TILINGS ASSOCIATED WITH NON-PISOT MATRICES

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Abstract. Suppose $A \in \mathrm{Gl}_d(\mathbb{Z})$ has a 2-dimensional expanding subspace E^u, satisfies a regularity condition, called "good star", and has $A^* \geq 0$, where A^* is an *oriented compound* of A. A morphism θ of the free group on $\{1, 2, \ldots, d\}$ is called a *non-abelianization* of A if it has structure matrix A. We show that there is a *tiling substitution* Θ whose "boundary substitution" $\theta = \partial(\Theta)$ is a non abelianization of A. Such a tiling substitution Θ leads to a self-affine tiling of $E^u \sim \mathbb{R}^2$ with $A_u := A|_{E_u} \in \mathrm{GL}_2(\mathbb{R})$ as its expansion. In the last section we find conditions on A so that A^* has no negative entries.

1. Introduction

A *tiling substitution* Θ is a mapping from tiles in $\mathbb{R}^2$ to finite tiling patches that has enough regularity to be applied to tiling patches. This permits the iteration of Θ, starting with a single tile, or a small tiling patch, to obtain larger tiling patches, and ultimately tilings of the plane.

A tiling substitution Θ induces a mapping $\theta = \partial\Theta$ on tile boundaries. Assuming there are finitely many polygonal prototiles, having d different boundary segments, this boundary map θ can be viewed as a 1-dimensional *substitution* on $\mathcal{B} := \{1, 2, \ldots, d\}$. The *structure matrix* of θ is the matrix A whose i, jth entry counts the number of times the edge j occurs in the substitution $\theta(i)$. Frequently, this matrix A will be hyperbolic, and have a 2-dimensional expanding subspace, corresponding to how the tiling substitution "expands" $\mathbb{R}^2$.

The process of iterting a (sufficiently nice) tiling substitution Θ yields an uncountable collection X_Θ of tilings of $\mathbb{R}^2$. This set X_Θ can be regarded as a compact metric space (see e.g., [16]), on which Θ acts as a homeomorphism. Typically Θ acts on a tiling $y \in X_\Theta$ as follows. First y is replaced by $A_u y$, for A_u a linear expansion of the plane. Then the new tiling $\Theta(y)$ is obtained by applying a *local mapping* (see [16]) to $A_u y$, which essentially tiles certain regions of $A_u y$ according to their shapes. The matrix A_u is the restriction of the structure matrix A of θ to its expanding subspace. In the best circumstances the action of Θ on X_Θ is semiconjugate to the action of A on $\mathbb{T}^d$ as a toral automorphism. In fact, the tilings $y \in X_\theta$ are closely related to Markov partitions for A (see [11], [9]).

In this paper we seek to reverse the process described above. Starting with a hyperbolic matrix $A \in \mathrm{GL}_d(\mathbb{Z})$ having a 2-dimensional expanding subspace, we want (1) to find a 1-dimensional substitution θ, or more generally a *free group endomorphism* θ, that has A as its structure matrix, and (2) find a tiling substitution Θ with $\partial\Theta = \theta$. As it turns out, not every A is works, at least not for the method we describe here. However, we find sufficient conditions: namely, that a certain "oriented compound" matrix, which we denote by A^*, satisfies $A^* \geq 0$. In the last part of the paper we give an easy condition on A to check for this.

Once we have a suitable A we need to carefully choose one of the many endomorphisms θ with structure matrix A. We describe a condition called *properly ordered* that guarentees θ is the boundary of a tiling substitution.

The idea of starting with a matrix, making a corresponding substitution, and using it geometrically first appears in G. Rauzy [15], where the "Rauzy fractal" is introduced (a nice update on this theory appears in [3], highlighting its connections to number theory, dynamics and fractal geometry). The idea of using A and a 1-dimensional substitution to construct a tiling substitution appears many works by P. Arnoux, S. Ito and their co-workers (see e.g., [11], [4], [6], and [7]). An approach related to the one described here appears in [9].

However, all of the works mentioned above assume some form of the *Pisot condition*, a codimension-1 condition, which when expressed in the context of this discussion would say A has a 1-dimensional *contracting*[1] subspace. Recall that a real algebraic integer $\lambda > 1$ is a *Pisot number* if all its algebraic conjugates satisfy $|\lambda'| < 1$. The role of the Pisot condition in most previous approaches to similar problems cannot be overstated. In this paper we also drop the common assumption that $A \geq 0$. A program simialar to ours, (in particular, without the Pisot assumption) appears in the dissertation [13] of R. Kenyon, but with few details. For other approaches the non-Pisot case see [12], [9] and [8].

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2. Linear theory

2.1. The non-Pisot property. Let $A \in \mathrm{Gl}_d(\mathbb{Z})$ have

$$|\lambda_1| \geq |\lambda_2| > 1 > |\lambda_3| \geq \cdots \geq |\lambda_d| > 0,$$

i.e., A is *nonsingular* and *hyperbolic* and has a 2-dimensional *expanding subspace*, denoted E^u. Let E^s be the *contracting subspace*, which has dimension $d-2$, and assume $d-2 \geq 1$. We say A satisfies a *Pisot condition* if $d = 1$. A matrix A satisfying (2.1) is called *non-Pisot of order n* if $n = d - 3 > 1$. Although we do not need to assume the non-Pisot condition, we don not need to exclude it either. In particular, our examples typically have $d = 4$, so that $n = d - 2 = 2$.

2.2. The good star property. Let $\mathbf{v}_1, \mathbf{v}_2$ be an ordered basis for E^u (for example, we could take $\mathbf{v}_1$ and $\mathbf{v}_2$ to be eigenvectors). We call a matrix $P \in \mathbb{R}^{2\times d}$ a "projection" to E^u if

$$P\mathbf{v}_1 = \mathbf{e}_1 \text{ and } P\mathbf{v}_2 = \mathbf{e}_2. \tag{2.1}$$

It will often be convenient to identify E^u with its P-image, which we think of as the plane $\mathbb{R}^2$. If P' is another projection, corresponding to a different basis for E^u, then

$$P' = MP \text{ for } M \in Gl_2(\mathbb{R}). \tag{2.2}$$

Let us write $P = (\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^d)$ in terms of its columns. Then $P\mathbf{e}_j = \mathbf{p}^j$. In other words, the columns $\mathbf{p}^j$ are projections of $\mathbf{e}_j$ to $\mathbb{R}^2$. When we draw these vectors in $\mathbb{R}^2$, we refer to the picture as a *star* of vectors (see Figure 1).

[1]In most cases (e.g, [4],[9]) the Pisot condition expressed in terms of what would be A^{-1} in our treatment, and would say that the expanding subspace is one dimensional.

Definition 2.1. *We say a matrix A that satisfies* (2.1) *has the* good star property *if*

$$\mathbf{p}^i = \omega \mathbf{p}^j \text{ for some real } \omega \neq 0 \text{ implies } i = j. \tag{2.3}$$

In other words, the vectors $\mathbf{p}^i$ are pairwise non-parallel.

By (2.2), the good star property depends on A, but not on the choice of P.

Definition 2.2. *Let A satisfy* (2.1) *and* (2.3)*, and let P be a projection to E^u. Define $A_u \in \mathbb{R}^{2\times 2}$ be the matrix conjugate to $A|_{E^u}$ via P. That is,*

$$A_u := PA(\mathbf{v}_1, \mathbf{v}_2). \tag{2.4}$$

It is easy to see that A_u has eigenvalues λ_1, λ_2.

2.3. **The compound.** Let $B \in \mathbb{C}^{p\times q}$ (the set of $p \times q$ complex matrices), with $p, q \geq n$. The n^{th} *compound* of B is the matrix $C_n(B) \in \mathbb{C}^{\binom{p}{n}\times\binom{p}{n}}$ whose entries are the $n \times n$ minors of B (see [2]). In this paper we will always assume $n = 2$. We index $C_2(B)$ by pairs $i \wedge j, k \wedge \ell$, where $i < j$ and $k < \ell$, and we arrange such pairs in lexicographic order.

Theorem 2.3 (Binet-Cauchy, see [2])**.** *If $B = B_1B_2$ then $C_n(B) = C_n(B_1)C_n(B_2)$. If B is non-singular then $C_n(B^{-1}) = C_n(B)^{-1}$.*

Let $P \in \mathbb{R}^{2\times d}$ be the projection for a matrix A satisfying (2.1) and (2.3). The compound $C_2(P) \in \mathbb{R}^{1\times\binom{d}{2}}$ has entries

$$p_{i\wedge j} := \det(\mathbf{p}^i, \mathbf{p}^j) = \sin(\angle \mathbf{p}^i\mathbf{p}^j) \neq 0, \tag{2.5}$$

where the last inequality follows $\angle \mathbf{p}^i\mathbf{p}^j \in (-\pi, 0) \cup (0, \pi)$ by (2.3).

For $p \neq 0$, let $\operatorname{sgn}(p) = p/|p|$. For a vector $\mathbf{a} \in \mathbb{C}^n$, let $\operatorname{diag}(\mathbf{a})$ denote the $n \times n$ matrix with the entries of $\mathbf{a}$ along the diagonal, and zeros everywhere else. Define

$$S(A) = \operatorname{diag}\big((\operatorname{sgn}(p_{1\wedge 2}), \operatorname{sgn}(p_{1\wedge 3}), \ldots, \operatorname{sgn}(p_{(d-1)\wedge d})\big) \in \{-1, 0, 1\}^{\binom{d}{2}\times\binom{d}{2}}. \tag{2.6}$$

The Binet-Cauchy Theorem and (2.2) imply that $S(A)$ is well defined up to a change of sign. Define

$$A^* = S(A)C_2(A)S(A). \tag{2.7}$$

This is the matrix with entries $a_{i\wedge j,k\wedge\ell}\operatorname{sgn}(p_{i\wedge j})\operatorname{sgn}(p_{k\wedge\ell})$, where $a_{i\wedge j,k\wedge\ell}$ are the entries of $C_2(A)$. We call A^* the *oriented compound* of A.

From now on, in additon to (2.1) and (2.3), we will assume:

$$A^* \geq 0, \tag{2.8}$$

i.e., A^* has no negative entries. It is easy to see that this does not depend on the choice of the basis for E^u.

2.4. **Example: The Ammann matrix.** Consider the matrix

$$A = \begin{pmatrix} -1 & 1 & 0 & -1 \\ 1 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix}, \tag{2.9}$$

which is related to the tiling sometimes called the Ammann-Beenker tiling (see [17], [12]). Note that A is symmetric and has characteristic polynomial $p(x) = (x^2 + 2x - 1)^2$. The eigenvalues are $\lambda_1 = \lambda_2 = -1 - \sqrt{2}$ and $\lambda_3 = \lambda_4 = \sqrt{2} - 1$, so A satisfies (2.1).

Let

$$Q = \big(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4\big) = \begin{pmatrix} \sqrt{2} & -1 & -\sqrt{2} & -1 \\ -1 & \sqrt{2} & -1 & -\sqrt{2} \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

where the columns $\mathbf{q}_j$ are eigenvectors for λ_j. Define the projection P to be the the first two rows of Q^{-1}:

$$P = \frac{\sqrt{2}}{4} \begin{pmatrix} 1 & 0 & 1 & \sqrt{2} \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix} = (\mathbf{p}^1, \mathbf{p}^2, \mathbf{p}^3, \mathbf{p}^4), \tag{2.10}$$

The columns of P are plottedin Figure 1. Clearly A satisfies the good star property (2.3).

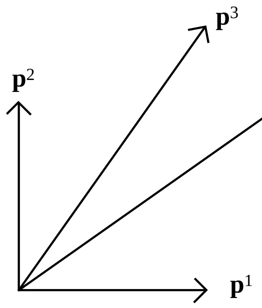


FIGURE 1. The good star of vectors corresponding to the Ammann matrix A in (2.9).

We have $A_u = \lambda_1 \mathrm{Id} = -(1 + \sqrt{2})\mathrm{Id}$, which we interpret as a composition of an expansion of $\mathbb{R}^2$ by $1 + \sqrt{2}$, and a rotation by π. Applying (2.6) we find

$$S(A) = \mathrm{diag}(1, 1, 1, -1, -1, -1), \tag{2.11}$$

so that by (2.7)

$$A^* = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \geq 0. \tag{2.12}$$

Thus A satisfies (2.8).

3. Proto-objects and objects

In the next few sections we are going to consider geometric objects g in $\mathbb{R}^2$, including *curves*, *closed curves*, *tiles*, *tiling patches* and *tilings*. Each object g will have a particular *location* in $\mathbb{R}^2$, and we can move an object by a *translation*. In particular, $g + \mathbf{w}$ denotes the translation of g by the vector $\mathbf{w} \in \mathbb{R}^2$. We call two objects g_1 and g_2 *translationally equivalent* if $g_1 = g_2 + \mathbf{w}$ for some $\mathbf{w} \in \mathbb{R}^2$. We denote the translational equivalence class of the object g by G, and refer to G as a *proto-object*. Thus, we will refer to *protocurves*, *prototiles*, etc..

It will be convenient to represent each proto-object G by an arbitrary fixed choice of a geometric object from its translational equivalence class. We think of this object as being "located at the origin" in $\mathbb{R}^2$. Let $\mathcal{G}$ denote the set of all proto-objects G. Each object g is then a translation $g = G + \mathbf{w} := (\mathbf{w}, G)$ of the corresponding proto-object $G \in \mathcal{G}$ by a unique vector $\mathbf{w} \in \mathbb{R}^2$.

3.1. Words and substitutions.

Let $\mathcal{B} = \{1, 2, \ldots, d\}$. Let $\mathcal{B}^*$ denote the free semigroup of nonempty finite words in $\mathcal{B}$. Let $\mathcal{B}^{\pm} = \{i^{\pm 1} : i \in \mathcal{B}\}$, and let $\mathcal{F}\langle\mathcal{B}\rangle$ denote the *free group* on $\mathcal{B}$, which we think of as the set of *reduced* words in $(\mathcal{B}^{\pm})^*$, together with the empty word ϵ.

A *substitution* is a mapping $\theta : \mathcal{B} \to \mathcal{B}^*$. Since $\mathcal{B}$ is a basis both for both $\mathcal{B}^*$ and $\mathcal{F}\langle\mathcal{B}\rangle$, a substitution uniquely defines both a semigroup and a group *endomorphism*. More genrally, any mapping $\theta : \mathcal{B} \to \mathcal{F}\langle\mathcal{B}\rangle$ defines an endomorphism of $\mathcal{F}\langle\mathcal{B}\rangle$. We will always assume an endomorphism is *non-erasing*, which means $\theta(i) \neq \epsilon$. We think of a non-erasing endomorphism a sort of "generalized substitution", but refrain from using this language to avoid confusion with its other uses.

The *abelianization homomorphism* $f : \mathcal{F}\langle\mathcal{B}\rangle \to \mathbb{Z}^d$ is defined on $\mathcal{B}$ by $f(i) = \mathbf{e}_i$, and extended to $\mathcal{F}\langle\mathcal{B}\rangle$. A endomorphism of $\mathbb{Z}^d$ is given by a matrix $M \in \mathbb{Z}^{d\times d}$. For any endomorphism θ of $\mathcal{F}\langle\mathcal{B}\rangle$, the abelianization f induces an (abelian group) endomorphism of $\mathbb{Z}^d$, denoted by L_θ, satisfying $L_\theta \circ f = f \circ \theta$. In particular,

$$L_\theta = (f(\theta(1)), f(\theta(2)), \ldots, f(\theta(d))) \,. \tag{3.1}$$

This matrix is also called the *structure matrix* of θ, or more formally the *abelianization* of θ. We note that if each $\theta(i)$ is an efficient word (see Definition 5.6 below), then entry $\ell_{i,j}$ of L_θ equals the signed number times that i appears in $\theta(j)$ (in particular, this always holds if θ is a substitution). For a given matrix A, we call any endomorphism θ that satisfies $L_\theta = A$ a *non-abelianization* of A.

3.2. Curves.

A *curve* is continuous, piecwise C^1 mapping $w : [a, b] \to \mathbb{R}^2$. Two curves are considered *the same* if they differ by an orientation preserving picewise C^1 change of parameterization. The reverse of a curve w, denoted $-w$, is given by $-w(t) = w(-t+a+b)$.

For $\mathbf{x}_0, \mathbf{x}_1 \in \mathbb{R}^2$ the *linear* curve from $\mathbf{x}_0$ to $\mathbf{x}_1$ is defined to be $w(t) = (1-t)\mathbf{x}_0 + t\mathbf{x}_1$, for $t \in [0, 1]$. Similarly, for a sequence of points $\mathbf{x}_0, \mathbf{x}_2, \ldots, \mathbf{x}_\ell$ we define the *piecwise linear curve* (or "broken line") connecting the points by $w : [0, \ell] \to \mathbb{R}^2$ where

$$w(t) = (1 - (t - j))\mathbf{x}_j + (t - j)\mathbf{x}_{j+1} \text{ for } t \in [j, j + 1].$$

We will usually restrict our attention to the following case. Let $P = (\mathbf{p}^1, \mathbf{p}^2, \ldots, \mathbf{p}^d)$ be a good star of vectors. For each $k = 1, \ldots, \ell$, suppose there exists $i_k \in \{1, \ldots, d\}$ and a nonzero $a_k \in \mathbb{R}$ so that

$$\mathbf{x}_k - \mathbf{x}_{k-1} = a_k \mathbf{p}^{i_k}. \tag{3.2}$$

In effect, we consider piecwise linear curves whose segments are parallel to the vectors $\mathbf{p}^1, \ldots, \mathbf{p}^d$.

Now suppose $W \in (\mathcal{B}^{\pm})^*$ and $\mathbf{x} \in \mathbb{R}^2$. In particular, $W = i_1^{a_1} i_2^{a_2} \ldots i_\ell^{a_\ell}$, where $i_k \in \mathcal{B}$ and $a_k \in \mathbb{Z}$, $a_k \neq 0$. Define a finite sequence of points $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\ell$ inductively, by putting $\mathbf{x}_0 = \mathbf{x}$, and applying (3.2) for $k = 1, \ldots, \ell$. We define the curve $w := (\mathbf{x}, W)$ to be the piecwise linear curve connecting the points $\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_\ell$. In this way a word is a *proto-curve*.

A curve $w : [a, b] \to \mathbb{R}^2$ is *closed* if $w(a) = w(b)$ (we sometimes think of it as mapping $w : \mathbb{S}^1 \to \mathbb{R}^2$). A *simple closed curve* is a closed curve that is injective on $\mathbb{S}^1$.

We say $W \in (\mathcal{B}^{\pm})^*$ is a *cyclic* if the corresponding curve $w = (\mathbf{x}_0, W)$ is closed. This is equivalent to $\mathbf{x}_\ell = \mathbf{x}_0$. Since

$$\mathbf{x}_\ell - \mathbf{x}_0 = f(W), \tag{3.3}$$

the curve w is closed, or equivalently the word W is a cyclic, if and only if $f(W) = 0$. A cyclic word is a *proto-closed curve*.

We define the *commutator* of two words W_1, W_2 by

$$W = [W_1, W_2] := W_1 W_2 W_1^{-1} W_2^{-1}. \tag{3.4}$$

Note that $f([W_1, W_2]) = f(W_1 W_2 W_1^{-1} W_2^{-1}) = 0$, so that a commutator is always cyclic.

3.3. **A topological proposition.** Unfortunately, it is not easy to tell if a cycle W corresponds to a *simple* close curve. However, in this section we obtain a partial result in the case $W = [W_1, W_2]$.

The *Jordan curve theorem* (see e.g., [1]) says that if $w : \mathbb{S}^1 \to \mathbb{R}^2$ is a simple closed curve then $\mathbb{R}^2 \backslash w(\mathbb{S}^1)$ consists of two open sets: one bounded, called the *inside* of w, and one unbounded, called the *outside*. More generally, if w is piecwise linear closed curve that is not necessarily simple, then $\mathbb{R}^2 \backslash w(\mathbb{S}^1)$ consists of a finite collection of open sets: the *outside*, which is unbounded, and and a fine number of bounded open sets, called the *components* of the *inside*.

A *chain* is a finite sum of the form

$$w = n_1 w_1 + n_2 w_2 + \cdots + n_k w_k, \tag{3.5}$$

where each w_k is a different (not necessarily closed) curve in the plane, and $n_k \in \mathbb{Z}$ (see [1]). There is a fairly obvious equivalence relation on chains, which in addition to orientation preserving reparameterization, allows the combination of curves by following a sucession of them. Line integrals of real or complx functions f in the plane are defined over chains. It can be shown that two chains are equivalent if and only if they yield the same line integral for every function f, (see [1]). Equivalent chains are considered to be *equal*.

A chain is called a *cycle* if (up to equivalence) each w_i is a closed curve. We assume in addition that each w_i is pieciewise linear satisfying (3.2). A cycle is a generalization of a closed curve, and like in that case, $\mathbb{R}^2 \backslash \cup_{i=1}^k w_k(\mathbb{S}^1)$ consists of a finite collection of open sets: an unbounded *outside*, and a finite number of bounded inside *components*. The union of the components is called the *inside* of the curve or the cycle. The set $\cup_{i=1}^{k} w_k(\mathbb{S}^1)$ is called the *trace.*

Given a cycle w, and $\mathbf{x} \in \mathbb{R}^2$ not in the trace, the *winding number* is defined

$$n_w(\mathbf{x}) = \frac{1}{2\pi i} \int_w \frac{d\mathbf{z}}{\mathbf{z} - \mathbf{x}}, \tag{3.6}$$

Here we think of $\mathbf{x}$ and $\mathbf{z}$ as a complex numbers. The integral is carried out separately over each closed curve in (3.5), and the results are added.

The basic property of the winding number function n_w is that it is is integer valued and constant on each component of the inside of w, ond zero outside (see [1]). It can be shown that two cycles are equivalent if and only if they assign the same winding numbers to all non-trace points (this follows from the generalized Cauchy integral formula, [1]). The change of variables formula for integrals shows that $n_{w+\mathbf{y}}(\mathbf{x} + \mathbf{y}) = n_w(\mathbf{x})$, and $n_{-w}(\mathbf{x}) = -n_w(\mathbf{x})$. It is easy to see that $n_{w_1+w_2}(\mathbf{x}) = n_{w_1}(\mathbf{x}) + n_{w_2}(\mathbf{x})$ provided $\mathbf{x}$ is not in the trace of w_1 or w_2.

Let w be a simple closed curve. Then $n_w(\mathbf{x}) = \pm 1$ for $\mathbf{x}$ inside w (see [1]). If $n_w(\mathbf{x}) = 1$ for $\mathbf{x}$ inside w, we say w is *positive* (i.e., it is positively oriented). More generally, we say a (not necessarily simple) closed curve w, or even a cycle w, is *positive* if $n_w(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$ not in the trace. We call a closed curve or cycle w *positive semi-simple* if $n_w(\mathbf{x}) \in \{0, 1\}$ for all $\mathbf{x} \in \mathbb{R}^2$ not in the trace.

Proposition 3.1. *Let $W = [W_1, W_2]$ be the commutator of $W_1, W_2 \in (\mathcal{B}^{\pm})^*$ and define the closed curve $w = (0, W)$. Assume w is positive and*

$$\det\big([Pf(W_1), Pf(W_2)]\big) > 0. \tag{3.7}$$

Then w is positive semi-simple.

Proof. For simplicity we identify $W = w$. Let $\mathbf{v}_1 = Pf(W_1)$ and $\mathbf{v}_2 = Pf(W_2)$, which are linearly independent vectors by (3.7). Let R be the piecwise linear curve in $\mathbb{R}^2$ connecting the points $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2, \mathbf{0}$, and let R_1, R_2, R_3, and R_4 denote the four linear segments that make up R. It follows from (3.7) that R is a positive simple closed curve, and thus $n_R(\mathbf{x}) = 1$ for $\mathbf{x}$ inside R and $n_R(\mathbf{x}) = 0$ outside. We also divide W into four segments:

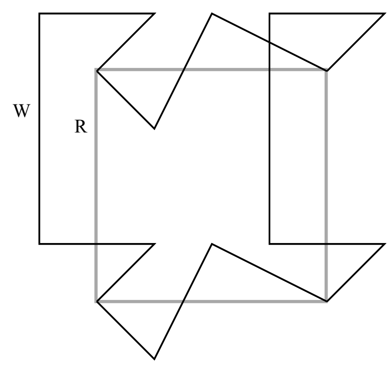


FIGURE 2. The curve $W = [W_1, W_2]$ and its "linearization" R.

W_1, W_2, W_1^{-1}, and W_2^{-1}, corresponding to the four factors of W with the same names. Note that for each i, the curves W_i and R_i start end end at the same place.

Starting at 0, follow W_1. For a while it may follow R_1, but assuming, $W_1 \neq R_1$, the two curves eventually part. Continue following W_1 until its first return to R_1. Call this point $\mathbf{z}_1$. Then follow R_1 or $-R_1$ back to $\mathbf{0}$. Call the resulting closed curve Z_1. Let $Z_1' = -Z_1 + \mathbf{v}_2$ be the *reflection* of Z_1 across R.

Next, starting at $\mathbf{z}_1$, repeat the previous construction. Follow W_1 until it leaves R_1 and then returns for the first time at $\mathbf{z}_2$. Then follow R_1 or $-R_1$ back to $\mathbf{z}_1$ Call the resulting closed curve Z_2, and define its reflection $Z_2' = -Z_2 + \mathbf{v}_2$. Continuing in this fashion we get a sequence $Z_1, Z_2, \ldots, Z_{\ell_1}$ of closed curves. We stop when $\mathbf{z}_{\ell_1} = \mathbf{v}_1$. We also obtain their reflections $Z_1', Z_2', \ldots, Z_{\ell_1}'$

Repeat construction with W_2 and R_2 to obtain simple closed curves $Z_{\ell_1+1}, \ldots, Z_{\ell_2}$, and their reflections $Z_{\ell_1+1}', \ldots, Z_{\ell_2}'$, where $Z_k' = -Z_k - \mathbf{v}_1$ for $k = \ell_1 + 1, \ldots, \ell_2$.

Let $\Lambda = \{n\mathbf{v}_1 + m\mathbf{v}_2 : n, m \in \mathbb{Z}\}$ be the lattice in $\mathbb{R}^2$ generared by $\mathbf{v}_1$ and $\mathbf{v}_2$. Consider the tesselation $\tilde{R}$ of $\mathbb{R}^2$ by the parallelograms $\Lambda + R$. We can assume without loss of generality that each closed curve Z_i lies inside one of the parallelograms in $\tilde{R}$. If not, we can subdivide Z_i into an sum of closed curves (i.e., a cycle) that follow Z_i and the lines in $\tilde{R}$. Simultaneously, we do the same thing to each Z_i' in reverse.

Now for convenience, we renumber all the reflections

$$Z_{\ell_1}', Z_{\ell_1-1}', \ldots, Z_1', -Z_{\ell_2+\ell_1}', Z_{\ell_2+\ell_1-1}', \ldots Z_{\ell_1+1}'$$

as

$$Z_{\ell_1+\ell_2+1}, Z_{\ell_1+\ell_2+2}, \ldots, Z_{2(\ell_1+\ell_2)}.$$

Thus the closed curves $Z_1, \ldots, Z_{2(\ell_1+\ell_2)}$ come in pairs of reflections (with opposite orientations), and each lies in just one parallelogram of $\tilde{R}$.

Define the chain

$$Z := \sum_{j=1}^{2(\ell_1+\ell_2)} Z_j. \tag{3.8}$$

Because of the way the curves Z_i were constructed, $W = R + Z$. It follows that

$$n_W(\mathbf{x}) = n_R(\mathbf{x}) + n_Z(\mathbf{x}) \tag{3.9}$$

for (non-trace) $\mathbf{x} \in \mathbb{R}^2$. By (3.8)

$$n_Z(\mathbf{x}) = \sum_{j=1}^{2(\ell_1+\ell_2)} n_{Z_j}(\mathbf{x}) \tag{3.10}$$

for (non-trace) $\mathbf{x} \in \mathbb{R}^2$.

Let $\mathbf{x}_0 \in \mathbb{R}^2$ be such that no $\mathbf{x} \in \Lambda + \mathbf{x}_0$ is in the trace of W, Z, R or $\tilde{R}$. Let

$$I_{\mathbf{x}_0} = \{j : \exists \mathbf{x} \in \Lambda + \mathbf{x}_0 \text{ with } n_{Z_j}(\mathbf{x}) \neq 0\}.$$

For each $j \in I_{\mathbf{x}_0}$ there is a unique $\mathbf{x}_j \in \Lambda + \mathbf{x}_0$ so that $n_{Z_j}(\mathbf{x}_i) \neq 0$. This is because each Z_j lies in a single parallelogram form $\tilde{R}$. Using (3.10) we have

$$\begin{aligned} \sum_{\mathbf{x}\in\Lambda+\mathbf{x}_0} n_Z(\mathbf{x}) &= \sum_{\mathbf{x}\in\Lambda+\mathbf{x}_0} \sum_{j=1}^{2(\ell_1+\ell_2)} n_{Z_j}(\mathbf{x}) \\ &= \sum_{j=1}^{2(\ell_1+\ell_2)} \sum_{\mathbf{x}\in\Lambda+\mathbf{x}_0} n_{Z_j}(\mathbf{x}) \\ &= \sum_{j\in I_{\mathbf{x}_0}} n_{Z_j}(\mathbf{x}_j). \end{aligned} \tag{3.11}$$

For each $j \in I_{\mathbf{x}_0}$ there exists j' so that $Z_{j'}$ is the reflection of Z_j. Since $Z_{j'} = -Z_j + (\mathbf{x}_{j'} - \mathbf{x}_j)$, it follows that $n_{Z_{j'}}(\mathbf{x}_{j'}) = -n_{Z_j}(\mathbf{x}_j)$. This implies the last sum in (3.11) is zero, so that

$$\sum_{\mathbf{x}\in\Lambda+\mathbf{x}_0} n_Z(\mathbf{x}) = 0. \tag{3.12}$$

Suppose that $n_W(\mathbf{x}_0) > 1$, and consider two cases: (i) $\mathbf{x}_0$ is inside$R$ and (ii) $\mathbf{x}_0$ is not inside R.

In case (i), (3.9) and the fact that R is a positive simple closed curve implies $n_Z(\mathbf{x}_0) \geq 1$. It also follows that $n_W(\mathbf{x}) = n_Z(\mathbf{x})$ for $\mathbf{x} \in \Lambda + \mathbf{x}_0$, $\mathbf{x} \neq \mathbf{x}_0$. By (3.12) there exists such an $\mathbf{x}$ so that $n_W(\mathbf{x}) = n_Z(\mathbf{x}) \leq 1 < 0$. This contradicts the fact that W is positive.

In case (ii), $n_Z(\mathbf{x}_0) = n_W(\mathbf{x}_0) \geq 2$. It follows that either: (a) there exists $\mathbf{x} \in \Lambda + \mathbf{x}_0$, $\mathbf{x} \neq \mathbf{x}_0$, so that $n_Z(\mathbf{x}) \leq 2$, or (b) there exist two distinct $\mathbf{x}_1, \mathbf{x}_2 \in \Lambda + \mathbf{x}_0$, not equal to $\mathbf{x}_0$, so that $n_Z(\mathbf{x}_1) \leq 1$ and $n_Z(\mathbf{x}_2) \leq 1$. In case (a), (3.9) implies $n_W(\mathbf{x}) < 1$ since $n_R(\mathbf{x}) \leq 1$. In case (b) we have $n_R(\mathbf{x}_i) = 1$ for $i = 1$ or $i = 2$ only if $\mathbf{x}_i$ is inside R. This can happen for at most one of the two, since they differ by Λ. Otherwise $n_R(\mathbf{x}_i) = 0$. Then (3.9) implies that $n_W(\mathbf{x}_i) < 0$ for at least one of the $\mathbf{x}_i$. In both cases we again contradict the fact that W is positive. □

3.4. **Tiles, tilings and tiling patches.** Let $\mathcal{B}^2$ denote the set of all pairs $i\wedge j$ for $i, j \in \mathcal{B}$. Define

$$\operatorname{sgn}(i \wedge j) := \operatorname{sgn}(p_{i\wedge j}) = \operatorname{sgn}(\sin(\angle \mathbf{p}^i \mathbf{p}^j)). \tag{3.13}$$

By (2.5), $\operatorname{sgn}(i \wedge j) \in \{-1, 1\}$ if $i \neq j$ and $\operatorname{sgn}(i \wedge i) = 0$. For $i < j$, (3.13) shows that the entries of the matrix $S(A)$ in (2.6) are $\operatorname{sgn}(i\wedge j)$. In this casewe write $j \wedge i = -(i \wedge j)$, and more generally, for $a, b \in \{-1, 1\}$ we simplify $i^a \wedge j^b = ab(i \wedge j)$. We call $i \wedge j$ a *positive prototile* if $\operatorname{sgn}(i \wedge j) = 1$, and a *negative* prototile if $\operatorname{sgn}(i \wedge j) = -1$. In the latter case, $j \wedge i = -(i \wedge j)$ is positive. The set of all positive prototiles is denoted by $\mathcal{B}^2_+$.

Prototiles $i \wedge i$ are neither positive or negative, and are called *trivial*. We simplify $i^a \wedge i^b = i \wedge i$. Thus $\mathcal{B}^2$ denotes the set of *all* prototiles.

Definition 3.2. *Let $W \in \mathcal{F}\langle\mathcal{B}\rangle$ be a positive semi-simple word (i.e., any $(\mathbf{x}, W)$ is a positive semi-simple curve). For $\mathbf{x} \in \mathbb{R}^2$, define the* tile *$t = (\mathbf{x}, \overline{W})$ to be the union of the the inside of the curve $(\mathbf{x}, W)$ with its trace. We write $\partial(\mathbf{x}, \overline{W}) = (\mathbf{x}, W)$ for the* boundary *of the tile.*

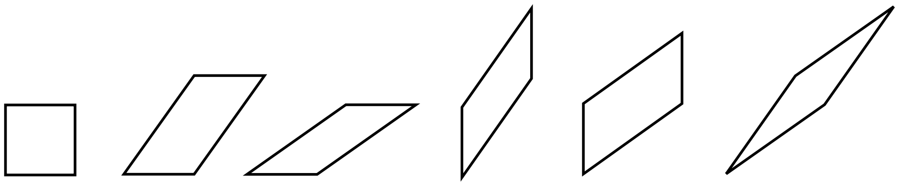


FIGURE 3. The positive prototiles for the matrix A in (2.9): $1\wedge 2$, $1\wedge 3$,$1\wedge 4$, $3\wedge 2$, $4\wedge 2$, $4\wedge 3$.

Now let $i\wedge j\in\mathcal{B}^2$. To define the geometric realization of the tiles corresponding to $i\wedge j$, we first define the boundary of $i\wedge j$ by

$$\partial(i\wedge j)=[i,j]=iji^{-1}j^{-1}.$$

We define the tile

$$t=(\mathbf{x},i\wedge j):=(\mathbf{x},\overline{[i,j]}).$$

We define the boundary $\partial(t)=\partial(\mathbf{x},i\wedge j):=(\mathbf{x},iji^{-1}j^{-1})$, which is is a piecewise linear simple closed curve (i.e., a parallelogram). The four oriented segments of this curve are called the *edges* of the tile, and are denoted $\mathcal{E}\big((\mathbf{x},i\wedge j)\big)$. Two tiles t_1,t_2 are said to be *adjacent* if $-\mathcal{E}(t_1)\cap\mathcal{E}(t_2)\neq\emptyset$, i.e., the two tiles share an oppositely oriented edge.

Now consider a finite sum[2]

$$y=n_1t_1+n_2t_2+\dots n_\ell t_\ell \tag{3.14}$$

of different tiles $t_k=(\mathbf{x}_k,i_k\wedge j_k)$. Assume that if any two tiles in (3.14) intersect at more than a vertex, they are adjacent. Define $\partial(y)=\sum n_k\partial(t_k)$, which is a cycle. If the cycle $\partial(y)$ is positive semi-simple, then we say y is a (positive) *tiling patch.* Note that this implies that $n_1=n_2=\cdots=n_k=1$. Tiling patches, modulo translation are, called *tiling protopatches*, and are denoted by $(\mathcal{B}^2_+)^*$.

3.5. **Positive tiling substitutions.** Our goal in this section is to give a definition of a positive tiling substitution analogous to the definition $\theta:\mathcal{B}\to\mathcal{B}^*$ of an ordinary substitution. Basically it will be a mapping $\Theta:\mathcal{B}^2_+\to(\mathcal{B}^2_+)^*$. But, whereas in the case of a substitution, essentially any mapping works, we have two additional requirements for a tiling substitution: (a) Θ needs to be able to apply to tiles (i.e., translations of prototiles), and (b) Θ needs to apply consistently to adjacent tiles in a tiling patch, and stil output a tiling patch. We will achieve these two goals using the next definition.

Definition 3.3. *A* tiling substitution *is a mapping* $\Theta:\mathcal{B}^2_+\to(\mathcal{B}^2_+)^*$ *such that there is a morphism* $\theta:\mathcal{B}\to\mathcal{F}\langle\mathcal{B}\rangle$ *with the property that*

$$\partial(\Theta(i\wedge j))=\theta([i,j])\text{ for each }i\wedge j\in\mathcal{B}^2_+. \tag{3.15}$$

We abbreavaite (3.15) $\theta=\partial(\Theta)$. In effect, this says that $\Theta(i\wedge j)$ should be a tiling of the inside of the curve $\theta([i,j])$. Since $\Theta(i\wedge j)$ is a positive tiling patch, it follows that its boundary $\theta([i,j])$ is a positive semi-simple curve. More enerally, we define $\Theta(\mathbf{x},i\wedge j):=(A_u\mathbf{x},\Theta(i\wedge j))$.

[2]This is in the same spirit as a cycle as a sum of closed curves.

Lemma 3.4. *Let Θ be a positive tiling substitution. Suppose $y = t_1 + t_2 + \dots t_\ell \in (\mathcal{B}^2_+)^*$, i.e., y is a positive tiling patch. Define $\Theta(y) = \Theta(t_1) + \Theta(t_2) + \cdots + \Theta(t_\ell)$. Then $\Theta(y) \in (\mathcal{B}^2_+)^*$.*

Corollary 3.5. *The iterates $\Theta^n(i \wedge j)$ for $i \wedge j \in \mathcal{B}^2_+$, and $\Theta^n(y)$, for $y \in (\mathcal{B}^2_+)^*$, are well defined for all $n \geq 0$.*

Proof of Lemma 3.4. For $i \wedge j$ define $A_u(i \wedge j)$ to be the tile whose boundary is the curve connecting the points $\mathbf{0}, \mathbf{p}^i, \mathbf{p}^i + \mathbf{p}^j, \mathbf{p}^j, \mathbf{0}$. For $t_k = (\mathbf{x}_k, i_k \wedge j_k)$ in y, let $A_u t_k = (A_u \mathbf{x}_k, A_u(i_k \wedge j_k))$, and let $A_u y = A_u t_1 + \dots A_u t_\ell$. Then $A_u y$ is a tiling patch.

Similarly, define $\theta(t_k) = \theta(\mathbf{x}_k, i_k \wedge j_k) = (A_u \mathbf{x}_k, \overline{[\theta(i_k), \theta(j_k)]})$, using Definition 3.2, and put $\theta(y) = \theta(t_1) + \dots \theta(t_\ell)$. Each of these tiles has four "edges", which along the boundary are labeled $\theta(i_k)$, $\theta(j_k)$, $\theta(i_k^{-1})$ and $\theta(j_k^{-1})$.

We have

$$
\begin{aligned}
\partial(\theta(t_k)) &= (A_u \mathbf{x}_k, [\theta(i_k), \theta(j_k)]) \\
&= (A_u \mathbf{x}_k, \theta([i_k, j_k])) \qquad (3.16)\\
&= \partial(\Theta(\mathbf{x}_k, i_k \wedge j_k)).
\end{aligned}
$$

Since Θ is a positive tiling substitution, its boundary is a positive semi-simple closed curve. Thus (3.16) shows $\partial(\theta(t_k))$ is a positive semi-simple closed curve, and $\theta(t_k)$ is a tile.

If t_h and t_k are adjacent in y then $A_u t_h$ and $A_u t_k$ are adjacent in $A_u y$. Moreover, both endpoints of each edge of $A_u t_k$ are endponts of the edges of $\theta(t_k)$. Thus $\theta(t_h)$ and $\theta(t_k)$ are adjacent in $\theta(y)$ whenever t_h and t_k are adjacent in y. It follows that $\theta(y)$ is a "positive tiling patch" by the tiles $\theta(t_k)$.

Now each term $\Theta(t_k)$ in $\Theta(y)$ is a tiling patch with boundary $\partial(\Theta(\mathbf{x}_k, i_k \wedge j_k))$, which by (3.16) is $\partial(\theta(t_k))$. Since they only overlap on their edges (with opposite orientations), their sum is positive semi-simple closed curve, and it follows that $\Theta(y)$ is a positive tiling patch. $\square$

4. de Bruijn diagrams

4.1. **The definition of a de Bruijn diagram.** Starting with a cyclic word $W \in (\mathcal{B}^\pm)^*$, we will construct an object Y, called a *de Bruijn diagram*, which depends (in a nonunique way) on W. We will think of such a diagram as a combinatorial representation of a tiling patch with W as its boundary. However, we will need to drop the restriction that a patch be a tiling by positive tiles, and allow negative and trivial tiles as well.

Suppose $W = i_1^{a_1} i_2^{a_2} \dots i_n^{a_n}$. Let $c : [0,1] \to \mathbb{R}^2$ be a positive simple closed curve. Starting at $c(0)$ we follow c and attach n arrows normal to c, labeled $i_1, i_2, \dots, i_n$. We make the kth arrow *point in* if $a_k = 1$ and *point out* if $a_k = -1$ (see Figure 4(a)). This is called the *frame* of the (still to be defined) de Bruijn diagram Y. We denote the frame by $\partial(Y)$. Up to a diffeomorphism of $\mathbb{R}^2$, $\partial(Y)$ is uniquely defined by W.

Now we describe how to get from the frame $\partial(Y)$ to the diagram Y. Since W is cyclic, $f(W) = 0$, and thus for each $i \in \mathcal{B}$ there are the same number of i labeled in- and out-arrows. We choose some arbitrary matching of these and connect each matched pair by a non-self intersecting curve through the inside of c, called a *pseudo-line*. Assume that

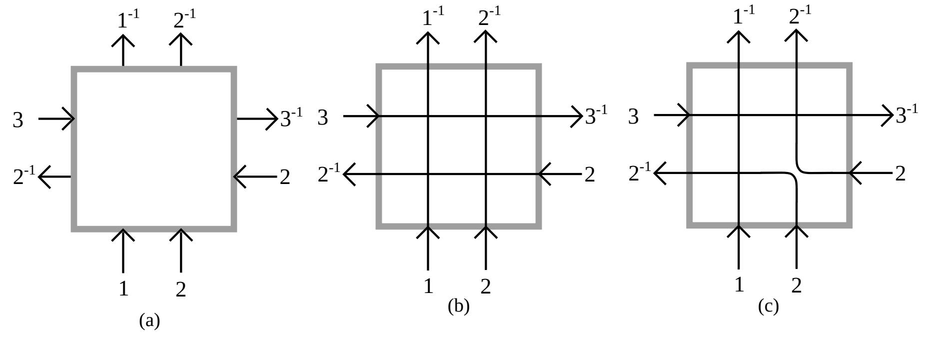


FIGURE 4. (a) The frame for $W = [W_1, W_2] = 1223^{-1}2^{-1}1^{-1}32^{-1}$ where c is a square. Two corresponding de Bruijn diagrams: (b) the product $W_1 \wedge W_2$, and (c) the only other possibility in this case.

at most two pseudo-lines cross at any point, and any two pseudo-lines cross transversally. The resulting picture is a de Bruijn diagram Y (see Figure 4).

Each pseudo-line in Y is labeled by some $i \in \mathcal{B}$, and is oriented by its arrows. Two diagrams that differ by an orientation preserving diffeomorphism of $\mathbb{R}^2$ are considered the same. The frame $\partial(Y)$ of a de Bruijn diagram Y determines the cyclic word W up to a cyclic permutation. We write $\partial(Y) = W$.

The *vertices* of Y are defined to be the crossings of its pseudo-lines. The pseudo-lines divide the inside of c into finitely many *faces* F, the edges E of which are either pseudo-line segments or segments of c. A face with no c segment edges is called an *internal* face. A vertex that is an intersection of two pseudo-lines is called an *internal vertex.*

Given a de Bruijn diagram Y, we can draw a neighborhood of any internal vertex as shown in Figure 5, using a (possibly orientation reversing) change of coordinates. In

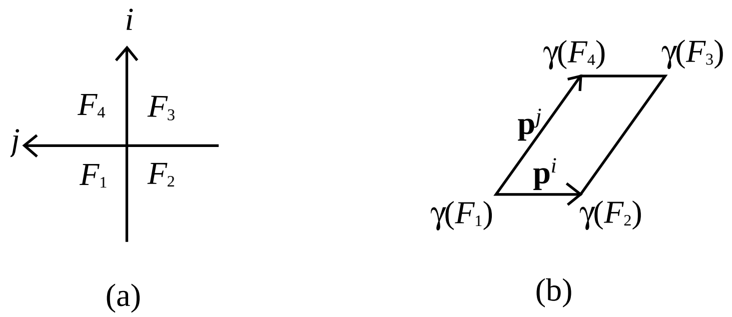


FIGURE 5. (a) A vertex $\mathbf{v}$ of type $i \wedge j$ in canonical coordinates, showing the four faces F_1, F_2, F_3, F_4. (b) The γ image of F_1, F_2, F_3, F_4 in the case $i \wedge j$ is positive. In particular, this illustrates the case $i \wedge j = 1 \wedge 3$ in the Ammann matrix example.

particular, we have a crossing of a straight pseudo-line segment labeled i, pointing up, and a straight pseudo-line segment labeled j, pointing left. We say this vertex is of type $i \wedge j$.

Definition 4.1. *A vertex of type $i \wedge j$ in a de Bruijn diagram Y is called* positive *if* $\mathrm{sgn}(i \wedge j) > 0$. *A de Bruijn diagram Y is* positive *if all its vertices are positive.*

4.2. **The tiling patch corresponding to a diagram.** Here we describe how to go from a nonsingular de Bruijn diagrams Y to the corresponding tiling patch y. The idea—to use planar graph duality—comes from de Bruijn's algebraic theory of Penrose tilings [5]. In particular, an internal type $i \wedge j$ vertex in Y corresponds to a tile t of type $i \wedge j$. Similarly, each face F in Y corresponds, under duality, to a tile vertex in y. We will start with this correspondence.

Let $\mathcal{F}$ denote the set of all faces in Y. Given a vector $\mathbf{x}_0$ and some $F_0 \in \mathcal{F}$, we define $\gamma : \mathcal{F} \to \mathbb{R}^2$ as follows. Put $\gamma(F_0) = \mathbf{x}_0$, and proceed by induction. Suppose γ has been defined on a set $\mathcal{F}_k \subseteq \mathcal{F}$ of $k < \#(\mathcal{F})$ faces. Let $F \in \mathcal{F} \backslash \mathcal{F}_k$ be adjacent to some $F' \in \mathcal{F}_k$ across a pseudo-line segment labeled i. Put $a = 1$ or $a = -1$ depending on whether segment i is oriented to the left or to the right in crossing from F' to F. Define $\gamma(F) = \gamma(F') + a\mathbf{p}^i$, and put $\mathcal{F}_{k+1} = \mathcal{F}_k \cup \{F\}$.

Lemma 4.2. *Suppose Y is a nonsingular de Bruijn diagram. Let F_1, F_2, F_3 and F_4 be faces surrounding a type $i \wedge j$ vertex in Y, as shown in* Figure 5(a)*. Then $\gamma(F_1)$, $\gamma(F_2)$, $\gamma(F_3)$ and $\gamma(F_4)$ are the vertices of the tile* $(\gamma(F_1), i \wedge j)$ (Figure 5(b))*, and the boundary of this tile is the piecewise linear curve corresponding to the sequence of points $\gamma(F_1)$,$\gamma(F_2)$, $\gamma(F_3)$, $\gamma(F_4)$, $\gamma(F_1)$.*

Proof. We can assume without loss of generality that $\gamma(F_1) = \mathbf{0}$. Then $\gamma(F_2) = \mathbf{p}^i$, $\gamma(F_4) = \mathbf{p}^j$, $\gamma(F_3) = \mathbf{p}^i + \mathbf{p}^j$ (see Figure 5(b)). □

Lemma 4.2 says that a positive de Bruijn diagram describes a sort of "locally correct" tiling patch. It gives each tile a precise location, and tiles corresponding to adjacent vertices meet across complete edges. However, there is no *a priori* guarantee that different parts of the tiling do not overlap. The next result shows this cannot happen if the boundary choice is correct.

Proposition 4.3. *Let $W = [W_1, W_2]$ with $W_1, W_2 \in \mathcal{F}\langle\mathcal{B}\rangle$ satisfy*

$$\det\big([Pf(W_1), Pf(W_2)]\big) > 0.$$

Suppose Y is a positive de Bruijn diagram with $\partial(Y) = W$. Then Y corresponds to a positive tiling patch y with $\partial(y) = (\mathbf{x}, W)$ for some $\mathbf{x} \in \mathbb{R}^2$.

Proof. By Lemma 4.2, the vertices of Y correspond to positive tiles $t_1, t_2, \ldots t_\ell$. Let $T_i = \partial(t_i)$. Then T_i is a simple closed curve, which is positive since Y is positive. Thus $n_{T_i}(\mathbf{x}) = 1$ if $\mathbf{x}$ is inside T_i, and 0 otherwise. Also

$$\sum_{i=1}^{\ell} T_i = \partial(Y) = W, \tag{4.1}$$

since two adjacent tiles have oppositely oriented boundaries.

Let $\mathbf{x}$ be inside W. Then (4.1) implies

$$n_W(\mathbf{x}) = \sum_{i=1}^{\ell} n_{T_i}(\mathbf{x}) \geq 0. \tag{4.2}$$

Since $\det\big([Pf(W_1), Pf(W_2)]\big) > 0$, Proposition 3.1 implies $n_W(\mathbf{x}) = 1$, from which it follows that the sum in (4.2) is ≤ 1 for all non-trace $\mathbf{x}$. Thus any $\mathbf{x}$ inside W can be in at most one tile t_i, so that two tiles only can intersect along their boundaries. This means $y = t_1 + t_2 + \cdots + t_\ell$ is a positive tiling patch. □

Conversely, if y is a positive tiling patch with $\partial(y) = (\mathbf{x}, W)$, then there exists a de Bruijn diagram Y corresponding to y such that $\partial(Y) = W$. This fact, which we do not use, is discussed in [5]. In general, we think of a not necessarily positive de Bruijn diagram as a *generalized* tiling protopatch by positive, negative and trivial tiles. This is illustrated in Section 6.1. We denote the set of all these protopatches by $(\mathcal{B}^2)^*$.

4.3. **Product diagrams and tiling substitutions.** From now on we will consider only diagrams Y where $\partial(Y) = [V, W]$ for $V, W \in \mathcal{F}\langle\mathcal{B}\rangle$. For the frame, we take c to be the unit square. We put V along the bottom, put W going up the right side, put V^{-1} going backwards along the top, and put W^{-1} going down along the left side (see Figure 4(a)).

The simplest diagram of this type is a *product diagram*, denoted $Y = V \wedge W$. To obtain it, we start with the frame described above and connect the arrows by vertical and horizontal lines (see Figure 4(b)). It follows that

$$\partial(V \wedge W) = [V, W]. \tag{4.3}$$

The following "matrix notation" for a product diagram will be useful. Suppose $V = v_1 v_1, \ldots v_\ell$ and $W = w_1 w_2 \ldots w_n$. We write

$$V \wedge W = \begin{bmatrix} v_1 \wedge w_n & v_2 \wedge w_n & \ldots & v_\ell \wedge w_n \\ \vdots & \vdots & \ldots & \vdots \\ v_1 \wedge w_1 & v_2 \wedge w_1 & \ldots & v_\ell \wedge w_1 \end{bmatrix}$$

5. Cancellation

5.1. **The three moves on de Bruijn diagrams.** Let Y be a de Bruijn diagram. In this section we will describe three "moves" μ that can be applied to a diagram Y to obtain a new diagram Y'. Since the moves are reversible denote them $Y \overset{\mu}{\leftrightarrow} Y'$.

The first move μ_1 is called the *flip*. In it, we slide a a pseudo-line across a vertex. It

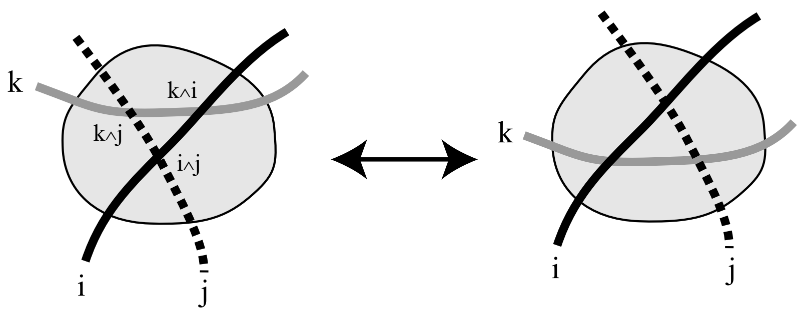


Figure 6. The flip move μ_1.

is called the flip move because in a (positive) tiling it implements the "Necker cube flip" (see Figure 7).

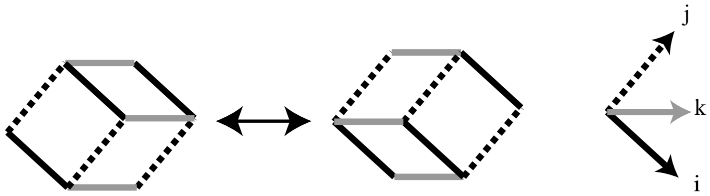


FIGURE 7. The Necker cube flip, implemented by the flip move μ_1.

The second move μ_2 is called *cancellation*. It takes a pair of opposite sign adjacent vertices, $i \wedge j$ and $-(i \wedge j)$, and cancels them by uncrossing two loops (see Figure 8).

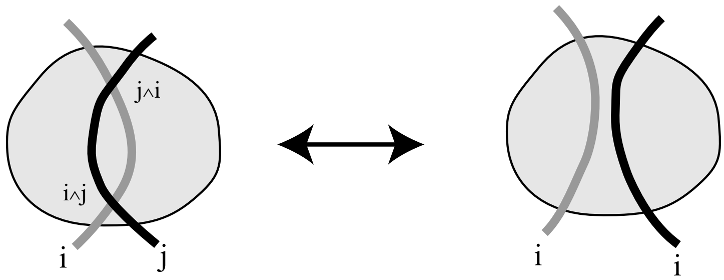


FIGURE 8. The cancellation move μ_2.

The third move μ_3 is called *trivial tile elimination*. It may be used to cancel a trivial tile $i \wedge i$, which occurs when two pseudo-lines with the same label cross (see Figure 9). Move μ_3 is different form the other two for two reasons: (i) pseudo-lines are cut and reconnected in a different way, and (ii) it depends on the orientation.

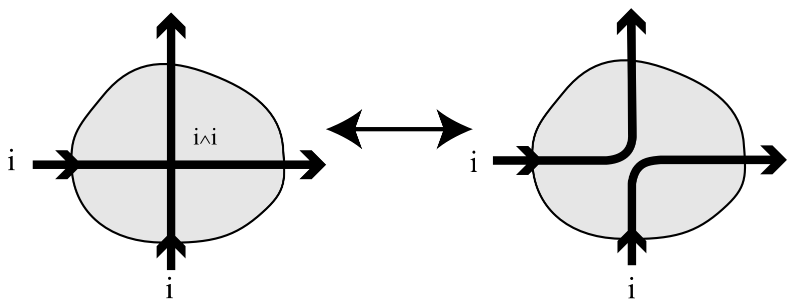


FIGURE 9. Trivial tile elimination move μ_3.

Definition 5.1. *Two de Bruijn diagrams are called μ-equivalent if one can be obtained form the other by a finite series of moves.*

Lemma 5.2. *If $Y \overset{\mu}{\leftrightarrow} Y'$ the $\partial(Y') = \partial(Y)$.*

5.2. **Counting tiles.** For a de Bruijn diagram Y based on $\mathcal{B} = \{1, 2, \ldots, d\}$, let $M(Y)$ be the $d \times d$ matrix with i, jth entry $m_{i\wedge j}$ equal to the number of type $i \wedge j$ vertices that occur in Y. Let $\mathbf{e}_{i\wedge j}$ be the $d \times d$ matrix with $(i \wedge j)$th entry equal to 1 and all other entries 0.

Lemma 5.3. *Let Y and Y' be de Bruijn diagrams with $\partial(Y) = \partial(Y')$.*

(i) If $Y \overset{\mu_1}{\leftrightarrow} Y'$ (i.e., a flip) then $M(Y') = M(Y)$.
(ii) If $Y \overset{\mu_2}{\leftrightarrow} Y'$, where μ_2 implements a cancellation of $i \wedge j$ and $j \wedge i$, then $M(Y') = M(Y) - \mathbf{e}_{i\wedge j} - \mathbf{e}_{j\wedge i}$.
(iii) $Y \overset{\mu_3}{\leftrightarrow} Y'$, where μ_3 implementselimination of a trivial tile $i \wedge i$, then $M(Y') = M(Y) - \mathbf{e}_{i\wedge i}$.

For $i \wedge j \in \mathcal{B}^2$, $i \neq j$ let $|i \wedge j| = \mathrm{sgn}(i \wedge j)(i \wedge j)$. That is to say, for a nontrivial tile $i \wedge j$, $|i \wedge j|$ is its *positive* version. Now suppose Y is a de Bruijn diagram and define the vector $f^*(Y) \in \mathbb{Z}^{\binom{d}{2}}$ to have entries

$$f^*(Y)_{i\wedge j} = \mathrm{sgn}(i \wedge j)(m_{|i\wedge j|} - m_{-|i\wedge j|}), \tag{5.1}$$

$i < j$ in lexicographic order.

Corollary 5.4. (of Lemma 5.3) *The vector $f^*(Y)$ is an invariant of μ-equivalence.*

For an endomorphism θ, define the *product "tiling substitution"* by

Proposition 5.5. *Suppose θ is an endomorphism on $\mathcal{B} = \{1, \dots, d\}$, and let and let $A = L_\theta$. Then the* "product tiling substitution" *$\theta \wedge \theta : \mathcal{B}^2_+ \to (\mathcal{B}^2)^*$, defined by*

$$(\theta \wedge \theta)(i \wedge j) := \theta(i) \wedge \theta(j), \tag{5.2}$$

satisfies

$$A^* = (f^*(\Theta(1 \wedge 2)), f^*(\Theta(1 \wedge 3)), \dots, f^*(\Theta((d-1) \wedge d))),$$

where $\Theta = \theta \wedge \theta$.

Proof. First note that by (2.7) and (3.13) that A^* has entries

$$a^*_{i\wedge j,k\wedge \ell} = \mathrm{sgn}(i \wedge j)\mathrm{sgn}(i \wedge j) \det \begin{pmatrix} a_{i,k} & a_{i,\ell} \\ a_{j,k} & a_{j,\ell} \end{pmatrix},$$

where $a_{i,j}$ are the entries of A. By (5.1)

$$\begin{aligned} f^*(\Theta(k \wedge \ell))_{i\wedge j} &= \mathrm{sgn}(k \wedge \ell)\big(m_{|i\wedge j|} - m_{-|i\wedge j|}\big) \\ &= \mathrm{sgn}(i \wedge j)\mathrm{sgn}(k \wedge \ell)(m_{i\wedge j} - m_{j\wedge i}), \end{aligned}$$

where the $m_{i\wedge j}$ are the entries of $M(\Theta(k \wedge \ell))$.

Since $\Theta(k \wedge \ell) = \theta(k) \wedge \theta(\ell)$ and $L_\theta = A$, it follows that $m_{i\wedge j} = a_{i,k}a_{j,\ell}$ and $m_{j\wedge i} = a_{j,k}a_{i,\ell}$. Thus

$$m_{i\wedge j} - m_{j\wedge i} = \det \begin{pmatrix} a_{i,k} & a_{i,\ell} \\ a_{j,k} & a_{j,\ell} \end{pmatrix},$$

and $f^*(\Theta(k \wedge \ell))_{i\wedge j} = a^*_{i\wedge j,k\wedge\ell}$. □

5.3. **Properly ordered words.** Let

$$W = i_1^{a_1} i_2^{a_2} \dots i_\ell^{a_\ell} \in \mathcal{F}\langle\mathcal{B}\rangle. \tag{5.3}$$

We say W has the *natural order* if $W = 1^{a_1}2^{a_2}\dots d^{a_d}$, where $a_k \in \mathbb{Z}$. (In effect, we have imposed an arbitrary order on $\mathcal{B}$). We say W is *unipotent* if $|a_j| = 1$ for $j = 1, \dots, \ell$, We say W is *positive* if $a_k > 0$ for $j = 1, \dots, \ell$, A positive word W has $f(W) \geq 0$ (where f is the abelianization mapping).

Definition 5.6. *We say W is* efficient *if whever $i_j = i_k$, then a_j and a_k are either both positive or both negative.*

Two efficient words W_1 and W_2 are called *disjoint* if $f(W_1)$ and $f(W_2)$ have different entries nonsero. We say a word W_2 is a *subword* of W_1 if W_2 is obtained from W_1 by deleting some letters. We write $W_2 \subseteq W_1$.

Definition 5.7. *A word $W \in \mathcal{F}\langle\mathcal{B}\rangle$ is said to be* properly ordered *if any of the following hold:*

1. *W is positive, unipotent, and has the natural order,*
2. *W^{-1}, where W is positive, unipotent and has the natural order,*
3. *$W = (W_1)^{-1}W_2$, where W_1 and W_2 are positive, unipotent, have the natural order, and are disjoint.*
4. *$W = W_1W_2\dots W_n$, where each W_i satisfies (1), (2) or (3) above, and there is a permutation $i_1, i_2, \dots, i_n$ of $1, \dots, n$ so that*

$$W_{i_{k+1}} \subseteq W_{i_k}, \;\; k < n. \tag{5.4}$$

Lemma 5.8. *For any $\mathbf{a} \in \mathbb{Z}^d$, $\mathbf{a} \neq \mathbf{0}$, there exists a properly ordered word W with $f(W) = \mathbf{a}$.*

This follows directly from Definition 5.7.

We say a endomorphism θ is properly ordered if $\theta(i)$ is a properly ordered word for each $i \in \mathcal{B}$.

Corollary 5.9. *Suppose A is a $d \times d$ integer matrix so that no row has all entries zero. This holds, in particular, if A satisfies* (2.1)*. Then there exists a properly ordered endomorphism θ so that $L_\theta = A$.*

This follows from Lemma 5.8.

We say a non-singular de Bruijn diagram Y is *efficient* if there are no two vertices $i \wedge j$ and $-(i \wedge j) = j \wedge i$.

Proposition 5.10. *Let $V, W \in \mathcal{F}\langle\mathcal{B}\rangle$ be properly ordered. Then there exists an efficient de Bruijn diagram Y that is μ-equivalent to $V \wedge W$.*

Proof. We first prove a basic case in which we assume V and W are positive, unipotent and naturally ordered. Thus

$$\begin{aligned} V &= 1^{a_1}2^{a_2}\dots d^{a_d} \\ W &= 1^{b_1}2^{b_2}\dots d^{b_d} \end{aligned}$$

where $a_k, b_k \in \{0, 1\}$.

Let

$$U = 1^{c_1} 2^{c_2} \dots d^{c_s}, \text{ where } c_k = \min\{a_k, b_k\},$$

and let ℓ be the length of U.

Think of $\mathcal{V}(U \wedge U) \subseteq \mathcal{V}(V \wedge W)$. Denote the set of ℓ vertices along the diagonal of $U \wedge U$ by S. All of them are trivial. Let S_1 denote the set of vertices $i \wedge j$ of $U \wedge U$ above the diagonal and let S_2 denote the corresponding set of vertices $j \wedge i$ below the diagonal. Let $\psi : S_1 \to S_2$ be the "transpose," map, which takes a vertex to its flip across the diagonal.

A vertex $i \wedge j \in S_1$ is connected to its transpose $j \wedge i = \psi(i \wedge j) \in S_2$ by a a sequence of two pseudo-line segments labeled i that meet at a trivial vertex $i \wedge i$. The same pair of vertices is also connected by a sequence of two pseudo-line segments labeled j that meet at a trivial vertex $j \wedge j$ (see Figure 10(a)).

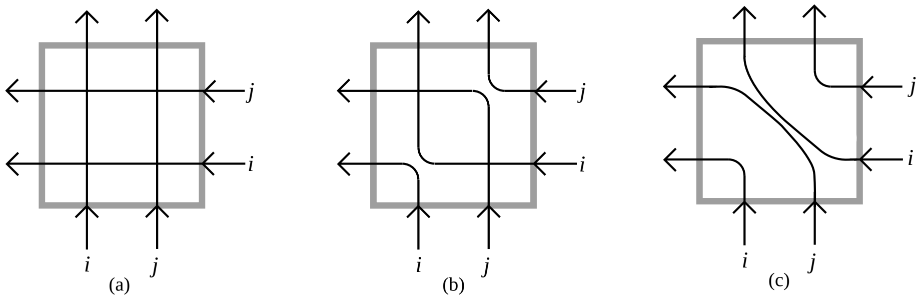


FIGURE 10. (a) A vertex configuration showing $S = \{i \wedge i, j \wedge j\}$, $S_1 = \{i \wedge j\}$ and $S_2 = \{j \wedge i\}$. (b) The result of applying move μ_3 to the trivial vertices in S. (c) Here $i \wedge j$ and $j \wedge i$ are canceled using μ_2.

After performing move μ_3 (trivial tile elimination) on $i \wedge i$ and on $j \wedge j$, there will be a single curve labeled i and a single curve labeled j connecting the two remaining vertices, $i \wedge j$ and $j \wedge i$, common to the two curves (see Figure 10(b)).

If these two vertices are adjacent along these curves, they can be canceled by move μ_2 (Figure 10(c)). However, there may be other vertices between the two loops (see Figure 11(a)). In case the two vertices must first be brought together by a series of μ_1 (flip) moves (see Figure 11(b)).

In general, we proceed as follows:

(i) We remove all the trivial tiles in S, using move μ_3 a total of ℓ times.
(ii) We make all the trivial tiles adjacent using the flip move μ_1, as many times as necessary.
(iii) We cancel ℓ pairs of trivial tiles, matched by ψ.

Let us call the resulting diagram Y. Step (ii) requires some comments.

(1) There is no loss of generality in assuming that $S, S_1, S_2 \subseteq \mathcal{V}(U \wedge U) \subseteq \mathcal{V}(V \wedge W)$. To reduce $V \wedge W$ to Y may require additional flip moves μ_3, but no additional moves μ_1 or μ_2 are needed. Thus we can assume Y is μ-equivalent to $V \wedge W$.
(2) The sequence of flip moves is highly non-unique, so the resulting diagram Y is not unique.

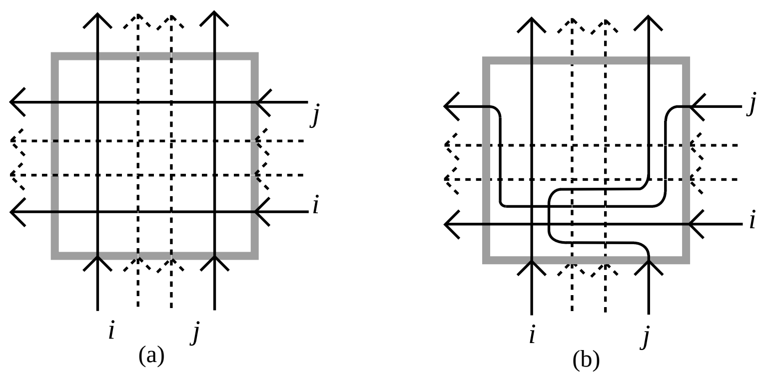


FIGURE 11. (a) Foreign vertices inside the loop. (b) The result of applying eight flip moves μ_1 as well as a few cancellations.

(3) It is crucial in this part of proof that V and W are naturally (i.e., properly) ordered. In particular, if we take the diagram $V \wedge W$ where $V = ij$ and $W = ji$, the cancellation will fail (see Figure 12).

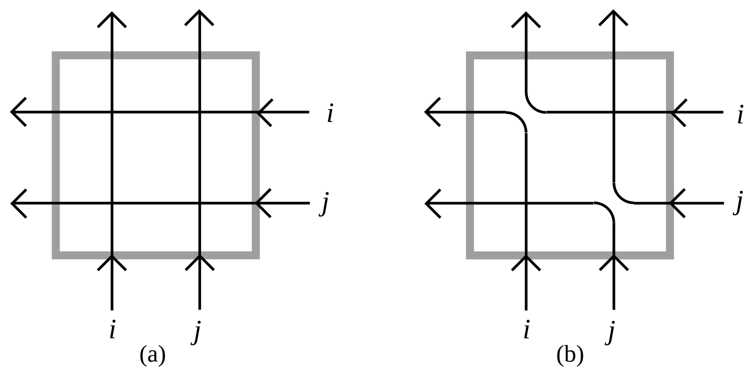


FIGURE 12. Badly ordered words lead to a diagram where cancellation fails.

Now let us consider the vertices of Y. Let V' and W' be the subwords of V and W that remain after removing U. Since all the nontrivial vertices that were in $U \wedge U$ have been canceled, all the nontrivial vertices in Y come from $V' \wedge U$, $U \wedge W'$, or $V' \wedge W'$. Since U, V' and W' are all disjoint, there are no pairs $i \wedge j$ and $j \wedge i$ in these sets. It follows that Y is efficient, and the lemma is proved in this basic case.

There are three more basic cases: $V^{-1} \wedge W$, $V \wedge W^{-1}$ and $V^{-1} \wedge W^{-1}$, where V and W positive, unipotent, and naturally ordered. The proofs, in each of these cases, proceed exactly as above.

Now suppose $V = V_1^{-1}V_2$ and $W = W_1^{-1}W_2$ are simply properly ordered as in part (3) of Definition 5.7. That is V_1, V_2, W_1, W_2 are positive, unipotent, naturally ordered words, with V_1 and V_2 disjoint and W_1 and W_2 disjount. Then

$$V \wedge W = \begin{bmatrix} V_1^{-1} \wedge W_2 & V_2 \wedge W_2 \\ V_1^{-1} \wedge W_1^{-1} & V_2 \wedge W_1^{-1} \end{bmatrix}.$$

Note that the four entries in the "matrix" $V \wedge W$ are de Bruijn diagrams that are "disjoint" in the sense that they have no vertices in common. We apply the previous argument to each of thes four entries to obtain μ-equivalent diagrams Y_1, Y_2, Y_3, Y_4 that are efficient. Then

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}.$$

is efficient and μ-equivalent to $V \wedge W$.

Finally, suppose $V = V_1 V_2 \ldots V_n$ and $W = W_1 W_2 \ldots W_\ell$ are properly ordered, each satisfying (5.4), where each V_m and W_k are unipotent properly ordered. Then

$$V \wedge W = \begin{bmatrix} V_1 \wedge W_\ell & \ldots & V_n \wedge W_\ell \\ \vdots & \ldots & \vdots \\ V_1 \wedge W_1 & \ldots & V_n \wedge W_1 \end{bmatrix}.$$

We complete the proof by applying the previous argument to each entry $V_m \wedge W_k$. □

Consider a non erasing endomorphism θ such that $L_\theta = A$. We say θ is *efficient* if the word $\theta(i)$ is efficient for each $i \in \mathcal{B}$. We say θ is the *natural order endomorphism* if each $\theta(i)$ has the natural order. We say θ is a *properly ordered endomorphism* if $\theta(i)$ is properly ordered for each $i \in \mathcal{B}$.

6. The main result

Theorem 6.1. *Suppose A is a $d \times d$ matrix satisfying* (2.1), (2.3) *and* (2.8)*. Then there exists a properly ordered non endomorphism θ with $L_\theta = A$, such that there is a positive tiling substitution Θ with $\partial(\Theta) = \theta$.*

Proof. By Corollary 5.9, (2.1) implies there exists a properly ordered non erasing endomorphism θ with $L_\theta = A$.

For each $i \wedge j \in \mathcal{B}_+^2$, there exists an efficient de Bruijn diagram $Y(i \wedge j)$ that is μ-equivalent to $\theta(i) \wedge \theta(j)$ by Proposition 5.10.

By Corollary 5.4, $f^*(Y(i \wedge j)) = f^*(\theta(i) \wedge \theta(j))$. By Proposition 5.5, the vector $f^*(\theta(i) \wedge \theta(j))$ is column $i \wedge j$ of A^*. Thus $A^* \geq 0$ implies $f^*(Y(i \wedge j)) \geq 0$, which since $Y(i \wedge j)$ is efficient, shows that it is positive.

Define $\Theta(i \wedge j)$ to be the geometric realization of $Y(i \wedge j)$. By Proposition 4.3, $\Theta(i \wedge j) \in (\mathcal{B}_+^2)^*$, i.e., it is a positive tiling patch.

Finally we have

$$\begin{aligned} \partial(\Theta(i \wedge j)) &= \partial\big((\theta \wedge \theta)(i \wedge j)\big), && \text{by Lemma 5.2,} \\ &= \partial\big(\theta(i) \wedge \theta(j)\big), && \text{by (5.2),} \\ &= [\theta(i), \theta(j)], && \text{by(4.3),} \\ &= \theta([i, j]), && \text{endomorphism,} \\ &= \theta(\partial(i \wedge j)), \end{aligned}$$

so $\partial(\Theta) = \theta$. It follows that Θ is a positive tiling substitution. □

We define the *structure matrix* $\mathcal{L}_\Theta$ of a positive tiling substitution θ to be the $\binom{d}{2} \times \binom{d}{2}$ integer matrix whose $i \wedge j, k \wedge \ell$ entry gives the number of times $k \wedge \ell$ occurs inside $\Theta(i \wedge j)$.

Corollary 6.2. *The positive tiling substitution* Θ *constructed above satisfies* $\mathcal{L}_\Theta = A^*$, *i.e.*, $(L_\theta)^* = \mathcal{L}_\Theta$.

6.1. **Example. The tiling substitution for the Ammann matrix.** Let A be the Ammann matrix (2.9), which we know from Section 2.4 satisfies (2.1), (2.3) and (2.7). Let P be the projection (2.10).

The first task is to define a properly ordered endomorphism θ. For each i we want $\theta(i) = W_i$, where W_i is a properly ordered word such that $f(W_i)$ is the ith column of A. Using cases (2) and (3) of Definition 5.7, we obtain the following endomorphism

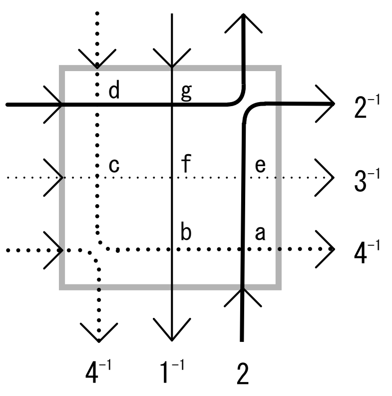
$$\begin{array}{ll} \theta(1) = 4^{-1}1^{-1}2 & \theta(2) = 3^{-1}2^{-1}1 \\ \theta(3) = 4^{-1}3^{-1}2^{-1} & \theta(4) = 4^{-1}3^{-1}1^{-1}. \end{array}$$

Next define the product substitution $(\theta \wedge \theta)(i \wedge j) = \theta(i) \wedge \theta(j)$ whose value on each of the six positive prototiles we represent as product de Bruijn diagram.

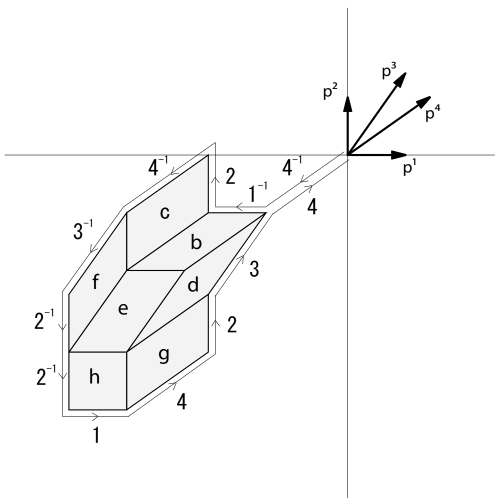
In two cases, $(\theta \wedge \theta)(1 \wedge 3)$ and $(\theta \wedge \theta)(4 \wedge 2)$, the product diagrams consist of just positive and trivial tiles. For example,

$$(\theta \wedge \theta)(1 \wedge 3) = (4^{-1}1^{-1}2) \wedge (4^{-1}3^{-1}2^{-1}) = \begin{bmatrix} 4\wedge 2 & 1\wedge 2 & 2\wedge 2 \\ 4\wedge 3 & 1\wedge 3 & 3\wedge 2 \\ 4\wedge 4 & 1\wedge 4 & 4\wedge 2 \end{bmatrix},$$

in matrix form, where the entries have been simplified to positive prototiles. The resulting μ-equivalent positive de Bruijn diagram $Y(1 \wedge 4)$ is shown below (left), as well as its realization $\Theta(1 \wedge 4)$ as a tiling patch (right).

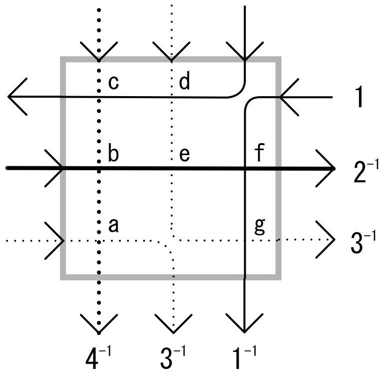


$Y(1 \wedge 3)$

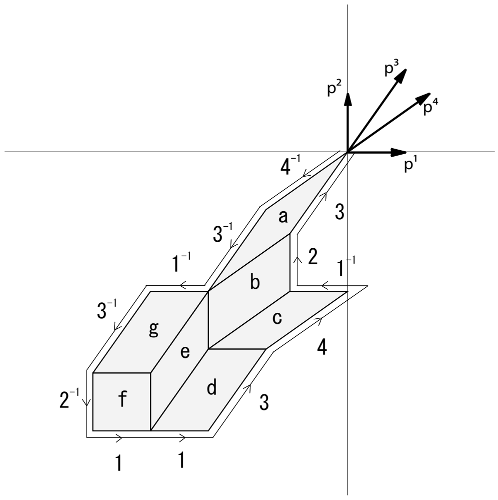


$\Theta\,(1 \wedge 3)$

Similarly, for $(\theta \wedge \theta)(4 \wedge 2)$ we obtain:



$Y(4 \wedge 2)$

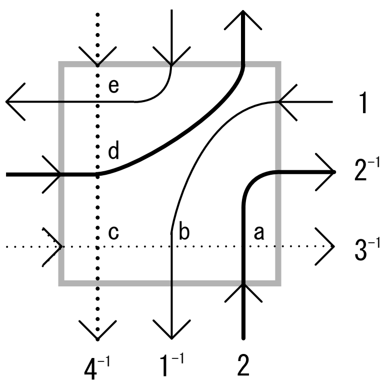


$\Theta\,(4 \wedge 2)$

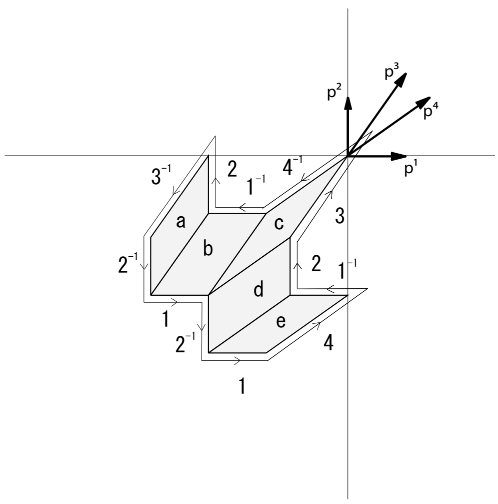
Now consider the case

$$(\theta \wedge \theta)(1 \wedge 2) = (4^{-1}1^{-1}2) \wedge (3^{-1}2^{-1}1) = \begin{bmatrix} 1 \wedge 4 & 1 \wedge 1 & -(1 \wedge 2) \\ 4 \wedge 2 & 1 \wedge 2 & 2 \wedge 2 \\ 4 \wedge 3 & 1 \wedge 3 & 3 \wedge 2 \end{bmatrix}.$$

Notice that the upper right 2×2 block has the form shown in Figure 10(a), and can be cacelled as shown in Figure 10(b) and (c) to obtain the positive diagram $Y(1 \wedge 2)$ shown below (left). The resulting geometric realization $\Theta(1 \wedge 2)$ is shown (right).

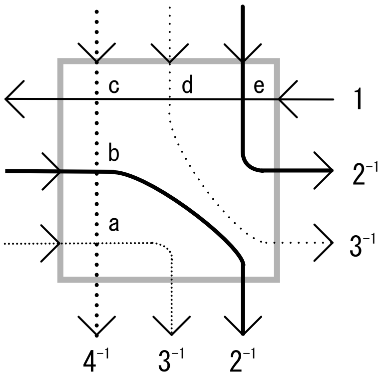


$Y(1 \wedge 2)$

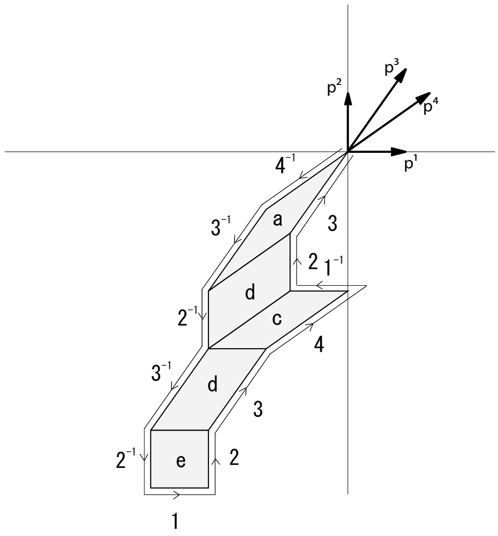


$\Theta\,(1 \wedge 2)$

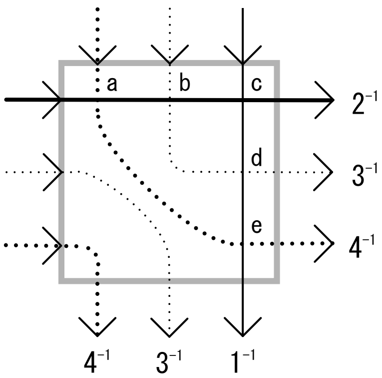
There are two more cases that work exactly the same way: $(\theta\wedge\theta)(3\wedge 2)$ and $(\theta\wedge\theta)(4\wedge 3)$. These are shown below.



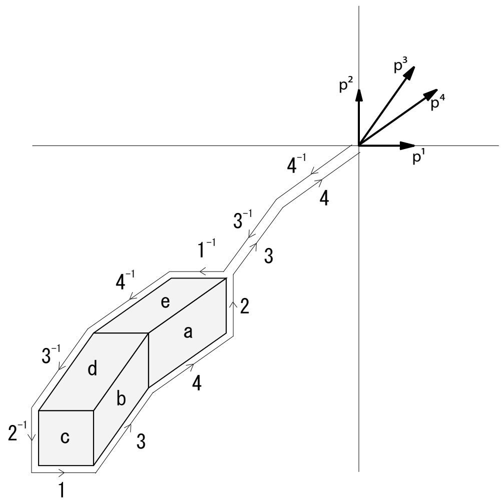
$Y(3 \wedge 2)$



$\Theta\,(3 \wedge 2)$



$Y(4 \wedge 3)$

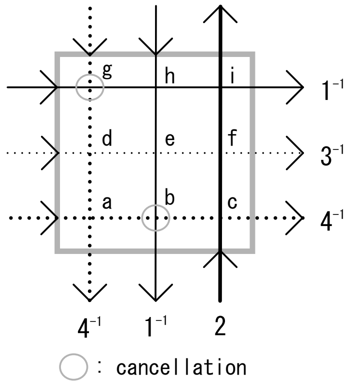


$\Theta\,(4 \wedge 3)$

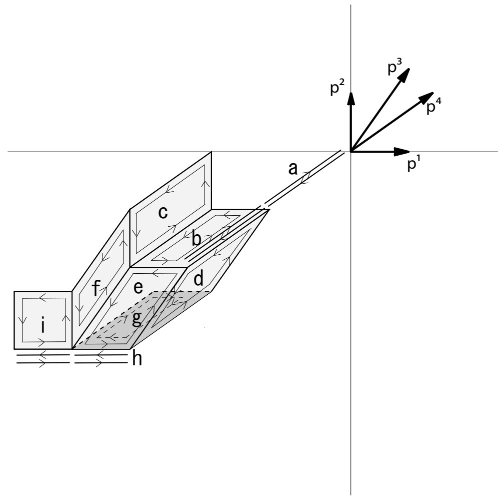
The most complicated case is

$$(\theta \wedge \theta)(1 \wedge 4) = (4^{-1}1^{-1}2) \wedge (3^{-1}2^{-1}1) = \begin{bmatrix} 1 \wedge 4 & 1 \wedge 1 & -(1 \wedge 2) \\ 4 \wedge 2 & 1 \wedge 2 & 2 \wedge 2 \\ 4 \wedge 3 & 1 \wedge 3 & 3 \wedge 2 \end{bmatrix}.$$

Notice that the required cancellations are not adjacent. We use the method in the proof of Proposition 5.10 as illustrated in Figure 10. The steps are shown in detail below.

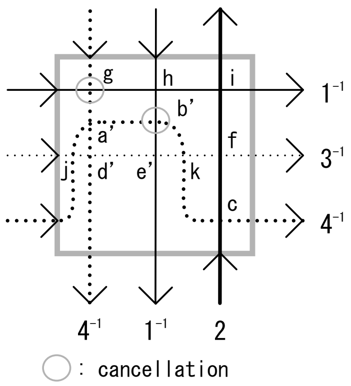


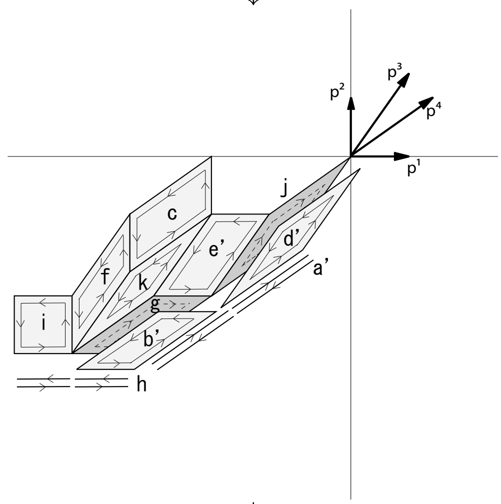
Product diagram $(\theta \wedge \theta)(1 \wedge 4)$.



$(\theta \wedge \theta)\,(1 \wedge 4)$ as a signed tiling patch.

↓ ↓





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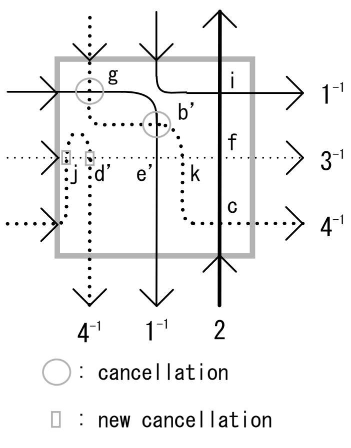

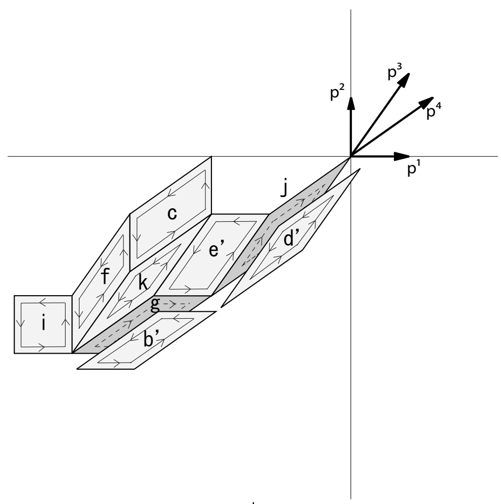

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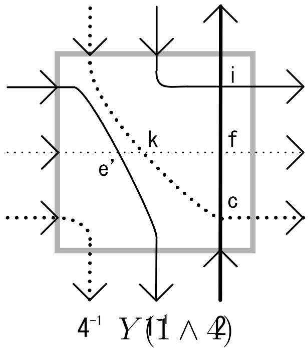
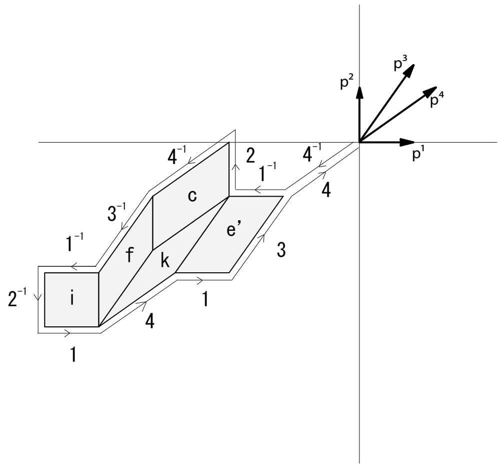


$Y(1 \wedge 4)$



$\Theta\,(1 \wedge 4)$

Now we will show how to iterate Θ to obtain tiling patches and tilings. We start with the tiling patch y, shown below, and apply Θ to obtain a sequence of patches $\Theta^n(y)$. The

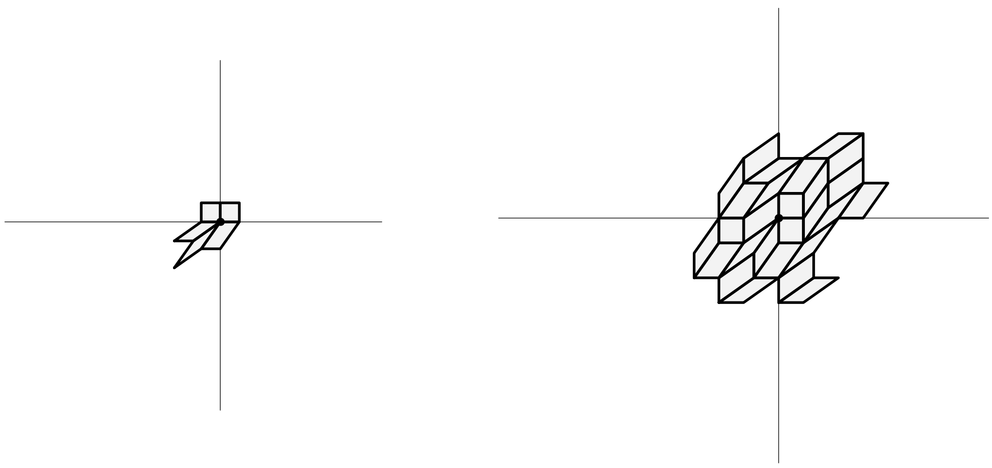


FIGURE 13. The patches y and $\Theta(y)$.

patch y has been carefully chosen so that for each $i \wedge j \in \mathcal{B}^2_+$, a translation of y appears as a subpatch of each $\Theta^n(i \wedge j)$ for all n sufficiently large. The fact that such a patch y exists follows from the fact that $(A^*)^N > 0$ for all $N \geq 2$. We say in such a case that Θ is a *primitive* tiling substitution (this means that each $k \wedge \ell$ appears in each $\Theta^N(i \wedge j)$ for all N sufficiently large). Another property of the patch y is that it appears as a subpatch of $\Theta^2(y)$, completely surrounded by other tiles, as shown in Figure 13. It follows that $\Theta^{2N}(y)$ is a subpatch of $\Theta^{2M}(y)$ for all $M > N$. Thus the "infinite sum" $\sum_{N\geq 0} \Theta^{2N}(y)$ is na increasing sum of patches, and defines a *tiling* of $\mathbb{R}^2$. A swatch of this tiling is shown in Figure 16. It is interesting to note that one obtains a different tiling with the sum $\sum_{N\geq 0} \Theta^{2N+1}(y)$ (there is a different patch y' around the origin). However, since Θ is primitive, both of these belong to the same tiling space (see [16]) X_Θ. We call this an "Ammann tiling" although it is much less symmetric than the classical Ammann-Beenker tiling (see [17]).

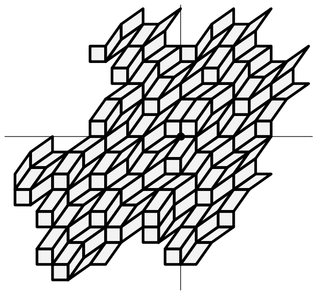


FIGURE 14. The patch $\Theta^2(y)$, showing y as a sub-patch around $\mathbf{0}$.

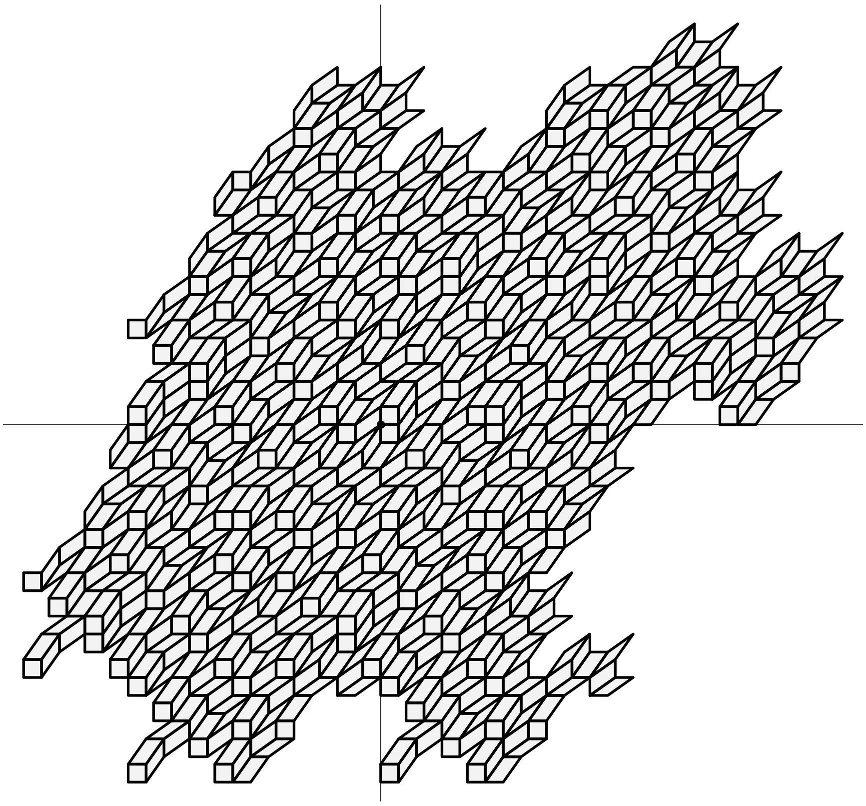


FIGURE 15. The patch $\Theta^3(y)$.

7. The positivity of $(A^*)^N$

In addition to (2.1) and (2.3), all of the tiling results in this paper require that A satisfy (2.8): $A^* \geq 0$. We begin by noting that A and A^N, $N > 1$, have the same expanding subspace E^u and the same projections P to E^u. Thus A^N satisfies (2.1) and (2.3) if and only if A does. In general, we are not so concerned with the difference between A and A^N. If θ is a non-abelianization of A, then θ^N is a non-abelianization of A^N, and we tend to think of θ^N and θ as the same "substitution". The main result in this section is that A^N may satisfy (2.8) (or rather a slightly stronger condition) even if A does not.

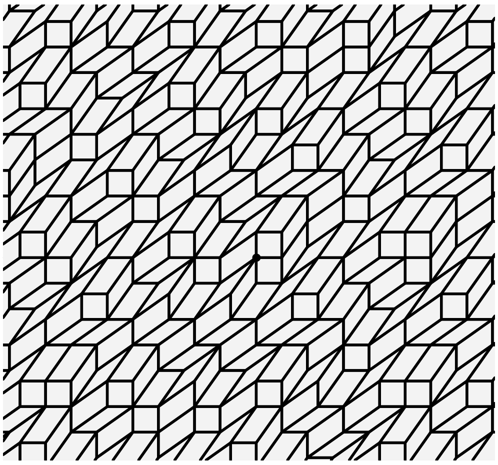


FIGURE 16. A swatch of the "Ammann" tiling.

It follows easily from the Binet-Cauchy theorem (Theorem 2.3) that if A satisfies (2.1) then

$$(A^*)^N = (A^N)^* \text{ for all } N \geq 0. \tag{7.1}$$

Instead of looking for matrices A where $A^* \geq 0$, it will suffice to find A so that $(A^*)^N = (A^N)^* \geq 0$. Unfortunately this is not so easy. However, it turns out to be possible to find good conditions on A for the stronger conclusion $(A^*)^N > 0$. In this case, we say A^* is *eventually positive*.

Even in the case where $A^* \geq 0$, the situation that A^* is eventually positive is important because it implies a corresponding tiling substitution Θ is a primitive tiling substitution. In any case, if a matrix A satisfying (2.1) and (2.3) is such that $(A^*)^N > 0$ then (7.1) implies $B = A^N$ satisfies (2.1), (2.3) and (2.8).

7.1. **The eventual positivity of A^*.** A matrix A is called *eventually positive* if $A^N > 0$ for all N sufficiently large. A matric A is *primitive* if $A \geq 0$ and A is eventually positive. Our main result in this section is the following.

Theorem 7.1. *Let A satisfy* (2.1) *and* (2.3). *The matrix A^* is eventually positive if and only if $S(A) = \pm S(A^T)$, where $S(A)$ is the matrix defined in* (2.6).

Since A and A^T have the same eigenvalues, A satisfies (2.1) if and only if A^T does. The proof of Theorem 7.1 will come at the end of this section.

Corollary 7.2. *If $A^T = A$, then A^* is eventually positive.*

If $A = A^T$, then A satisfies (2.3) if and only if A^T does (they share the projection P). As we will show, however, there are also many non-symmetric examples. In the 3×3 case we get the following.

Corollary 7.3. *Let $B \in \mathrm{Sl}_3(\mathbb{Z})$. Suppose B is eventually positive and satisfies the following Pisot condition:* $\dim(E^u(B)) = 1$. *Then $A := B^{-1}$ satisfies (2.1), (2.3) and (2.8).*

Combining this Theorem 6.1, we obtain a new proof of the result from [9] that under the consitions of Corollary 7.3 there exists a tiling substitution Θ with boundary θ whose structutre matrix is A. In [9] an additional connectivity result ia obtained as well.

Proof of Corollary 7.3. Assume without loss of generality that $B > 0$. The Pisot property for B implies that B has three distinct eigenvalues satisfying $\omega_3 > 1$ and $|\omega_2|, |\omega_1| < 1$. It follows that the corresponding eigenvalues of A, $\lambda_i = \omega_i^{-1}$, satisfy $|\lambda_1|, |\lambda_2| > 1$ and $\lambda_3 < 1$, so A satisfies (2.1).

Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be the corresponding eigenvectors. Since $B > 0$ we can assume $\mathbf{v}_3 = (1, a, b)$ where $a, b > 0$. Let $\mathbf{w}_1 = (-a, 1, 0)$ and $\mathbf{w}_2 = (-b, 0, 1)$. Since $\mathbf{w}_1 \cdot \mathbf{v}_3 = \mathbf{w}_2 \cdot \mathbf{v}_3 = 0$,

$$P = (\mathbf{w}_1, \mathbf{w}_2)^T = \begin{pmatrix} -a & 1 & 0 \\ -b & 0 & 1 \end{pmatrix} \tag{7.2}$$

and $C_2(P) = (b, -a, 1)$. It follows that $S(A) = \operatorname{diag}(1, -1, 1)$. Since $B^T > 0$, and since $A^T = (B^T)^{-1}$, the same argument shows $S(A^T) = S(A)$, and the corollary follows from Theorem 7.1. ☐

7.2. **Subspaces and projections.** We say $\mathbf{v} \in \mathbb{C}^d$ is *real* if $\gamma \mathbf{v} \in \mathbb{R}^d$ for some nonzero $\gamma \in \mathbb{C}$; otherwise we say $\mathbf{v}$ is *complex*. Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^d$ be linearly independent over $\mathbb{C}$. If both $\mathbf{v}_1$ and $\mathbf{v}_2$ are real then $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \mathbb{R}^d$ denotes span over $\mathbb{R}$. Otherwise $\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \mathbb{C}^d$ denotes span over $\mathbb{C}$.

In general, we let $\mathbb{F} = \mathbb{C}$ or $\mathbb{F} = \mathbb{R}$. Given a two dimensional subspace $E = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \subseteq \mathbb{F}^d$, we call a $2 \times d$ matrix P satisfying (2.1) a *projection* to E. As before, (2.2) holds, except now for $M \in \mathrm{Gl}_2(\mathbb{F})$.

A subspace E is called *real* if it has a real basis, or equivalently, a real projection P. In particular, this means there is a nonsingular M so that MP has all real entries. Defining $R = (\mathbf{v}_1, \mathbf{v}_2)$, we have that (2.1) implies $PR = I$. We put $A_E = PAR$, which satisfies $P|_E A = A_E P|_E$.

7.3. **Jordan forms.** Let $V \in GL_d(\mathbb{C})$ be such that $VJ = AV$, where J is the *upper triangular Jordan Canonical Form* of A, with $\operatorname{diag}(J) = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d$. The columns $V = (\mathbf{v}_1, \ldots, \mathbf{v}_d)$ are a basis of *ordered generalized eigenvectors* for A. If A is diagonalizable, these are actually eigenvectors.

Proposition 7.4. *Let A satisfy* (2.1) *and let $\mathbf{w}_1$ and $\mathbf{w}_2$ be the first two ordered generalized eigenvectors for A^T (i.e., corresponding to $E^u(A^T)$). Then $P = (\mathbf{w}_1, \mathbf{w}_2)^T$ is a projection to $E^u(A)$.*

Proof. The unique *dual basis* $\mathbf{w}'_1, \ldots \mathbf{w}'_d$ corresponding to $\mathbf{v}_1 \ \ldots \ \mathbf{v}_d$ is defined to satisfy $\mathbf{v}_i \cdot \mathbf{w}'_j = \delta_{i,j}$. Thus $W' := (\mathbf{w}'_1, \mathbf{w}'_2, \ldots, \mathbf{w}'_d)^T$ satisfies $W' = V^{-1}$.

We have $A^T (W')^T = (W')^T J^T$ because $W' A (W')^{-1} = V^{-1} A V = J$ implies $W'A = JW'$. But J^T is the lower-triangular Jordan form for A.

There exists a permutation matrix U so that $U^T J^T U = J$ where the first two rows (and columns) of U are either: (a) $\mathbf{e}_1, \mathbf{e}_2$, or (b) $\mathbf{e}_2, \mathbf{e}_1$, depending on whether or not λ_1 and λ_2 are in a non-trivial Jordan block (this can only happen if $\lambda_1 = \lambda_2$).

Let $W = U^T W'$, and express $W = (\mathbf{w}_1, \ldots, \mathbf{w}_d)^T$. Then

$$W^{-1} A^T W = (U^T W')^{-1} A^T (U^T W') = J.$$

This shows $\mathbf{w}_1, \ldots, \mathbf{w}_d$ are the ordered generalized eigenvectors for A^T. Let $P = (\mathbf{w}_1, \mathbf{w}_2)^T$. Then $P = MP'$, where $M = I$ in case (a), or $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the case (b). ☐

Corollary 7.5. *The expanding subspace $E^u(A)$ is real.*

Proof. This is clear if λ_1 is real, since this is equivalent to λ_2 also being real.

If λ_1 is complex, then $\lambda_2 = \overline{\lambda_1}$. Put $P = (\mathbf{w}_1, \overline{\mathbf{w}_1})^T$. Let $P' = MP = (\mathrm{Re}\{\mathbf{w}_1\},\ \mathrm{Im}\{\mathbf{w}_1\})^T$ where $M = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. Then P' is a real projection to E^u. □

7.4. **Perron-Frobenius theory for A^*.** Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^d$. Define $\mathbf{v}_1 \wedge \mathbf{v}_2 \in \mathbb{R}^{\binom{d}{2}}$ by

$$\mathbf{v}_1 \wedge \mathbf{v}_2 = C_2\big((\mathbf{v}_1, \mathbf{v}_2)\big). \tag{7.3}$$

Lemma 7.6. *If A is $d \times d$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^d$ then*

$$C_2(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) = (A\mathbf{v}_1) \wedge (A\mathbf{v}_2).$$

Proof.

$$\begin{aligned} C_2(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) &= C_2(A)C_2\big((\mathbf{v}_1, \mathbf{v}_2)\big) \\ &= C_2\big((A\mathbf{v}_1, A\mathbf{v}_2)\big) \\ &= (A\mathbf{v}_1) \wedge (A\mathbf{v}_2). \end{aligned}$$

□

A matrix $B \in \mathbb{Z}^{p\times p}$ is called *spectrally Perron* (see [14]) if it has a real eigenvalue $\omega > 0$, such that for any other eigenvalue ω' of B, $|\omega'| < \omega$. The eigenvector $\mathbf{u}$ corresponding to ω is called a *Perron eigenvector.*

Theorem 7.7. (Lind and Marcus, [14]) *A matrix B is eventually positive if and only if it is spectrally Perron and the Perron eigenvectors $\mathbf{u}$ and $\mathbf{u}'$ for B and B^T respectively are both positive.*

The "only if" direction follows from the the Perron-Frobenius Theorem (see [14]). The "if" direction is Lind and Marcus [14], Exercise 11.1.9.

Lemma 7.8. *Let A satisfy (2.1) and let $E = \mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}$ be any real or complex 2-dimensional A-invariant subspace. Let P be a projection to E, and let A_E be the induced matrix on $\mathbb{F}^2$. Then $\mathbf{u} = S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2)$ is an eigenvector for A^* corresponding to the eigenvalue $\omega = \det(A_E)$. Moreover, every eigenvalue of A^* is obtained this way.*

Proof. We have

$$RA_E = RPAR = AR.$$

Using this with (7.3) and the Binet-Cauchy Theorem (Theorem 2.3)

$$\begin{aligned} C_2(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) &= C_2(A)C_2(R) \\ &= C_2(AR) = C_2(RA_E) \\ &= \det(A_E)C_2(R) = \omega\,(\mathbf{v}_1 \wedge \mathbf{v}_2), \end{aligned}$$

where $\omega = \det(A_E)$.

The fact that every eigenvalue of $C_2(A)$ is obtained this way follows from the Jordan canonical form for A.

Now put $\mathbf{u} = S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2)$. Then

$$\begin{aligned} A^*\mathbf{u} &= S(A)C_2(A)S(A)^2(\mathbf{v}_1 \wedge \mathbf{v}_2) \\ &= S(A)C_2(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) \\ &= \omega\, S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) = \omega\, \mathbf{u}. \end{aligned}$$

□

Proposition 7.9. *Let A be order 2 non-Pisot and satisfy the good star property. Then A^* is spectrally Perron. In particular, if $E^u = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\mathbf{u} = S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2)$ is the Perron eigenvector and $\omega = \det(A_u)$ is the Perron eigenvalue.*

Proof. Since $A^* = S(A)C_2(A)S(A)$ and $A = V^{-1}JV$, the Binet-Cauchy theorem implies

$$A^* = (C_2(V)S(A))^{-1}C_2(J)(C_2(V)S(A)).$$

Now $C_2(J)$ is upper triangular so its eigenvalues are its diagonal entries. These are pairwise products of the diagonal entries of J, which are the eigenvalues of A.

Since E^u is a real 2-dimensional invariant subspace there are three possibilities for A_u: (i) A_u is diagonalizable over $\mathbb{R}$ and has two real eigenvalues λ_1, λ_2 with $|\lambda_1|, |\lambda_2| > 1$. If necessary, by replacing A with A^2, we can assume $\lambda_1, \lambda_2 > 0$. Then $\omega = \lambda_1\lambda_2 > 1$. (ii) A_u is a nontrivial Jordan block with eigenvalue λ. As above, assume by replacing A with A^2 if necessary $\lambda > 0$. Then $\omega = \lambda^2 > 1$. (iii) A_u has complex eigenvalues λ and $\overline{\lambda}$, and $\omega = \lambda\overline{\lambda} = |\lambda|^2 > 1$.

Now let E be any other real or complex 2-dimensional A-invariant subspace, and let ω' be the corresponding eigenvalue for A^*. If $E \subseteq E^-$ then $\omega' = \det(A_E) = \lambda_i\lambda_j$, where $i, j > 2$, so $|\omega'| = |\lambda_i||\lambda_j| \leq (\lambda_-)^2 < 1 < \omega$. Otherwise A_u is diagonalizable and has eigenvalues λ_i, λ_j, $i \leq 2$,$j > 2$. Then $|\omega'| = |\lambda_i||\lambda_j| < |\lambda_1||\lambda_2| = \omega$. The result now follows from an application of Lemma 7.8. □

Proposition 7.10. *Suppose A satisfies (2.1). Let $\mathbf{v}_1, \mathbf{v}_2$ be a real basis for $E^u(A)$ and let $\mathbf{w}_1, \mathbf{w}_2$ be a real basis for $E^u(A^T)$. Then A^* is eventually positive if and only if $S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2)$ and $S(A^T)(\mathbf{w}_1 \wedge \mathbf{w}_2)$ are positive.*

Proof. This follows from Theorem 7.7, Proposition 7.9 and Corollary 7.4. □

Proof of Theorem 7.1. Let

$$\mathbf{u} = S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) \text{ and } \mathbf{u}' = S(A^T)(\mathbf{w}_1 \wedge \mathbf{w}_2).$$

By proposition 7.10, Theorem 7.1 follows once we show

$$\mathbf{u}, \mathbf{u}' > 0 \text{ if and only if } S(A) = \pm S(A^T). \tag{7.4}$$

By Proposition 7.4, we have

$$P = P_{E^u(A)} = (\mathbf{w}_1, \mathbf{w}_2)^T \text{ and } P' = P_{E^u(A^T)} = (\mathbf{v}_1, \mathbf{v}_2)^T$$

where

$$E^u(A) = \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2\} \text{ and } E^u(A^T) = \operatorname{span}\{\mathbf{w}_1, \mathbf{w}_2\}.$$

By (7.3), $C_2(P) = \mathbf{w}_1 \wedge \mathbf{w}_2$ and $C_2(P') = \mathbf{v}_1 \wedge \mathbf{v}_2$. It follows from (2.6) that

$$S(A) = \operatorname{diag}(\operatorname{sgn}(C_2(P)) = \operatorname{diag}(\operatorname{sgn}(\mathbf{w}_1 \wedge \mathbf{w}_2))$$

and

$$S(A^T) = \operatorname{diag}(\operatorname{sgn}(C_2(P')) = \operatorname{diag}(\operatorname{sgn}(\mathbf{v}_1 \wedge \mathbf{v}_2)).$$

Thus $S(A) = \pm S(A^T)$ if and only if $\operatorname{sgn}(\mathbf{v}_1 \wedge \mathbf{v}_2) = \operatorname{sgn}(\mathbf{w}_1 \wedge \mathbf{w}_2)$.

Now suppose $S(A) = S(A^T)$. Then

$$\begin{aligned}\mathbf{u} \;\; = S(A)(\mathbf{v}_1 \wedge \mathbf{v}_2) \;\; &= \operatorname{sgn}(\mathbf{w}_1 \wedge \mathbf{w}_2) * (\mathbf{v}_1 \wedge \mathbf{v}_2) \\ &= \operatorname{sgn}(\mathbf{v}_1 \wedge \mathbf{v}_2) * (\mathbf{v}_1 \wedge \mathbf{v}_2) > 0,\end{aligned}$$

and similarly $\mathbf{u}' > 0$, where $*$ denotes entry-wise multiplication.

Conversely, suppose $\mathbf{u}, \mathbf{u}' > 0$. Then

$$\operatorname{sgn}(\mathbf{w}_1 \wedge \mathbf{w}_2) * (\mathbf{v}_1 \wedge \mathbf{v}_2) > 0,$$

which implies

$$\operatorname{sgn}(\mathbf{v}_1 \wedge \mathbf{v}_2) = \pm \operatorname{sgn}(\mathbf{w}_1 \wedge \mathbf{w}_2).$$

It follows that $S(A) = S(A^T)$. □

8. Example: Irreducible characteristic polynomial

Consider the matrix

$$A = \begin{pmatrix} -1 & -1 & -1 & 0 \\ -1 & 0 & -1 & 0 \\ -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

The characteristci polynomial $p(x) = x^4 + x^3 - 4x^2 + 1$ is irreducible. The eigenvalues of A are $\lambda_1 = -2.5231$, $\lambda_2 = 1.44129$, $\lambda_3 = 0.566889$, and $\lambda_4 = -0.485084$, so A satisfies (2.1)

Starting with

$$Q = (\mathbf{q}_1, \mathbf{q}_2 \mathbf{q}_3, \mathbf{q}_4) = \begin{pmatrix} -3.12676 & -0.252532 & -2.19713 & 0.576415 \\ -2.23925 & -0.824788 & 2.87576 & 0.188279 \\ -2.5231 & 1.44129 & 0.566889 & -0.485084 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

as the matrix with eigenvectors as columns, the projection P to E^u is given by

$$P = \begin{pmatrix} -0.141119 & -0.101064 & -0.113874 & 0.0451327 \\ -0.0660841 & -0.215836 & 0.377166 & 0.261686 \end{pmatrix}.$$

The columns of P are shown in Figure 17.

A^* is given by

$$A^* = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & -1 & -1 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 \end{pmatrix}$$

which satisfies $A^* \leq 0$ condition. This is sufficient for our method, since it follows from (7.1) that A^2 satisfies $(A^*)^2 = (A^2)^* \geq 0$.

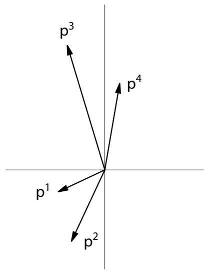
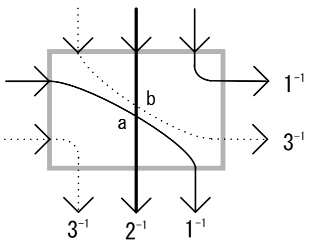


FIGURE 17. A good star of vectors. The positive prototiles are $1\wedge 2$, $3\wedge 1$, $4\wedge 1$, $3\wedge 2$, $4\wedge 2$, and $4\wedge 3$.

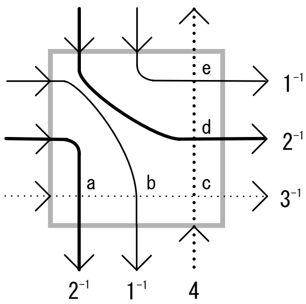
For the matrix A, the properly ordered morphism θ is

$$\theta : \begin{cases} 1 & \rightarrow & 3^{-1}2^{-1}1^{-1} \\ 2 & \rightarrow & 3^{-1}1^{1} \\ 3 & \rightarrow & 2^{-1}1^{-1}4 \\ 4 & \rightarrow & 3 \end{cases} .$$

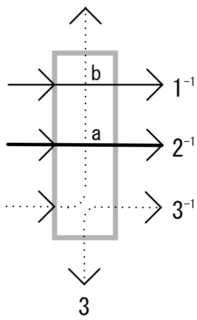
Next we simplify all six product se Bruijn diagrams to obtain the followng efficient diagrams (all of which are negative):



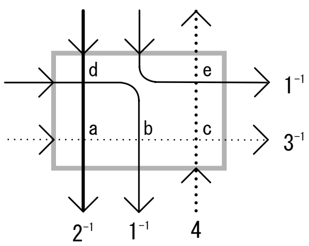
$Y(1\wedge 2)$



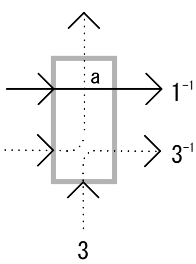
$Y(1\wedge 3)$



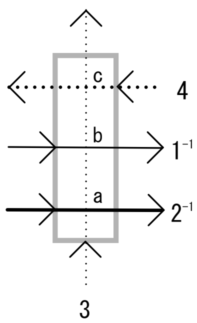
$Y(4\wedge 1)$



$Y(3\wedge 2)$

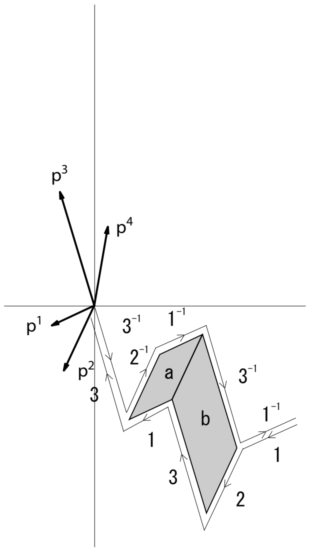


$Y(4\wedge 2)$

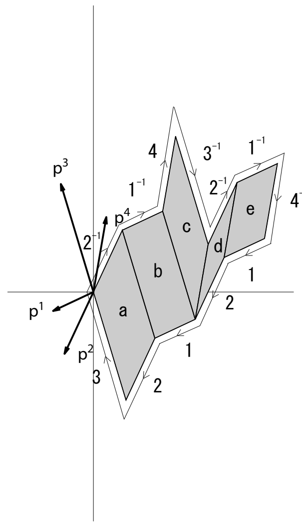


$Y(4\wedge 3)$

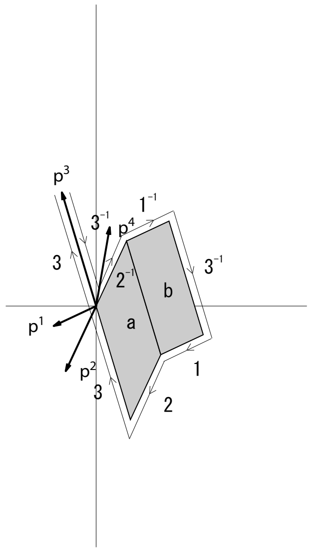
These lead to the corresponding geometric realiations of the tiling patches in the tiling substitution. The grey color indicates *negative* tiles.



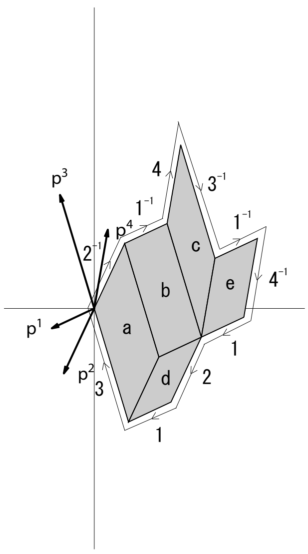
$\Theta(1 \wedge 2)$



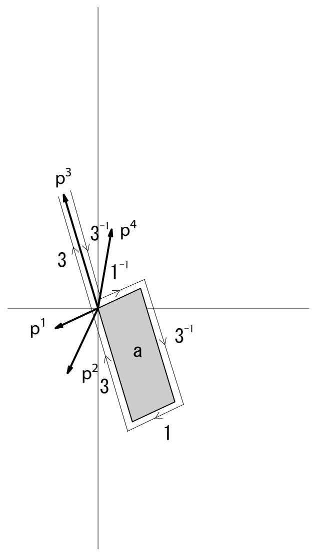
$\Theta(1 \wedge 3)$



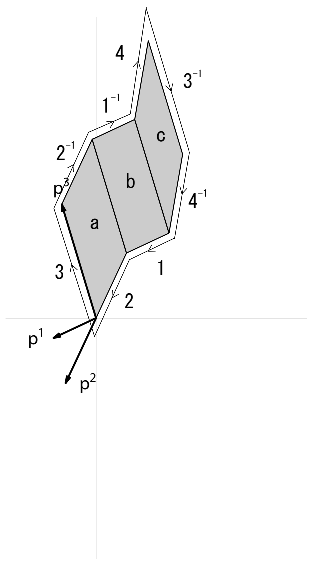
$\Theta(4 \wedge 1)$



$\Theta(3 \wedge 2)$

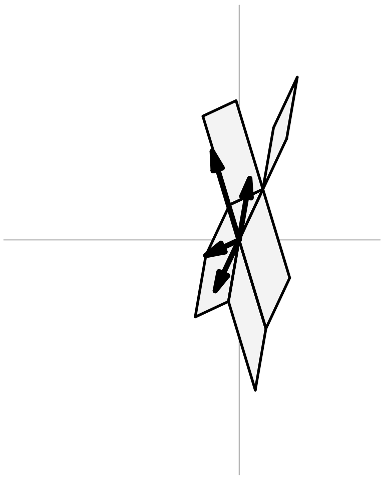


$\Theta(4 \wedge 2)$

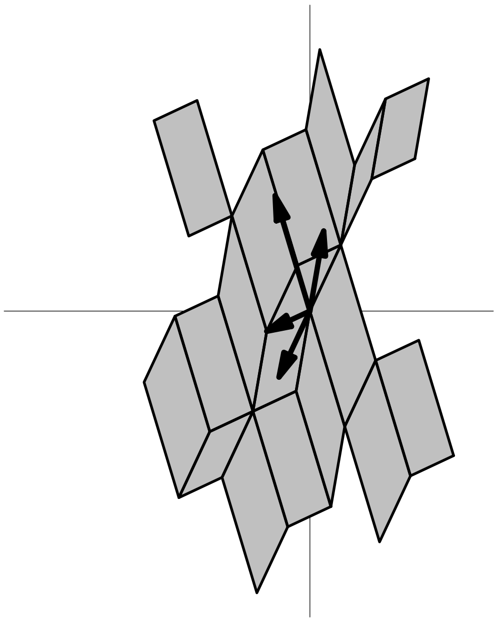


$\Theta(4 \wedge 3)$

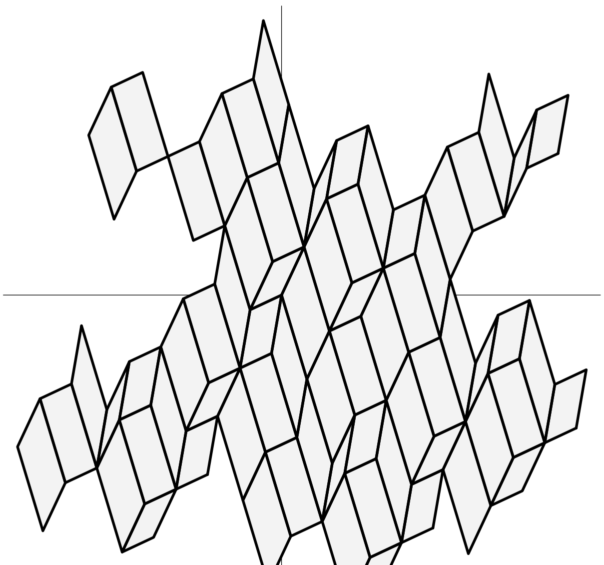
Starting with a tiling patch y we find patches $\Theta^N(y)$, where the patch is by positive tiles for each N odd.



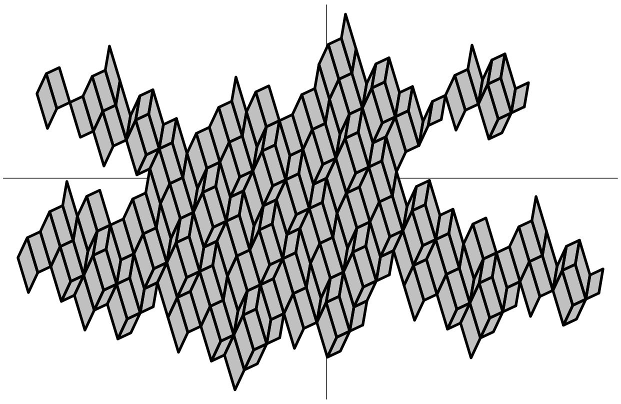
The patch y



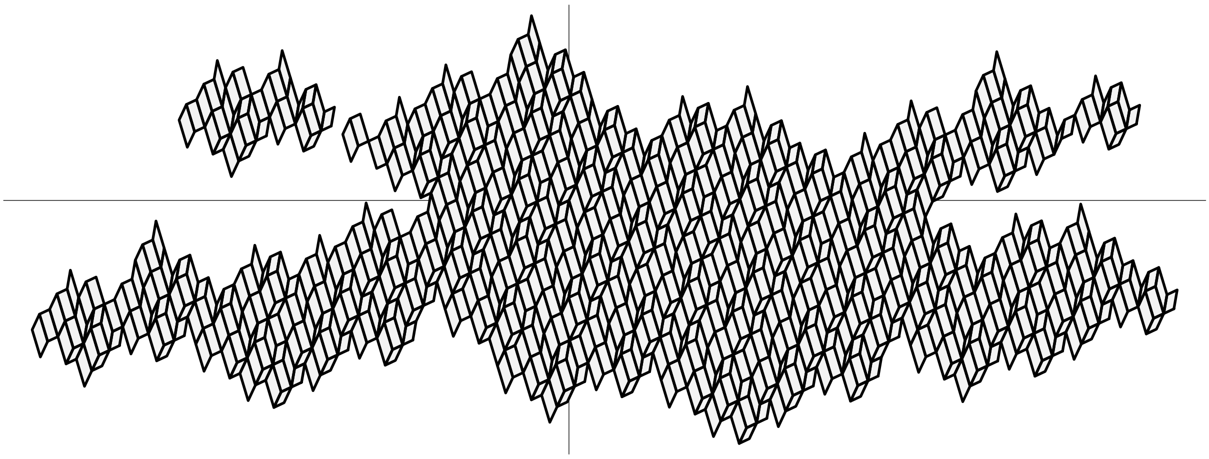
$\Theta(y)$



$\Theta^2(y)$



$\Theta^3(y)$



$\Theta^4(y)$

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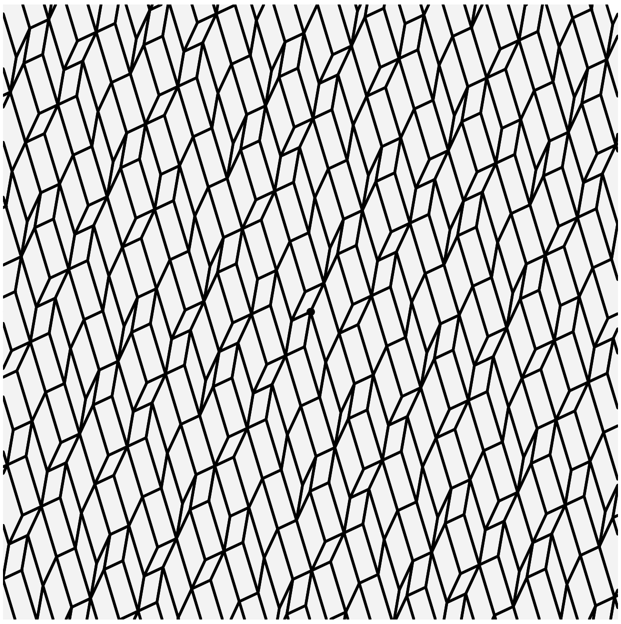

FIGURE 18. The positive tiling $\sum_{N=0}^{\infty} \Theta^{2N+1}(y)$.

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