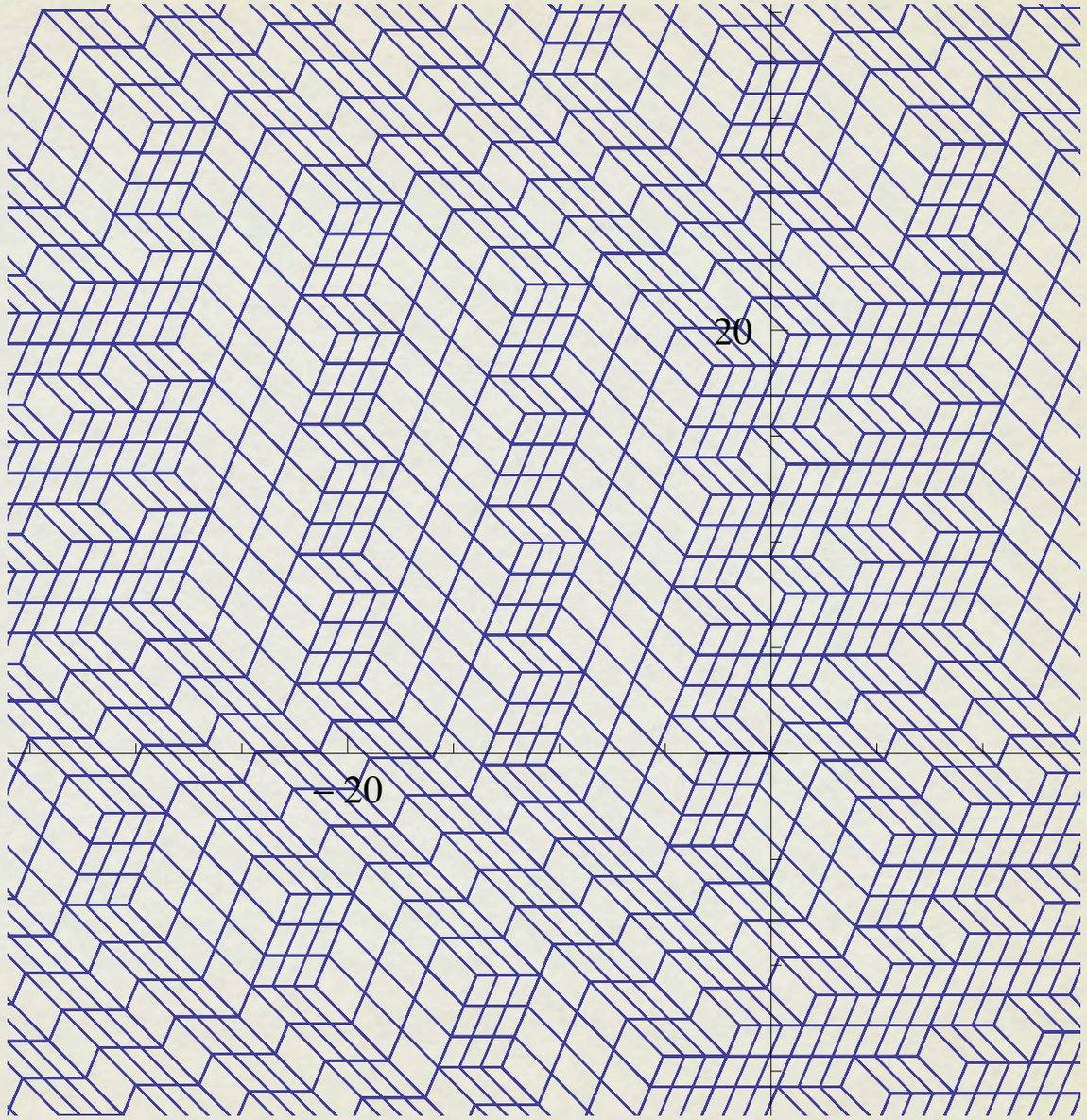


Zoo of inflation tilings

"**Mixing**": Tyler White
(March 2012 Dissertation)



Topological mixing



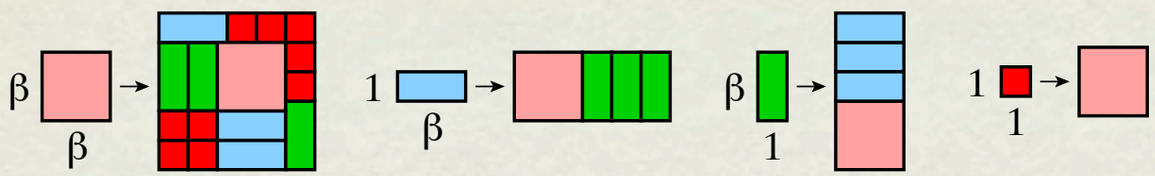
(Case: q=3, r=1)

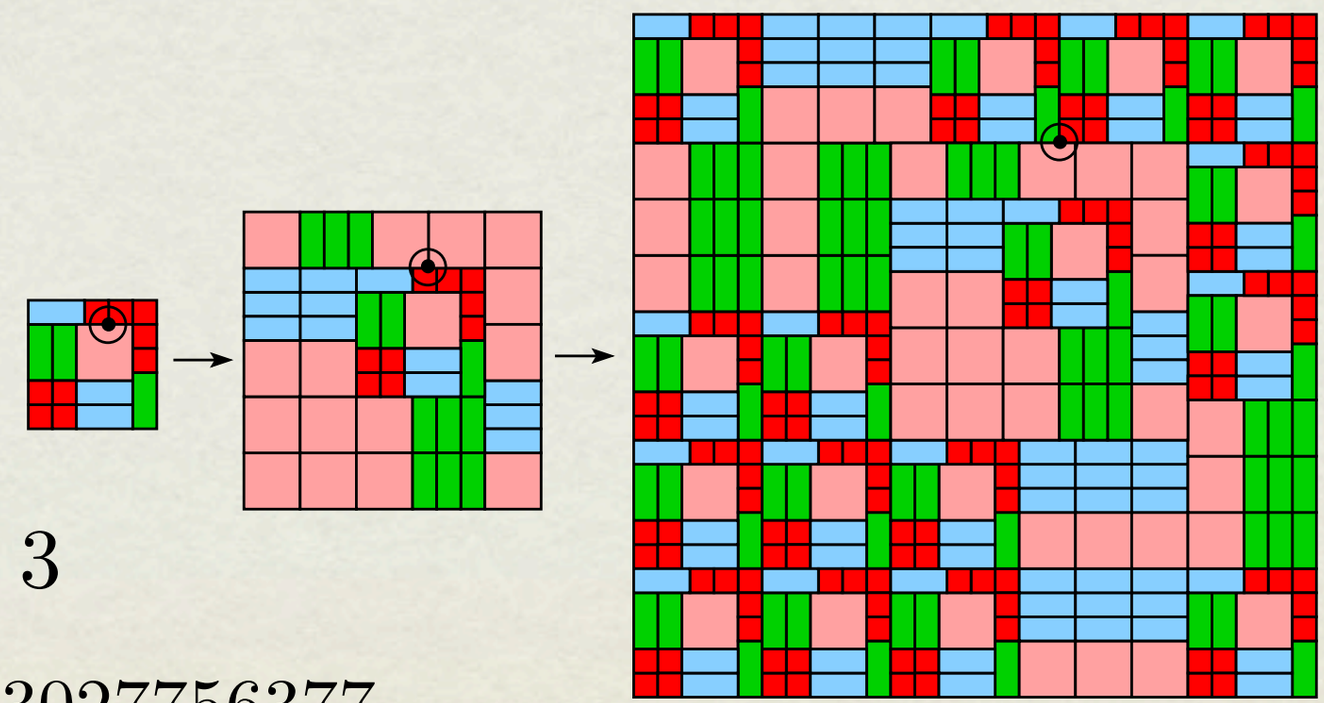
Zoo of inflation tilings-4

Table cloth

Natalie Frank & RR (2008)

Not finite local complexity (FLC).





$$p(x) = x^2 - x - 3$$

$$\beta = (1 + \sqrt{13})/2 \approx 2.3027756377$$

Zoo of inflation tilings-3

Pinwheel tiling

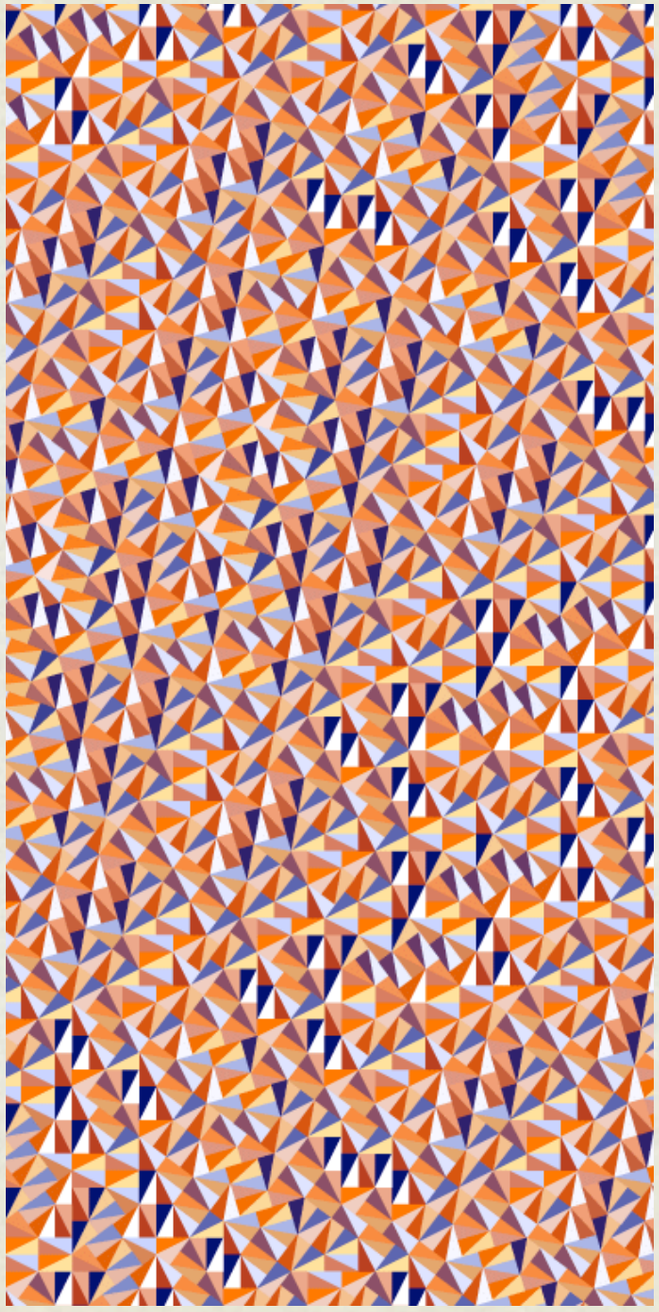
Conway/Radin

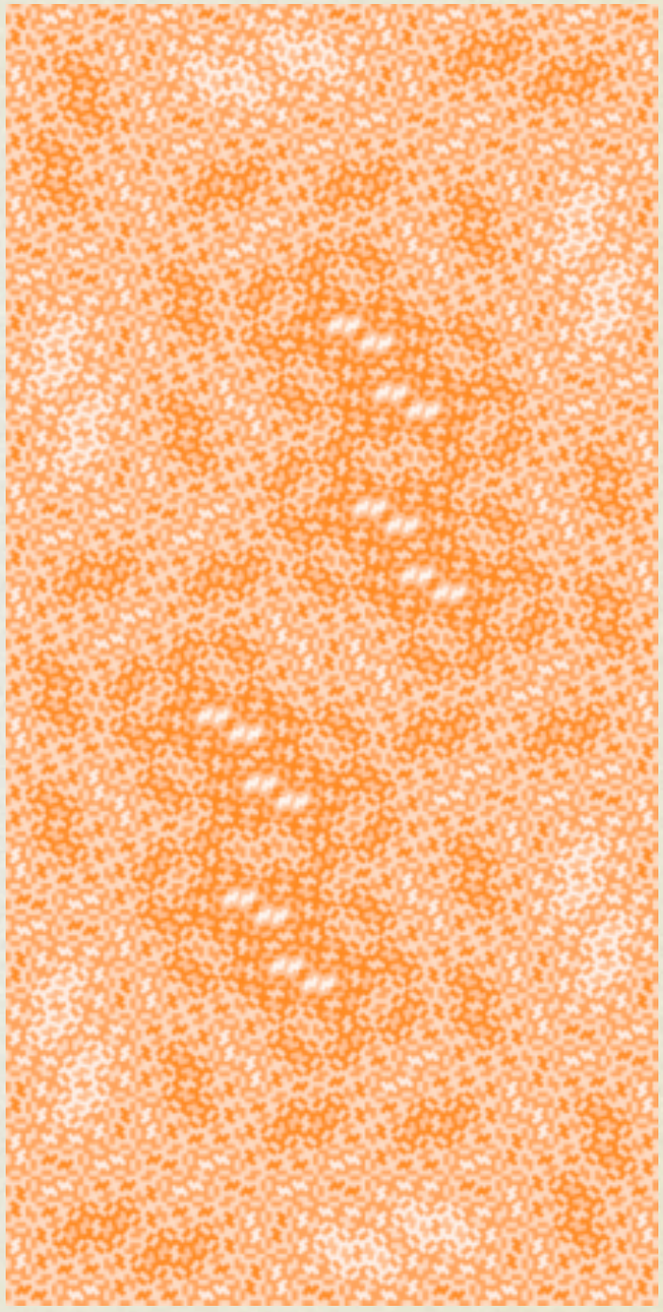
1994

Rotational symmetry **T**.

Not finite local complexity (FLC).

$$\tan(\theta) = \frac{1}{2}$$





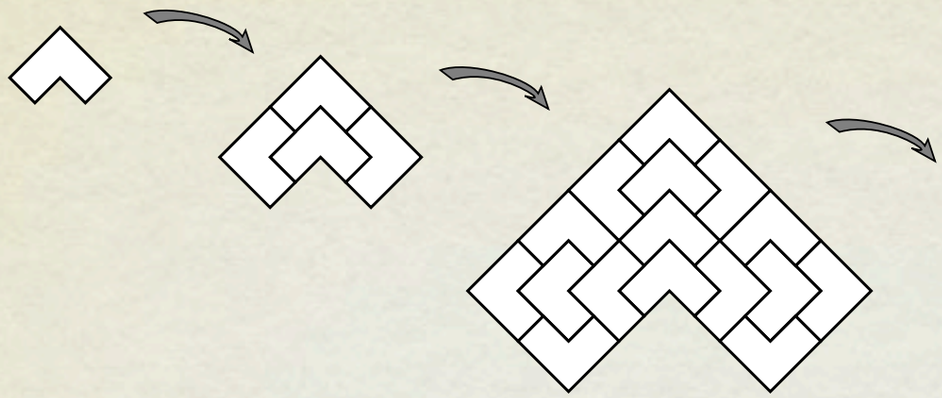
Inflation implies matching

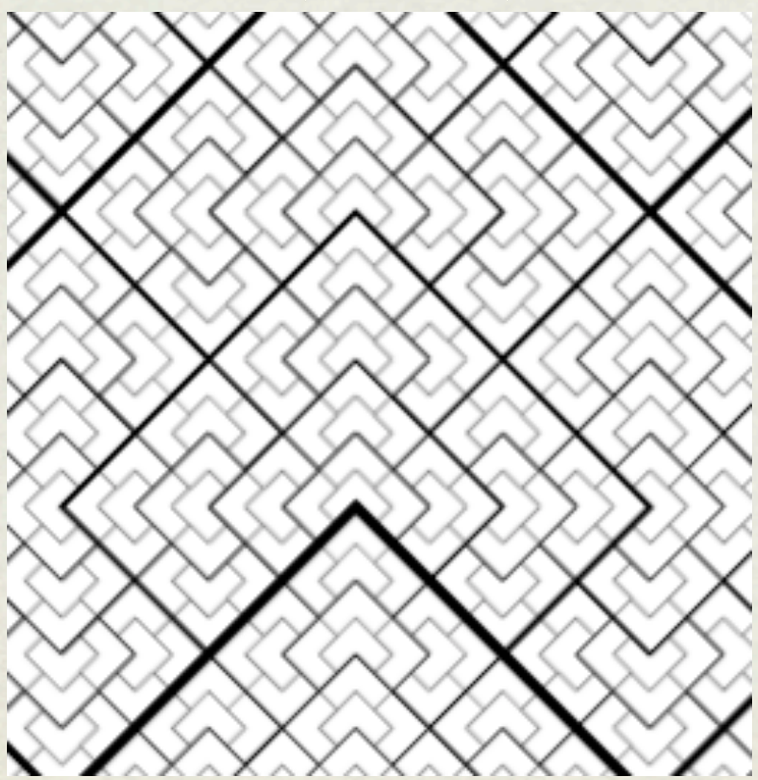
- **Theorem.** (S. Mozes, 1989) Any $\mathbf{Z}^2$ substitution can be "realized" as a Wang tiling (2-dim SFT).
- **Theorem.** (C. Radin, 1994 C. Goodman-Strauss 1998) Any substitution tiling can be "realized" as a tiling with matching rules.

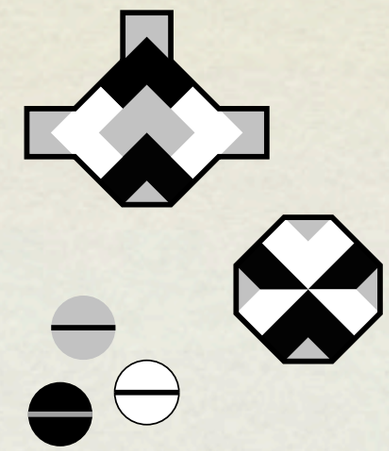
Comment 1: Nothing like this can happen in d=1!

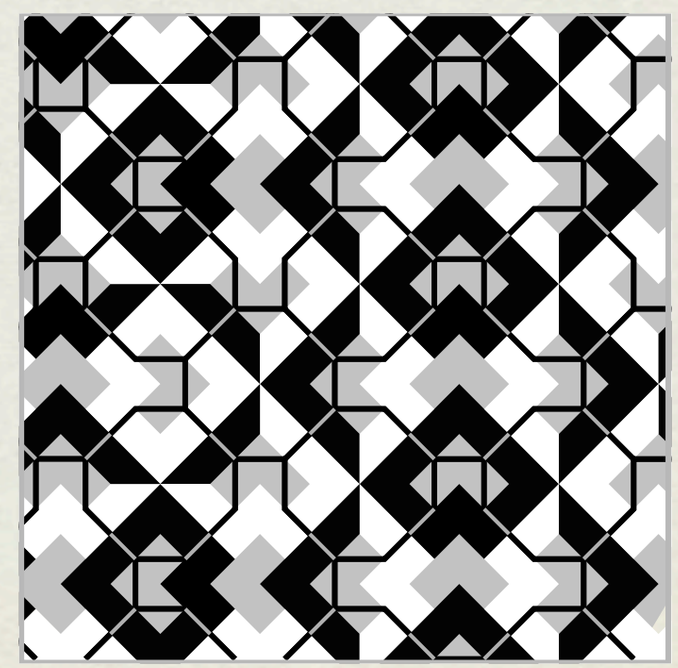
Comment 2: "Realized" means some tiles may need to be replaced by multiple **remarked** copies. So the result really says a substitution is *sofic*.

Matching rule for chair









Goodman-Strauss, Europe. J. Comb. **20**, 1999, 375-384

✤ Projection

Sturmian sequence: Morse & Hedlund, 1938

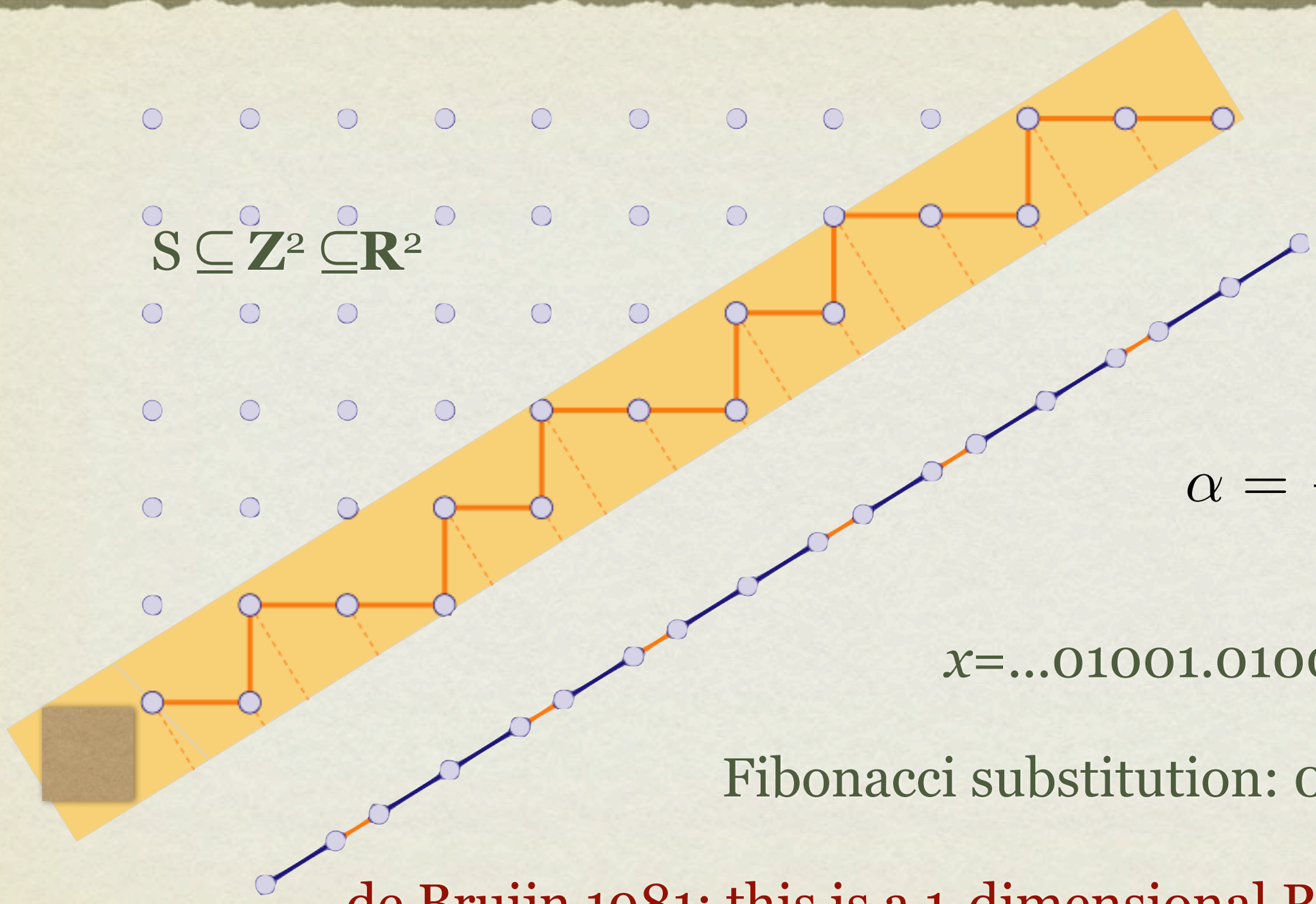
- Fix $0 \le \alpha < 1$ irrational, $0 \le \beta < 1$.
- **Sturmian seq.** $x_n = \lfloor (n+1)\alpha+\beta \rfloor - \lfloor n\alpha+\beta \rfloor \in X \subseteq \{0,1\}^{\mathbb{Z}}$

  (mechanical line, cutting sequence, Beatty sequence).

  $x = \ldots 0100101 00.100101001010 \ldots$

- Approximates line $y=\alpha x+\beta$.
  - ***Properties***: aperiodic, repetitive (minimal), uniquely ergodic, *infinite* type, discrete spectrum, $h=0$.
  - Only sometimes hierarchical, (always weakly hierarchical).

Mechanical Lines



$S \subseteq \mathbf{Z}^2 \subseteq \mathbf{R}^2$

$$\alpha = \frac{\sqrt{5}-1}{2}$$

x=...01001.010010100100...

Fibonacci substitution: 0→01, 1→0

de Bruijn 1981: this is a 1-dimensional Penrose tiling.

de Brujin's parameterization

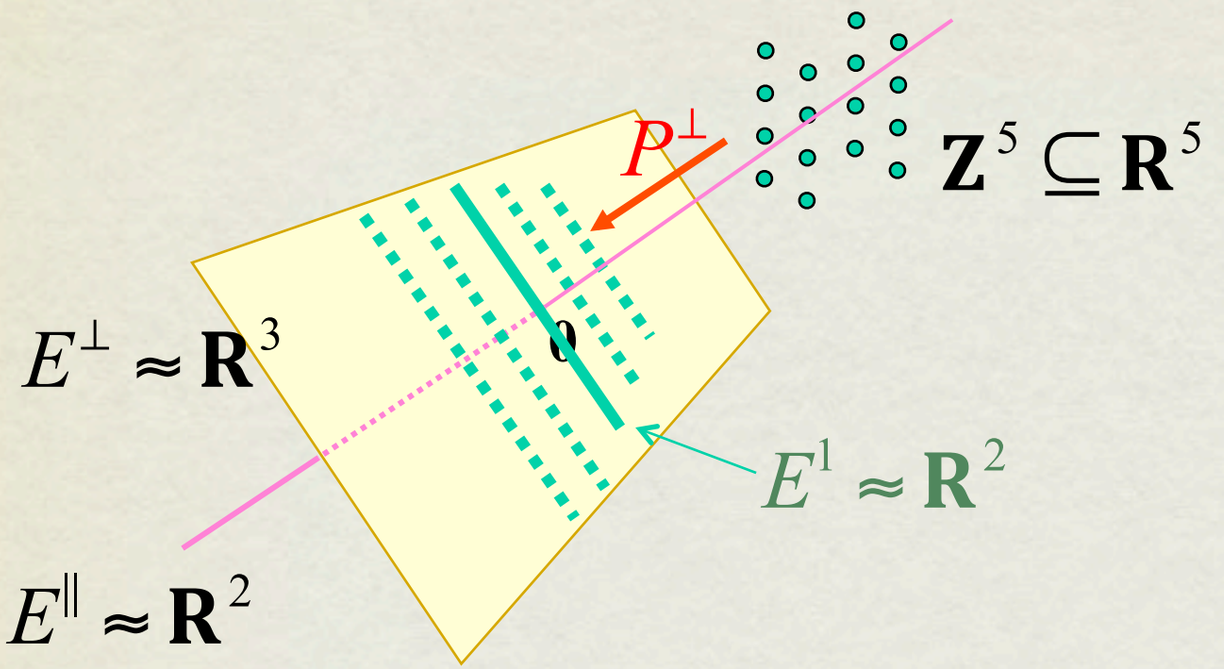
Theorem. (de Bruijn, 1981) Penrose tiles come from projecting a hypercubic slice of the lattice $\mathbf{Z}^5$ in $\mathbf{R}^5$ to a symmetric irrational affine plane $L+\mathbf{v}$, $L \subseteq \mathbf{R}^2$.

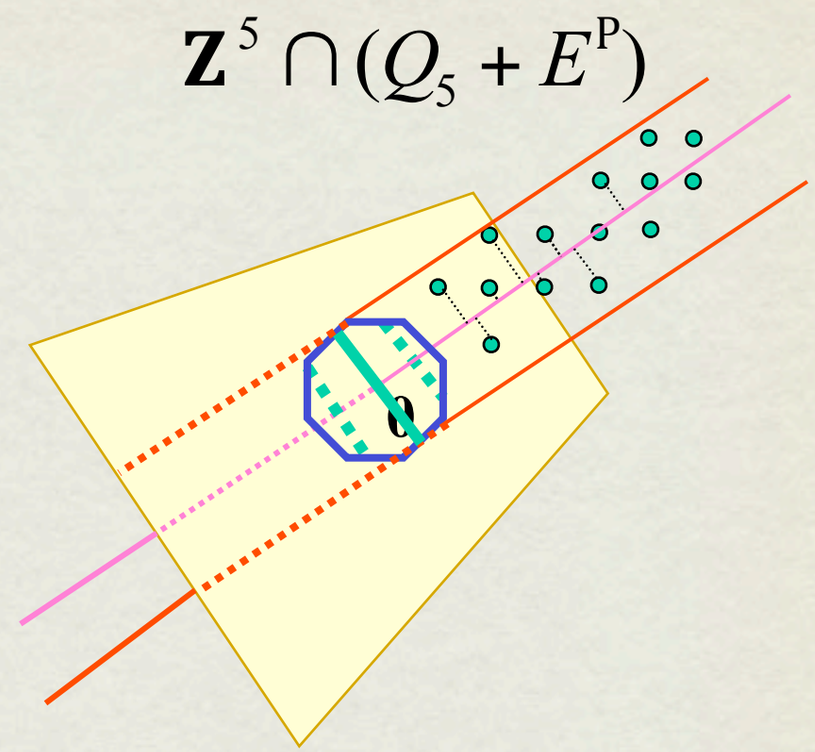
The set of allowed shifts $\mathbf{v}$ has codimension-4, so there is actually a 1-parameter family of tilings, only one of which is true Penrose tilings. Each can be obtained as a projection of $\mathbf{Z}^5$ in $\mathbf{R}^5$.

In particular, Penrose tilings are parameterized by $\mathbf{T}^4$ up to a "few" parameter values corresponding to more than one tiling.

de Brujin's parameterization-2

The geometry of de Bruijn's idea





Projection explains diffraction (see next topic) and (quasi-)symmetry. But maybe not all quasicrystals come from projection.

Rauzy Substitution

$0 \to 01,\ 1 \to 02,\ 2 \to 0$

u=0102010010200101020100102101 20101...

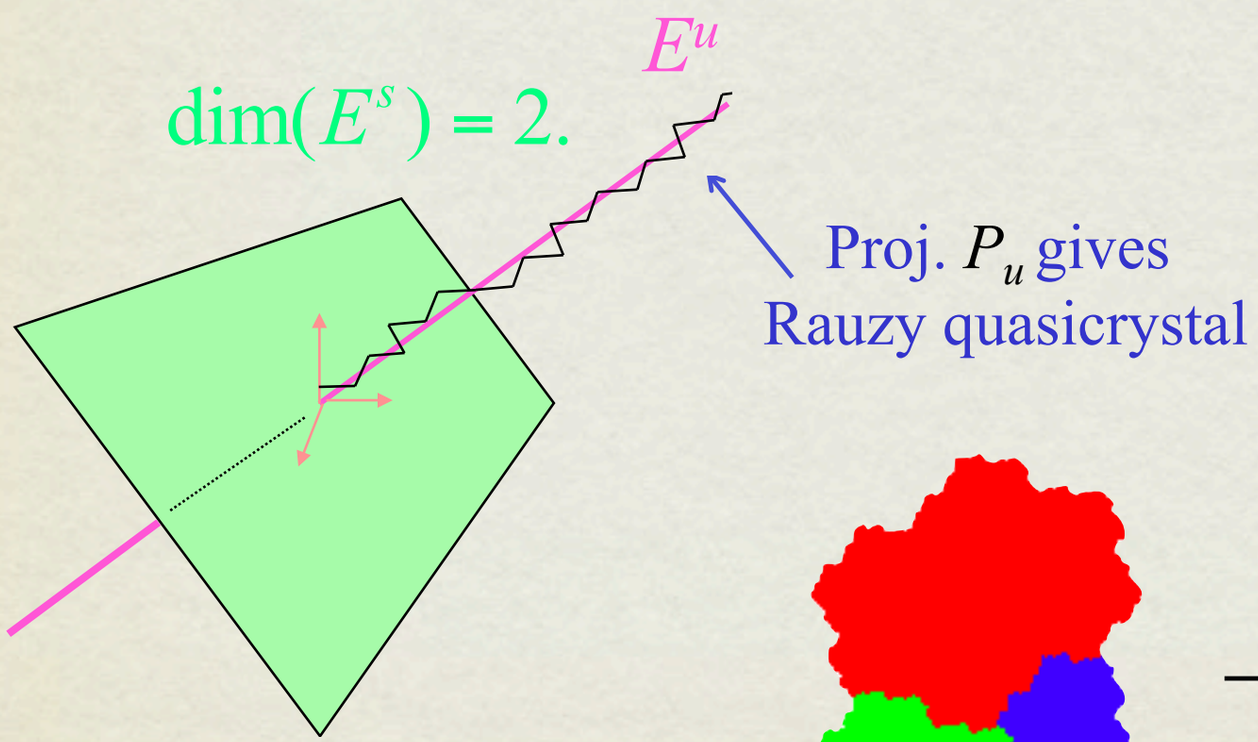
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

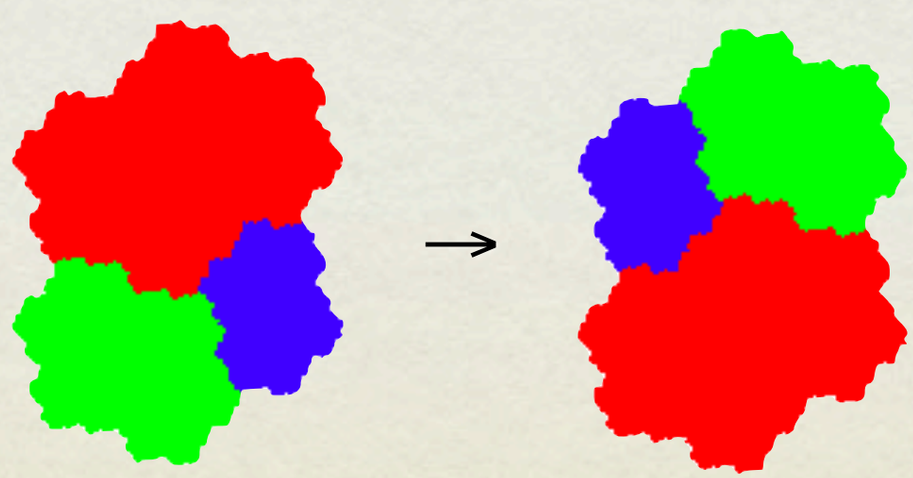
$z^3 = z^2 + z + 1$

Pisot property

$$\lambda = \frac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right) \approx 1.83929$$

$|\lambda'| = |a \pm ib| < 1$





T^3

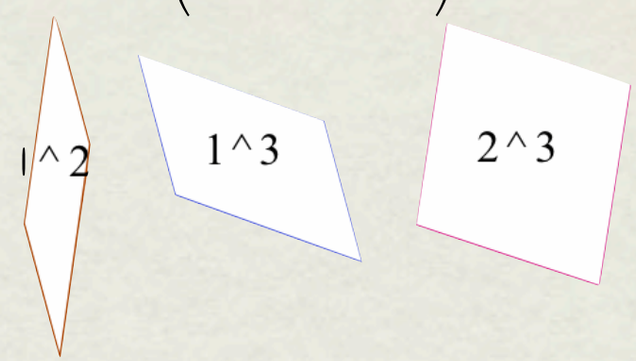
Gerard Rauzy, 1982

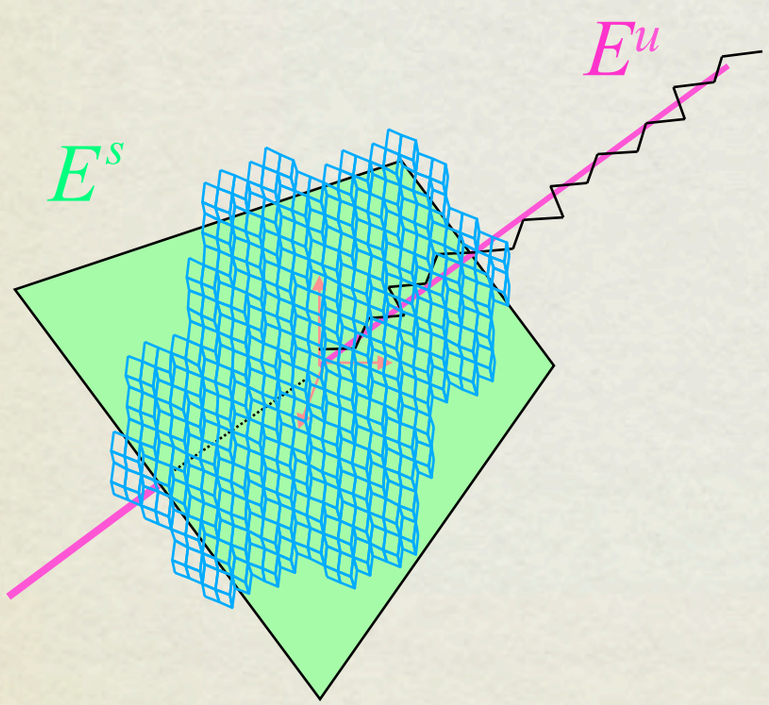
Rauzy Substitution

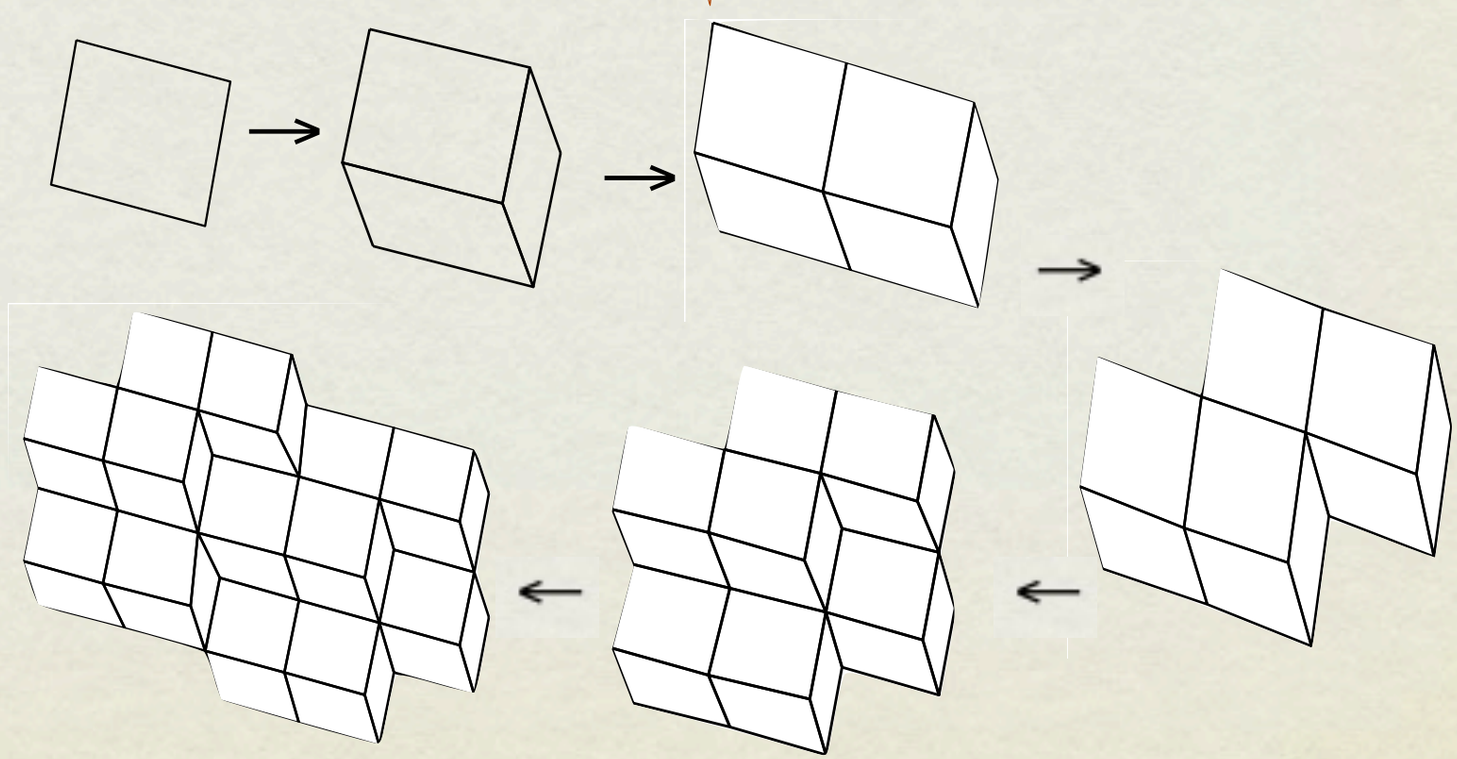
1→12, 2→13,3→1

u=121312112131212131211213121 3121...

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$







Gerard Rauzy, 1982

Pisot conjecture

- **Theorem.** (Dekking, 1977) A substitution of constant length has discrete spectrum if and only if it has a *coincidence*:

$0 \to 01$	$0 \to 01$
$1 \to 10$	$1 \to 00$
Morse	Period doubling

- **Pisot conjecture.** (Arnoux &Ito, 2001): An irreducible Pisot substitution has discrete spectrum.
  (Such a substitution automatically satisfies a generalization of Dekking's coincidence.)

Pisot conjecture-2

- **Example.** Fibonacci 0→01, 1→0 has **incidence matrix**

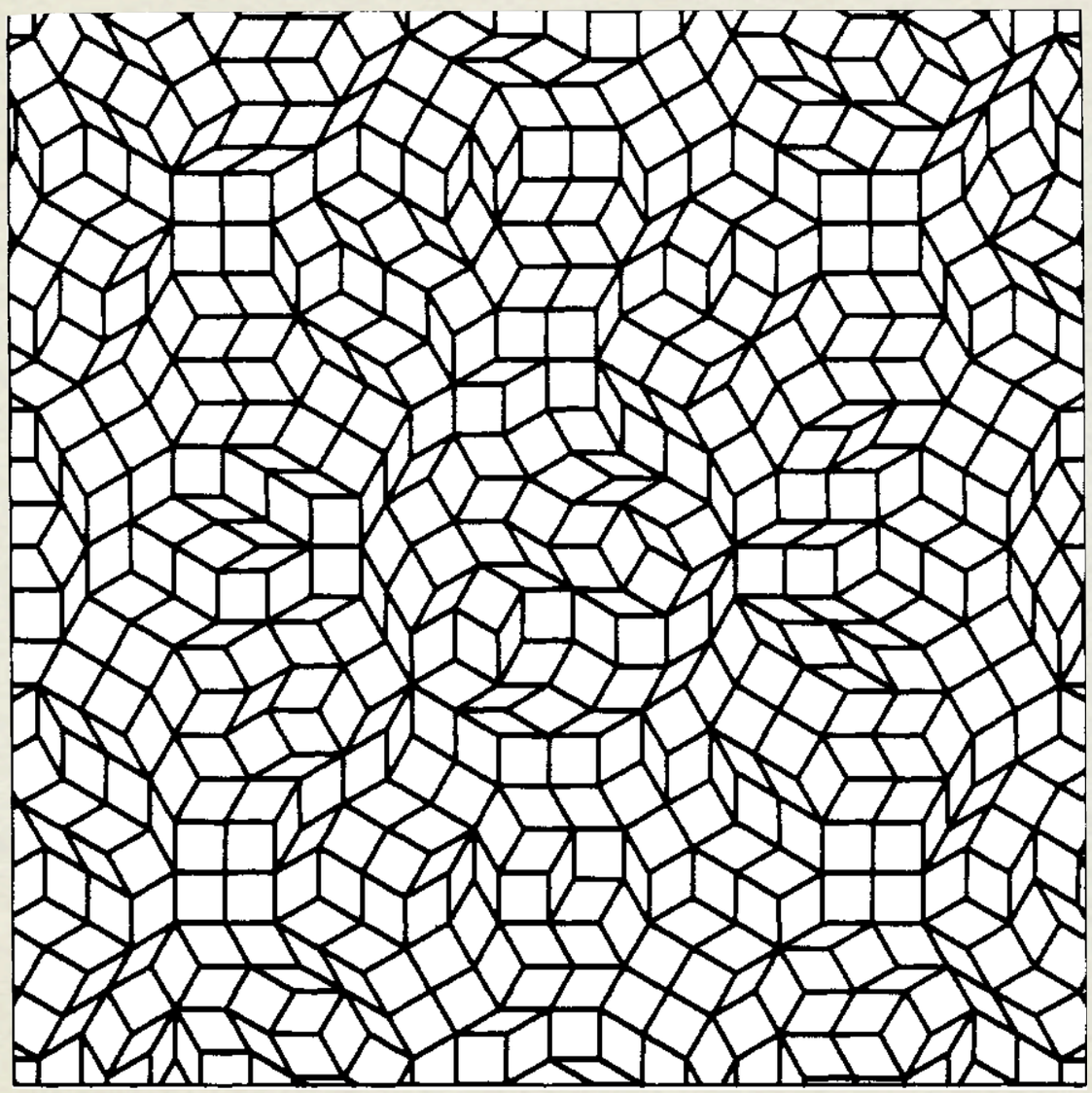
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \; p_A(\lambda) = \lambda^2 - \lambda - 1. \quad \lambda = \frac{1 \pm \sqrt{5}}{2} \approx 1.61, -.61$$

- **Pisot number.** Real algebraic integer > 1 with all conjugates < 1 in modulus. **Irreducible** means $p_A(x)$ irreducible.
- Pisot conjecture is **True** for substitutions on two symbols (Hollander & Solomyak 2003).
- Holds for Rauzy substitution 0→01, 1→02, 2→0 as well.
- Essentially says when a substitution is actually a projection.

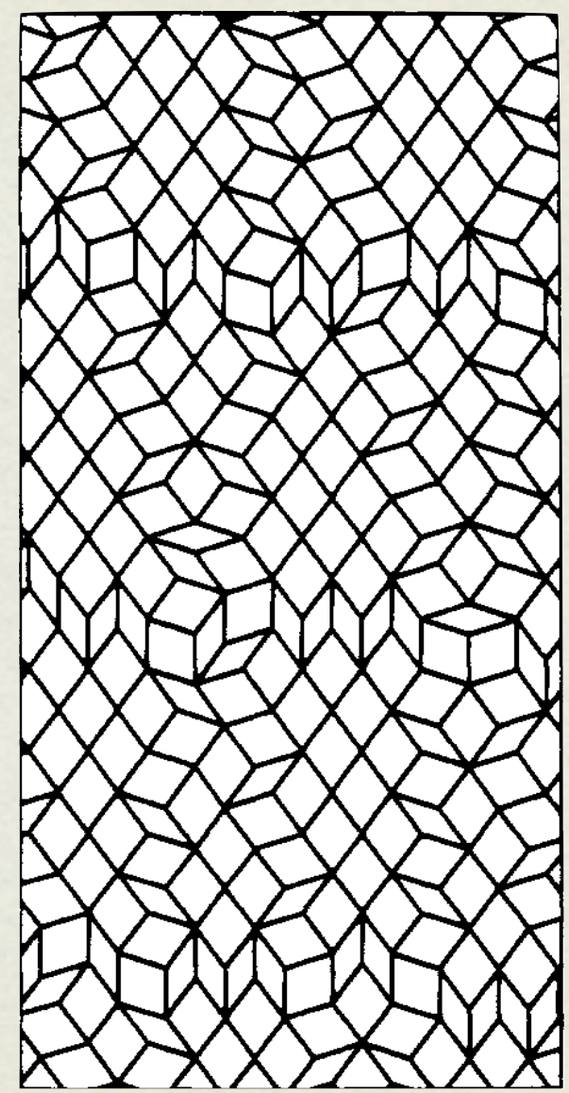
We will see later that discrete spectrum is related to diffraction.

Zoo of projection tilings

Gahler&Rayner, J. Phys. A, Math Gen. **19**, 1986



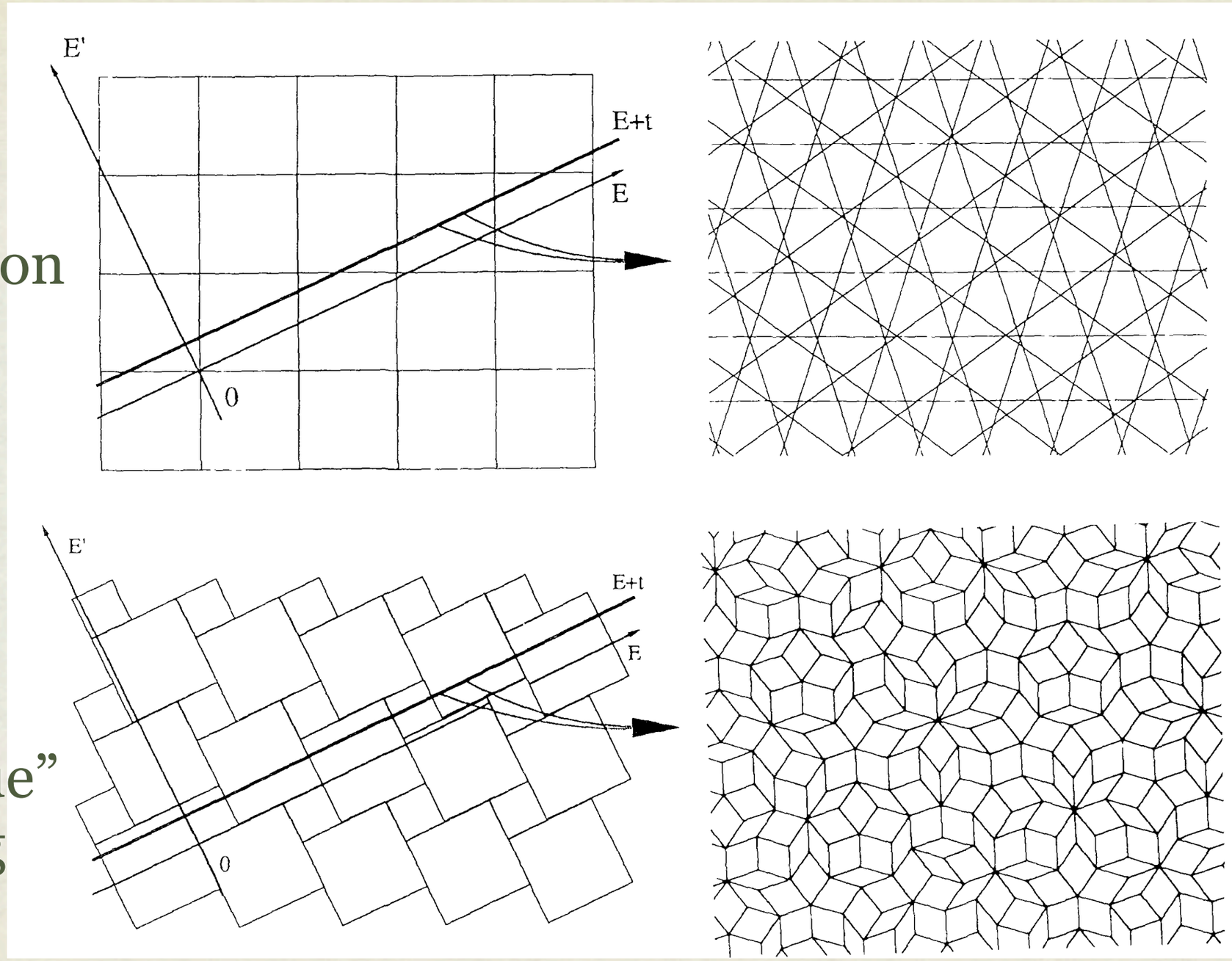
12-fold quasisymmetry



No rotational symmetry

Zoo of projection tilings

Projection

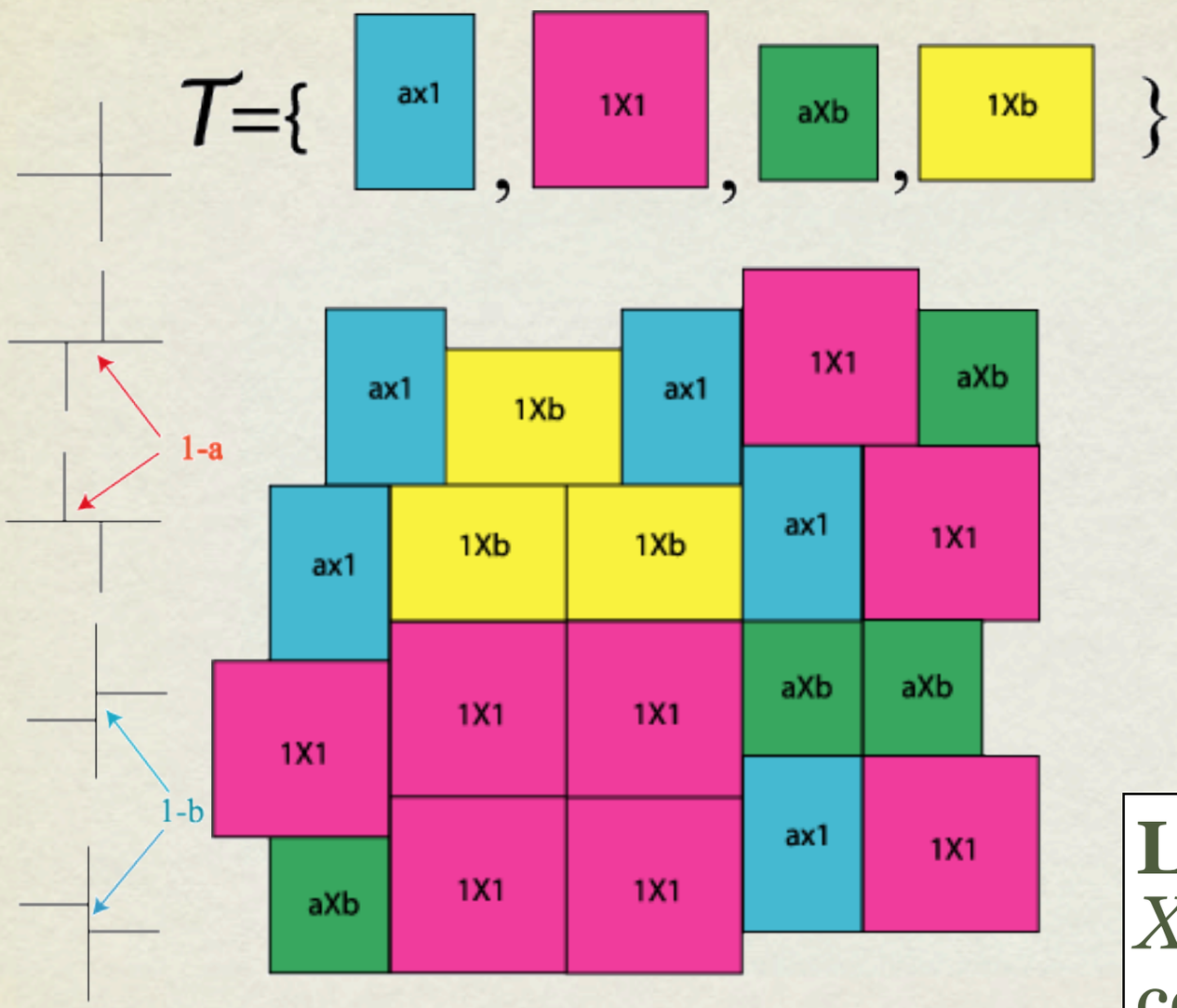


Dual to grid

"Oblique" tiling

Oguey, Duneau & Katz, Comm Math Phys **118** 1998

VI. TILING SPACES & DYNAMICS



Rudolph tiles of $\mathbf{R}^2$.

Tiling topology: x,y close if, after small translation, agree on large ball around 0.

Lemma: (Rudolph, 1988). Set X of all locally finite tilings is compact metric and translation action T is continuous.

Penrose & Chair tilings

Theorem. (RR, 1996) Penrose tiling is almost 1:1 extension of a rotation of $\mathbf{T}^4$. Spectrum $\mathbf{Z}[\xi]$, $\xi^5-1=0$. Inflation map realized as hyperbolic toral automorphism (tiles are unstable manifolds intersect Markov partition).

Theorem. (RR, 1999) Chair tiling is almost 1:1 extension of a rotation on $(\mathbf{Z}_2)^2$ ($\mathbf{Z}_2$ = 2-adic integers = "adding machine"). Spectrum $\mathbf{Z}[1/2]^2$.

Hyperbolic toral automorphism & Markov partition ⇒ tiling . But what do partition elements look like?

- Polygonal? Usually no: Bowen 1978, Cawley 1991.
- Connected? Yes in "Pisot" case: Furukado 2005, otherwise open

Model Sets-1 (algebraic part)

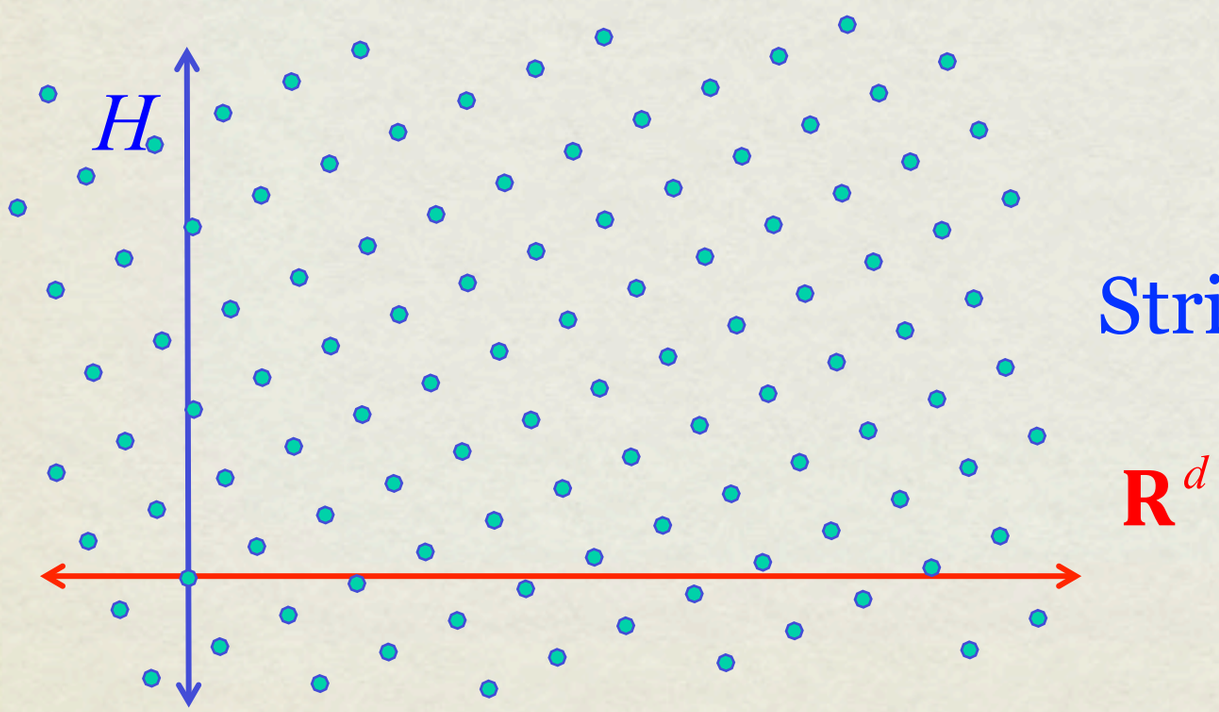
Cut and project scheme:

$$\mathbf{R}^d = \text{“physical” space}$$

$$H = \text{“internal” space}$$

Yves Meyer (1972), Hof (1998), Schlottman (1998), Moody (2002)

$$L \subseteq \mathbf{R}^d \oplus H = \text{lattice}$$



Kronecker action: $\mathbf{R}^d$

$$G = (\mathbf{R}^d \oplus H)/L \text{ compact}$$

Strictly ergodic, discrete spectrum.

Can get essentially any countable subgroup of $\mathbf{R}^d$ as spectrum RR, 2004

Model Sets-2 (Geometric part)

$W \subseteq H$ window

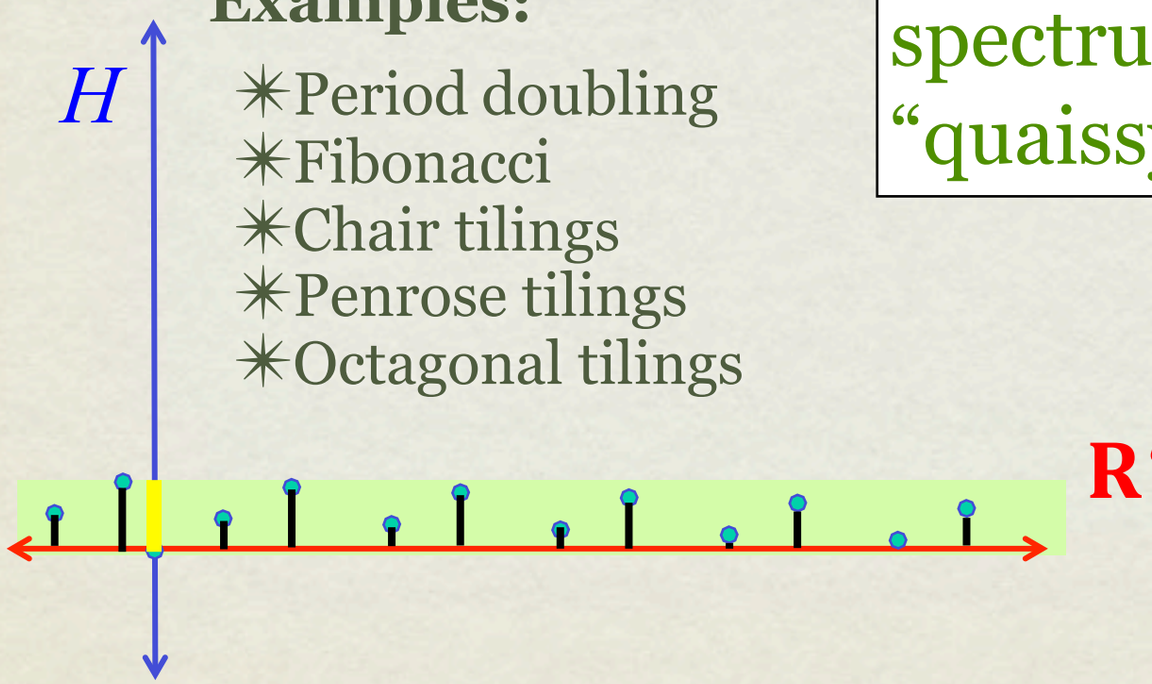
Regular:

$W = \overline{W^\circ}$

& $\mu_H(\partial W) = 0$

Examples:

- ✳Period doubling
- ✳Fibonacci
- ✳Chair tilings
- ✳Penrose tilings
- ✳Octagonal tilings



Model sets x are **Delone** sets and **Meyer** sets: x-x Delone.

Theorem. The model set shift T is a strictly ergodic almost 1:1 extension of R with dynamical & diffraction spectrum Σ. Any rotational "quaissymmetry" can be achieved.

An ***uncountable*** class of examples, mostly not substitutions and no matching rules (but conditions in some cases given by T. Le (1997), Julien (2010)).

Mixing Properties

- **Weak mixing** means $\Sigma=\{0\}$ (no discrete spectrum). False for model sets and projection tilings. Sometimes for substitutions and matching rules.

- **Mixing** means $\gamma_f\hat{}(\mathbf{v}) \to 0$ as $|\mathbf{v}| \to \infty$. False for model/projections and FLC substitutions. Unknown in other cases.

- **Topological mixing**. False for model/projection tilings. Sometimes for substitutions.

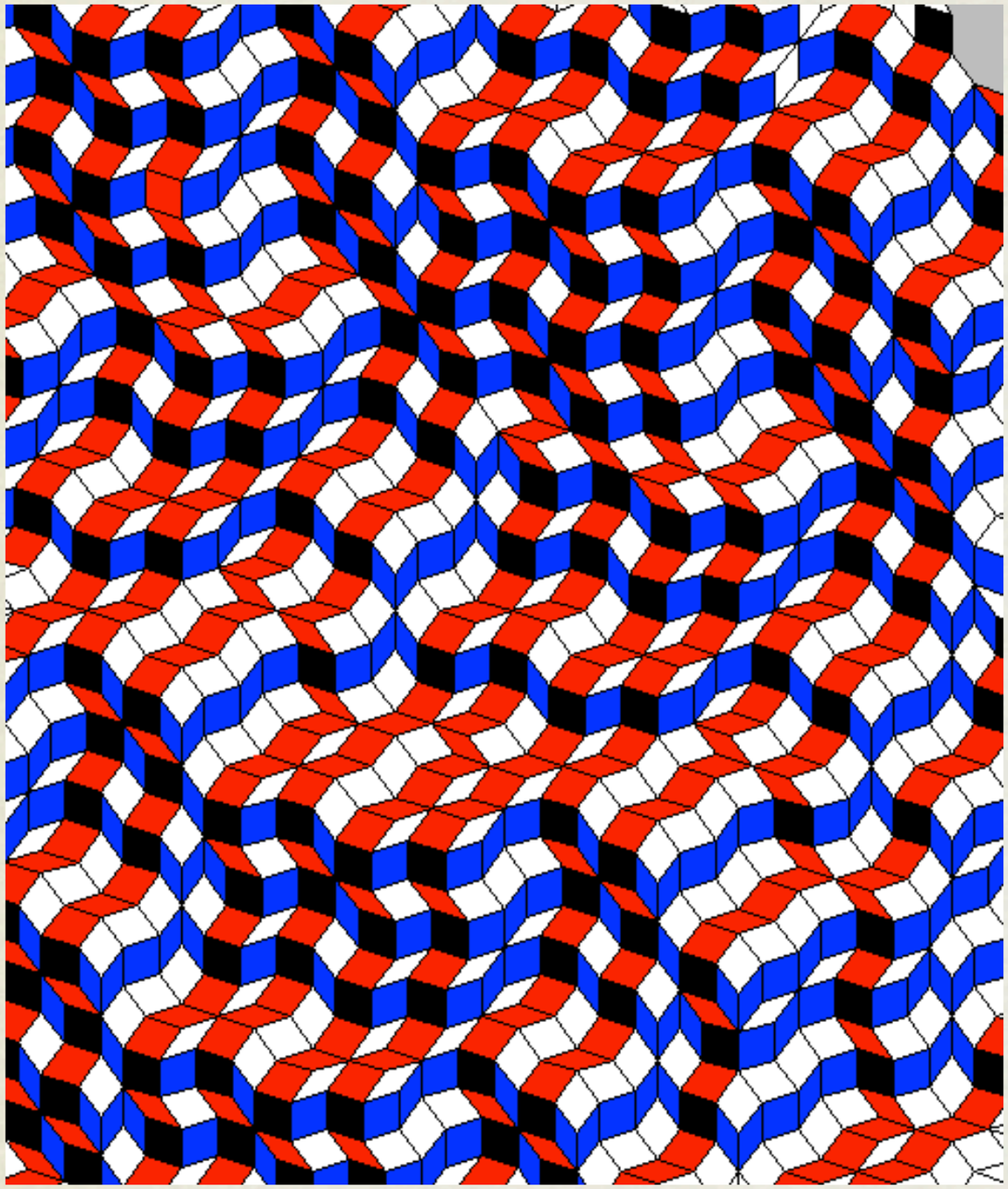
- **Mixed spectrum** (discrete & continuous or Lebesgue). Lebesgeue by itself?

Should these count as quasicrystals?

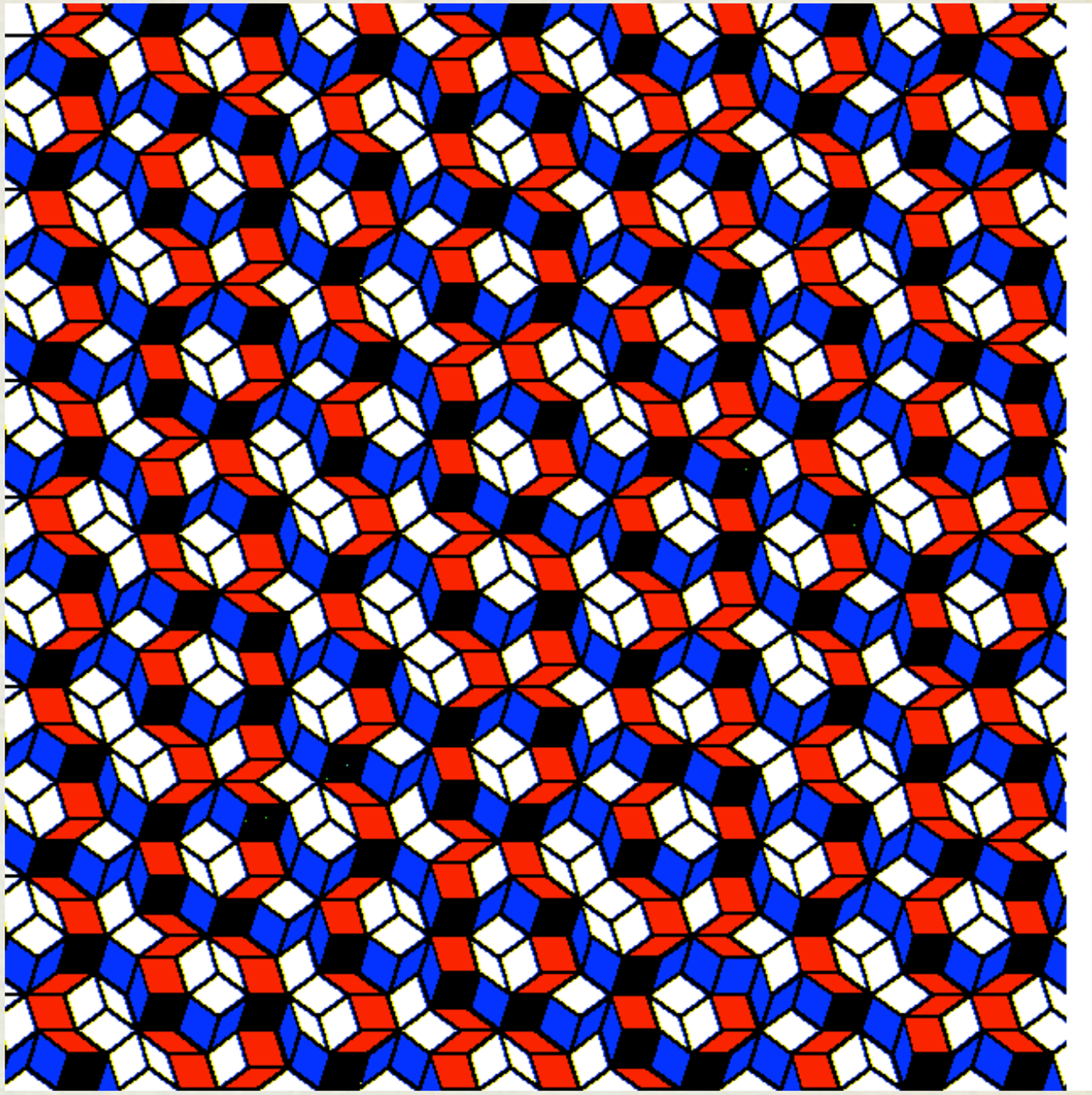
Weak mixing

- Substitution tilings (self-similar) with expansion $\lambda \in \mathbf{C}$.
- Say algebraic integer λ is **complex-Pisot** if $|\lambda|>1$, and all conjugates $\theta \neq \lambda, \overline{\lambda}$ satisfy $|\theta|<1$.
- **Theorem.** (Solomyak, 1997) T is weakly mixing iff λ is complex-Pisot.
- Complete characterization of discrete spectrum still open (the Pisot Conjecture in dimension d).
- Basically, this would say that all substitutions with discrete spectrum are model sets.

Visual comparison of discrete spectrum & weak mixing.



Binary: weakly mixing.



Penrose: discrete spectrum.

Entropy & Complexity

Complexity: C_n = #$n \times n$ patches, up to small translation.

Entropy: $h = \lim_{n\to\infty} \frac{\log C_n}{n^d}$.

- For substitution tiling with expansion L,

$$C_n \sim n^\alpha \text{ for } \alpha = \frac{\log|\det L|}{\log|\lambda_{\min}|} \text{ (Hansen,RR 2004).}$$

- For aperiodic model sets in $\mathbf{R}^d$ with H=$\mathbf{R}^e$ and an "almost canonical" window: $C_n \sim n^\alpha$ for $d \leq \alpha \leq de$ (Julien, 2010).

In both cases h =0.

More entropy

- 1-dimensional SFTs, if not finite sets, have $h>0$.
- Tiling spaces X with matching rules can have $h=0$ or $h>0$. Even if X is aperiodic, $h>0$ is possible.
- If X is a uniquely ergodic tiling space $C_n \leq K e^{\beta(d-1)}$.
- Again $h = 0$.

Tiling dynamical systems as symbolic dynamics

Tiling	Symbolic
Tiles	Symbols
Tiling topology	Product topology
Action	Shift map
Local derivability	Sliding block code
Mutual local derivability	Topological conjugacy
Periodic tiling	Periodic orbit

Geometric	Dynamical
Periodic	Transitive
Repetitive	Minimal
Patch statistics	Ergodicity
Patch growth	Entropy/ complexity
Diffraction	Discrete spectrum
Location independence	Mixing

VII. Spectral Measures

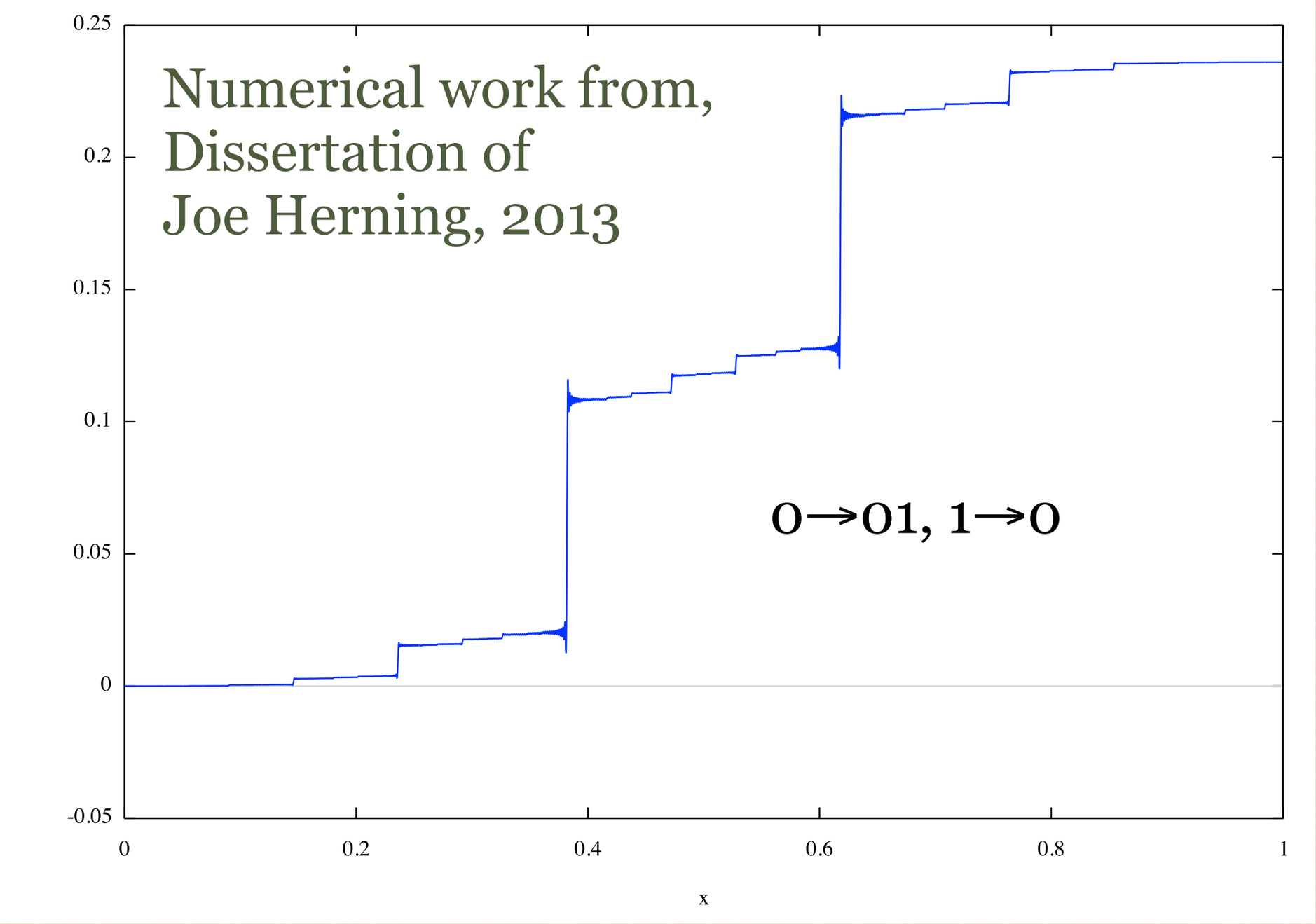
- $\mathbf{R}^d$ action T on tiling space X.
- Eigenvalue-eigenfunction $f(T^{\mathbf{v}}x) = e^{\mathbf{v}\cdot\omega} f(x)$.
- Spectral measure $\gamma_f(\mathbf{v})\hat{} = \langle f \circ T^{\mathbf{v}}, f\rangle$ (positive definite). Choose f so γ_f is maximal. Dynamical spectrum.
- Lebesgue decomposition: $\gamma_f = \delta_0 + \gamma_{\mathrm{dis}} + \gamma_{\mathrm{sc}} + \gamma_{\mathrm{ac}}$
- How is this related to diffraction spectrum σ?

Theorem. (S. Dworkin, 1993) $\sigma \ll \gamma_f$. Diffraction is only possible if T has nontrivial discrete spectrum.

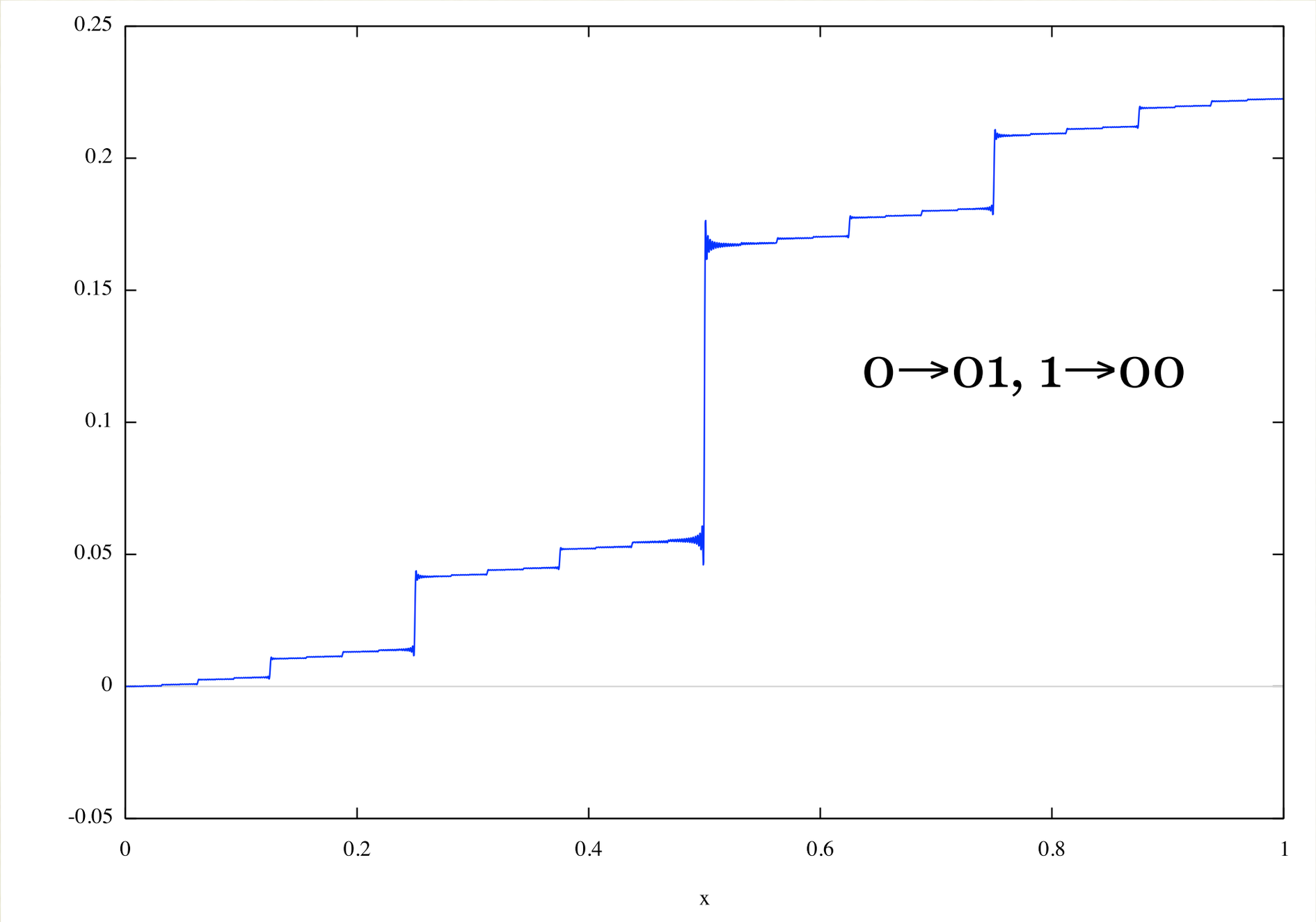
Some examples

Tiling	*d*	*Discrete*	*Singular*	*Lebesgue*
Period doubling	1	$\mathbb{Z}[1/2]$	**No**	**No**
Morse	1	$\mathbb{Z}[1/2]$	**Yes**	**No**
Rudin-Shapiro	1	$\mathbb{Z}[1/2]$	**No**	**Yes**
Fibonacci	1	$\mathbb{Z}[\lambda] \cong \mathbb{Z}^2$	**No**	**No**
Penrose	2	$\mathbb{Z}[\xi_5] \cong \mathbb{Z}^5$	**No**	**No**
Binary	2	**No**	**Yes**	**No**
Chair	2	$\mathbb{Z}[1/2]^2$	**No**	**No**
Table	2	$\mathbb{Z}[1/2]^2$	**Yes**	**No**
Pinwheel	2	**No**	**?**	**?**

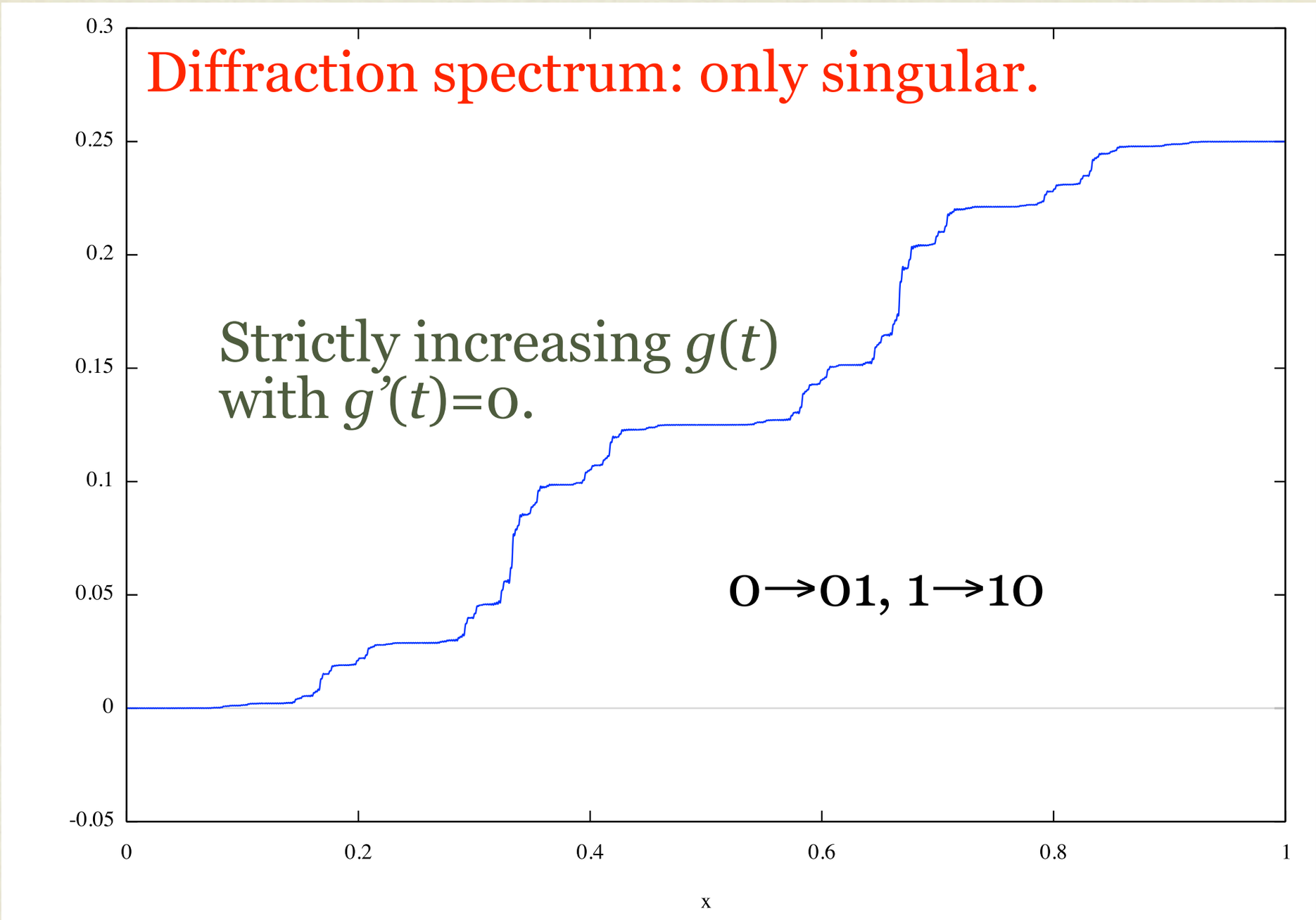
Fibonacci (1-dim. Penrose)



Period doubling



Morse Sequence



Morse Sequence & period doubling

Morse: 0→01, 1→10

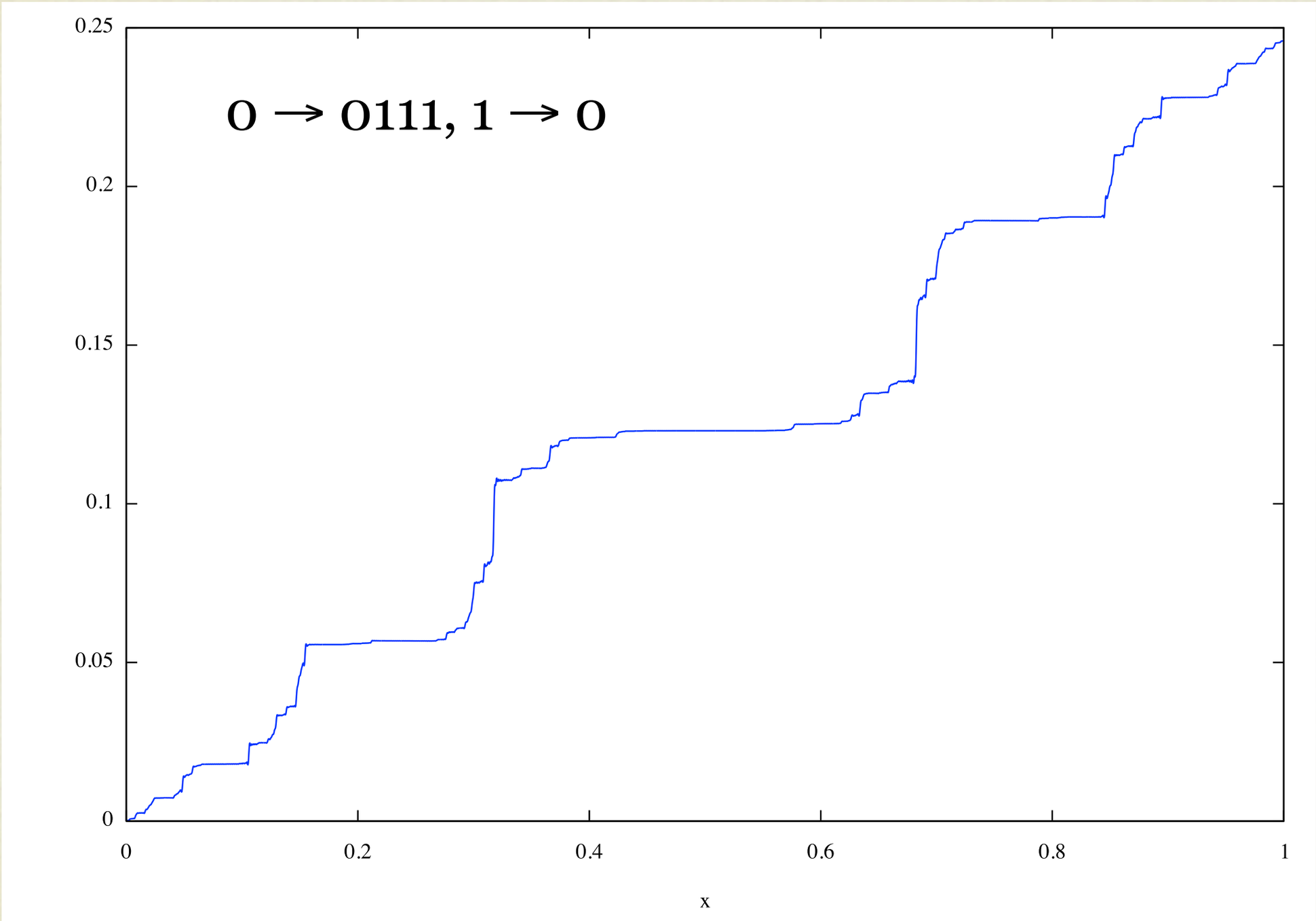
u **= 01101001100101101001011001101001...**

2:1 factor map: $v_i = -(u_i + u_{i+1}) \bmod 1$

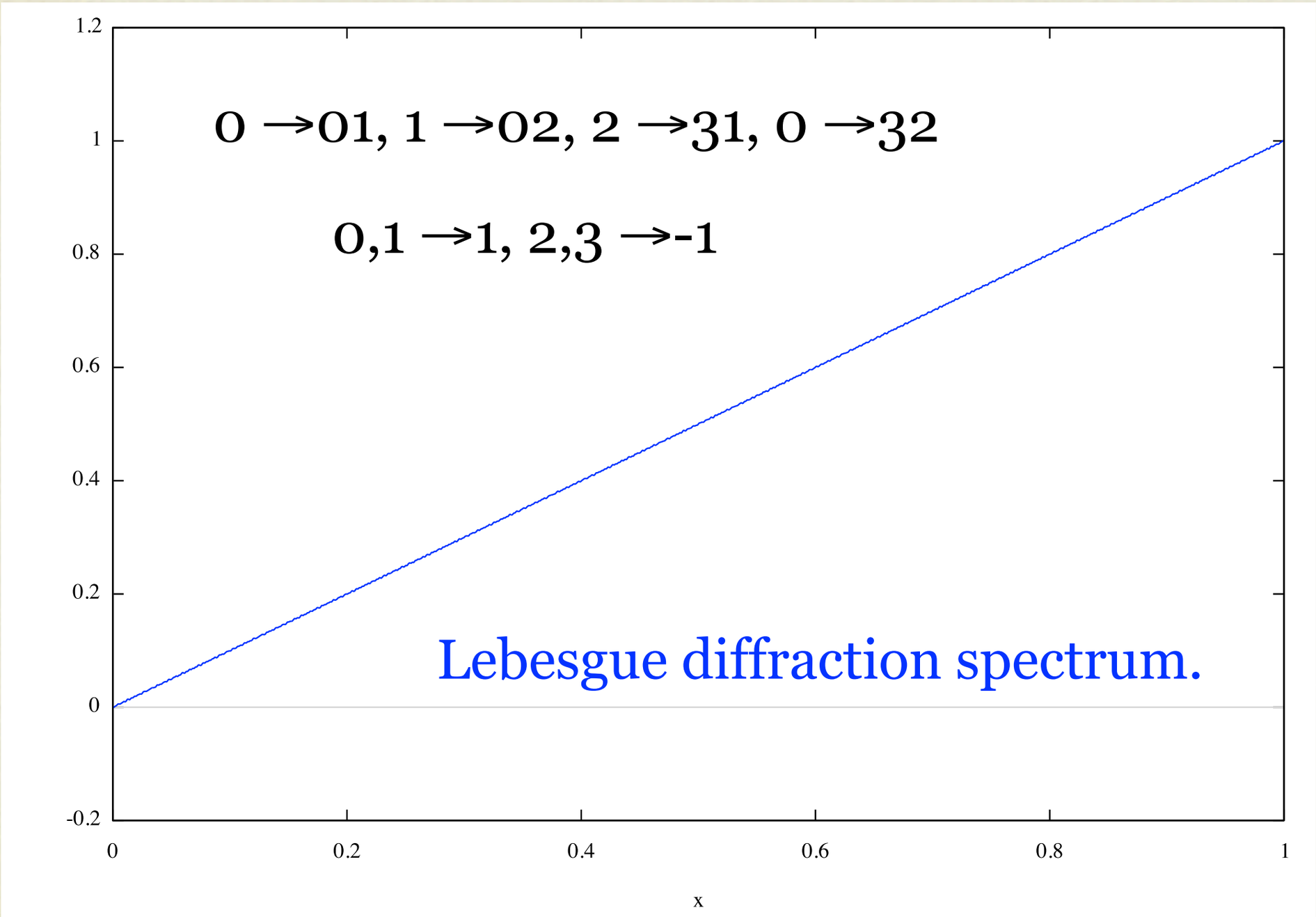
v **= 0100010101000100010001010100010 1....**

Period doubling: 0→01, 1→00

Mixing



Rudin-Shapiro

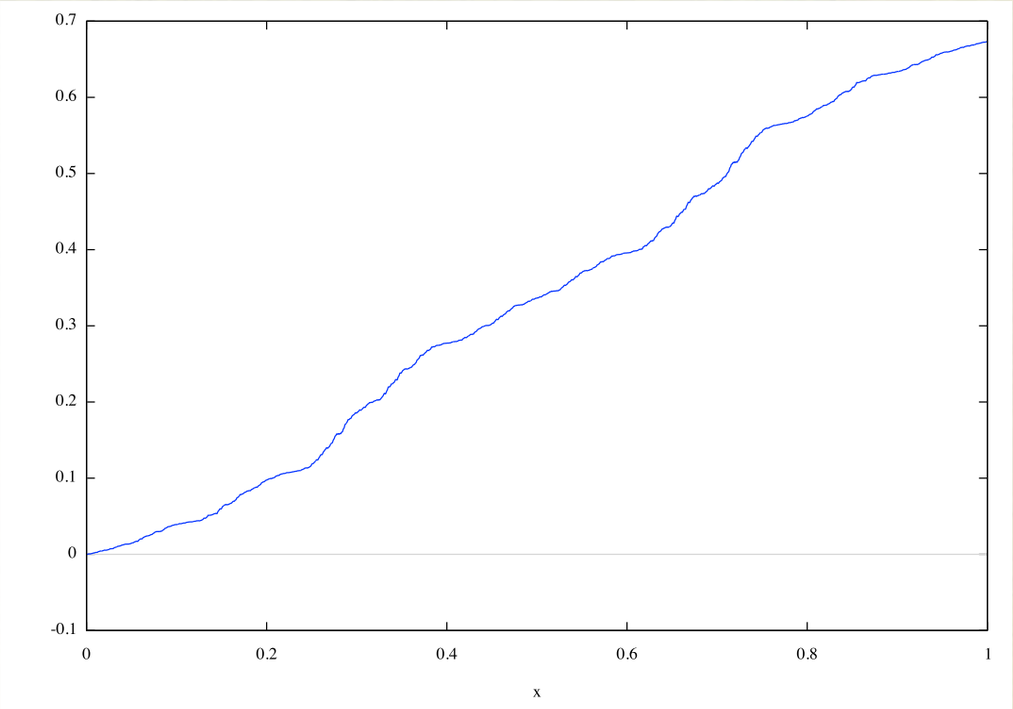


Seeing the discrete spectrum?

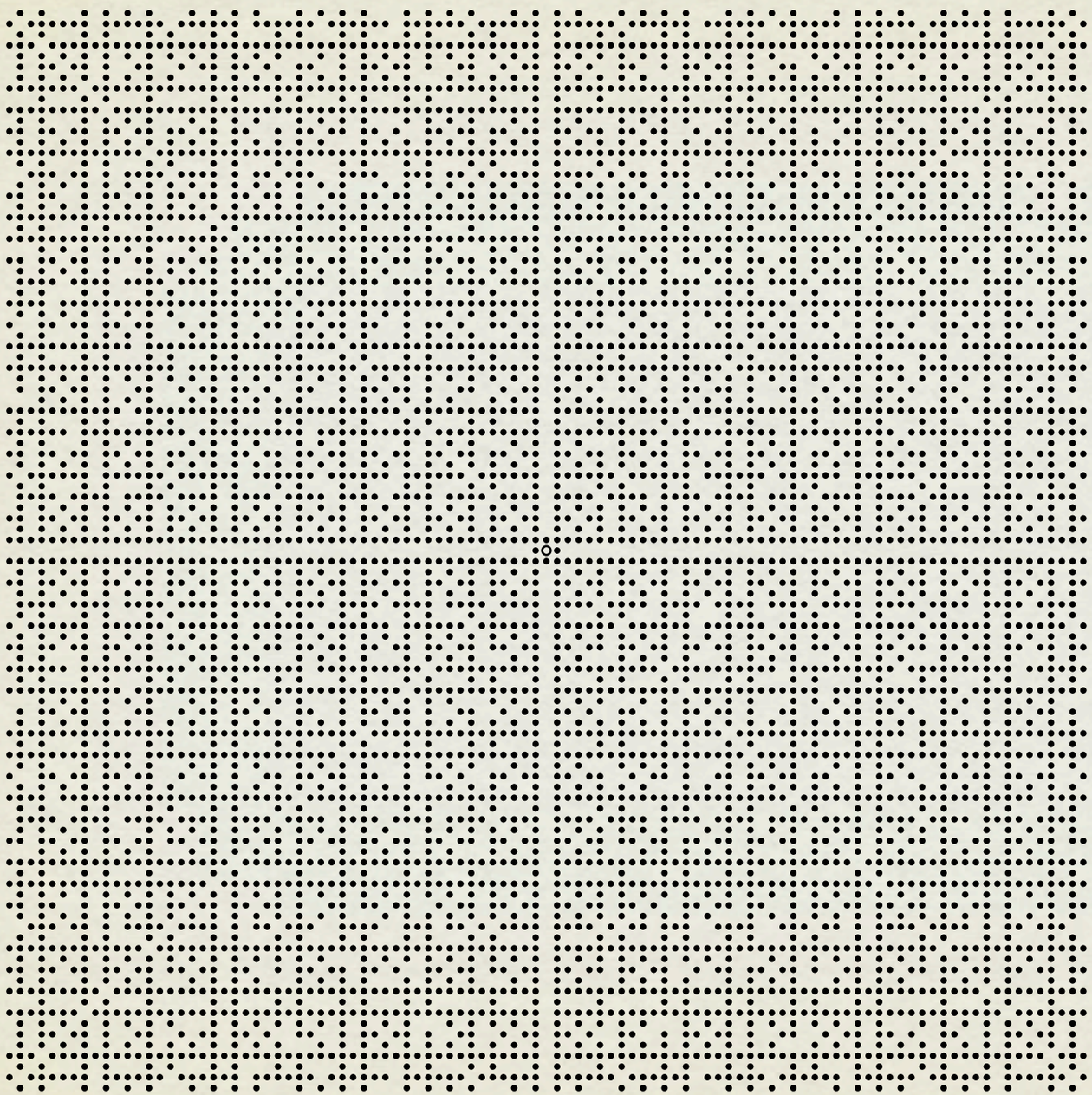
Conjecture. (Michael Baake, 2010) Every substitution has a substitution factor isomorphic to its discrete spectrum part.

Theorem. (Joe Herning, 2013 Dissertation) The substitution 0→00120, 1→21011, 2→12202 has no discrete spectrum substitution factor.

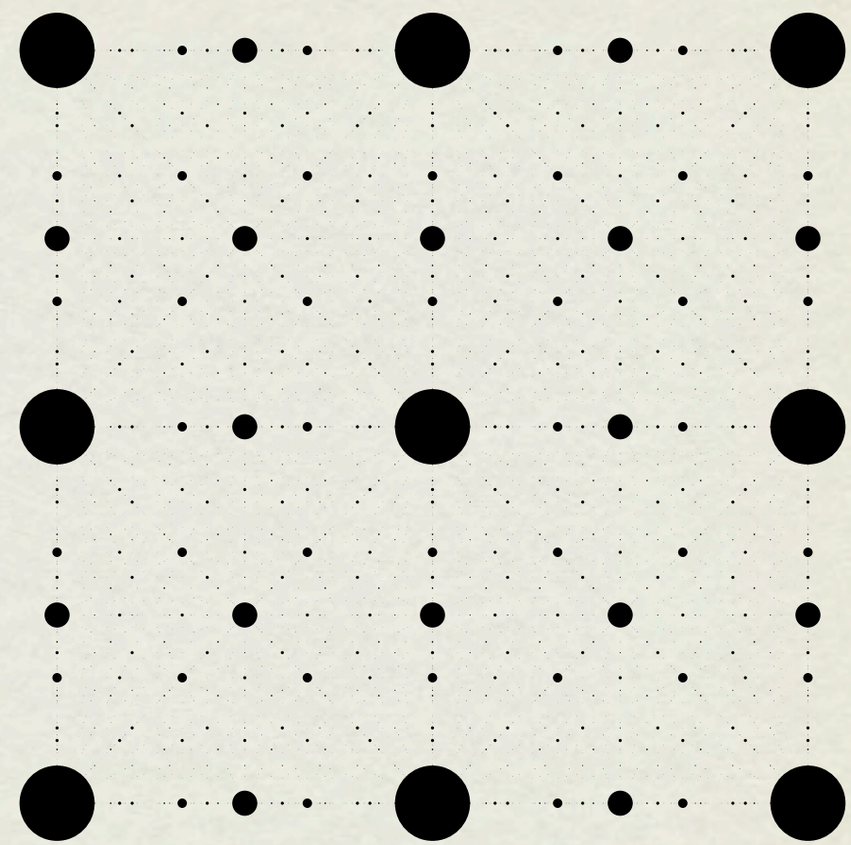
Note. The discrete part of the spectrum is **Z**[1/5]. It cannot be seen in any factors.



Visible Points

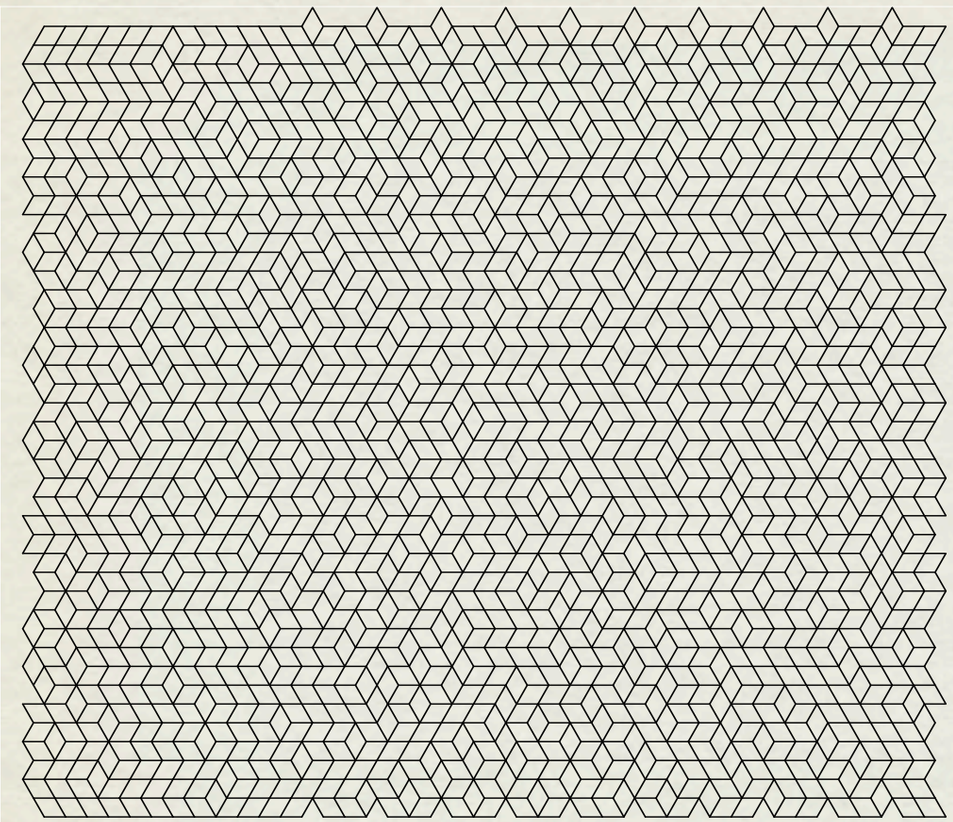


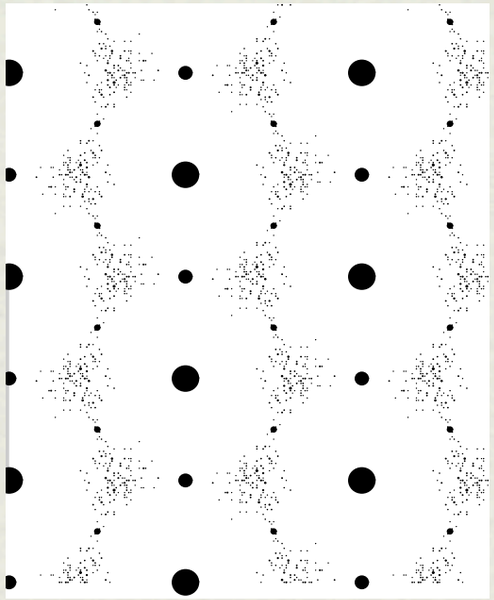
Baake & Grimm, 2013



Random lozenge tiling

Baake & Grimm, 2013





Note: The discrete spectrum comes from the underlying triangular lattice (a classical crystal).

VII. Algebraic topology

- A tiling space (of FLC tilings) is a ***lamination***: neighborhoods are homeomorphic to $\mathbf{R}^d \times C$. The space is a d-dimensional, connected compact metric space.
- The Čech cohomology $\check{H}^*(X)$ of X is well defined. FLC tiling spaces are all inverse limits of CW-complexes (easy to see in case of inflation), and are all $\mathbf{T}^d$ fibered spaces (Sadun and Williams, 2005)
- Cohomology first computed by Anderson & Putnam, Gahler
- Non FLC. Cohomology not yet well understood (but see Frank & Sadun)

Čech cohomology

Tiling	d	$\check{H}^0(X)$	$\check{H}^1(X)$	$\check{H}^2(X)$	$\check{H}^3(X)$
Morse	1	$\mathbf{Z}^1$	$\mathbf{Z}[1/2]\oplus\mathbf{Z}$	0	0
Fibonacci	1	$\mathbf{Z}^1$	$\mathbf{Z}^2$	0	0
Penrose	2	$\mathbf{Z}^1$	$\mathbf{Z}^5$	$\mathbf{Z}^8$	0
Octagonal (remarked)	2	$\mathbf{Z}^1$	$\mathbf{Z}^8$	$\mathbf{Z}^{23}$	0
Chair	2	$\mathbf{Z}^1$	$\mathbf{Z}[1/2]^2$	$\mathbf{Z}[1/4]\oplus\mathbf{Z}[1/2]^2$	0

Gap Labeling

- $\check{H}^1(X)$ is related to point spectrum (Andress & RR 2012).
- $\check{H}^1(X)$ is related to $K_0(C(X)\ltimes_T \mathbf{R}^d)$. This has a (unique) trace τ (unique ergodicity).
- The group $\Gamma=\tau(K_0(C(X)\ltimes_T \mathbf{R}^d)) \subseteq \mathbf{R}$ is called the **gap labeling group.** It is related to Schrödinger equation with x as potential.
- Define **frequency module** Φ the group generated by the patch frequencies.

Gap Labeling theorem. $\Gamma=\Phi$. (Conjectured Bellisard 1992, proved independently by three groups 2004).

Schrödinger Equations

Schrödinger operator: self-adjoint

$$(Hu)_n := u_{n+1} + u_{n-1} - v_n u_n$$

$$v_n = f(x_n)$$

$$\ldots x_{-1}.x_0 x_1 x_2 \cdots \in X$$

with substitution (quasicrystal) potential

$$u_n \in \ell^2(\mathbb{Z}) \qquad Hu = \lambda u$$

For what $\lambda \in \mathbf{R}$ is there solution?

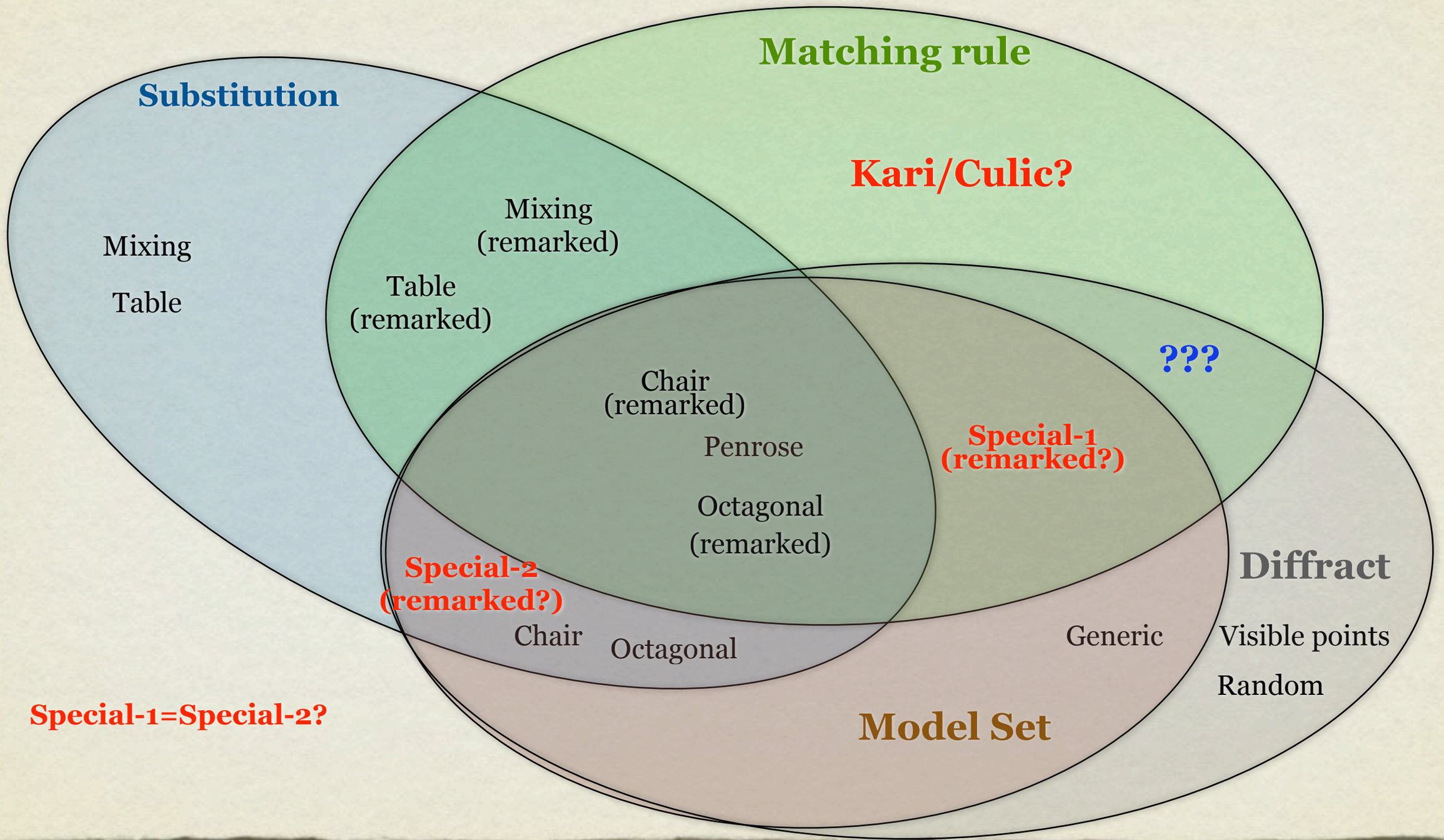
More generally study the spectrum Σ of H.

In this case, $\Sigma \subseteq \mathbf{R}$ is a Cantor set, and the lengths of the gaps (removed intervals) come from the frequency module Φ.

Note: Previously, only discrete spectrum and AC spectrum were observed.

(Bellisard, Damanik, Simon).

Summary


Substitution
Matching rule
Kari/Culic?
Mixing
Table
Mixing
(remarked)
Table
(remarked)
???
Chair
(remarked)
Penrose
Special-1
(remarked?)
Octagonal
(remarked)
Special-2
(remarked?)
Diffract
Chair
Octagonal
Generic
Visible points
Random
Special-1=Special-2?
Model Set

THE END

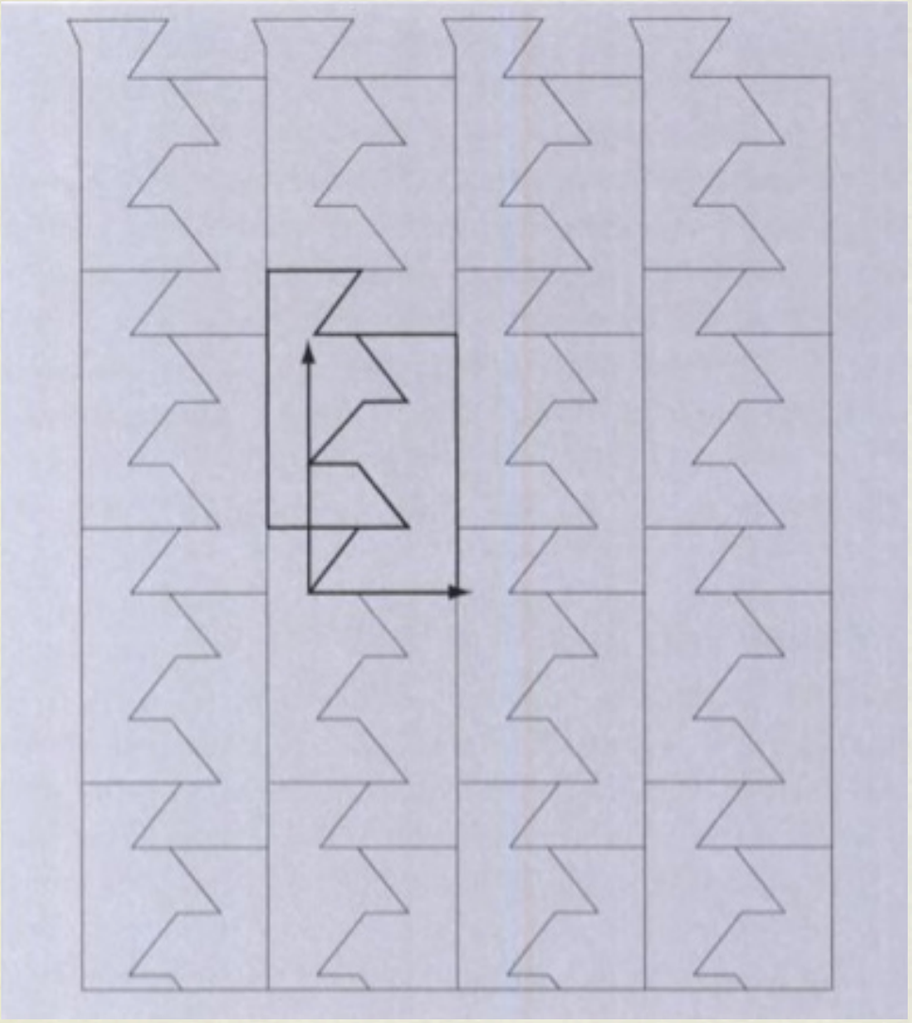
Dan Schectman in Stockholm 2011

APPENDIX

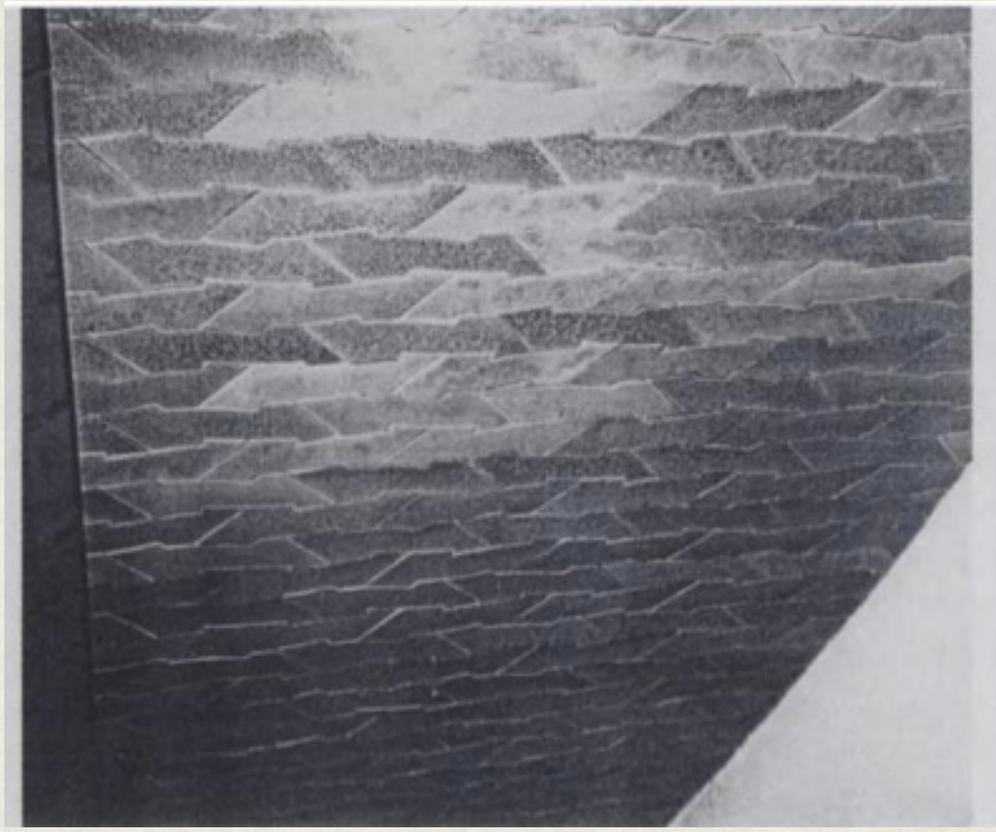
- Parts 2 & 3 of Hilbert's 18th problem.
- Algorithm for 2-dimensional space groups.
- Penrose's derivation of Penrose tilings
- Architectural quasicrystals
- Quasicrystals in industry & nature

Hilbert's 18th, part 2:

No. Reinhardt 1928 (n=3), Heesch (n=2) 1935



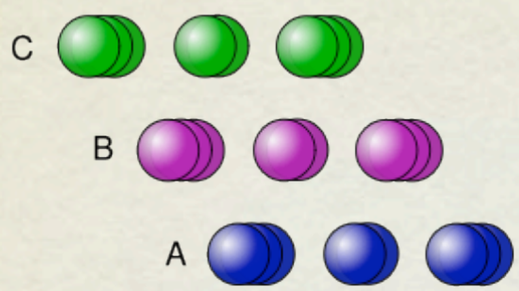
Heesch: $S(X)$ action not transitive on tiles.

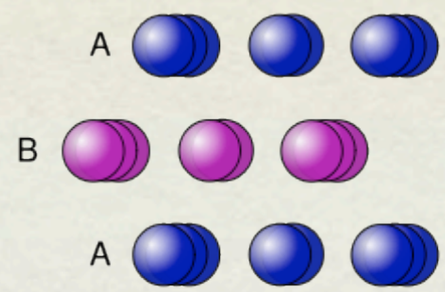


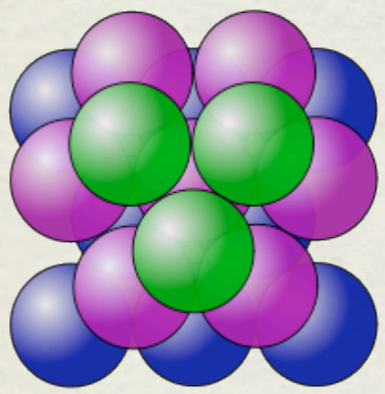
Ceiling of Gottingen City Hall.

From: Jeremy Gray, *The Hilbert Challenge*, Oxford University Press 2001

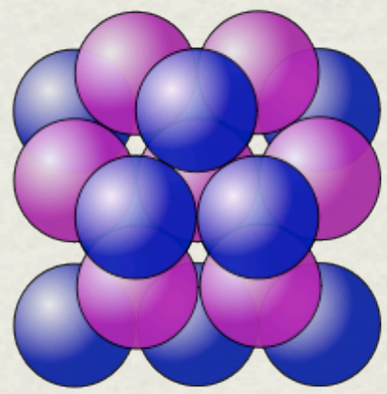
Hilbert's 18th, part 3.







Cubique à faces centrées ABC



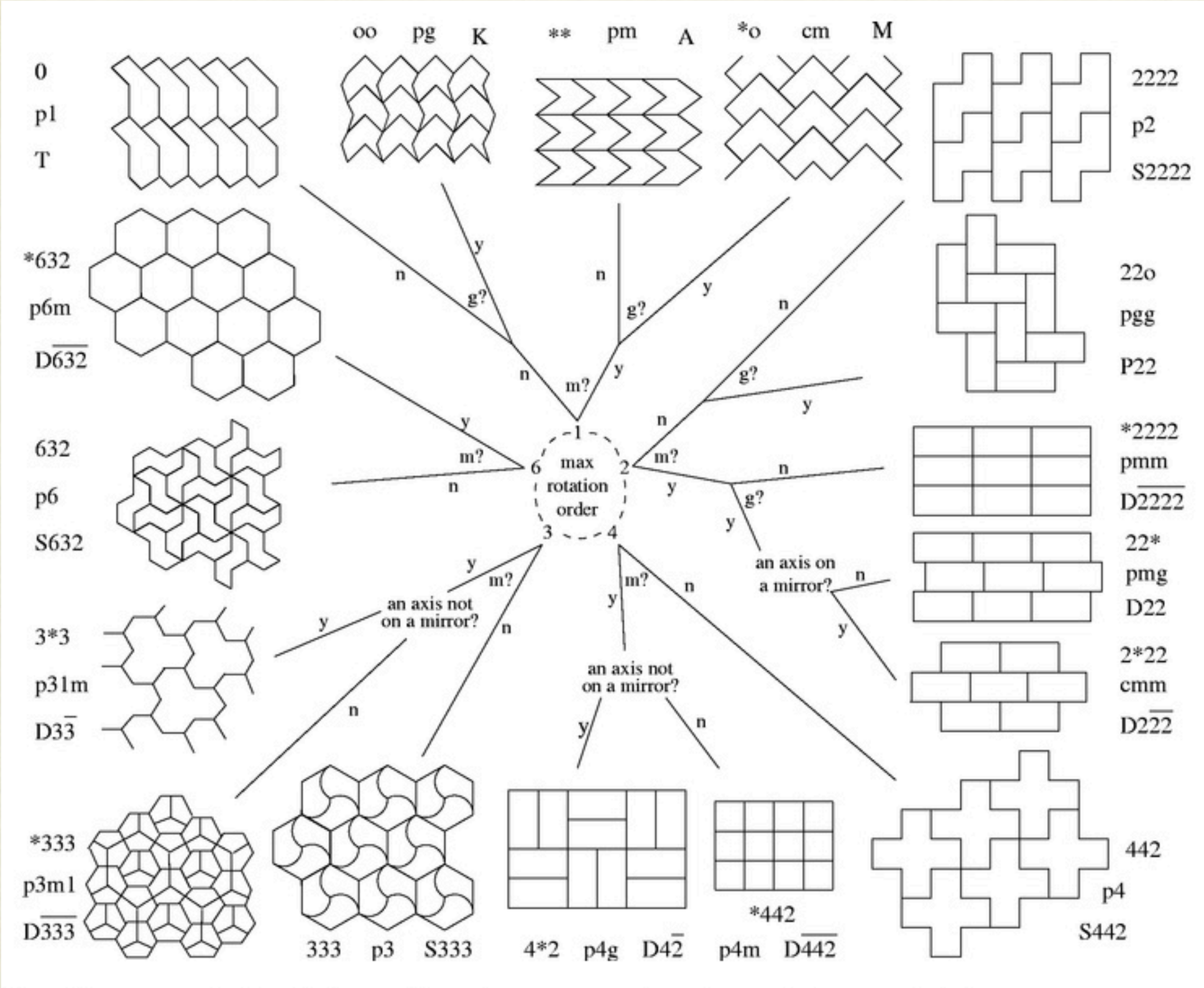
Hexagonal compact ABA

Kepler conjecture,
Johannes Kepler, 1611.

$$D = \frac{\pi}{18} \approx 0.74048$$

http://en.wikipedia.org/wiki/Kepler_conjecture

Proved by ThomasHales 1998.

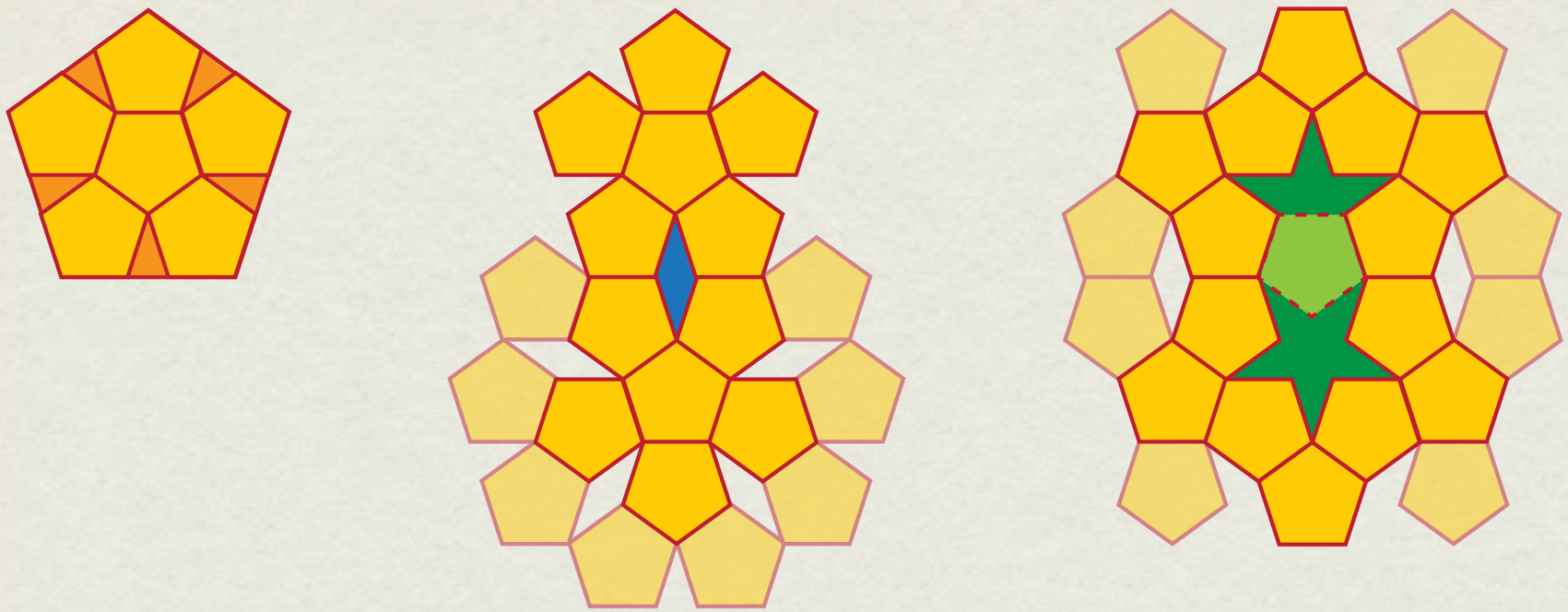


Brian Sanderson's Pattern Recognition Algorithm

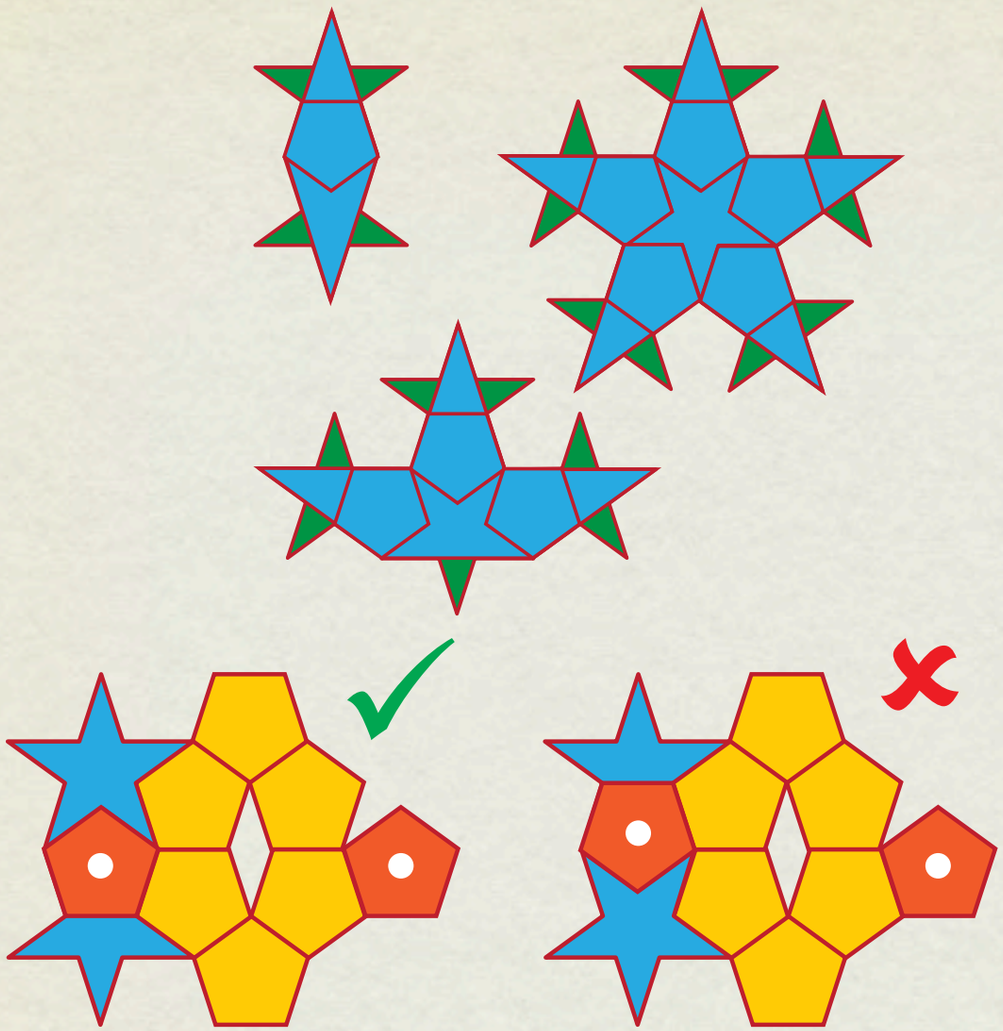
http://www.math.toronto.edu/drorbn/Gallery/Symmetry/Tilings/Sanderson/index.html

Origin in pentagonal symmetry

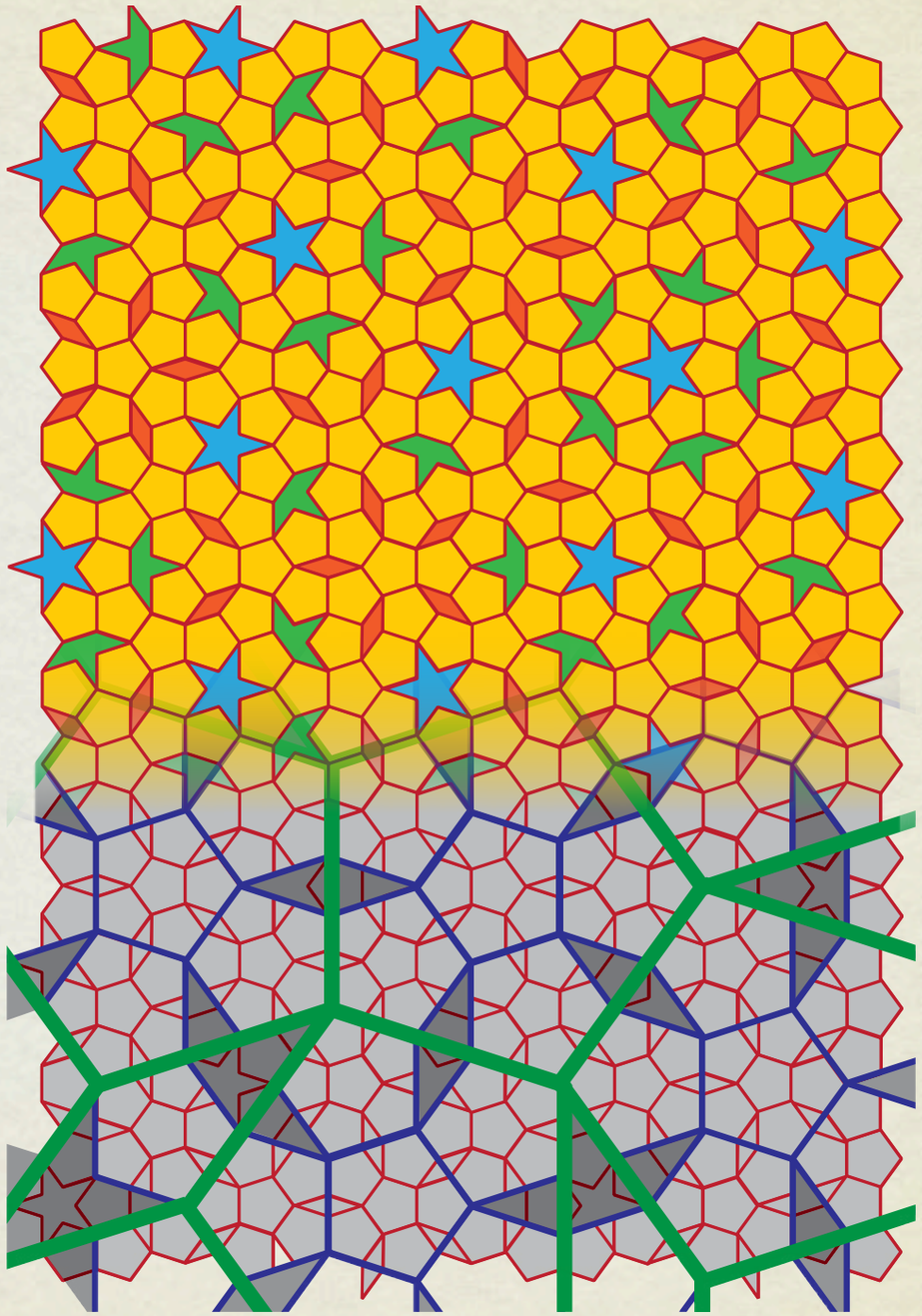
- **Penrose:** even though plane can not be tiled by pentagons, how close can you get?



"Pentaplexy" version of Penrose tilings

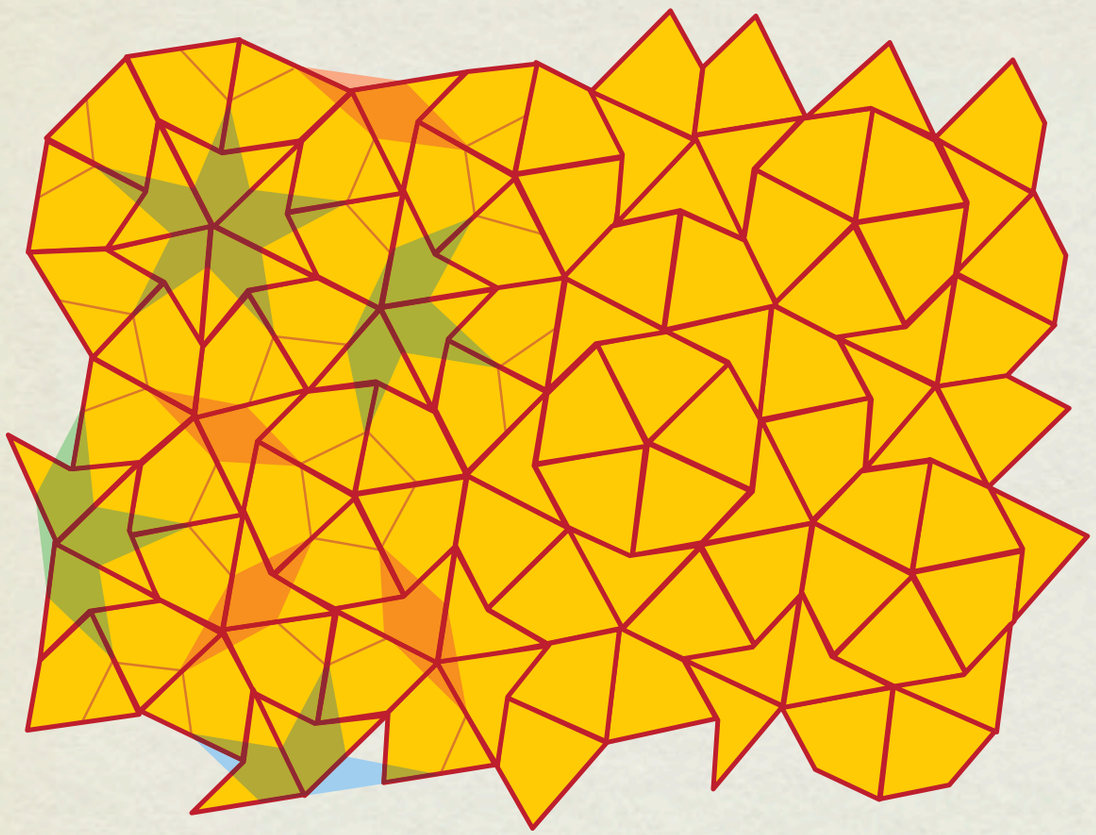


Four different **prototiles**.

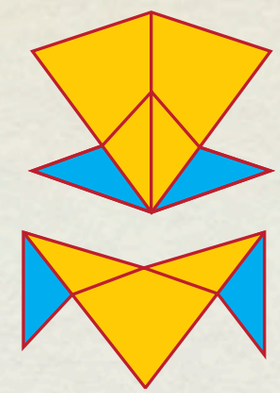


Pentaplexy to "Kites & Darts"

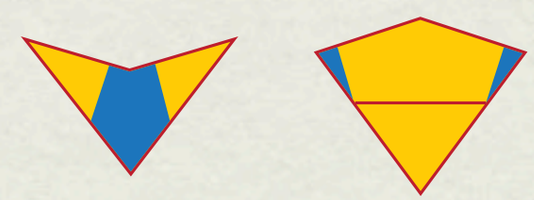
Two prototiles, (20 after rotation).



Mutually locally derivable (MLD) to "pentaplexy" tiling.

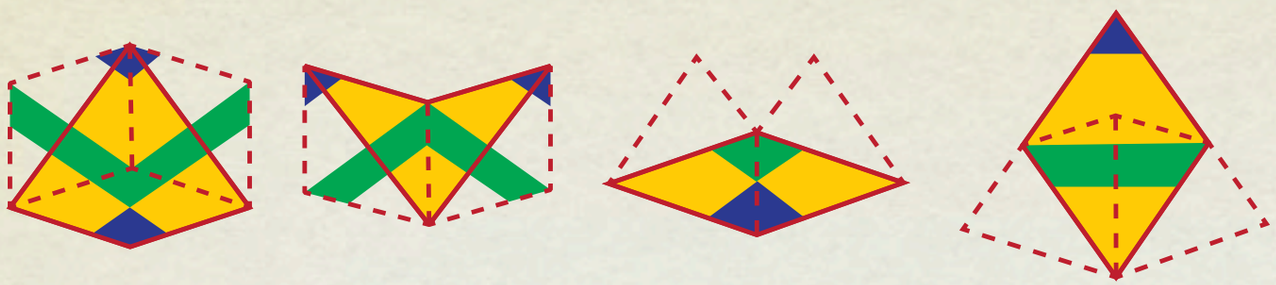


Inflation.

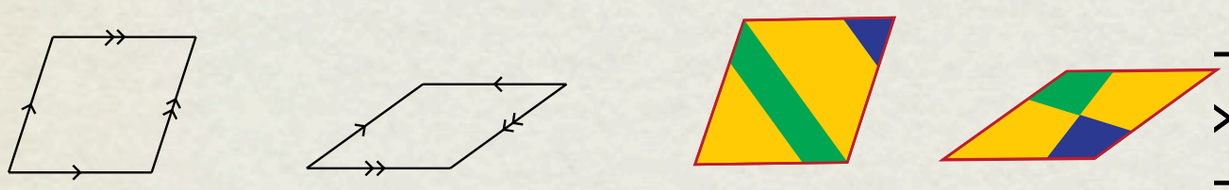


Local matching rule.

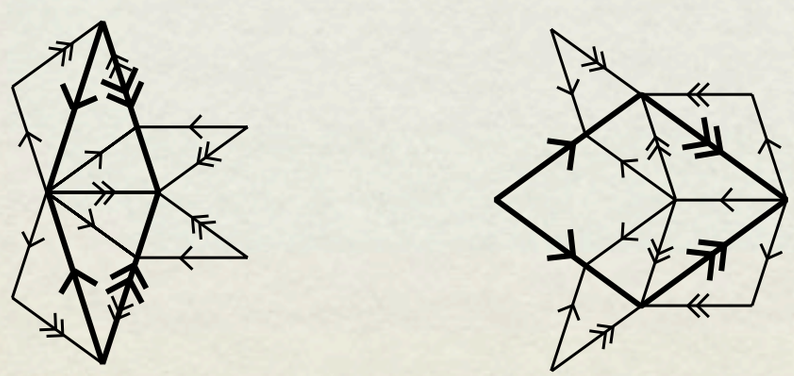
"Kites & Darts" to "Rhombic"

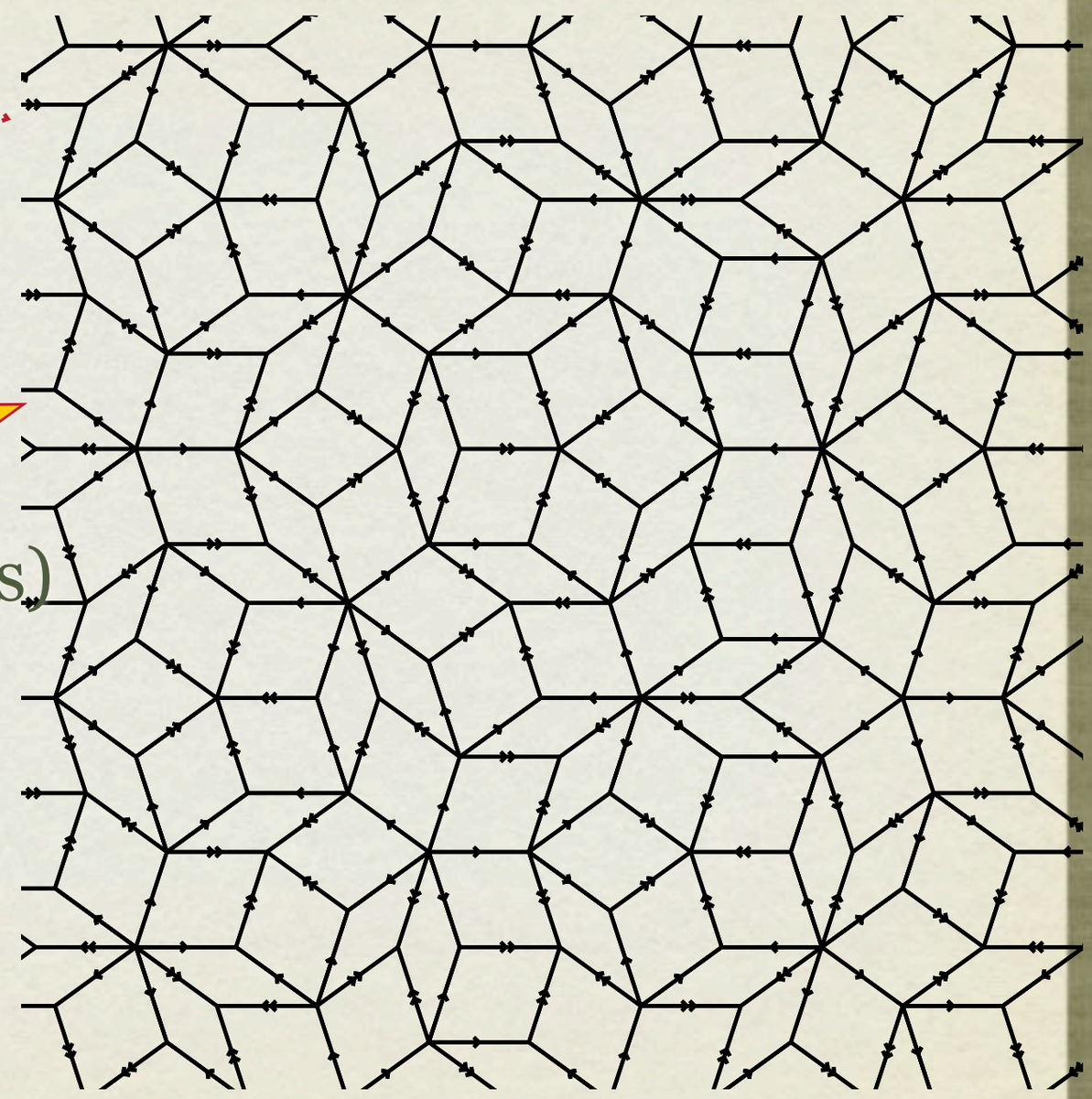


MLD to kites & darts.



Matching rule (decorated two ways)



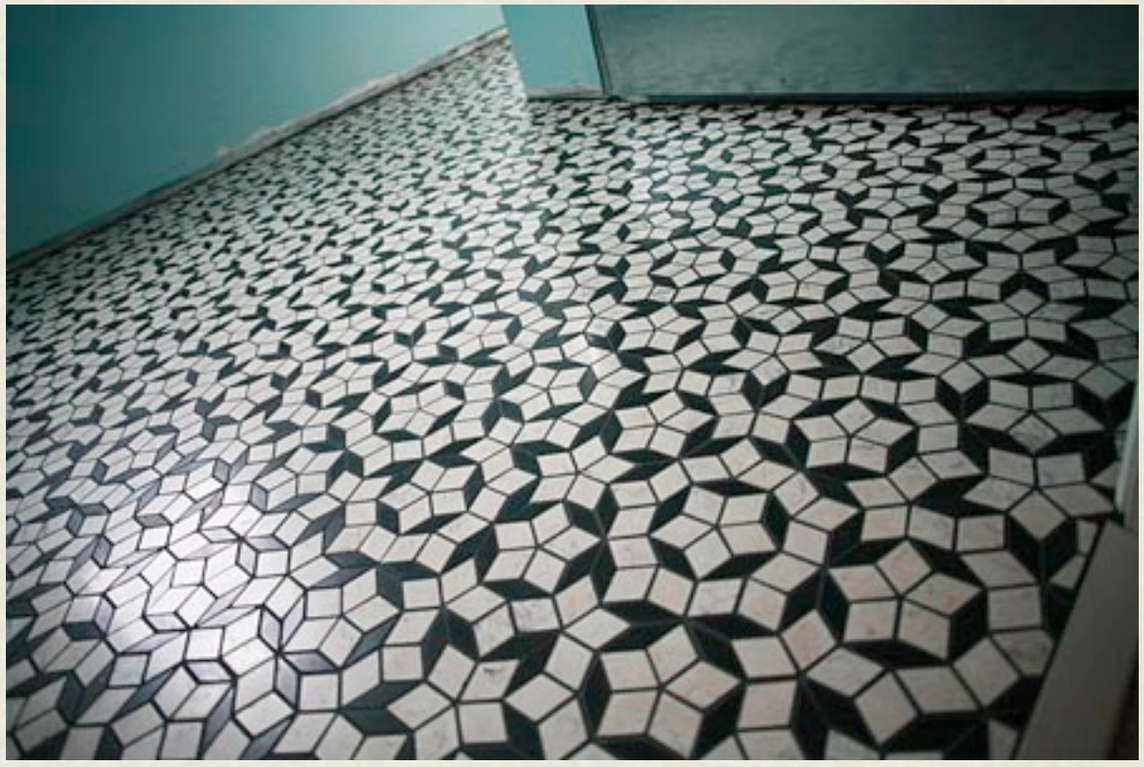


Penrose tilings as architecture

http://www.tessellations.org/real-materials-tessellations-16.shtml

Kites & Darts kitchen.

http://www.liefies.com/?p=366

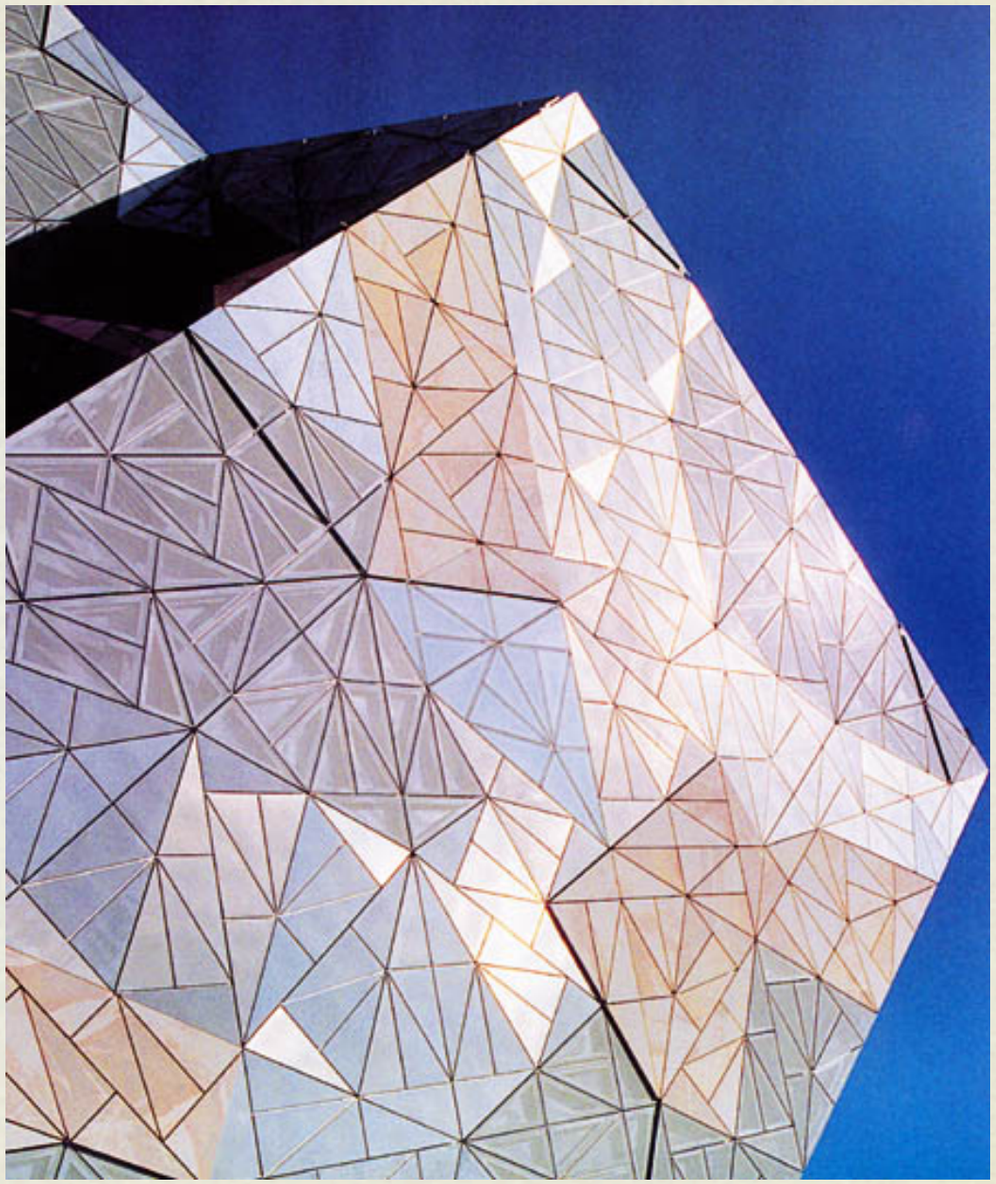


Rhombs bathroom

More architecture

Federation Square

Melbourne, Australia



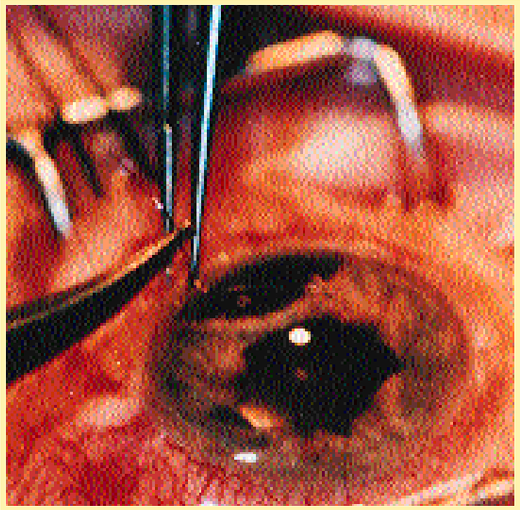
Industrial quasicrystals

MATERIALS by Diana Lutz

Putting Quasicrystals to Work

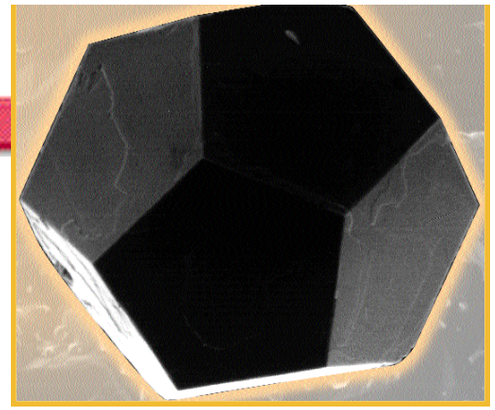
At a conference held August 19-23 in Ames, Iowa, "New Horizons in Quasicrystals: Research and Applications," Jan-Olof Nilsson, manager of physical metallurgy for the R&D Centre of Sandvic Steel, Sandviken, Sweden, said: "Three years ago I thought that quasicrystals were something that existed only in the academic world, but I have been forced to change my mind. As a matter of fact, we can't avoid quasicrystalline precipitates in our commercial materials, and furthermore, we can make use of them."

In the past several years many people have changed their minds about quasicrystals, the new form of matter discovered in 1982. (Quasicrystals differ from glasses in that their atoms are arranged in orderly patterns, and from crystals in that these patterns do not repeat periodically; see Figure 3, page 29.) At first quasicrystals were considtals, for example, are extremely poor electrical and thermal conductors. The thermal conductivity of quasicrystals containing more than 70 atomic percent aluminum is two orders of magnitude below that of aluminum and roughly equivalent to that of zirconia, which is used as a refractory material. Quasicrystals are also exceptionally hard, and their surfaces have very low coefficients of friction, good wear resistance, and good oxidation and corrosion resistance. Depending on how they are prepared, quasicrystals can have coefficients of friction so low they are comparable to the



Sandvic Steel

Figure 2. The first bona fide application of quasicrystals is a steel alloy produced by Sandvic Steel that is made into medical instruments. These must be easy to form, yet very strong, to avoid breakage during use.



Ames Laboratory

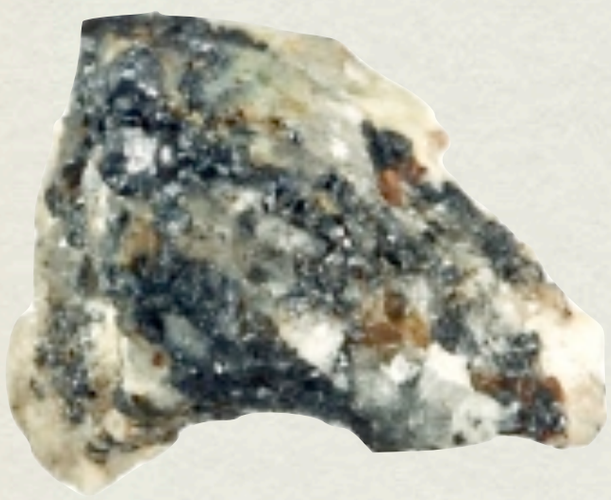
Figure 1. This aluminum–copper–iron quasicrystal, shown in a secondary electron microscope image, belongs to the icosahedral family of quasicrystals, and has 12 pentagonal faces.

[Conference proceedings will be published by World Scientific Publishing, which can be reached by telephone (800–227–7562) or on the Web (http://www.wspc.com).] It was chaired by Alan Goldman, Daniel Sordelet, and Patricia Thiel of the Department of Energy's Ames Laboratory, in Ames, Iowa, and by Jean-Marie Dubois of the École des Mines in Nancy, France.

Pinpointing the first instance of the commercial use of quasicrystals is a bit tricky, because there were some early "stealth" applications. Dan Shechtman of Technion-Israel Institute of Technology (Haifa, Israel), who discovered quasicrystals, remarked in passing that 1890—an aluminum–lithium alloy, developed by Alcan Aluminium Limitée in Montreal, that is now one of three standard aluminum–lithium alloys used in the aerospace industry—sometimes contains qua-

The Industrial Physicist, AIP, December 1996.

Naturally occurring quasicrystal



Nature, January 3, 2012.

Luca Bindi, Museum of Natural History, Florence, Italy. Found quasicrystal in meteor from Koryak Mountains in Russia.