Thirty-six views of the Rauzy Fractal

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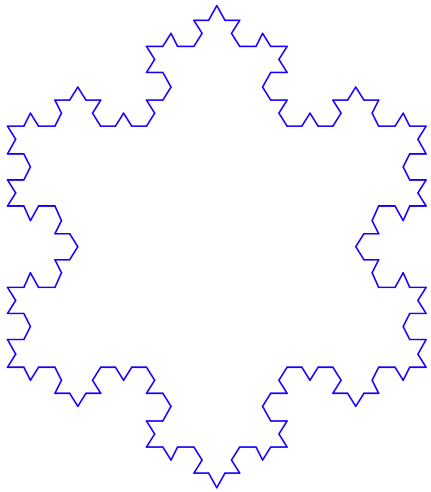
1. Two common methods for constructing fractals



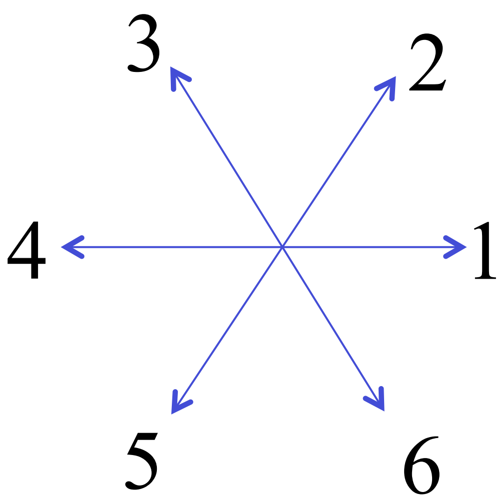
I. *L*-systems
(Lindenmayer 1968, Dekking, 1982)

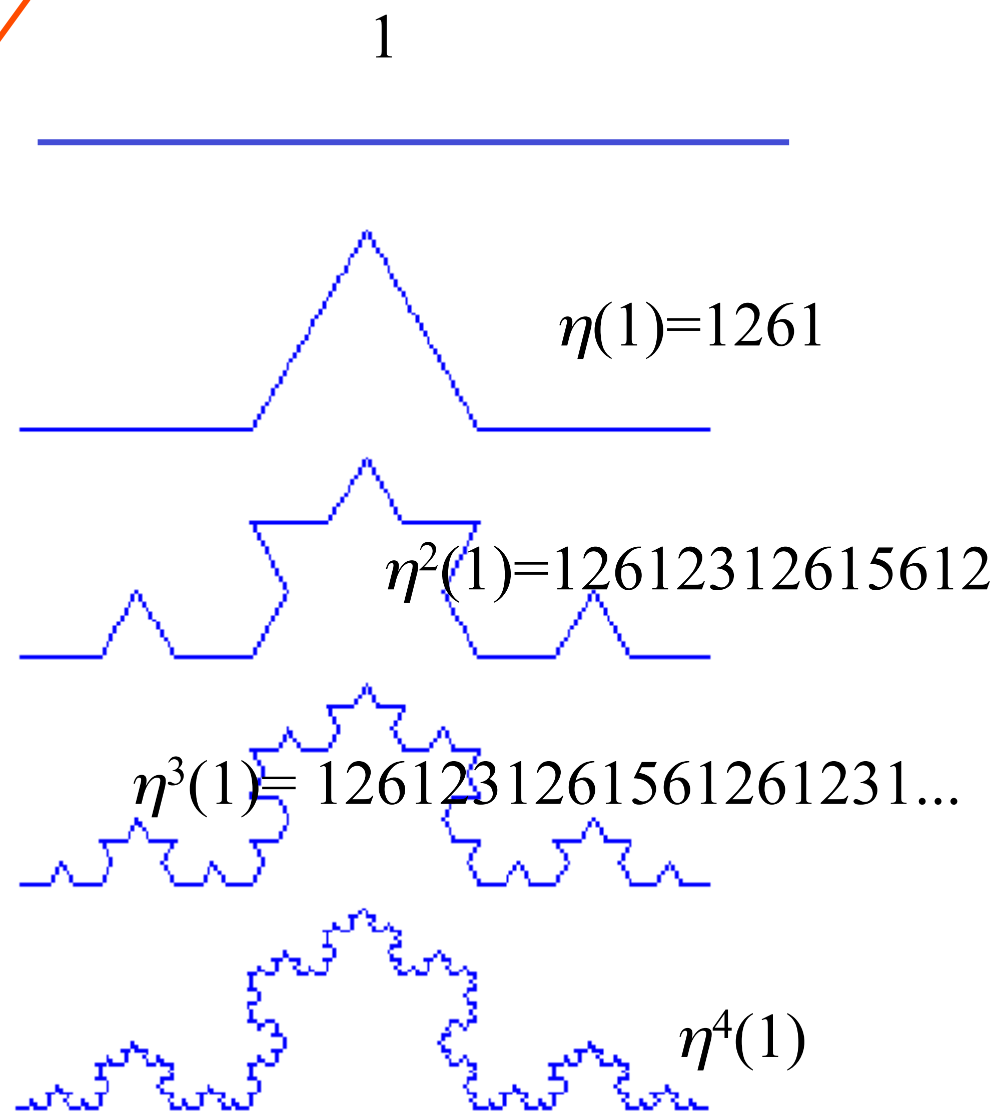
II. Iterated function systems
(Hutchenson 1981, Barnsley 1988)

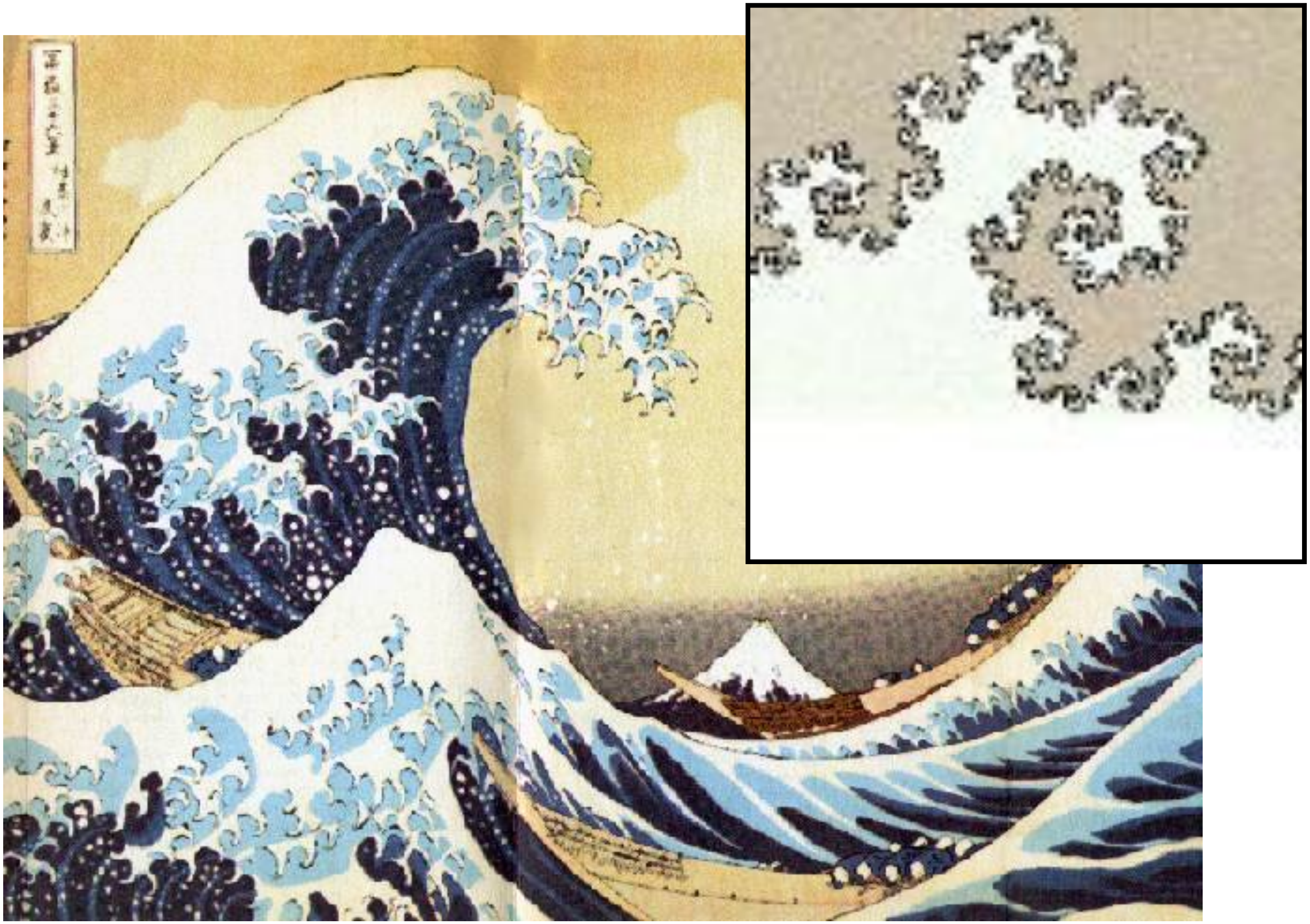
- *L*-systems

Substitution

$$\eta : \begin{cases} 1 \to 1261 \\ 2 \to 2312 \\ \quad \cdots \\ 6 \to 6156 \end{cases}$$







Great Wave off Kangawa
from **Thirty-six views of Mt. Fuji**
by Hokusai circa 1800

II. Iterated function systems

$$\mathcal{I} = \{C_1, C_2, \ldots, C_n\}.$$

$$C_i : X \to X.$$

Uniform contractions of a complete metric space.

Example: $X = \mathbf{C}$:

$$C_1(z) = \tfrac{1}{2} z$$

$$C_2(z) = \tfrac{1}{2} z + 1$$

$$C_3(z) = z + \left(\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}\right)$$

Define : $\mathcal{R}_{\mathcal{I}}(\Psi) = C_1(\Psi) \cup C_2(\Psi) \cup \ldots \cup C_n(\Psi).$

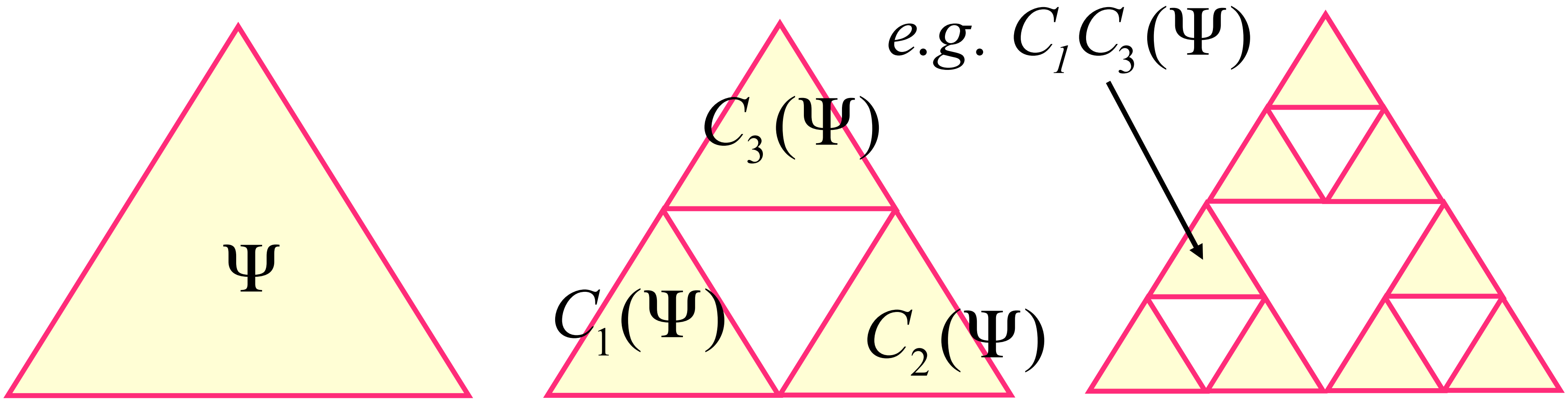
Hutchenson's Theorem: For any compact Ψ

$$\Theta = \lim_{n\to\infty} R_I^n(\Psi)$$

exists and is the unique fixed point $\Theta = R(\Theta)$.

$$\Theta = C_1(\Theta) \cup C_2(\Theta) \cup C_3(\Theta)$$

MCRM algorithm:



2. Fractals in number theory

0,1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144...

Leonardo Pisano Fibonacci
Liber Abaci, 1202

$$f_{n+1} = f_n + f_{n-1}$$

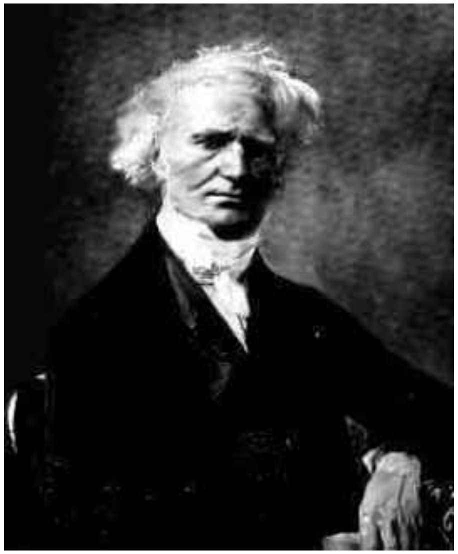
$$f_0 = 0, f_1 = 1$$

Define: $v_n = \begin{pmatrix} f_{n+1} \\ f_n \end{pmatrix}$ $\quad v_{n+1} = F v_n$ $\quad F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$\lambda_u \overset{\text{def}}{=} \gamma = \tfrac{1+\sqrt{5}}{2} \approx 1.6180\ldots$$

$$\lambda_s = -\tfrac{1}{\gamma} = \tfrac{1-\sqrt{5}}{2} \approx -.6180\ldots$$

$$v_n = F^n v_0 = P \cdot \begin{pmatrix} \gamma^n & 0 \\ 0 & \frac{-1}{\gamma} \end{pmatrix} \cdot P^{-1} v_0$$



Jacques Binet,
(1786-1856)

***Binet's formula*:**

$$f_n = \tfrac{1}{\sqrt{5}}\left(\gamma^n - \left(\tfrac{-1}{\gamma}\right)^n\right) = \tfrac{1}{\sqrt{5}}\left(\left(\tfrac{1+\sqrt{5}}{2}\right)^n - \left(\tfrac{1-\sqrt{5}}{2}\right)^n\right)$$

An *easy* version of Binet's formula:

The "Pisot Property"

Note $\lambda_u > 1$ and $|\lambda_s| < 1.$

So $\lim_{n\to\infty} \lambda_s^n = 0.$

Charles Pisot, c 1938

"Easy" Binet formula:

$$f_n = \frac{1}{\sqrt{5}}\left(\gamma^n - \left(\frac{-1}{\gamma}\right)^n\right) \approx \frac{1}{\sqrt{5}}\gamma^n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n$$

for n large.

n	1	2	3	4	5	6
f_n	1	1	2	3	5	8
ϕ_n	0.72361	1.17082	1.89443	3.06525	4.95967	8.02492

7	8	9	10	11	12
13	21	34	55	89	144
12.9846	21.0095	33.9941	55.0036	88.9978	144.0014

13	14	15	16	17	18
233	377	610	987	1597	2584
232.9991	377.0005	609.99967	987.0002	1596.99987	2584.00008

2. Pisot numbers

Charles Pisot, c 1938

Definition 1. A number λ is called a ***Pisot number*** if it is an algebraic integer with $\lambda > 1$ such that all its Galois conjugates λ' satisfy $|\lambda'|<1$.

Pisot's Theorem. An real algebraic number $\lambda>1$ is a Pisot number if and only if there exists α so that

$$\lim_{n\to\infty}\left(\alpha\lambda^{n} \bmod 1\right) = 0. \qquad (*)$$

This means $\alpha\lambda^n \approx M \in \mathbf{N}$ for n large.

Note: γ is a Pisot number since it satisfies $\lambda^2 - \lambda - 1 = 0$ and its conjugate satisfies $|\gamma| < 1$.

- Recall that an *algebraic integer* is a root of a monic irreducible integer polynomial.
- Its Galois conjugates are the other roots of the polynomal.

Comment: It is still unknown whether any transcendental number can satisfy (*).

3. Fibonacci sequence on a free group

Let $\mathcal{A} = \{0,1\}$.

Let $\mathcal{A}^*$ be the set of words in $\mathcal{A}$.

Then $\mathcal{A}^* \subset \mathcal{F}(\mathcal{A})$ = the free group on $\mathcal{A}$.

- The non-abelian Fibonacci sequence:

Define $u_0 = 0, \quad u_1 = 01, \quad u_{n+1} = u_n u_{n-1}$

• *Iterate* to obtain an infinite sequence..

$u_0 = 0$

$u_1 = 01$

$u_2 = u_1 u_0 = 01\,0$

$u_3 = u_2 u_1 = 010\,01$

$u_4 = u_3 u_2 = 01001\,010$

$u_4 = u_3 u_2 = 01001010\,01001$

$$\sigma := \begin{cases} 0 \to 01 \\ 1 \to 0 \end{cases}$$

$$\sigma(u) = u$$

$$u = 0100101001001010010100100101001001\ldots \in \mathcal{A}^{\mathbb{Z}}$$

- What is σ?

$$\sigma := \begin{cases} 0 \to 01 \\ 1 \to 0 \end{cases}$$

$$\sigma \in \mathrm{End}(\mathcal{F}(\mathcal{A}))$$

- But...

$$\sigma^{-1} := \begin{cases} 0 \to 1 \\ 1 \to 1^{-1}0 \end{cases}$$

$$\sigma \in \mathrm{Aut}(\mathcal{F}(\mathcal{A}))$$

Abelianize (i.e., linearize):

$$\begin{array}{ccc} \mathcal{F}(\mathcal{A}) & \xrightarrow{\sigma} & \mathcal{F}(\mathcal{A}) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \mathbf{Z}^2 & \xrightarrow{F} & \mathbf{Z}^2 \end{array}$$

$F = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ is the *abelianization* of σ.

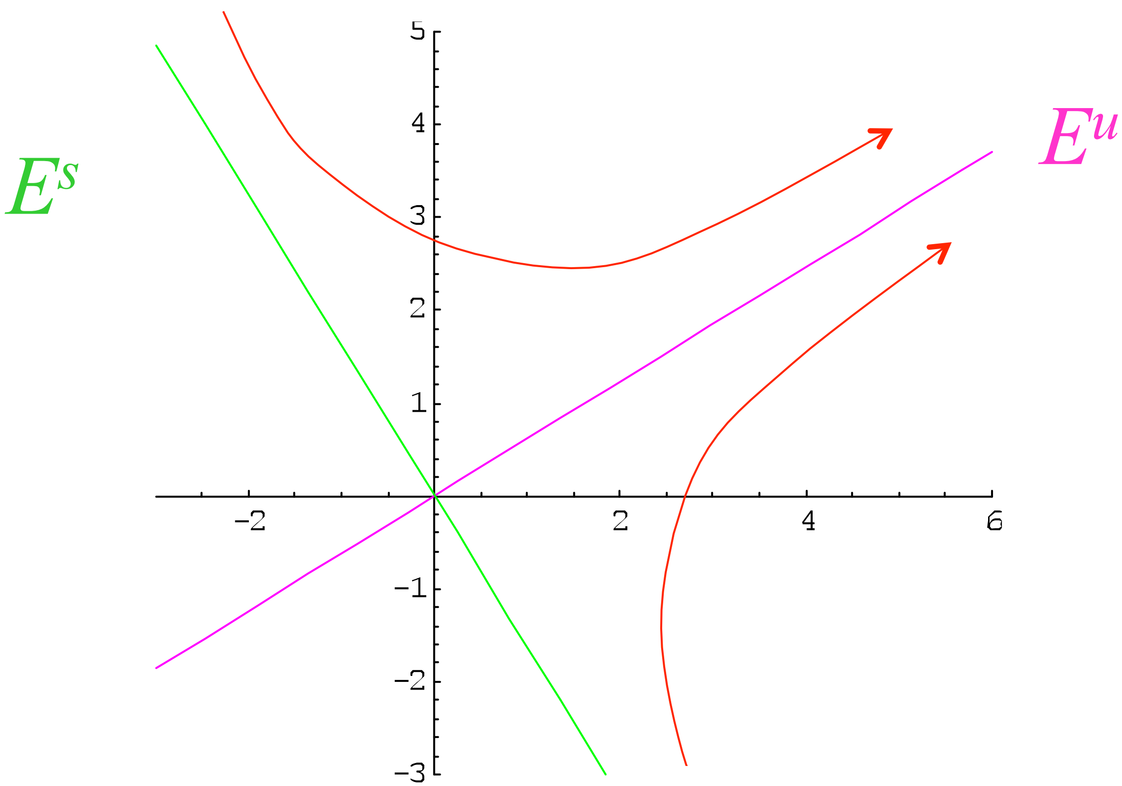
- How does F act (on $\mathbf{R}^2$)?

$$P^{-1}FP = \operatorname{diag}\{\gamma, (\tfrac{-1}{\gamma})\}$$

$$E_u = \text{span}\{w_u\} \text{ : unstable subspace.}$$

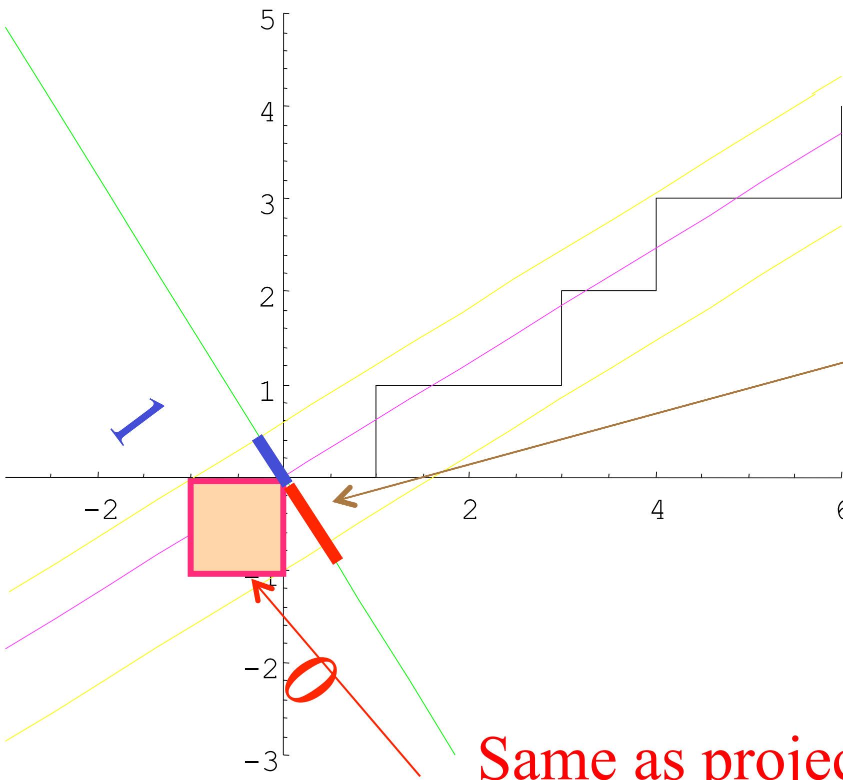
$$E_s = \text{span}\{w_s\} \text{ : stable subspace.}$$

$$P = [w_u, w_s]$$



4. Atomic surfaces and quasicrystals

Plot u = 01001010010 0...



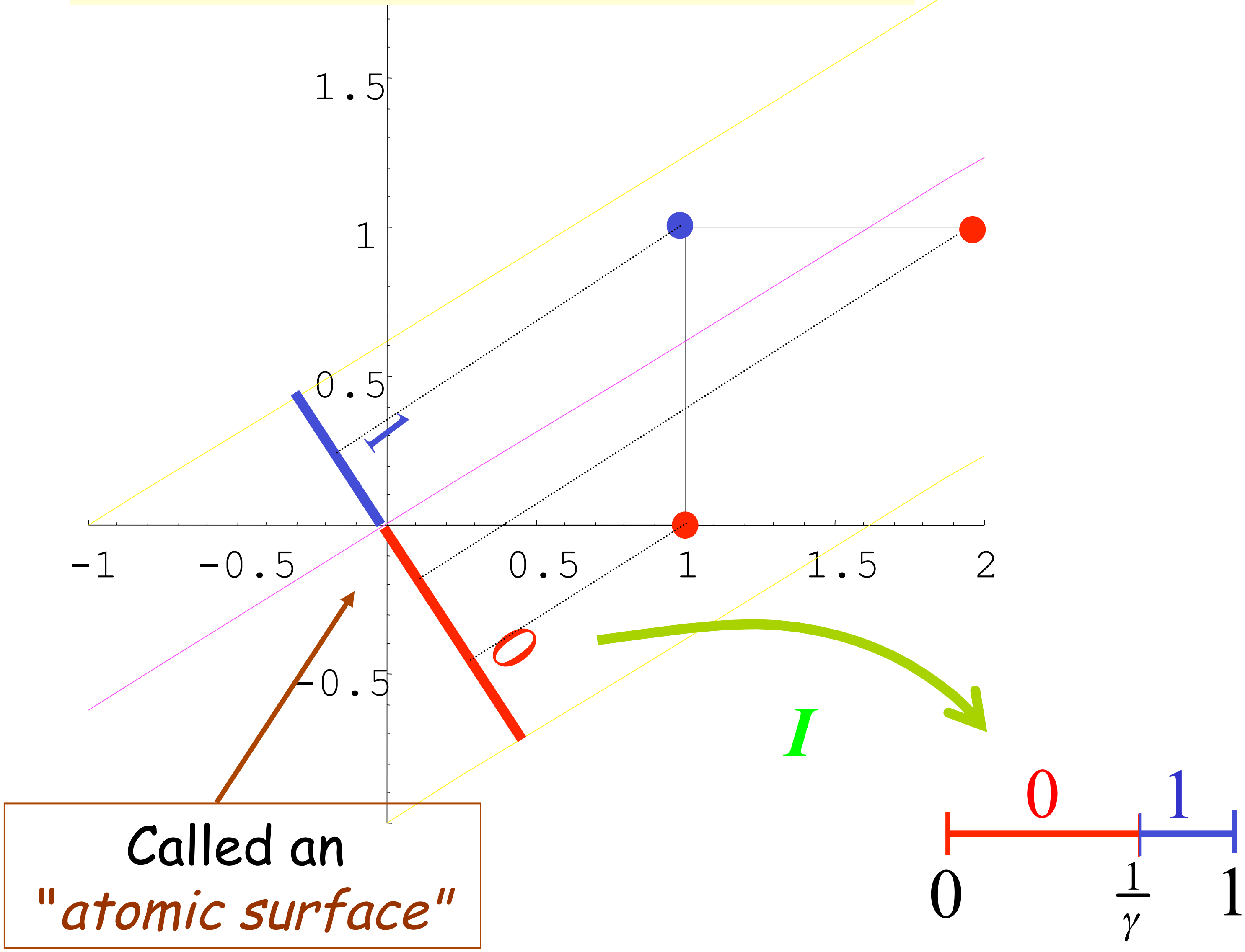
Projection to E^s
called an
atomic surface

Same as projection
of unit square.

The *Pisot property.*

Video
!

Project to stable direction:

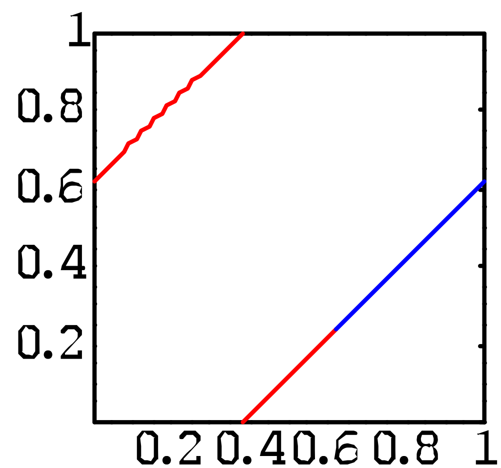


Dynamical systems in I

- Let τ be the *lef t shif* øf .

τu € Rx

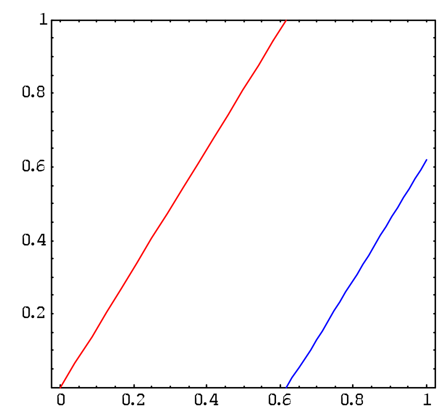
$$R : I \to I$$
$$Rx = x + \tfrac{1}{\gamma} \bmod 1$$



- How about σ?

$\sigma^{-1} u$ € $T_\gamma x$

$$T_\gamma : I \to I$$
$$T_\gamma x = \gamma x \bmod 1$$

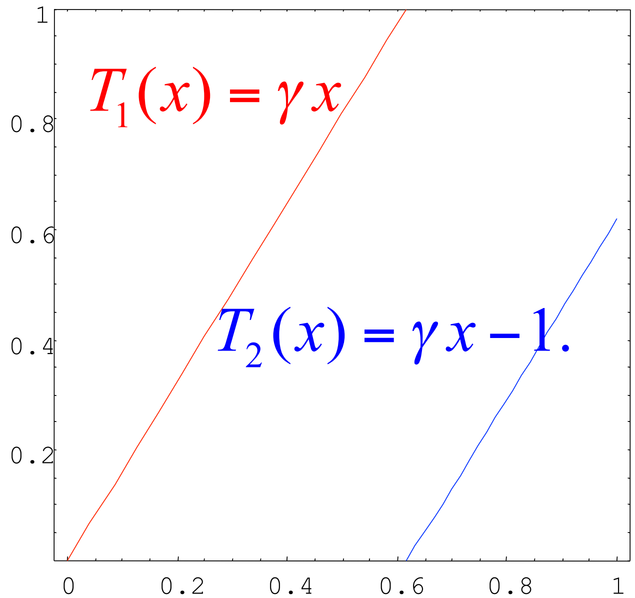


T_γ called β-transformation.

Iterated Function System (IFS):

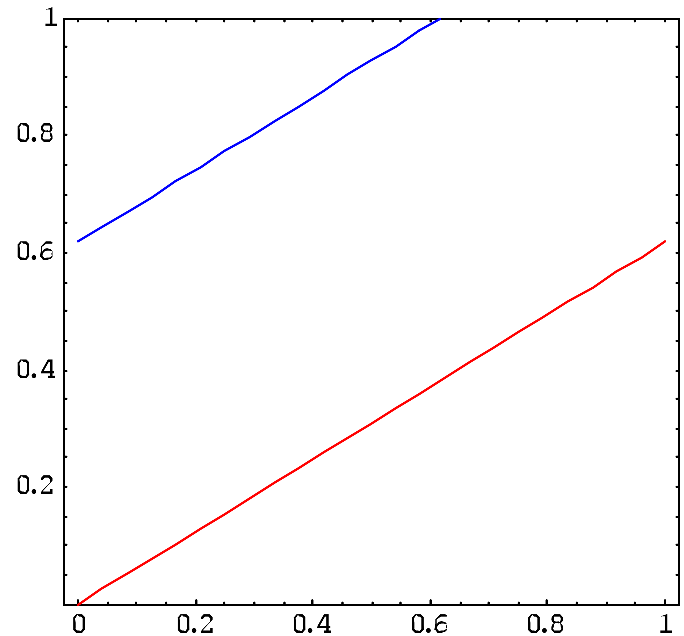
$$T_2^{-1}(x) = \tfrac{1}{\gamma}x + \tfrac{1}{\gamma}$$

$$T_1^{-1}(x) = \tfrac{1}{\gamma}x$$



$\sigma^{-1}u$ € $T_\gamma x$

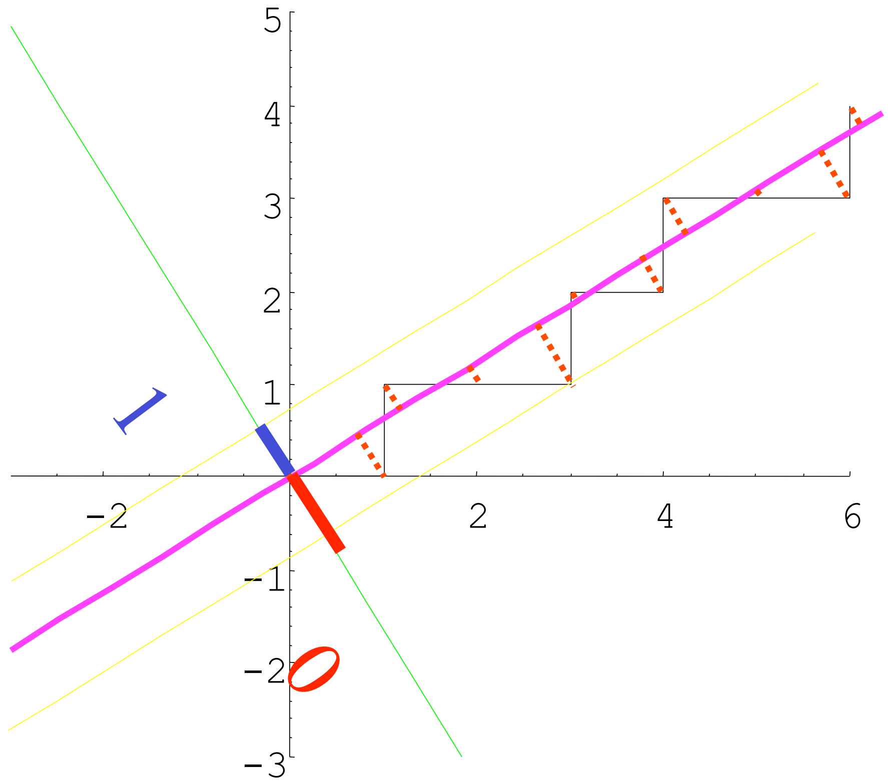
$$T_\gamma x = \gamma x \bmod 1$$



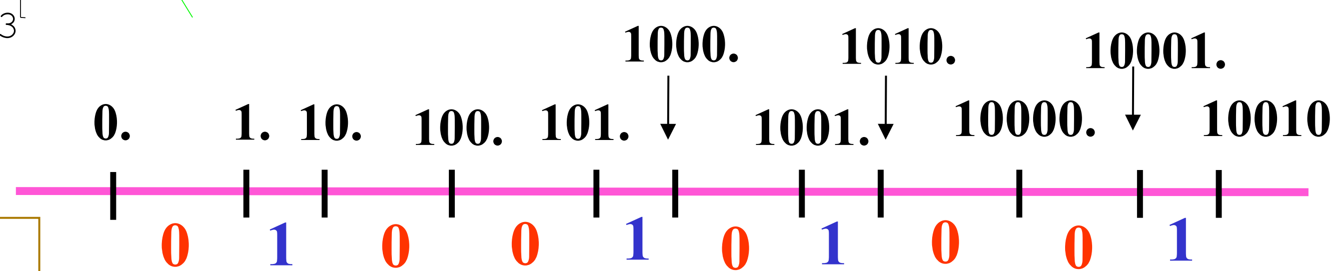
MCRM Algorithm: $I = T_1^{-1}(I) \cup T_2^{-1}(I).$

The Fibonacci quasicrystal

$$x_i x_{i+1} = 0$$



Duality between
atomic surface and
quasicrystal.



Atoms at "γ-integers"

• β-expansions

Given any β>1, we can express any positive real number x in the form

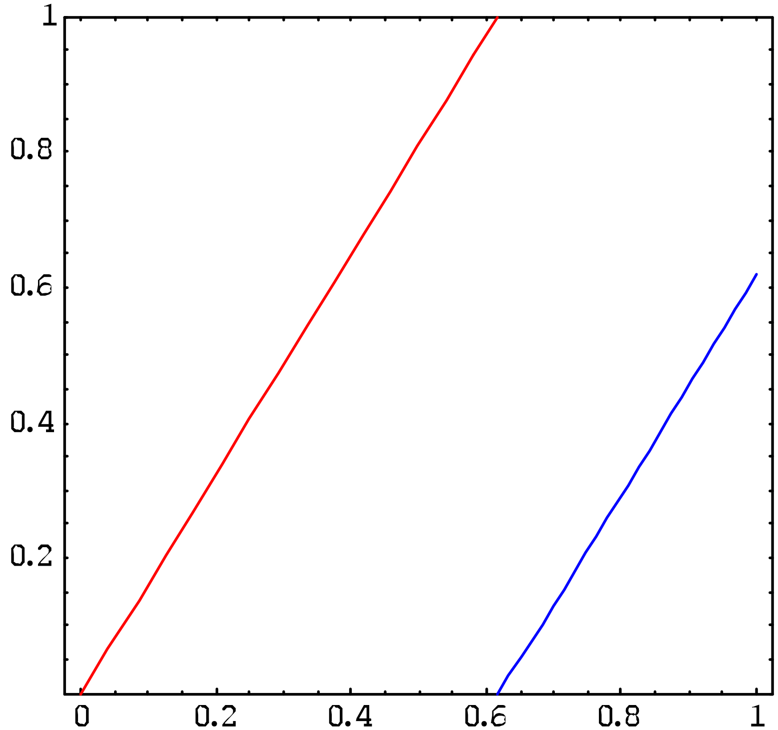
$$x = \sum_{k=-N}^{\infty} \frac{x_i}{\beta^i}$$

where $x_i \in \{0,1,\ldots,\lfloor \beta \rfloor\}$.

Write $x_\gamma = .x_1x_2x_3x_4x_5x_6x_7\ldots$

But there is no reason to require $\beta \in \mathbf{N}$!

Take $\beta = \gamma$. $T_\gamma(x) = \gamma \cdot x \bmod 1$.



$$D(x) = \begin{cases} 0 \text{ if } x \in [0, 1/\gamma) \\ 1 \text{ if } x \in [1/\gamma, 1) \end{cases}.$$

$$x_i = D(T_\gamma^i x)$$

This case is "nice" since $\beta = \gamma$ is a Pisot number.

The sequence 11 is forbidden.

5. The "tribonacci" sequence

0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149,...

$$t_{n+1} = t_n + t_{n-1} + t_{n-2}$$

$$t_0 = 0,\ t_1 = 0, t_2 = 1$$

$$v_n = \begin{bmatrix} t_{n+2} \\ t_{n+1} \\ t_n \end{bmatrix} \quad v_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_{n+1} = Av_n$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Eigenvalues:

$$\rho \overset{\text{def}}{=} \lambda_u = \frac{1}{3}\left(1 + \sqrt[3]{19 - 3\sqrt{33}} + \sqrt[3]{19 + 3\sqrt{33}}\right) \approx 1.83929$$

$$\lambda_s, \overline{\lambda_s} \in \mathbf{C} \quad \begin{cases} \lambda_s \approx -0.419643 + 0.606291 i \\ |\lambda_s| \approx 0.737353 \end{cases}$$

$$t_n = \alpha_1 \rho^n + \alpha_2 \lambda_s^n + \alpha_3 \overline{\lambda_s}^n$$

Binet-type formula

Consequence of Pisot property.

$$t_n \approx \alpha_1 \rho^n$$

6. Non-abelian tribonacci sequence

$$u_{n+1} = u_n u_{n-1} u_{n-2}$$

$$u_0 = 1,\ u_1 = 12,\ u_2 = 1213$$

On $\mathcal{A}^* \subset \mathcal{F}(\mathcal{A}),\quad \mathcal{A} = \{1,2,3\}$

u=12131211213121213121121312131 21...

(Gerard Rauzy, c.1982)

Institut de Mathématique de Luminy
March 2002

Corresponding free group endomorphism:

$$\theta := \begin{cases} 1 \to 12 \\ 2 \to 13 \\ 3 \to 1 \end{cases}$$

"Rauzy substitution"

$$\theta(u) = u$$

Abelianize:

$$\begin{array}{ccc} \mathcal{F}(\mathcal{A}) & \xrightarrow{\theta} & \mathcal{F}(\mathcal{A}) \\ p \downarrow & & p \downarrow \\ \mathbf{Z}^2 & \xrightarrow{A} & \mathbf{Z}^2 \end{array}$$

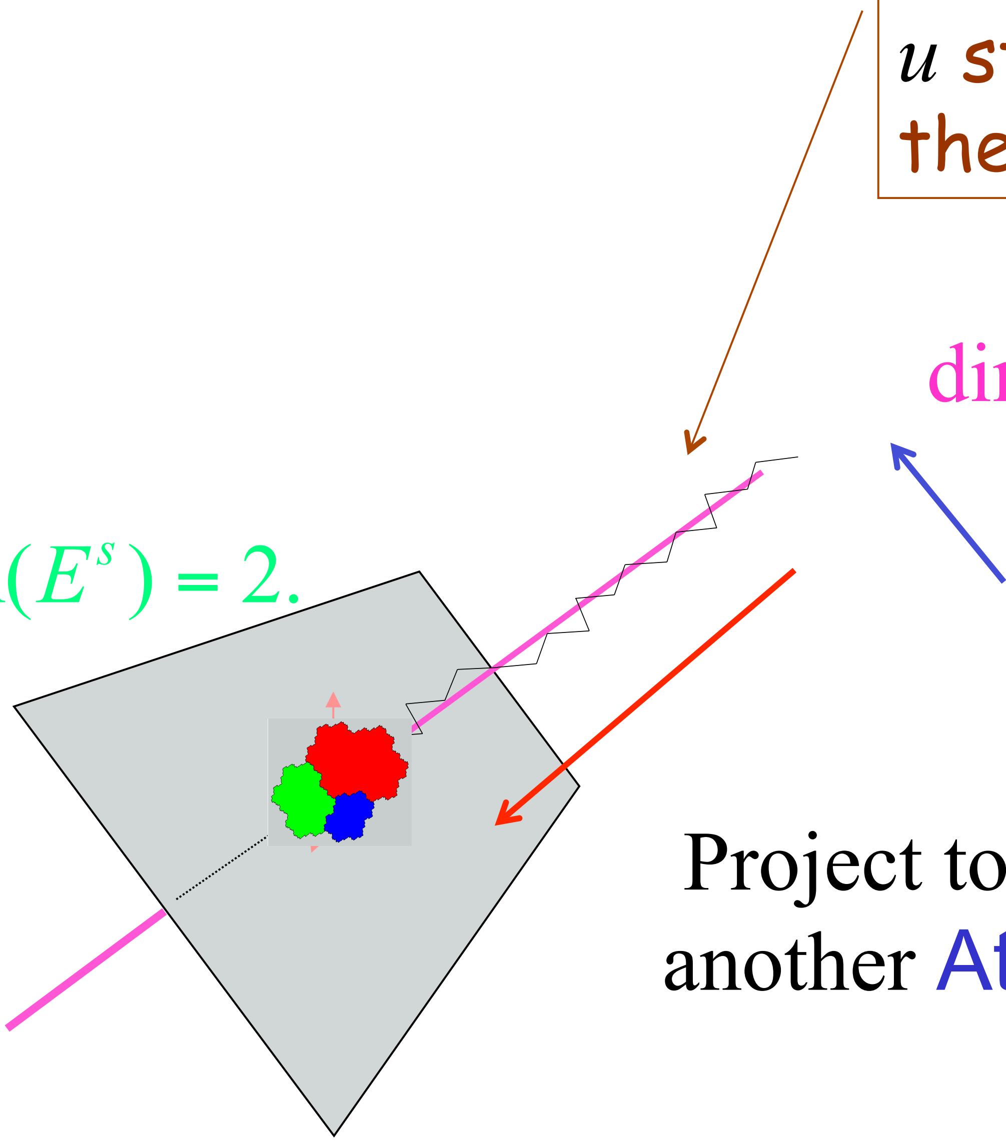
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

u stays near E^u by the Pisot property,

$\dim(E^u)=1$

Project to E^{u}. Get Rauzy quasicrystal

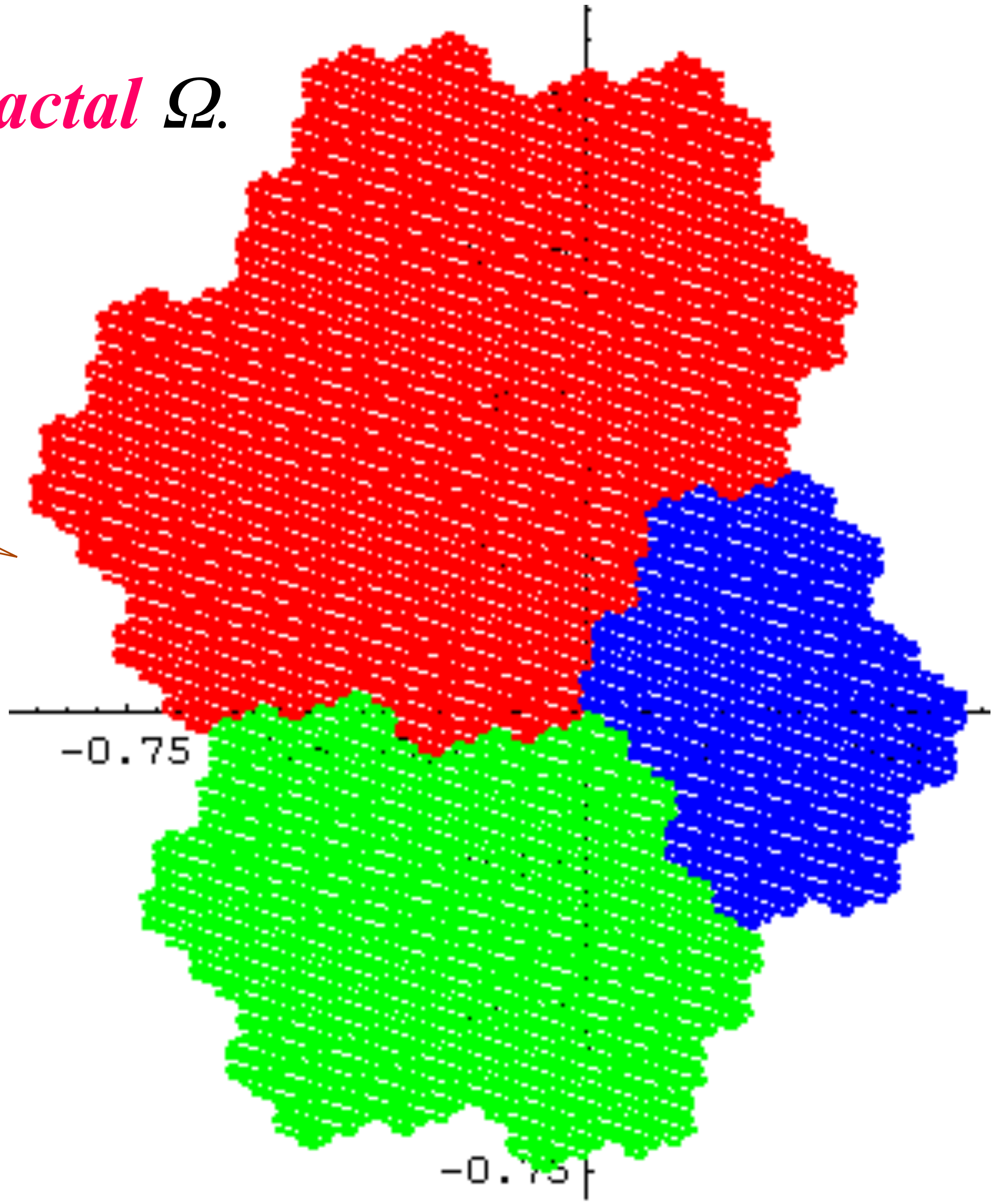
$\dim(E^s) = 2.$



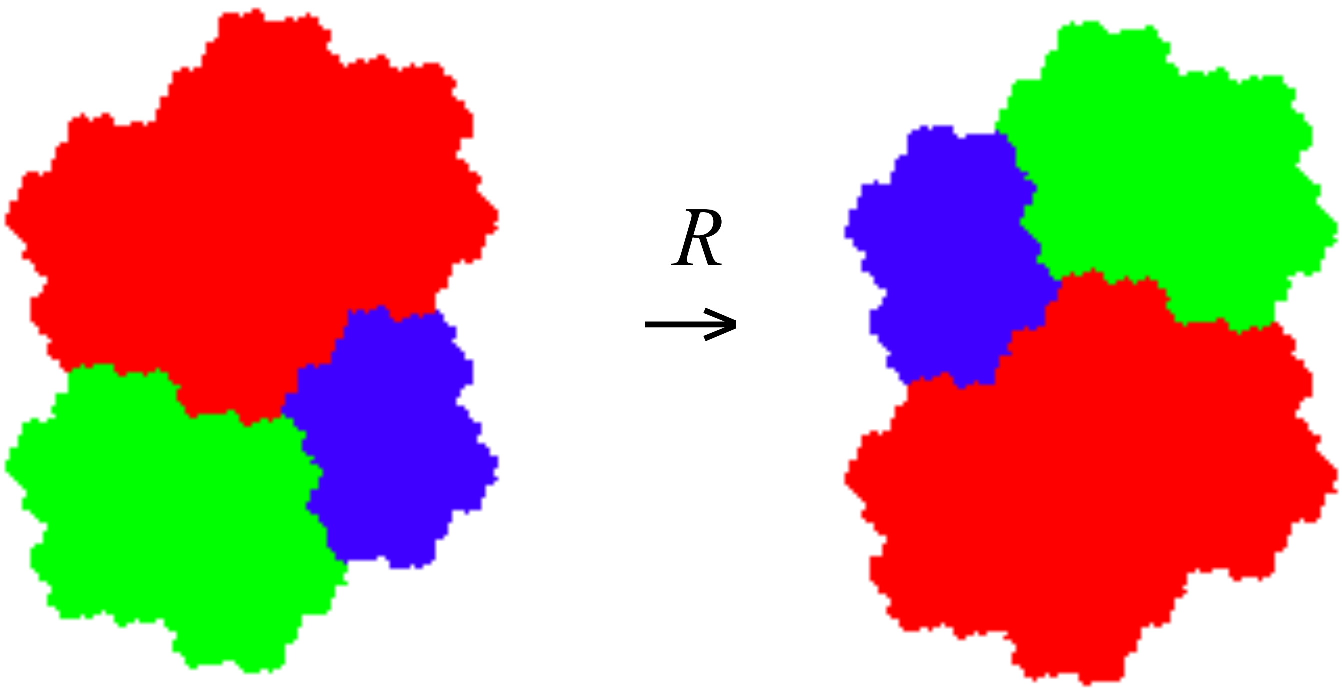
Project to E_{s} . Get another Atomic Surface...

...the ***Rauzy Fractal*** Ω.

click!

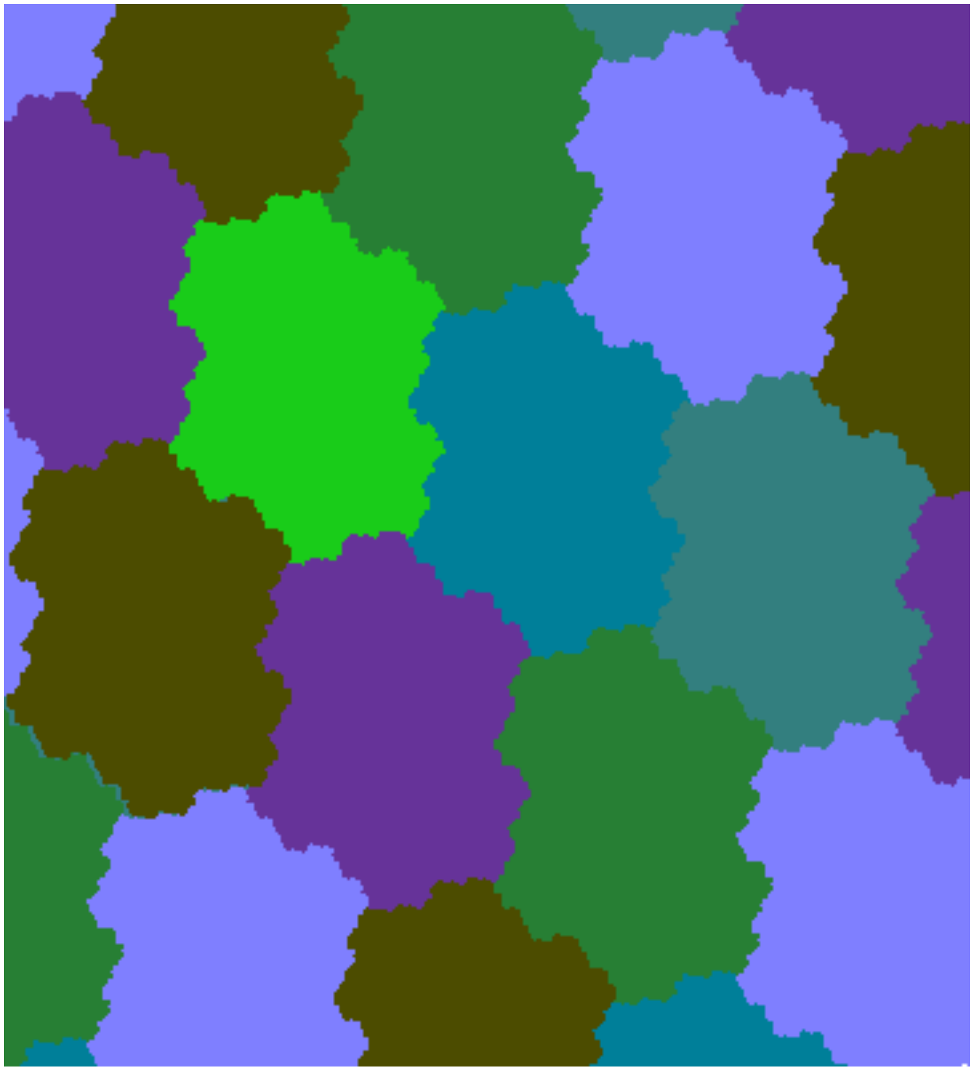


There is a *domain exchange* action R on Ω.



τu € Rx

•Understanding *R*



$$\Omega \approx \mathbf{T}^2 = \mathbf{R}^2 / \Gamma$$

$$\Gamma \approx \mathbf{Z}^2$$

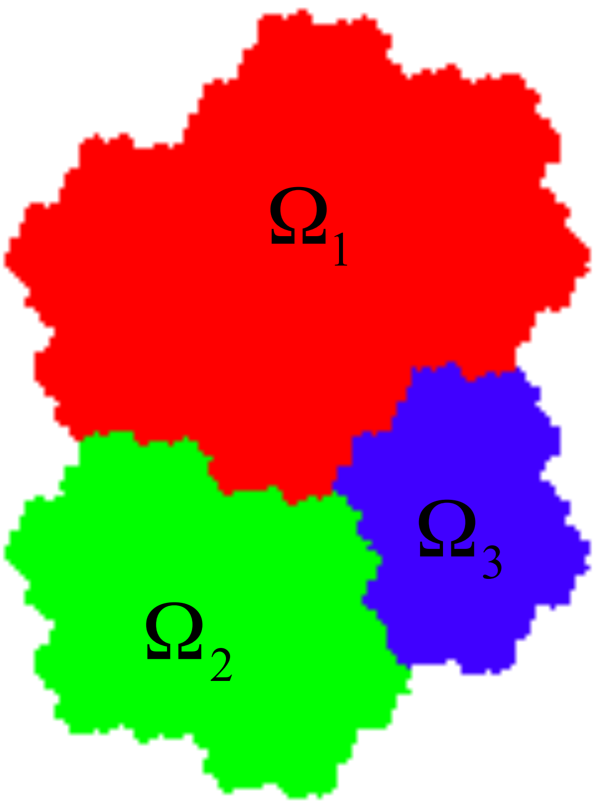
$$R : \mathbf{T}^2 \to \mathbf{T}^2$$

$$Rx = (x + w) \bmod 1$$

$$w = \begin{bmatrix} \lambda_u \\ \lambda_u^2 \end{bmatrix}$$

7. An IFS for Ω

Take $\beta = 1/\lambda_s \approx$ -0.771845 + 1.11514i



$|\beta| \approx 1.3562$

Identify $E^s \approx \mathbf{C}$

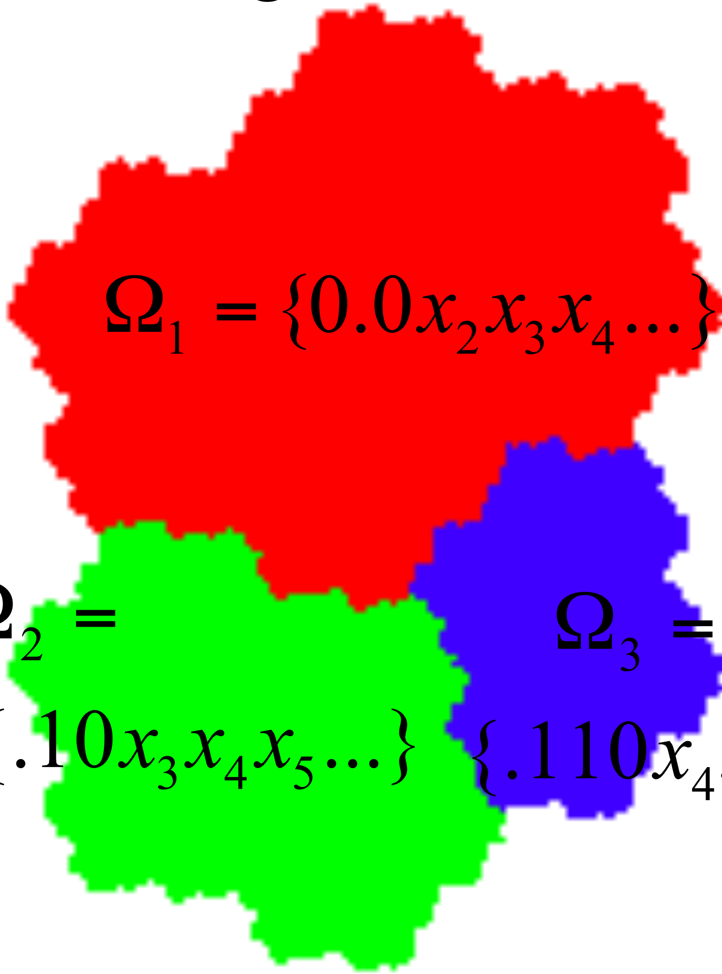
Define:

$$T_\beta(z) = \begin{cases} \beta z & \text{if } z \in \Omega_1, \\ \beta z - 1 & \text{if } z \in \Omega_2 \cup \Omega_3. \end{cases}$$

Addressing:

$$T_1(z) = \beta z$$

$$T_2(z) = \beta z - 1$$



$$T_1(\Omega_1) = \Omega$$

$$T_2(\Omega_2) = \Omega_1$$

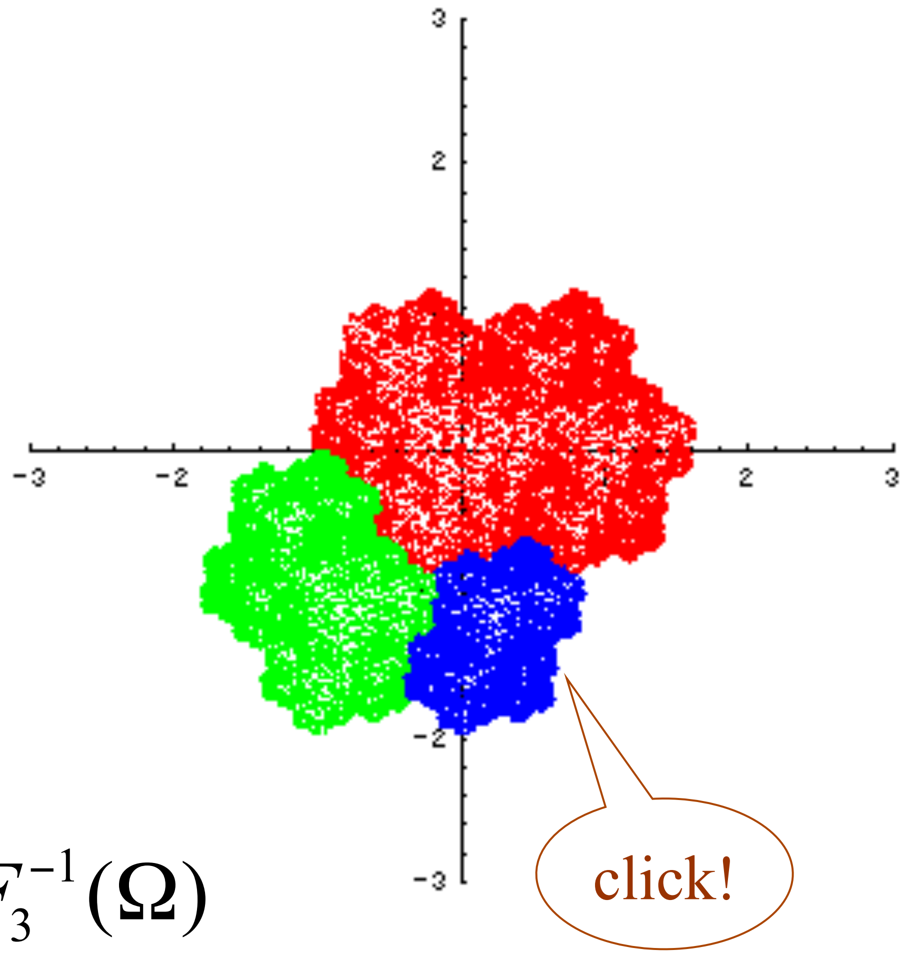
$$T_2(\Omega_3) = \Omega_2$$

$$F_1 = T_1,\ F_2 = T_1T_2,\ F_3 = T_1T_2^2$$

$$F_i(\Omega_i) = \Omega$$

Invert to get an IFS. Render using the MCR.

$$\begin{cases} F_1^{-1}(z) = \frac{1}{\beta} z \\ F_2^{-1}(z) = \frac{1}{\beta^2} z + \frac{1}{\beta} \\ F_3^{-1}(z) = \frac{1}{\beta^3} z + \frac{1}{\beta^2} + \frac{1}{\beta} \end{cases}$$



$$\Omega = F_1^{-1}(\Omega) \cup F_2^{-1}(\Omega) \cup F_3^{-1}(\Omega)$$

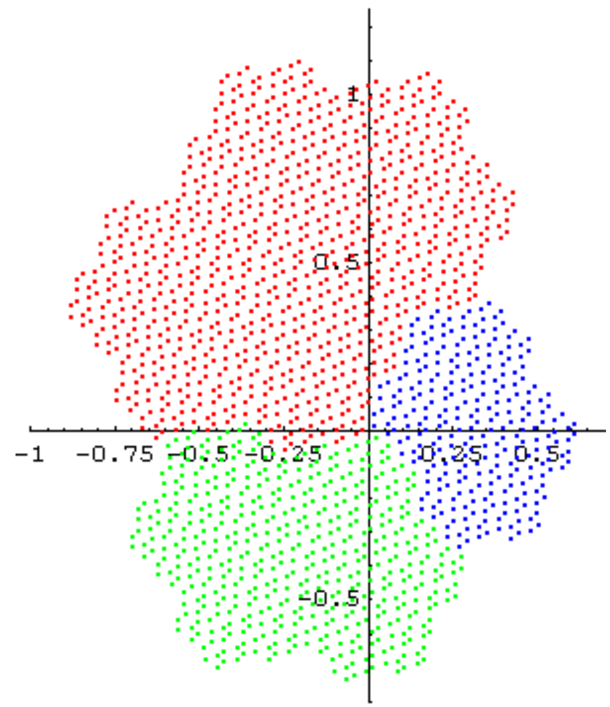
ρ-integers:

0.
1.
10.
11.
100.
101.
110.
1000.
...

β-van der Corput points:

0.0
0.1
0.01
0.11
0.001
0.101
0.011
0.0001
...

$x_{i-1}x_ix_{i+1} = 0$



Rauzy quasicrystal:
u projected to E^u

Rauzy fractal
u projected to E^s

8. The Rauzy fractal as an L-system

Recall that the *tribonacci matrix*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\theta := \begin{cases} 1 \to 12 \\ 2 \to 13 \\ 3 \to 1 \end{cases}$$

is the abelianization of the *Rauzy substitution*

But...there are actually 4 ways to define a substitution θ with abelianization A.

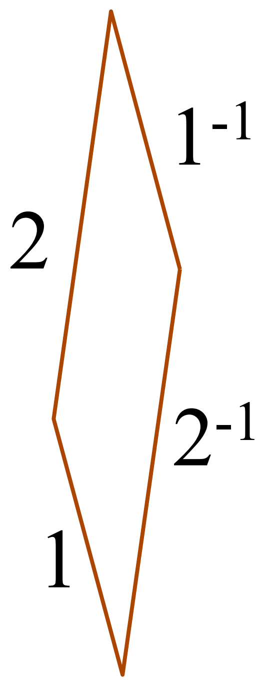
The *Rauzy substitution* is invertible!

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -1 \end{pmatrix} \qquad \theta^{-1}: \begin{cases} 1 \to 3 \\ 2 \to 3^{-1}1 \\ 3 \to 3^{-1}2 \end{cases}$$

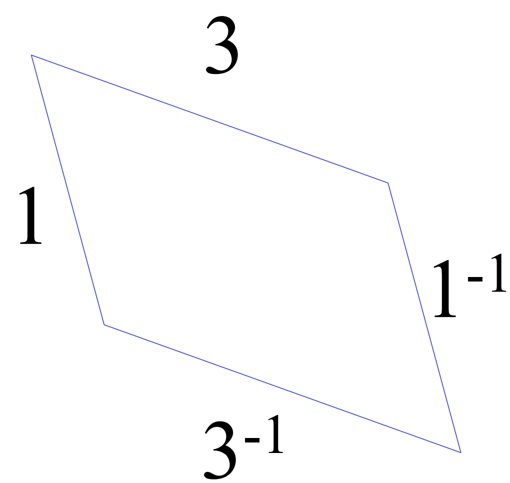
It is an automorphism of $\mathcal{F}(\mathcal{A})$.

Recall this was also the case with
the Fibonacci substitution.

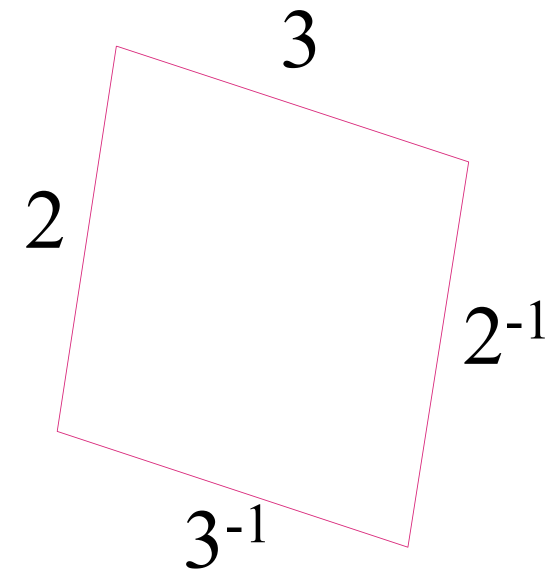
Cycles in $\mathcal{F}(\mathcal{A})$



$$[1,2] = 121^{-1}2^{-1}$$



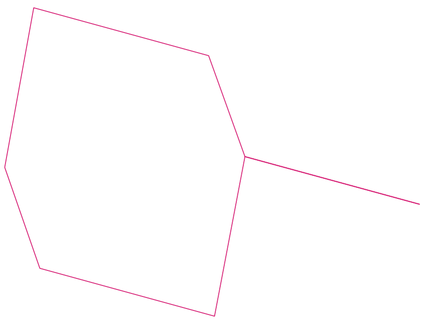
$$[1,3] = 131^{-1}3^{-1}$$

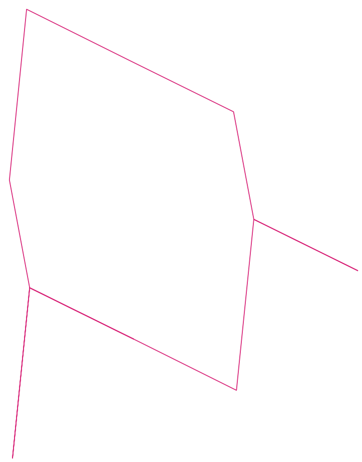


$$[2,3] = 232^{-1}3^{-1}$$

Edges are projections of standard basis vectors to E^s

$$\theta^{-1}[2,3] = \theta^{-1}(232^{-1}3^{-1}) = 3^{-1}13^{-1}21^{-1}32^{-1}3$$





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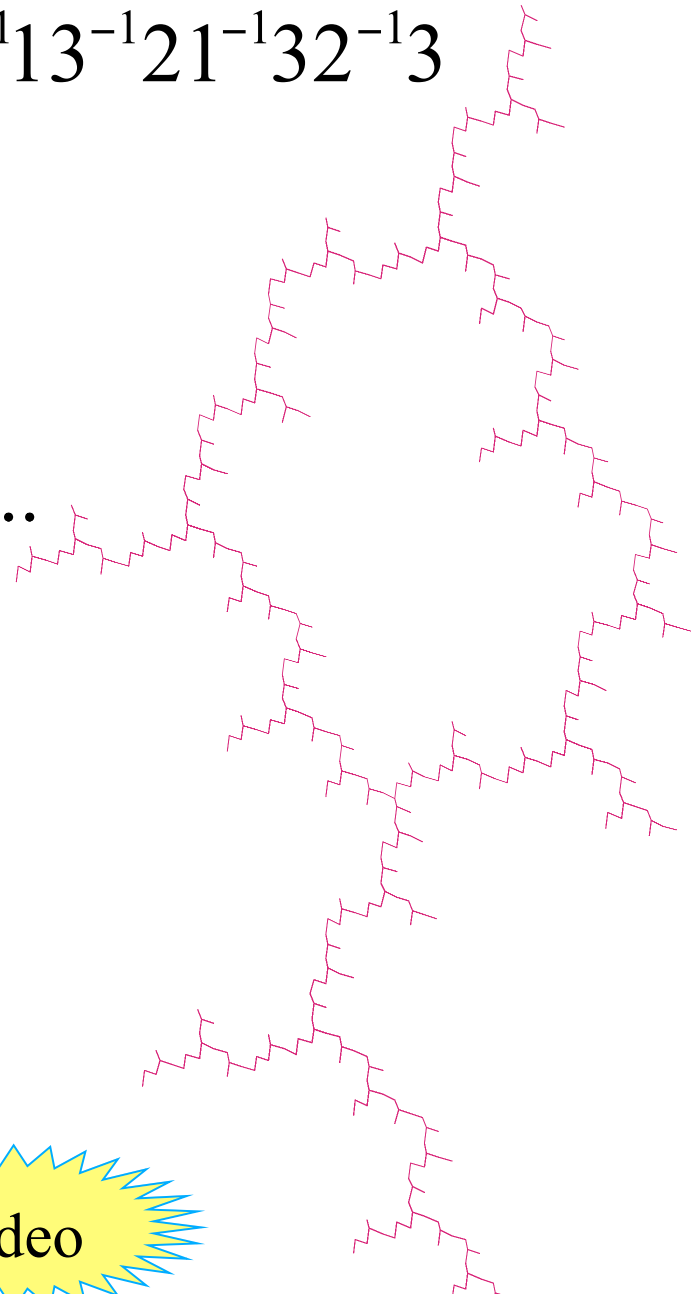
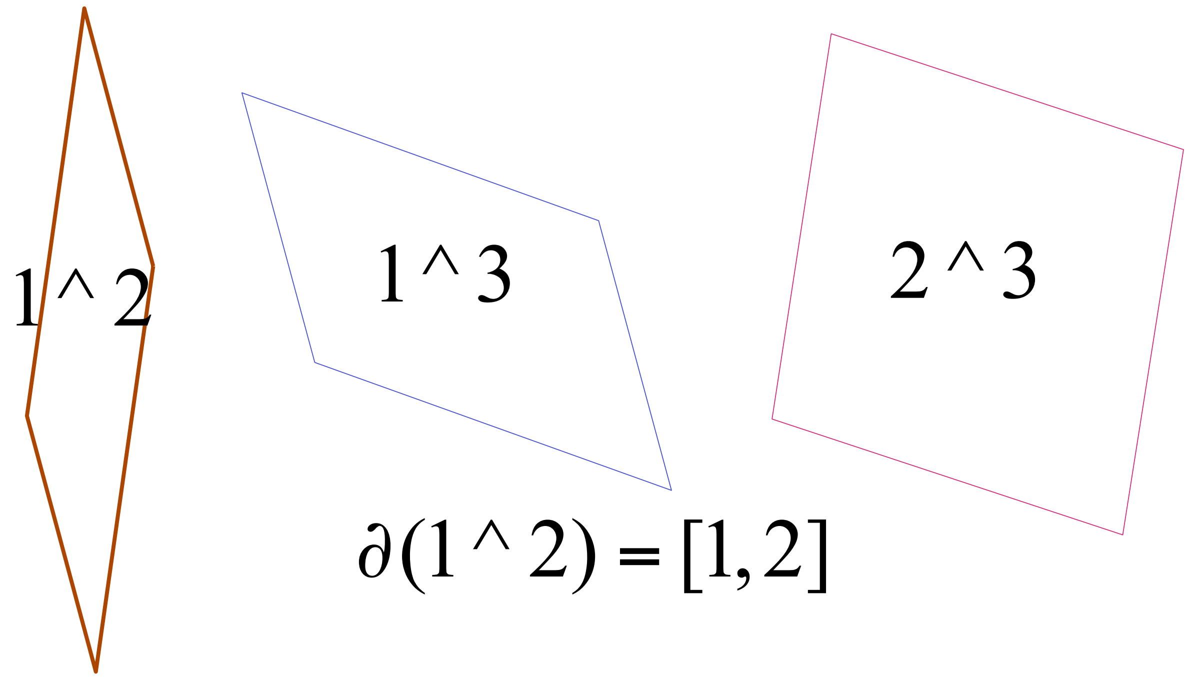


Iterate: $(\theta^{-1})^n[2,3] = \ldots$

!

As tiles:

$$\theta^{-1}: \begin{cases} 1 \to 3 \\ 2 \to 3^{-1}1 \\ 3 \to 3^{-1}2 \end{cases}$$



$$\partial(1 \wedge 2) = [1,2]$$

Tiling substitution induced by θ^{-1}:

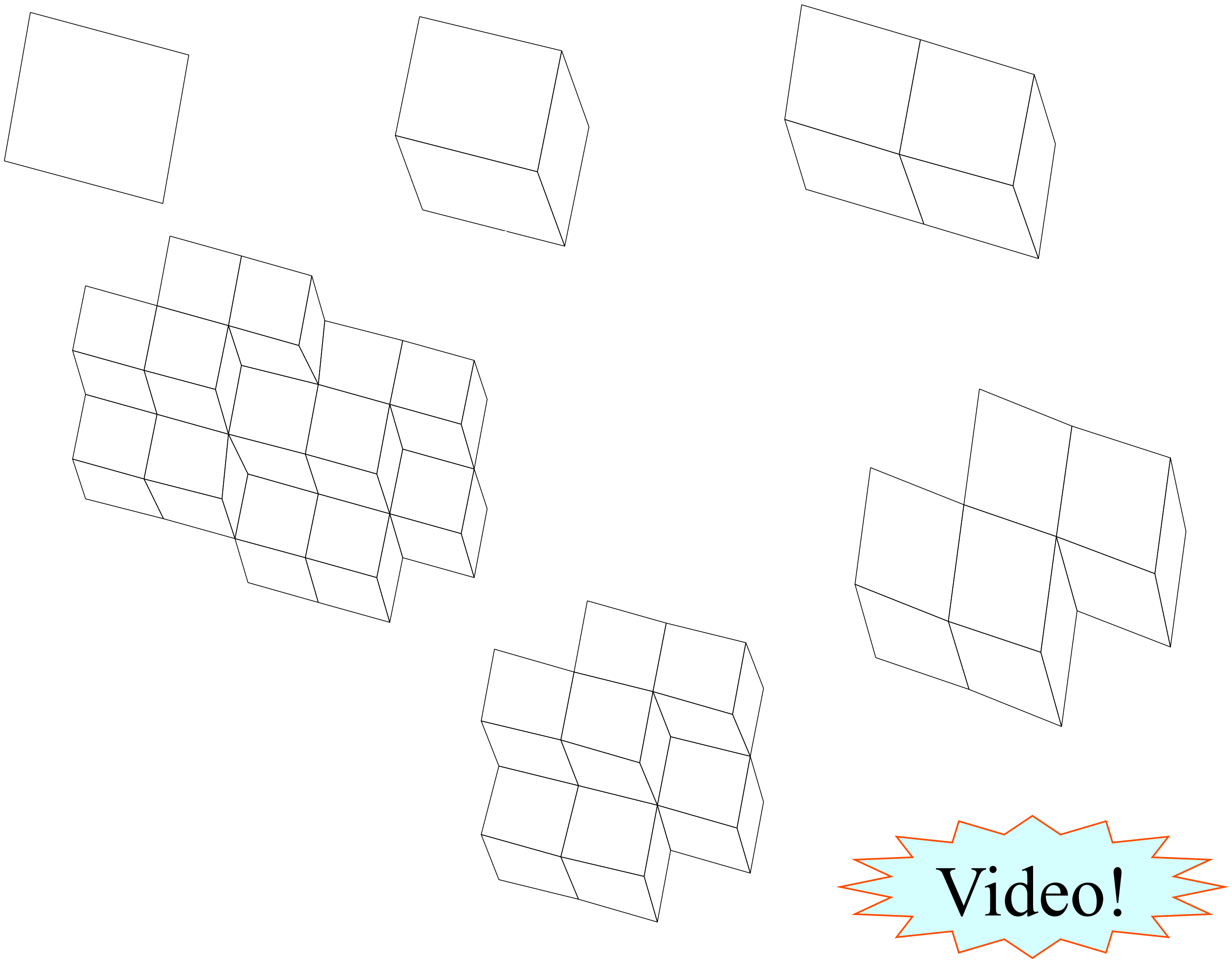
$$1 \wedge 2 \to 3 \wedge 3^{-1} + 3 \wedge 1 = 3 \wedge 1$$

$$1 \wedge 3 \to 3 \wedge 3^{-1} + 3 \wedge 2 = 3 \wedge 2$$

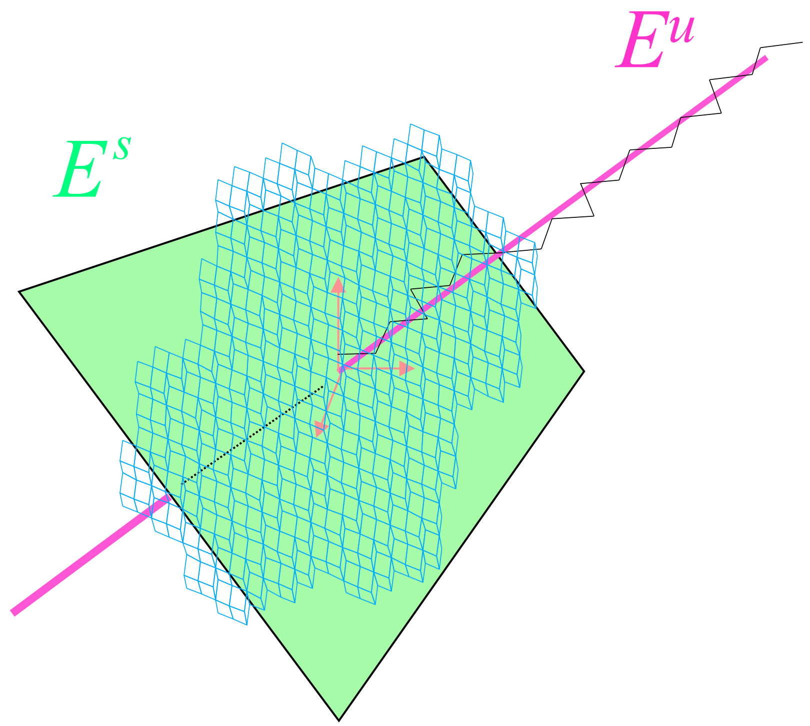
$$2 \wedge 3 \to 3^{-1} \wedge 3^{-1} + 1 \wedge 3^{-1} + 3^{-1} \wedge 2 + 1 \wedge 2$$

$$= 1 \wedge 3 + 2 \wedge 3 + 1 \wedge 2$$

The dual Rauzy quasicrystal



As a subset of $\mathbf{R}^2$ the *dual Rauzy quasicrystal* is discrete approximation of E^s.



9. Rauzy Fractal Gallery

Victor Sirvent

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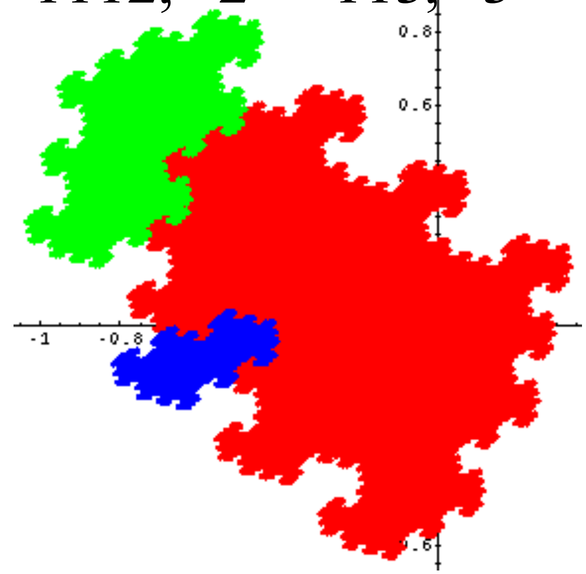
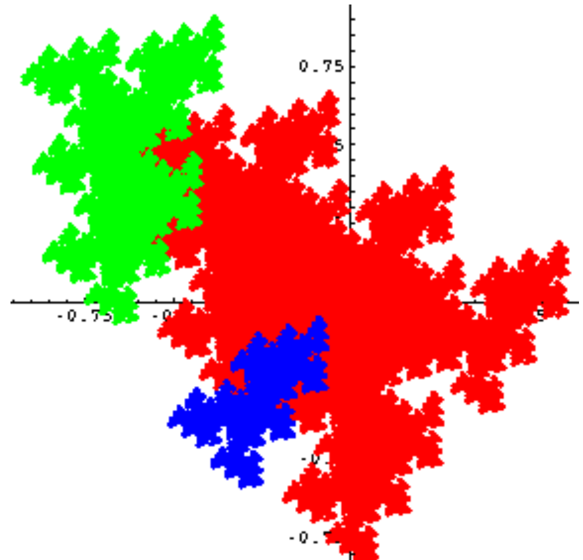
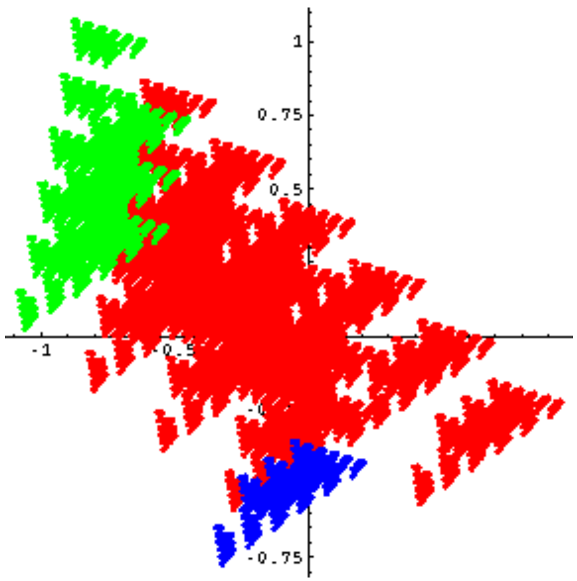


$1 \to 1112, \quad 2 \to 113, \quad 3 \to 1$



$1 \to 1112, \quad 2 \to 131, \quad 3 \to 1$



$1 \to 1112, \quad 2 \to 311, \quad 3 \to 1$

Video **http://www.ma.usb.ve/~vsirvent/gallery/rauzy.html**

$$\zeta_1 := \begin{cases} 1 \to 1112 \\ 2 \to 113 \\ 3 \to 1 \end{cases}$$

Invertible?

$$A_\zeta = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad A_\zeta^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -2 \end{pmatrix}$$

$$\zeta_1^{-1} := \begin{cases} 1 \to 3 \\ 2 \to 3^{-1}3^{-1}3^{-1}1 \\ 3 \to 3^{-1}3^{-1}2 \end{cases}$$

$$\zeta_2 := \begin{cases} 1 \rightarrow 1112 \\ 2 \rightarrow 131 \\ 3 \rightarrow 1 \end{cases}$$

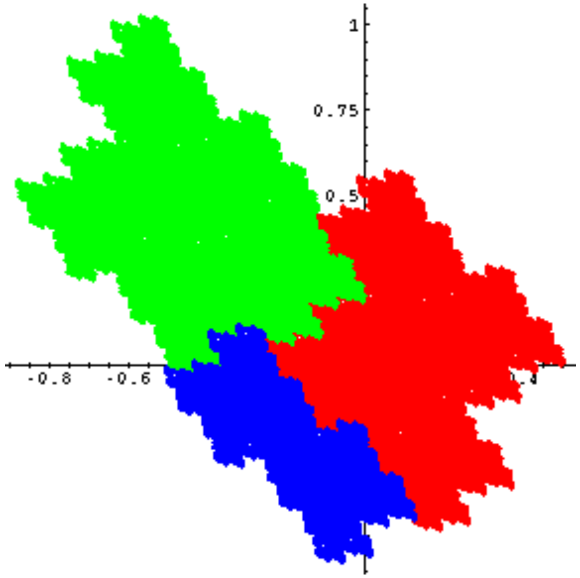
Invertible?

$$\zeta_1^{-1} := \begin{cases} 1 \rightarrow 3 \\ 2 \rightarrow 3^{-1}3^{-1}3^{-1}1 \\ 3 \rightarrow 3^{-1}\, 2\, 3^{-1} \end{cases}$$

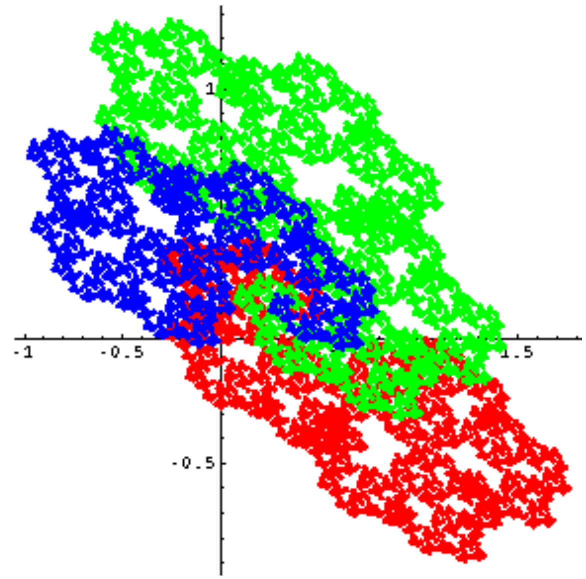
$$A_\zeta^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & -2 \end{pmatrix}$$

$$\zeta_3 := \begin{cases} 1 \rightarrow 1112 \\ 2 \rightarrow 311 \\ 3 \rightarrow 1 \end{cases}$$

Invertible?

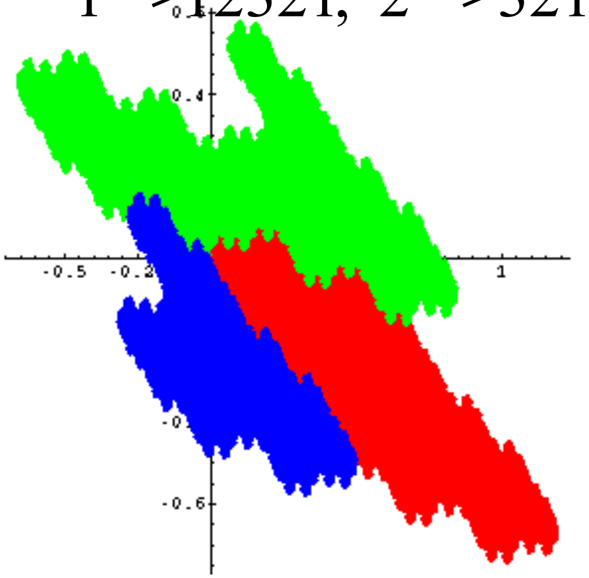


$1 \rightarrow 123,\ 2 \rightarrow 12,\ 3 \rightarrow 2$

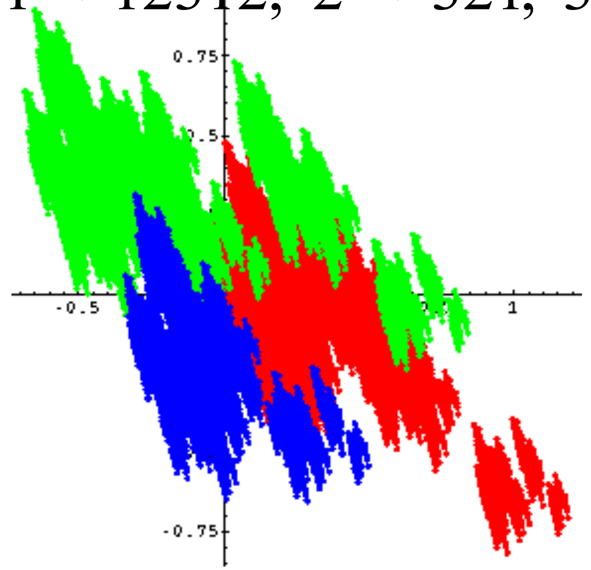


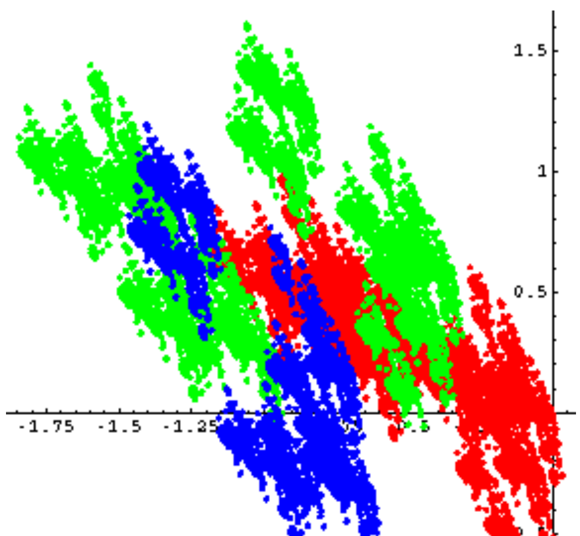
$1 \rightarrow 32,\ 2 \rightarrow 1,\ 3 \rightarrow 2$

$1 \to 12321,\ 2 \to 321,\ 3 \to 2$

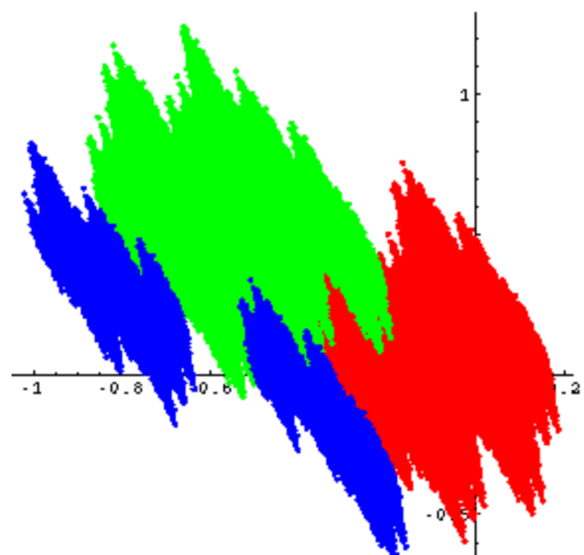


$1 \to 12312,\ 2 \to 321,\ 3 \to 2$





$1 \to 11223,\ 2 \to 321,\ 3 \to 2$



$1 \to 12312,\ 2 \to 123,\ 3 \to 2$