Complexity of self-affine tilings of $\mathbb{R}^d$

E. Arthur Robinson, Jr.
Department of Mathematics
George Washington University
Washington, DC 20052
robinson@gwu.edu

1. Tilings

- $D \subseteq \mathbb{R}^d$ is a *tile* if homeomorphic to B_1
- A *tiling* of $\mathbb{R}^d$ is a collection of tiles that
  - *pack* $\mathbb{R}^d$ (have disjoint interiors)
  - *cover* $\mathbb{R}^d$ (union or *support* = $\mathbb{R}^d$).
- $D_1 \sim D_2$ if they are *translates*.
- *Prototile:* equivalence class.
- $\mathcal{T}$: finite set of inequivalent prototiles.

Definition 1. The set X_T of all tilings by translates of tiles in T is called the *full tiling space*.

Definition 2. We denote the action of translation on X_T by T.

A *patch y* is a finite set of tiles with connected support. The set of all patches is denoted T^*.

Definition 3. X_T is *locally finite* if there are only finitely many 2-tile patches in all $x \in X_T$.

• For $x \in X_{\mathcal{T}}$ let $x[[B_n]]$ denote the set of all patches in x containing the ball or radius n.

• $x[B_n] \subseteq x$ denotes the patch of all tiles in x that intersect the ball.

Definition 4. The *tiling metric*

$$d(x,y) = \inf\{\tfrac{\sqrt{2}}{2}\} \cup \{r \in (0, \tfrac{\sqrt{2}}{2}) : \exists x' \in x[[B\,{}^{1}\!/_{r}]],\; y' \in y[[B\,{}^{1}\!/_{r}]], \text{ s.t. } T^{\mathbf{t}} x' = y' \text{ for } \|\mathbf{t}\| < r\}$$

Theorem 5. (Rudolph) If $X_{\mathcal{T}}$ is locally finite then $X_{\mathcal{T}}$ is compact.

- Let X_T be a full tiling space. A *tiling space* is a closed and T-invariant subset X Œ X_T.
- A pair (X,T) is called a *tiling dynamical system*.

We will also consider the situation where (X,T) is a $\mathbb{Z}^d$ symbolic dynamical system.

We think of a tiling dynamical system as a sort of $\mathbb{R}^d$ symbolic dynamical system.

2. Complexity

For simplicity we start with the case where (X,T) is a $\mathbb{Z}^d$ symbolic dynamical system.

Definition 6. The complexity $c(n)$ of (X,T) is the number of different n^d blocks in all the different $x \in X$.

We have

$$h(X,T) = \lim_{n\to\infty} \frac{\log(c(n))}{n^d}$$

Complexity is of interest in the case $h(X,T)=0$.

Now suppose (X,T) is a tiling dynamical system. We will generalize the previous definition as follows:

Definition 7. The complexity $c(n)$ of (X,T) is the number of equivalence classes of $x[B_n]$ for all the different $x \in X$.

3. Substitution tiling spaces

- Let $L \in \mathrm{Gl}(d, \mathbb{R})$ be a linear expansion (an *affinity*).
- $L = \lambda M$, where M is an isometry is called a *similarity*.

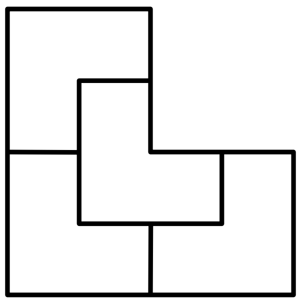
A *decomposition* is a mapping $C : \mathcal{T} \to L^{-1}\mathcal{T}^*$ that satisfies the perfect overlap condition

$$\mathrm{supp}(C(D)) = \mathrm{supp}(D)$$

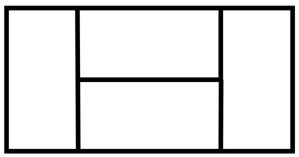
We call C *self-affine* or *self-similar* according to L

Some decompositions:

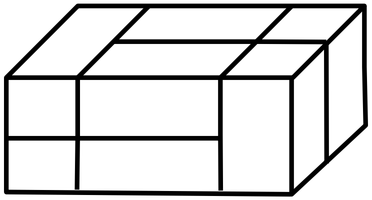
- A few self-similar "polyomino" decompositions



The chair



The table

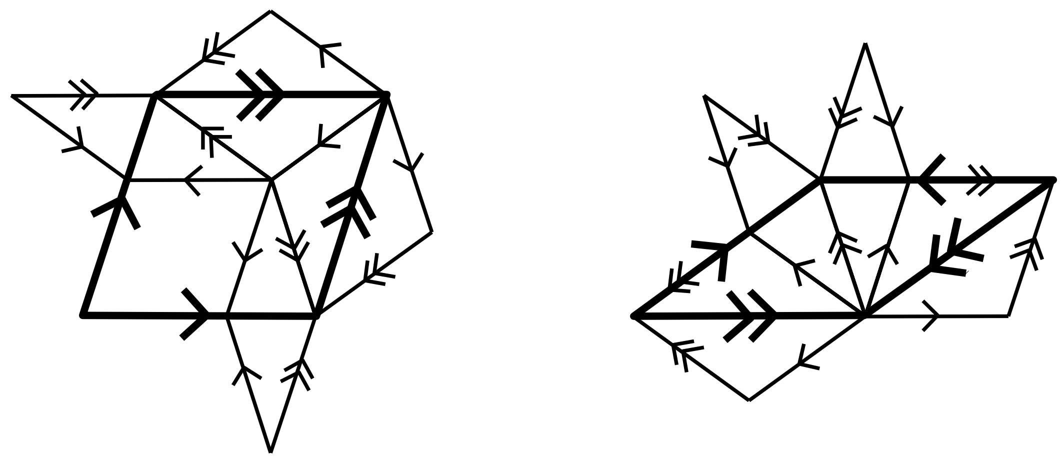


3-dimensional table

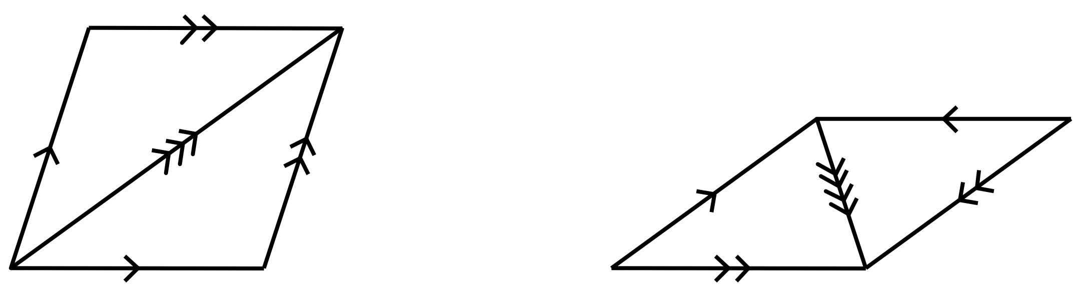
Definition 8. A mapping $S = LC$ is called a *tiling substitution*.

Penrose decompositions:

- The rhombic Penrose substitution (“imperfect”)

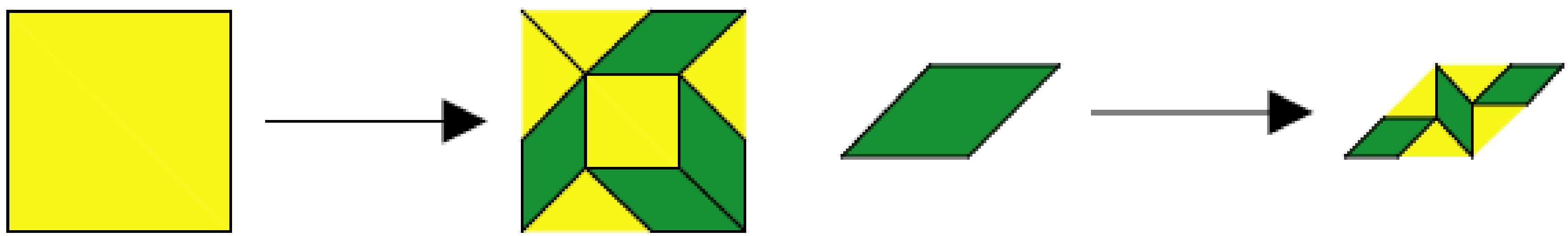


- Can be perfected using “half” tiles

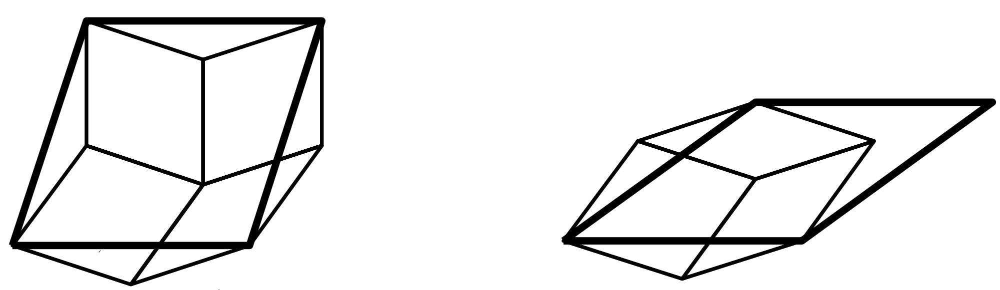


More “imperfect” self-similar substitutions

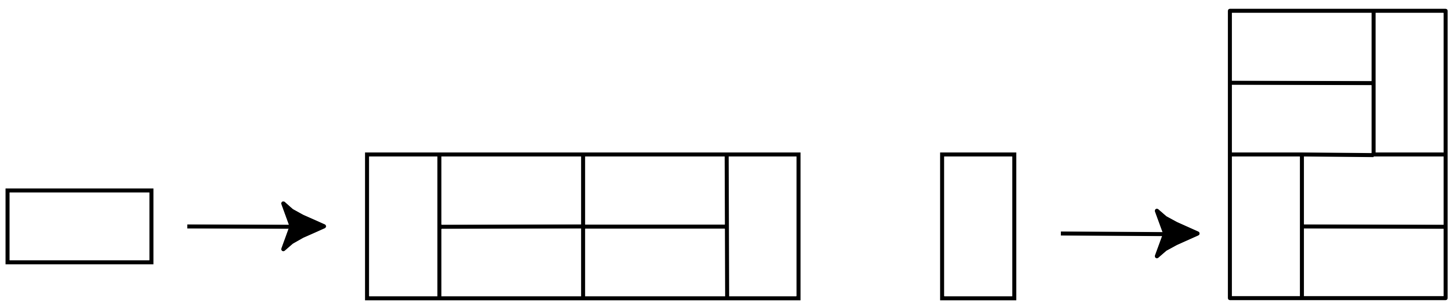
- The *octagonal* or *Ammann-Beenker* tiling



- The binary decomposition (note the required 2p/20 rotation)



A self-affine polygonal tiling substitution.



In this example $L = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ (i.e. it's non self-similar)

How to make substitution tiling spaces

- Let S be a tiling substitution on $\mathcal{T}$ and $D \in \mathcal{T}$
  - Put $x_1 = \{D\}$ & put $x_k = Sx_{k\text{-}1.}$
- Define X to be the set of all tilings so that every $x \in X_{\mathcal{T}}$ patch in x is a sub-patch of some x_k.

Definition 9. A tiling substitution S is primitive if there exists k so that for any $D_1, D_2 \in \mathcal{T}$ then $D_2 \in S^k D_1$.

Lemma 10. If S is a primitive substitution then X is a nonempty tiling dynamical system.

Proposition 11. The tiling dynamical system (X,T) corresponding to a primitive tiling substitution S is strictly ergodic.

Comment. So far everything corresponds to the familiar theory of (1-dimensional) discrete substitutions. It is easy to define d-dimensional discrete substitutions as well.

Note that S is continuous on X.

Definition 12. If $S : X \to X$ is invertible, we call S an *inevitable substitution*.

Proposition 13. If S is an inevitable tiling substitution then all the tilings $x \in X$ are *aperiodic*.

Solomyak proved the converse.

4. Complexity of self-affine tilings

Theorem 14. (Cliff Hansen, R.) Let (X,T) be the tiling dynamical system corresponding to a primitive inevitable tiling substitution S. Suppose L is diagonalizable and has eigenvalues, $\lambda_1,\ldots,\lambda_d$ where $|\lambda_d| \le |\lambda_j|$, $j=1\ldots d$.

$$\text{Let } c = \frac{\log|\det(L)|}{\log|\lambda_d|} = \frac{\log(|\lambda_1|\ldots|\lambda_d|)}{\log|\lambda_d|}$$

Then $c(n) \le K \cdot n^c$.

Comments:

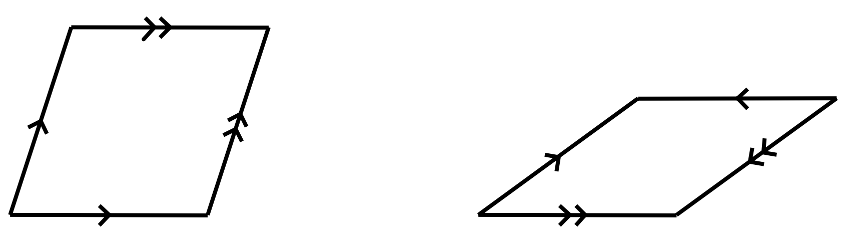
- If strictly self-affine $c>d$ ***(...but really?)***
- **Corollary:** (X,T) has entropy zero.
- For discrete 1-dimensional substitutions the result is $c(n) \leq k \cdot n$.
- Minimum complexity for aperiodic discrete 1-dimensional symbolic dynamical systems is $c(n) \leq n+1$. *Sturmian systems* (Gottschalk & Hedlund 1955)
- C. Hansen (Dissertation, 2000) proved Theorem 14 for multi-dimensional substitutions.

Why interesting?

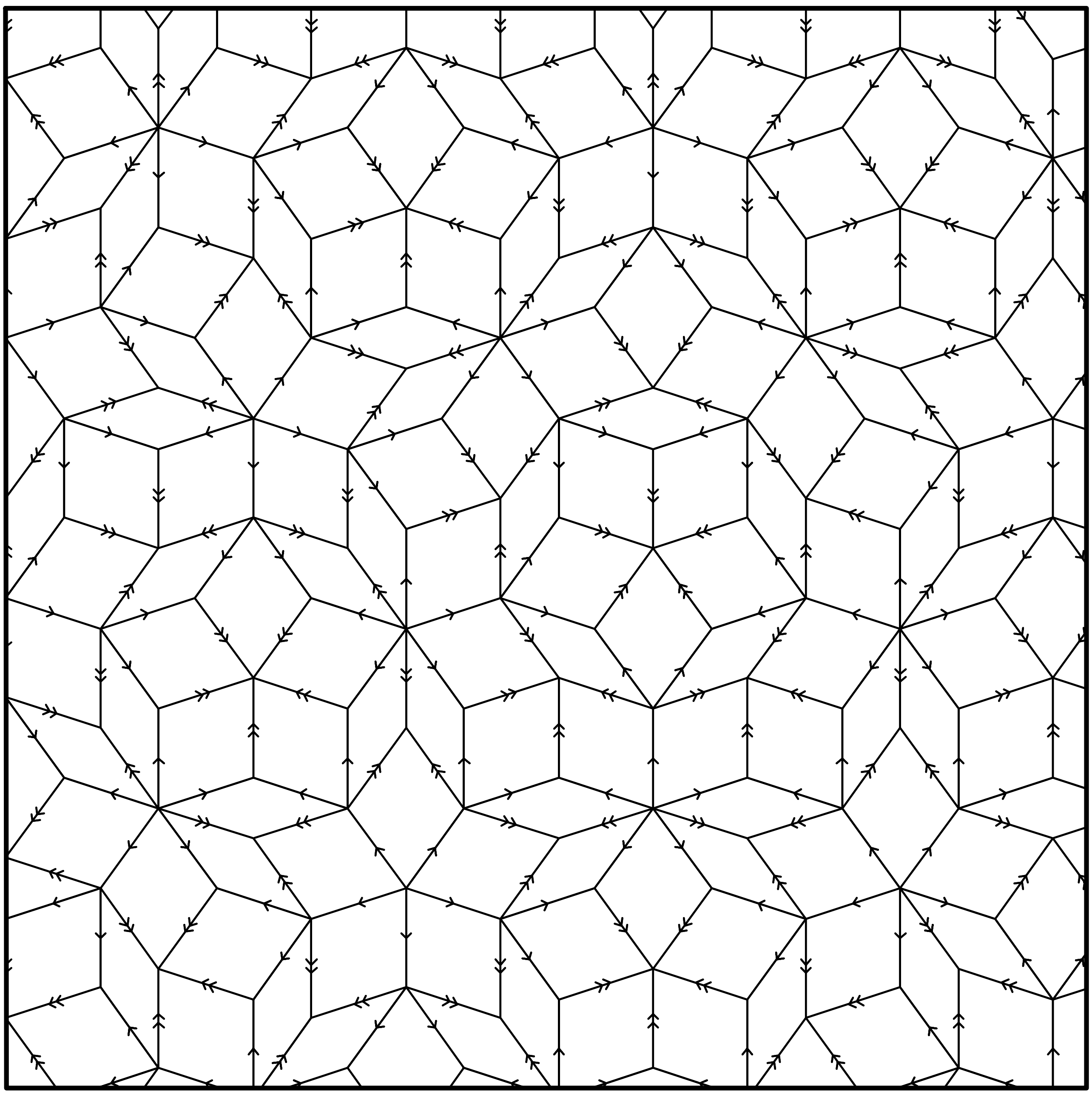
Any tiling space X Œ $X_{\mathcal{T}}$ can be defined by excluding a set $\mathcal{F}$ Œ $\mathcal{T}^*$ of *forbidden patches*.

Definition 15. X Œ $X_{\mathcal{T}}$ is *finite type* if $\mathcal{F}$ is finite.

Example 3.4. Penrose tiles ("local matching rule").



A patch of Penrose tiling



Theorem 16. (Goodman-Strauss) Suppose (X,T) is a self-similar substitution tiling dynamical system, d=2, with prototiles $\mathcal{T}$.

Then there exists a marking $\mathcal{T}_{\#}$ of $\mathcal{T}$, and a local matching rule such that the forgetful mapping $F: X_{\mathcal{T}\#}$ o $X_{\mathcal{T}}$ is almost 1:1, and satisfies $F(X_{\mathcal{T}\#}) = X$.

Definition 17. Call the strictly ergodic zero entropy shifts of finite type that arise this way *hierarchical*.

Amazing fact about *d*>1:

There exist aperiodic hierarchical $\mathbb{Z}^d$ shifts of finite type and $\mathbb{R}^d$ finite type tiling spaces!

Even more amazing:

All the known examples are of this type.

Question:

Are there non-hierarchical examples?

Perhaps complexity can help resolve this question….

Theorem 17. (Radin) Any strictly ergodic finite type shift has entropy zero.

The same result holds for strictly ergodic finite type tiling spaces.

Radin's proof actually shows $c(n) \leq Ke^{\gamma n}$.

This is larger than the complexity of any hierarchical example.

- Do any examples actually realize this?

There is a notion of a *quasiperiodic tiling*. Penrose tilings are examples of these.

One can think of quasiperiodic tiling systems as $\mathbb{R}^d$ Sturmian systems. They satisfy $c(n) \le kn^d$

Berthe and Vuillon defined a type of $\mathbb{Z}^2$ Sturmian shift, based on $\mathbb{R}^2$ quasiperiodic tiling system and proved $c(n) = n^2 + 2n$.

They conjecture this is the minimum possible.

T. Le showed that certain quasiperiodic tiling dynamical systems are almost 1:1 factors of finite type tiling systems.

In other cases, he shows this is not true.

• It is not known whether any of the finite type shifts obtained this way are not hierarchical.

However, complexity will not help here since these examples have $c(n) \le kn^d$.

5. The proof.

Theorem 14. (Cliff Hansen, R.) Let (X,T) be the tiling dynamical system corresponding to a primitive inevitable tiling substitution S. Suppose L is diagonalizable and has eigenvalues, $\lambda_1,\ldots,\lambda_d$ where $|\lambda_d| \leq |\lambda_j|$.

Let $$c = \frac{\log|\det(L)|}{\log|\lambda_d|} = \frac{\log(|\lambda_1|\ldots|\lambda_d|)}{\log|\lambda_d|}$$

Then $c(n) \leq K \cdot n^c$.

Let $\mathcal{T}^{(n)}$ be the set of all n-tile patches in $X_{\mathcal{T}}$.

Lemma 18. For all m>0 there exists J so that for all n sufficiently large

$$\#\{x \in \mathcal{T}^{(m)} : x \subseteq y \text{ for some } y \in \mathcal{T}^{(n)}\} \leq J \cdot n$$

Let A be the "structure matrix" for S:

- that is $a_{i,j}$ is the number of times tile i appears in the substitution of tile j.

Since S is primitive we have A^k >0 for some k.

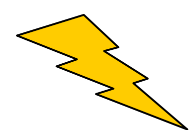
It follows from the **Perron Frobenius Theorem**

$$\lim_{p \to \infty} \frac{A^p \mathbf{v}}{\omega^p} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}$$

- Here $\omega>0$ is the Perron-Frobenius eigenvalue.
- And, $\mathbf{a},\mathbf{b}>\mathbf{0}$ are the left and right Perron_Frobenius eigenvectors.

Thus there is N so that for p sufficiently large:

$$\max_{D \in \mathcal{T}} \#(C^p D) \leq N \cdot \omega^p$$



For $p>0$ call a patch $y \in L^p\mathcal{T}^*$ a p-basic patch if for some $D \in y$ each $D' \in y$ satisfies $D \cap D' \neq \phi$.

- Denote these $y_1^p, \ldots, y_M^p$, where M is independent of p.
- Then (also independent of p)

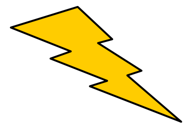
$$M' = \max\{\#(y_j^p) : j = 1, \ldots, M\}$$

- For a p-basic patch y_j^p we have $C^p y_j^p \in \mathcal{T}$.
- Let

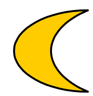
$$M_p = \max\{\#(C^p y_j^p) : j = 1, \ldots, M\}.$$

Since

$$\#(C^p y_j^p) \leq M' \cdot \max_{D \in \mathcal{T}} (C^p D),$$

it follows from  that

$$M_p \leq M' N \cdot \omega^p.$$



Let $\delta = \max\{\mathrm{diam}(D) : D \in \mathcal{T}\}$.

Let ε be the maximum radius of balls $B_\varepsilon \subseteq D$

Fix $q \geq (\delta + 1)/\varepsilon$.

Then $\varepsilon q n \geq n + \delta$ for all $n \geq 1$.

Define

$$p_n = \frac{\log(qn)}{\log|\lambda_d|}.$$

Then $\varepsilon |\lambda_d|^{p_n} = \varepsilon q n \geq n + \delta$. ★

Also $B_{n+\delta} \supseteq \mathrm{supp}(x[B_n])$ for all $n \geq 1, x \in X_T$.

Since S is inevitable we can define a *super-tiling* $C^{-p}x \in X_{L^p T}$ for all p>1.

By ★ $L^p B_\varepsilon \supseteq B_{n+\delta}$ for all $p \geq p_n$.

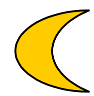
This means that each patch $(C^{-p}x)[B_{n+\delta}]$

is a sub-patch of some p-basic patch

$T^{\mathbf{t}} y_j^p$ in the super-tiling $C^{-p}x$.

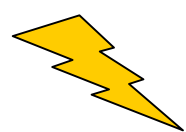
Hence

$$\mathrm{x}[\mathrm{B}_\mathrm{n}] \subseteq x[B_{n+\delta}] = C^p(C^{-p}x[B_{n+\delta}]) \subseteq T^{\mathbf{t}} C^p y_j^p .$$

Now we apply Lemma 18 and  to conclude

$$c(n) \leq JM_p \leq JM'N \cdot \omega^p \overset{\text{def}}{=} K' \cdot \omega^p$$

for all $p \geq p_n$ once n is sufficiently large.

We take n so large that  holds for $p=p_n$.

Then

$$c(n) \leq K'\omega^{p_n} = K'\omega^{\frac{\log(qn)}{\log|\lambda_d|}} = K'e^{\frac{\log(\omega)\log(qn)}{\log|\lambda_d|}}$$

$$= K'q^c n^c = K \cdot n^c$$