Entropy-zero f-expansions

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The New Math

Figure: Mullen-Hall Elementary School, Falmouth, Massuahusetts, USA.

In 1963, during the Cold War, my third grade math teacher was trained in the “New Math” and taught us base 2 and base 5.

Continued fractions

Any $x \in (0,1]$ is given as an expansion of the form

$$x = \cfrac{1}{d_1 + \cfrac{1}{d_2 + \cfrac{1}{d_3 + \cdots}}}.$$

where $\boldsymbol{d} = .d_1 d_2 d_3 \ldots$ is an arbitrary infinite sequence of positive integers. (We say $d_n \in \mathcal{D} = \mathbb{N}$, the digit set.)

This can be written

$$x = f(d_1 + f(d_2 + f(d_3 + \ldots))),$$

where $f(x) = \frac{1}{x}$.

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Base-r expansions

Let $r \in \mathbb{N}$, $r > 1$. Any $x \in [0, 1)$ is given as an expansion of the form

$$x = \sum_{n=1}^{\infty} \frac{d_n}{r^n},$$

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Base-r (continued)

Here is an alternative way to write this

$$x = \frac{d_1 + \frac{d_2 + \frac{d_3 + \dots}{r}}{r}}{r}.$$

Base-r (continued – 2)

Let $N(d : \boldsymbol{w})$ the number of digits d in word $\boldsymbol{w} \in \mathcal{D}^* = \cup_{n \geq 1} \mathcal{D}^n$.
For $x \in [0, 1)$ let $\boldsymbol{d} = .d_1 d_2 d_3 \ldots$ and let $\boldsymbol{d}_n = d_1 d_2 \ldots d_n \in \mathcal{D}^n$.
For a.e. x we can recover r, the base, by

$$1/r = \lim_{n \to \infty} \frac{1}{n} N(d : \boldsymbol{d}_n).$$

Let $C(n, \boldsymbol{d})$ be the number of words of length n in $\boldsymbol{d}$. Then for a.e. x

$$r = \lim_{n \to \infty} \frac{1}{n} \log C(n, \boldsymbol{d}).$$

β-expansions

Any $x \in [0,1)$ can be written in the form

$$x = \sum_{n=1}^{\infty} \frac{d_n}{\beta^n},$$

where $\boldsymbol{d} = .d_1 d_2 d_3 \ldots$ is a sequence from $\{0,1\}$, such that the sub-sequence 11 never occurs.

Again

$$x = \cfrac{d_1 + \cfrac{d_2 + \cfrac{d_3 + \ldots}{\beta}}{\beta}}{\beta}.$$

This works for any $\beta > 1$, with $\mathcal{D} = [0,\beta) \cap \mathbb{N}$, (in general, with a more complicated restriction on forbidden sub-words).

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Counting

Again each $d \in \mathcal{D}$ and a.e. $x \in [0,1)$,

$$\lim_{n\to\infty} \frac{1}{n} N(d : \boldsymbol{d}_n),$$

(in general, the value of the limit depends on β in a complicated way.)

Also, for a.e. x

$$\beta = \lim_{n\to\infty} \frac{1}{n} \log C(n, \boldsymbol{d}).$$

(this works for any $\beta > 1$).

Sturmian sequences

Let $\mathcal{D} = \{0, 1\}$. A Sturmian sequence

$$\boldsymbol{d} = .d_1 d_2 d_3 \cdots \in \mathcal{D}^{\mathbb{N}}$$

is a sequence so that

$$C(n, \boldsymbol{d}) = n + 1.$$

Strumian sequences were introduced by Morse and Hedlund (1940) who proved the following result:

Formula

Theorem (Morse, Hedlund: 1940)

Every Sturmian sequence $\boldsymbol{d}$ *is given by*

$$\begin{aligned} d_n &= \lfloor \alpha(n+1) + \beta \rfloor - \lfloor \alpha n + \beta \rfloor, \\ (\text{or } \tilde{d}_n &= \lceil \alpha(n+1) + \beta \rceil - \lceil \alpha n + \beta \rceil) \end{aligned}$$

for some $\alpha \in [0,1) \backslash \mathbb{Q}$ *and* $\beta \in [0,1)$.

A Sturmian sequence (or either kind) satisfies the uniqueness condition:

$$\boldsymbol{d} = \boldsymbol{d}' \implies \alpha = \alpha' \text{ and } \beta = \beta'.$$

Question: How can α and β be determined from $\boldsymbol{d}$?

Density

It is well know that any Sturmian sequence $\boldsymbol{d}$ satisfies

$$\alpha = \lim_{n\to\infty} \frac{1}{n} N(1 : \boldsymbol{d}_n),$$

Is there a similar formula for β?

Here is one answer (see Arnoux, Ferenczi and Hubert (1999)). Define two substitutions.

$$\begin{aligned} \sigma_0 0 &= 0 \\ \sigma_0 1 &= 10 \end{aligned} \qquad \begin{aligned} \sigma_1 0 &= 01 \\ \sigma_1 1 &= 1 \end{aligned}$$

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ONE ANSWER

Assume x is such that $d_n = \tilde{d}_n$ for $n \in \mathbb{Z}$ (a similar result holds in the opposite case).

THEOREM (ARNOUX, FERENCZI AND HUBERT (1999))

There are sequences $e_n \in \mathbb{N} \cup \{0\}$ and $a_n \in \mathbb{N}$ so that

$$\boldsymbol{d} = 0^{e_1}\ \sigma_0^{a_1}(1^{e_1})\ (\sigma_0^{a_1}\sigma_1^{a_2})(o^{e_3})\ (\sigma_0^{a_1}\sigma_1^{a_2}\sigma_0^{a_3})(1^{e_4})\dots$$

Then

$$\alpha = \cfrac{1}{(a_1+1) + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

and ...

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$$\alpha = \cfrac{1}{(a_1+1) + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

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ONE ANSWER

THEOREM (...CONTINUED)

$$\beta = 1 + (1-\alpha)\left(-e_1 + \sum_{n=1}^{\infty}(-1)^{n+1} e_{n+1}\,\alpha_1\alpha_2\ldots\alpha_n\right)$$

where

$$\alpha_n = \cfrac{1}{a_n + \cfrac{1}{a_{n+1} + \cdots}}.$$

- The formula for β is an example of Ostrowski numeration.
- The formula for d is an example of telescope form.
- The idea behind the proof is Rauzy induction.
- Similar results are due to Sidorov and Vershik (1993).

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Theorem (. . . continued)

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The coefficient algorithm

In the cases of both continued fractions ($f(x) = 1/x$) and radix representations ($f(x) = x/b$, $b = r \in \mathbb{N}$ or $b = \beta \notin \mathbb{N}$), there is an iterative algorithm to find the representation

$$\boldsymbol{d} = .d_1 d_2 d_3 \ldots$$

of $x \in (0, 1)$. Put $x_1 = x$.

- $d_n = \lfloor f^{-1}(x_n) \rfloor$.
- $x_{n+1} = f^{-1}(x_n) - d_n$.

Stop if $f^{-1}(x_n)$ is undefined and say x has a finite expansion.

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f-representations

Several authors (Kakeya (1924), Bissinger, (1944), Everett (1946), Rényi (1957), Parry (1964)) observed that $f(x) = 1/x$ and $f(x) = x/r$ can be replaced by different monotonic functions.

Rényi (1957) observed that a map of the interval:

$$T(x) = f^{-1}(x) \bmod 1.$$

can be used to obtain the digits

$$d_n = \lfloor f^{-1}(T^{n-1}(x)) \rfloor$$

by iteration.

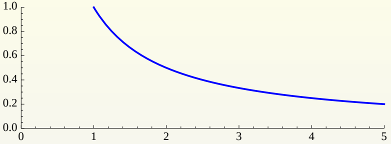

Figure: Continued fractions: $f(x) = 1/x$

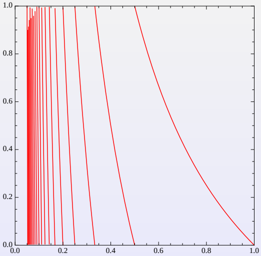

Figure: The Gauss map $T(x) = 1/x \bmod 1$.

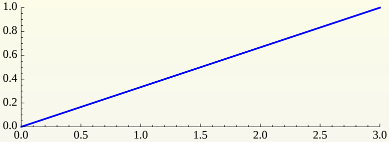

Figure: Base 3: $f(x) = x/3$ ($r = 3$).

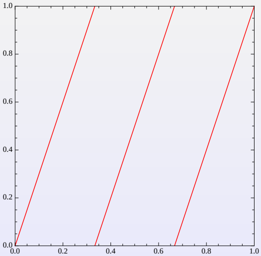

Figure: The "period tripling" map $T(x) = 3x \bmod 1$.

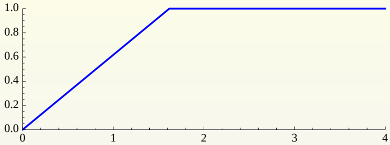

Figure: β-expansions: $f(x) = x/\beta$, $b = \beta = \frac{1+\sqrt{5}}{2}$

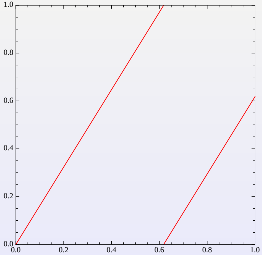

Figure: The β-transformation : $T(x) = \beta x \bmod 1$. Non-independent digits implies second branch not full (i.e., not onto).

Basic hypotheses

It will be useful to switch the viewpoint to T rather than f. The following hypotheses are essentially those of Parry (1957).

- Let $\mathcal{D} \subseteq \mathbb{N}$, $\#(\mathcal{D}) = 2$.
- Let $\Delta(d) = [c_d, c_{d+1}) \subseteq [0, 1]$
  - $c_d < c_{d+1}$ $\forall d$.
  - such that $\inf c_d = 0$, $\sup c_d = 1$.
- Let $I = \cup_{d \in \mathcal{D}} \Delta(d)$.
  - $I \in \{[0, 1], (0, 1], [0, 1), (0, 1)\}$.
- Let $T : I \to [0, 1]$ be strictly monotone on each $\Delta(d)$.
  - Type A/ Type B if all $T|_{\Delta(d)}$ decreasing/ increasing,
  - Else, mixed type.
- $T : B \to B$ where $B = \{x : T^{n-1}(x) \in I \ \forall n \in \mathbb{N}\}$.

An example of what Schweiger (1995) calls a fibered system.

THE FUNCTION f

- Let $\xi : I \to \mathcal{D}$ be $\xi(x) = d$ if $x \in \Delta(d)$.
- Also think of $\xi = \{\Delta(d) : d \in \mathcal{D}\}$ as a finite or countable partition.
- Define $T^*(x) = T(x) + \xi(x)$. Then $T^* : I \to \mathbb{R}$ is strictly monotone.
- Let $f : \mathbb{R} \to I$ be $f(x) = (T^*)^{-1}(x)$, extended to $\mathbb{R}$ to be (non-strictly) monotone (i.e., piecewise constant on the gaps).

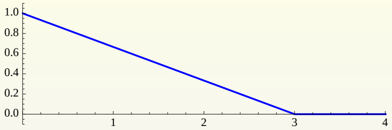

FIGURE: Base -3: $f(x) = 1 - x/3$.

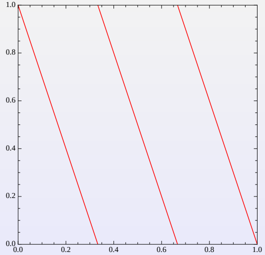

FIGURE: Reverse period tripling $T(x) = 4 - 3x \mod 1$.

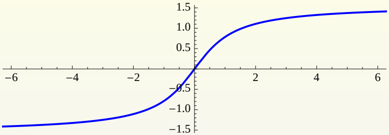

FIGURE: Arctangent expansions: $f(x) = \arctan(x)$, $D = (-\infty, \infty)$.

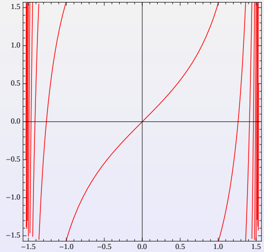

FIGURE: Tangent map: $T(x) = \tan(x) \bmod [-\pi/2, \pi/2]$.

Representations and expansions

For $x \in B$ define the f-representation

$$\delta(x) = \boldsymbol{d} = .d_1 d_2 d_3 \cdots \in \mathcal{D}^{\mathbb{N}}$$

by $d_n = \xi(T^{n-1}(x))$.

For any $\boldsymbol{d} = .d_1 d_2 d_3 \cdots \in \mathcal{D}^{\mathbb{N}}$, define the f-expansion

$$\varepsilon(\boldsymbol{d}) = f(d_1 + f(d_2 + f(d_3 \ldots))).$$

In particular,

$$\varepsilon(\boldsymbol{d}) = \lim_{n\to\infty} f(d_1 + f(d_2 + \cdots + f(d_n))),$$

provided the limit exists (we say the f-expansion converges).

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Kakeya's Theorem

Say f-representations satisfy uniqueness if $\delta(x) = \delta(y)$ implies $x = y$ for $x, y \in B$, i.e., δ is 1:1 on B.

Say f-expansions are valid if $x = \varepsilon(\delta(x))$ for all $x \in B$.

Theorem (Kakeya, 1924)

Let T be Type A or B, and suppose $|T'(x)| > 1$ almost everywhere. Then f-expansions are valid.

Bissinger (1944), Everett (1946), and Rényi (1957), and Parry(1964) have similar results, with the hypotheses $|f'(x)| < 1$, replaced by $|f(x) - f(y)| < |x - y|$ (or a little more).

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The f-shift

We say f-expansions have independent digits if each $T|_{\Delta(d)}$ is onto (each fiber is full in the terminology of Schweiger).

Give $\mathcal{D}^{\mathbb{N}}$ the product topology (compact if $\mathcal{D}$ is finite, Polish otherwise), and let S be the left-shift map.

Let $X_0 = \{\delta(x) : x \in B\}$ and let $X = \overline{X_0}$. Then X is called the f-shift.

Lemma (Renyi, 1957)

If f has independent-digits, then $X = \mathcal{D}^{\mathbb{N}}$ (the full shift).

More generally, let $\mathcal{L}_n$ be the set of all words of length n in X. Then $\mathcal{L} = \cup \mathcal{L}_n$ is called the language of the shift X.

Shifts

If $\#(\mathcal{D}) < \infty$, the complexity of X is defined

$$C(n, X) = \#(\mathcal{L}_n),$$

and topological entropy is given by

$$h_{\text{top}}(X) = \lim_{n\to\infty} \frac{1}{n} \log C(n, X).$$

- If $X = \mathcal{D}^{\mathbb{N}}$ and $\#(\mathcal{D}) = r < \infty$, then $C(n, X) = r^n$, and $h_{\text{top}}(X) = \log r$.
- If X a β-shift, $\#(\mathcal{D}) = \lfloor \beta \rfloor$, and $h_{\text{top}}(X) = \log \beta$.
- If X a Sturmian shift, $\#(\mathcal{D}) = 2$, then $C(n, X) = n + 1$, and $h_{\text{top}}(X) = 0$.

Ergodic theory

- A Borel probability measure γ on B is T-invariant if $\gamma(T^{-1}E) = \gamma(E)$ for every Borel set E.
- A Borel measure γ is absolutely continuous if there $\rho(x) \geq 0$, $\rho \in L^1(B, \lambda)$ (λ=Lebesgue) so that $\gamma(E) = \int_E \rho(x)\, d\lambda$.
- An absolutely continuous measure is equivalent to Lebesgue measure if $\rho(x) > 0$ a.e. Call this Lebesgue-equivalent.
- T is ergodic if $TE = E$ implies $\gamma(E) = 0$ or $\gamma(E) = 1$.

Ergodic theory

If T has ergodic Lebesgue-equivalent invariant measure, then the Birkhoff ergodic theorem implies f-representations are normal:
For $x \in B$ let $\delta(x) = \boldsymbol{d} = .d_1 d_2 d_3 \ldots$. Fix $d \in \mathcal{D}$. Then For λ a.e. $x \in B$

$$\lim_{n\to\infty} \frac{1}{n} N(d : \boldsymbol{d}_n) = \gamma(\Delta(d)) = \int_{\Delta(d)} \rho(x)\, d\mu.$$

(Similar formulas hold for d replaced by longer words).

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Partitions

Define

$$\Delta(d_1 d_2 \dots d_n) = \bigcap_{j=1}^{n} T^{-j+1}\Delta(d_j).$$

In ergodic theory, these are called n-cylinders, and $\delta(x) = .d_1 d_2 \dots$ is called the ξ-name of x.

Lemma

$\Delta(d_1 d_2 \dots d_n) \neq \emptyset$ iff $d_1 d_2, \dots d_n \in \mathcal{L}$, and in this case it is an interval $[a, b) \subseteq I$, called a fundamental interval.

Denote the partition into non-empty n-cylinders by

$$\xi^{(n)} = \xi \vee T^{-1}\xi \vee \cdots \vee T^{-n+1}\xi = \bigvee_{j=0}^{n-1} T^{-j}\xi$$

GENERATORS

Let $|\xi^{(n)}| = \max\{\lambda(\Delta) : \Delta \in \xi^{(n)}\}$.

LEMMA

f-expansions are valid if and only if $\lim_{n\to\infty} |\xi^{(n)}| = 0$.

In ergodic theory ξ is called a 1-sided generator. This is usually denoted something like $\xi^{(n)} \to \epsilon$.

If T is invertible, ξ is called a 2-sided generator if

$$T^n\xi \vee T^{n-1}\xi \vee \cdots \vee \xi \vee \cdots \vee T^{-n}\xi \to \epsilon$$

Entropy

The entropy of a (finite or countable) partition $\eta = \{E_1, E_2, \dots\}$ is $H(\nu) = -\sum_{E\in\nu} \gamma(E) \log \gamma(E)$.

Definition

If γ is an invariant probability measure for T, the metric entropy is defined

$$h_\gamma(T) = \sup_\eta h_\gamma(T, \eta),$$

where

$$h_\gamma(T, \eta) = \lim_{n\to\infty} \frac{1}{n} H(\eta^{(n)}).$$

Theorem (Kolmogorov-Sinai, 1958)

If η is a generator then $h_\gamma(T) = h(T, \eta)$.

Indepent-digits case

Suppose f satisfies Type A or B, has has independent digits, satisfies $|f(x) - f(y)| < |x - y|$ (and a little more along these lines), and satisfies one additional (difficult to verify) hypotheses. Rényi (1957) proved T has a Lebesgue-equivalent ergodic invariant measure.

- Gauss measure is ergodic invariant for continued fractions. It is given by $\rho(x) = \frac{1}{\log 2} \frac{1}{1+x}$, (Gauss, c 1800)
- Lebesgue measure is ergodic for base-r $(T(x) = rx \mod 1)$.

Adler's theorem

Theorem (Adler, 1973)

Suppose T satisfies

1. *Type A or B, with independent digits (i.e., $T|_{\Delta(d)}$ onto)*
2. $T|_{\Delta(d)}$ *is* C^2 *for each* d,
3. *There is* $n \in \mathbb{N}$ *so that* $\inf |(T^n)'(x)| > 1$,
4. $\sup_{x,y,z\in\Delta(d)} |T''(x)/(T'(y)T'(z))| < \infty$, $\forall d \in \mathcal{D}$.

Then

- *f-expansions are valid (ξ is a generator),*
- *$\exists$ ergodic Lebesgue-equivalent invariant measure (normal).*
- *T has a Bernoulli "natural extension" (this implies $h_\gamma(T) > 0$).*

β-expansions (Renyi, 1957)

Let $\beta > 1$, $\beta \notin \mathbb{N}$. Let $\mathcal{D} = \{0, 1, \dots, \lfloor \beta \rfloor\}$. Then the β-transformation $T(x) = \beta x \bmod 1$, has non-independent digits.

Theorem (Rényi)

The β-transformation T has an ergodic Lebesgue-equivalent ergodic invariant measure.

Note: $\gamma \neq \lambda$.

Corollary

β-expansions are valid, and for a.e. x the β-expansion is normal.

Some properties of β-expansions

- Parry (1960) found an explicit formula for density $\rho(x)$.
- Rényi (1957), Parry (1960), Schmidt (1980), determined structure of the β-shift X (in terms of the β-representation for $x = 1$, denoted $\mathbf{1} = .d_1 d_2 d_3 \dots$, where $d_n = \xi(T^{n-1}(1))$).
    - X SFT if $\mathbf{1} = .d_1 d_2 d_3 \dots d_n 0000 \dots$
    - If β is a Pisot number, X is sofic.
- T has a Bernoulli "natural extension" (Smorodinski, 1973)
- $h_\gamma(T) = h_{\text{top}}(X) = \beta > 0$.

Lasota-Yorke approach

Theorem (Lasota-Yourke, 1973)

Suppose T satisfies

1. *The partition ξ is finite. (There is no independent digits assumption),*
2. *T is C^2 on $\overline{\Delta(d)}$ for each d,*
3. *There is $n \in \mathbb{N}$ so that $\inf |(T^n)'(x)| > 1$ on $\cup \mathrm{int}(\Delta(d)$.*

Then T has an absolutely continuous invariant measure.

Lyapunov exponents

If γ is an invariant measure for T, then (Oseledec, 1978)the Lyapunov exponent

$$\ell(x) = \lim_{n\to\infty} \frac{1}{n} \log |(f^n)'(x)|.$$

Let $\ell_+(x) = \max(0, \ell)$ then $h_\gamma(T) \leq \int_I \ell_+(x)\, d\gamma$ (Margulis, 1968).

If γ is Lebesgue-equivalent, and $\ell_+(x) > 0$ a.e., $h_\gamma(T) = \int_I \ell_+(x)\, d\gamma$ (Katok, 1980).

If γ is ergodic and Lebesgue-equivalent, $\ell_+(x) = \ell_+$ γ a.e. is constant, and

$$h_\gamma(T) = \ell_+ > 0.$$

Parry's Theorem

Parry (1964) notes that validity is a dynamical rather than analytic property.
Let $O_+(x) = \{T^n(x) : n \geq 0\}$. Say T is topologically transitive if $O_+(x)$ is dense in B for some $x \in B$.

Theorem (Parry, 1964)

If T is Type B and topologically transitive, then f-expansions are valid.

Parry (1964) studies $T(x) = \beta x + \alpha \mod 1$. The case $\alpha = 0$ (Rényi, 1957; Parry, 1960) is the β-transformation.

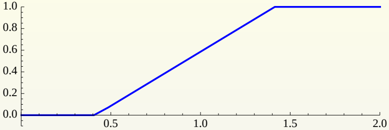

Figure: $\beta x + \alpha$ expansions: $f(x) = (x - \alpha)/\beta$.

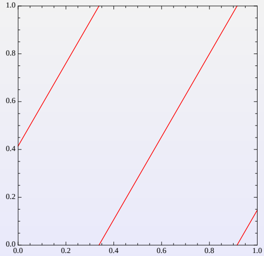

Figure: Map: $T(x) = \beta x + \alpha \bmod 1$. Here, $\alpha = \sqrt{2} - 1$ and $\beta = \sqrt[2]{3}$

The irrational rotation

The case $\beta = 0$, $\alpha \in [0,1)\backslash\mathbb{Q}$ gives the irrational rotation map.

$$T(x) = x + \alpha \bmod 1$$

Put

$$\xi(x) = \begin{cases} 0 & \text{if } x \in [0, 1-\alpha) \\ 1 & \text{if } x \in [1-\alpha, 1) \end{cases}$$

Parry mentions this case in passing, noting that expansions are valid, but saying it has been "studied elsewhere" (he cites Weyl, 1916).
The f-representations $\delta(x) = \boldsymbol{d} = .d_1 d_2 d_3 \ldots$

$$d_n = \lfloor (n+1)\alpha + x \rfloor - \lfloor n\alpha + x \rfloor = \xi(T^{n-1}(x)).$$

give Sturmian sequences.

Properties of irrational rotation

- $|T'(x)| = 1$ almost everywhere.
- T is invertible.
- Lebesgue measure is the unique T-invariant measure (unique ergodicity).
- $O_+(x)$ dense for all x, called minimal. Implies topologically transitive.
- $h_{\text{top}}(X) = h_\lambda(T) = 0$.

By Parry's theorem, Sturmian sequences are valid f-representations:

$$x = \epsilon(\delta(x)) = f(d_1 + f(d_2 + f(d_3 + \dots))).$$

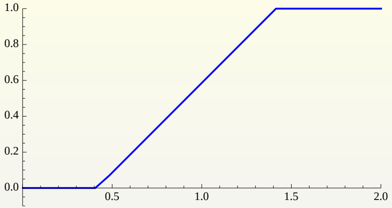

Figure: $f(x) = x - \alpha$ for $\alpha \leq x < \alpha + 1$.

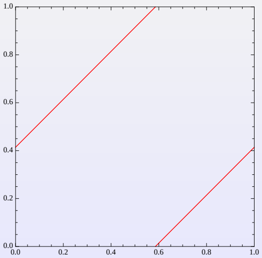

Figure: Irrational map: $T(x) = x + \alpha$.

An example of convergence

Let $\alpha = \frac{1+\sqrt{5}}{2} - 1$. Let $x = .322$. Then

$$\boldsymbol{d} = .0110110101101011011010110110110101101011011010110101\ldots$$

Here are the first 20 convergents:
$0,\ 3-\sqrt{5}+\frac{1}{2}(1-\sqrt{5}),\ 3-\sqrt{5}+\frac{1}{2}(1-\sqrt{5}),\ 3-\sqrt{5}+\frac{1}{2}(1-\sqrt{5}),\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5}, 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5},\ 7-3\sqrt{5}\ldots$

All belong to $\mathbb{N}+\alpha\mathbb{N}$ (in this case $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$).

Graph of convergence

The first 1000 has only a few values:
$7-3\sqrt{5}, \frac{1}{2}(61-27\sqrt{5}), 92-41\sqrt{5}, \frac{1}{2}(1027-459\sqrt{5})\ldots$

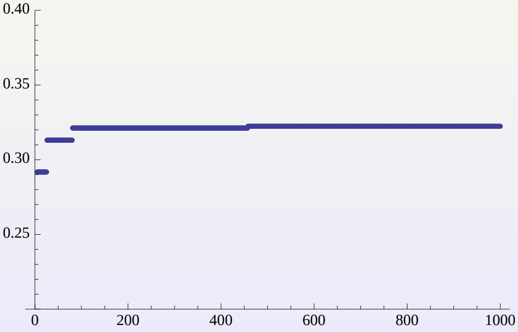

Figure: List Plot of first 1000 convergents

Interval Exchanges

- Irrational rotations maps are interval exchange transformations T (IET's) with 2 intervals.
- All IET's have Lebesgue measure λ invariant, and $h_\lambda(T) = 0$.
- A.e. interval exchange T (if based on a good permutation) is minimal and uniquely ergodic. In this case the corresponding f-expansions are valid and a.e x has normal f-expansions.
- However, $\exists$ IET T that are minimal but not unique ergodic. Parry's theorem implies f-expansions are valid, but normality may fail. (There is an bound on the number of ergodic invariant measures $\leq k-2$ where k=# intervals).

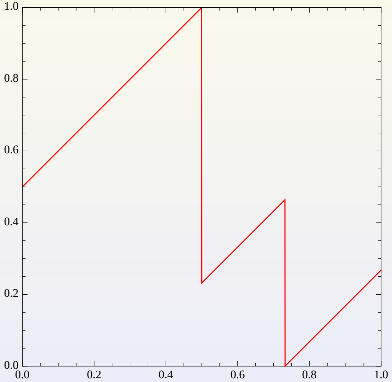

Figure: Map T is a 3-interval exchange

Homeomorphisms of the circle

- View a homeomorphism as a map $T : [0, 1) \to [0, 1)$ with one discontinuity.
- Assume the rotation number α is irrational. Poincare (1885) proved T is semi-conjugate to a rotation by α.
  1. If T is conjugate to rotation, f-expansions valid. (In this case there is a Lebesgue-equivalent invariant measure).
  2. If not, there is a wandering interval: $J = [a, b) \subseteq [0, 1)$ so that $T^n(J), n \in \mathbb{Z}$ are pairwise disjoint. Thus T is not topologically conjugate, and (one can show) f-expansions are not valid.

Infinite interval exchanges

Consider an "abstract" invertible ergodic measure preserving transformation τ on a Lebesgue probability space (Y, ν), and suppose $h_\nu(\tau) < \infty$.

One can construct model T of τ as an exchange of infinitely many intervals (i.e., so T and τ are isomorphic).

In particular there is a sequence of intervals $I_k = [c_k, c'_k)$, $k \in \mathbb{N}$, and $L_k = [e_k, e'_k)$ so that

- $e'_k - e_k = c'_k - c_k$ for all k,
- $T|_{I_k}(x) = x + (e_k - c_k)$, and
- $\cup_{k\in\mathbb{N}} I_k = \cup_{k\in\mathbb{N}} L_k = [0, 1)$.

Infinite interval exchanges (continued)

We want T to satisfy the Basic Hypotheses. For this we need to do the following.

1. Choose $\mathcal{D} \subseteq \mathbb{Z}$ (infinite), and for $d \in \mathcal{D}$, let $s(d)$ be the successor of d in $\mathcal{D}$ ($\mathcal{D}$ is well-ordered).
2. Choose a bijection

$$d \mapsto k_d : \mathcal{D} \to \mathbb{N}$$

so that $c_{k_{s(d)}} = c'_{k_d}$

This may or may not be possible.

Note that $T'(x) = 1$ a.e., so we should use Parry's Theorem instead of Kakeya's.

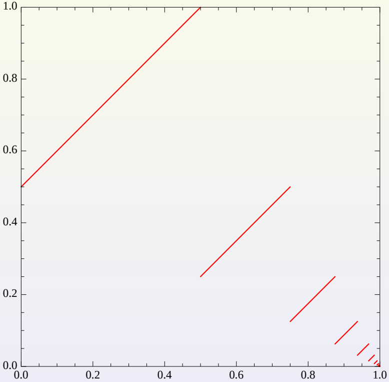

Figure: Map T is a 3-interval exchange

ENTROPY CONSTRAINT

Call such a T a good model for τ on (Y, ν). Let $\xi = \{[c_{k_d}, c_{k_{s(d)}})\}$. Note that $T'(x) = 1$ a.e.

Entropy theory places a strong limit on what we can really get.

LEMMA

If τ has a good model then $h_\nu(\tau) = 0$.

PROOF.

T is isomorphic to τ and ξ is a one sided generator. A theorem in entropy theory says if T is invertible and has a 1-sided generator, then $h_\lambda(T) = 0$ □

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