A survey of results and problems in tiling dynamical systems

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I. Basic definitions

II. Types of examples

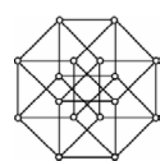
- Periodic tilings.
- Finite type tilings
- Substitution tilings
- Quasiperiodic tilings

III. Properties

- Types of isomorphism
- Issues surrounding finite type
- Entropy and complexity
- Spectral theory

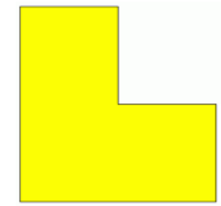
Key to *guide symbols*:

Theorem true
= due primarily to
continuous time.



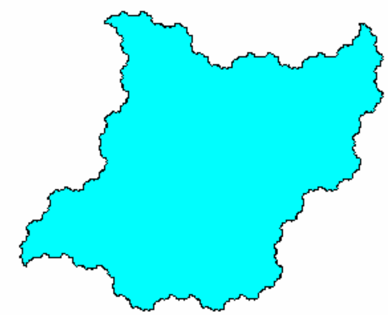
Theorem true
= due primarily to
multidimensional time.

I. Basic definitions



Tiles:

- A *tile* is a closed subset $D \subseteq \mathbb{R}^d$



Usual assumptions:

- D is homeomorphic to $\left\{ \mathbf{v} \in \mathbb{R}^d : \|\mathbf{v}\| \leq 1 \right\}$

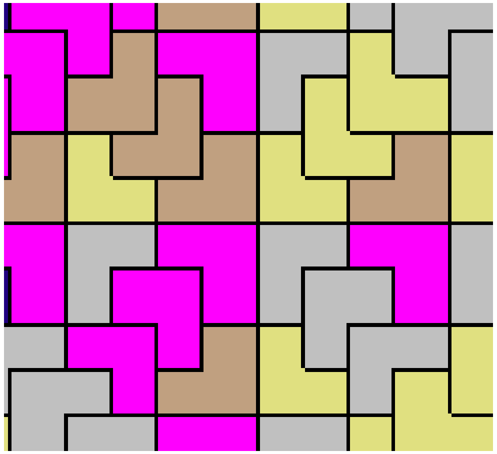
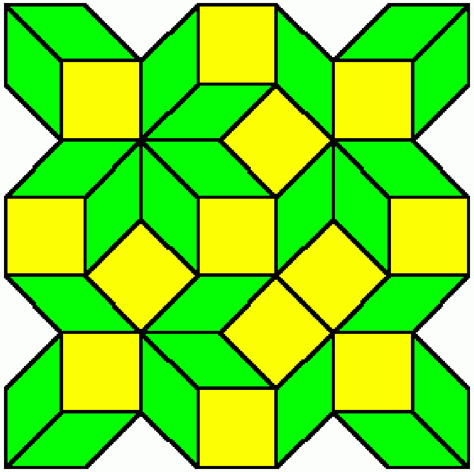
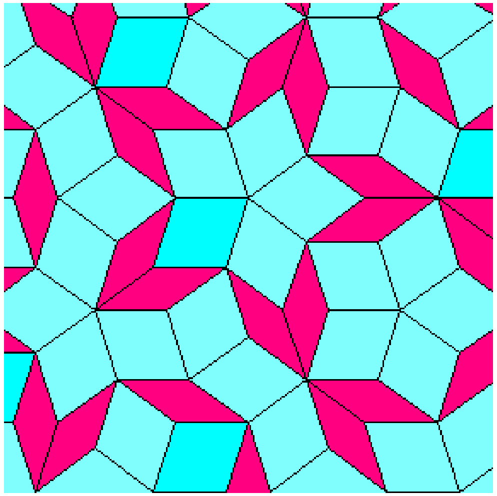
  (i.e., connected).

- ∂D has Lebesgue measure zero.

Tilings:

$x = \{D_1, D_2, D_3, \dots\}$ with:

- $D_i \cap D_j$ "essentially" empty.
- $\mathbb{R}^d = \bigcup_{D \in x} D$.

Definition. Translation action of $\mathbb{R}^d$ on x:

$$T^{\mathbf{t}}x = \{D - \mathbf{t} : D \in x\}$$

Usual assumptions on *tilings*:

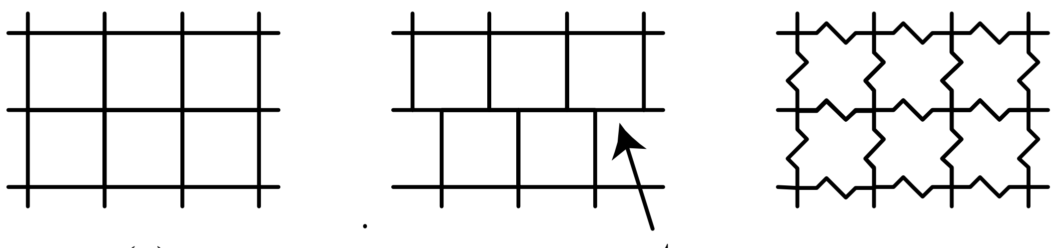
1. $\exists$ finite *protoset*

$$\mathcal{T} = \{D^1, D^2, \ldots, D^n\}$$

(i.e., every $D \in x$ satisfies $D = T^{\mathbf{t}} D^j$)

2. *Local finiteness* condition:

$\mathcal{T}^{(2)}$= finite collection of allowed 2-tile *patches* (usually implicit)



(a) (b) (c)

Full Tiling space:

$$X_{\mathcal{T}} = \{x : x \text{ a tiling by tiles from } \mathcal{T}\}.$$

...includes implicit choice of $\mathcal{T}^{(2)}$.

Tiling topology: Two tilings x,y are r-close if they agree completely on an $\frac{1}{r}$- square centered at **0**, after translating each by $\leq r$.

Theorem: X_T is compact metric. T is a continuous action of $\mathbb{R}^d$ (Rudolph, 1986).

Some variations on the theme:

- Allow larger groups to act on X_T.
- Allow *decorated* tiles.
- Relax local finiteness.
  - allow small rotations.
  - use the *Hausdorff metric.*
- Replace tilings with locally finite *Delone sets*.
  - keep the *tiling topology!*
  - *e.g.*, the **vertex set** of a tiling.
  - *Voronoi construction*: gets back a tiling.

Definition. A tiling space is $X \subseteq X_T$ that is closed and T-invariant.

• The pair (X,T) is called a *tiling dynamical system* (it is an action of $\mathbb{R}^d$).

Theorem. If (X,T) is topologically transitive, not transitive then each $x \in X$ has a neighborhood homeomorphic to $K \times \mathbb{R}^d$, K *Cantor* (Putnam, R).

絹

Theorem. If T consists of polygons, then X is a Cantor bundle over $\mathbb{T}^d$ (Sadun, Williams 2001).

Theme 1. Tiling dynamical systems are generalized symbolic dynamical systems.

Tiling DS	**Symbolic DS**
Tiles	Symbols
Tiling topology	Product topology
Action T	The shift
Mutual local derivability	Block map
Periodic tiling	Periodic orbit

Theme 2. Dynamical properties correspond to "interesting" geometric properties of tilings.

Dynamics	Tilings
Transitivity	Periodicity
Minimality	Repetivity
Topo. Conjugacy	Mutual local derivability
Ergodicity	Patch statistics
Entropy/Complexity	Patch-type growth
Discrete spectrum	Diffraction
Mixing properties	Patch-location independence

For example:

- Minimality: all tilings *repetitive*, and belong to the same *local isomorphism class*.

...for any patch y in x, $\exists\, R=R(y)$ so that a copy of y occurs within R of an arbitrary location in x.

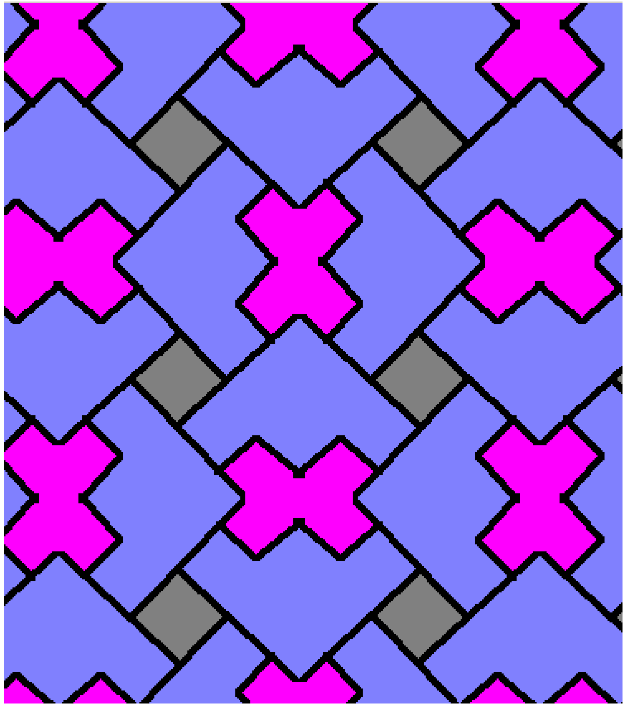
- Unique ergodicity: *uniform patch statistics*:

$$\frac{\#(y \text{ in } x[B_n])}{\mathrm{vol}(B_n)} \to K_y$$

II. Types of examples

- Periodic tilings.
  - M. C.Escher tilings
- Finite type tilings
  - Penrose tilings
    - Kari-Culic tiles
      - Rudolph tilings
- Substitution tilings
  - Penrose tilings
    - “Binary” tilings
- Quasiperiodic tilings
  - Model sets
    - Penrose tilings (again)

• **Periodic tilings.**



- Crystallographic restriction.
- Hilbert's 18th problem.
- Space (symmetry) groups.

- $X_T \cong \mathbb{R}^d$ and T is transitive action.
- "Rescale topological conjugacy" equivalent to having same symmetry group.

• Finite type tilings.

- Start with $X_{\mathcal{T}}$.
- Let $\mathcal{F}$ = *finite* collection of "forbidden" patches.
- Define $X_{\mathcal{F}} = \{x \in X_{\mathcal{T}}$: not containing $\mathcal{F}$ patches$\}$.

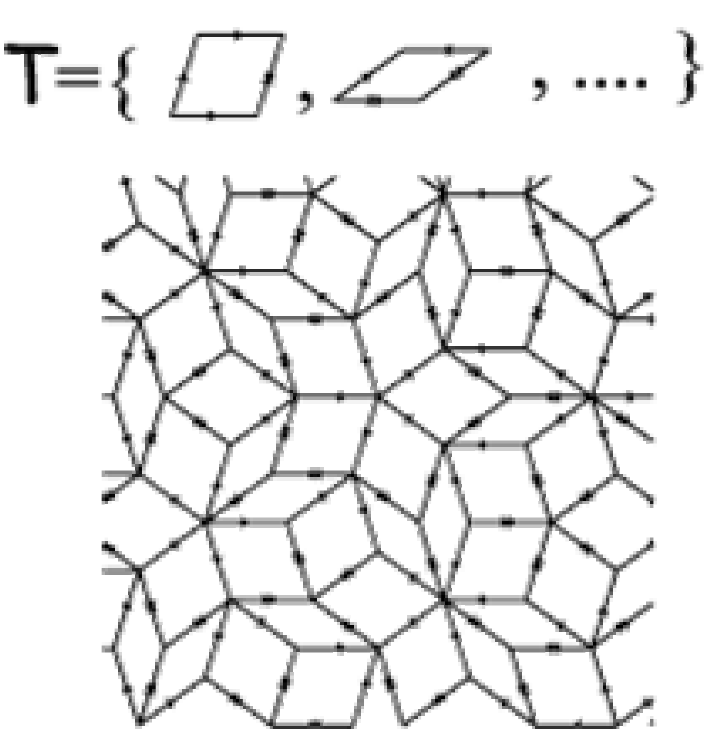
.... Includes all periodic tiling spaces

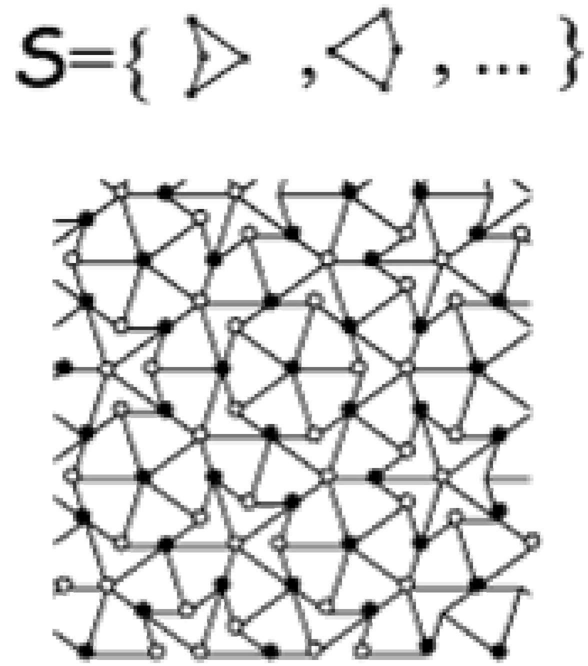
Usual assumption: $(X_{\mathcal{F}}, T)$ non-transitive
but topologically transitive.

positive entropy "high temperature"	v.s.	zero-entropy "low temperature"

Zero entropy:

Example. Penrose tilings





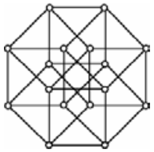
How do we know Penrose tilings actually exist ???

Wang's Conjecture---

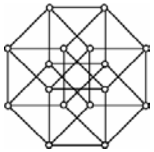
Berger's Undecidability Theorem.

Theorem: $\exists$ finite type tiling spaces with no periodic tilings (Berger, 1966).

Corollary: There exist *strictly-ergodic* non-transitive zero entropy finite type tiling dynamical systems.



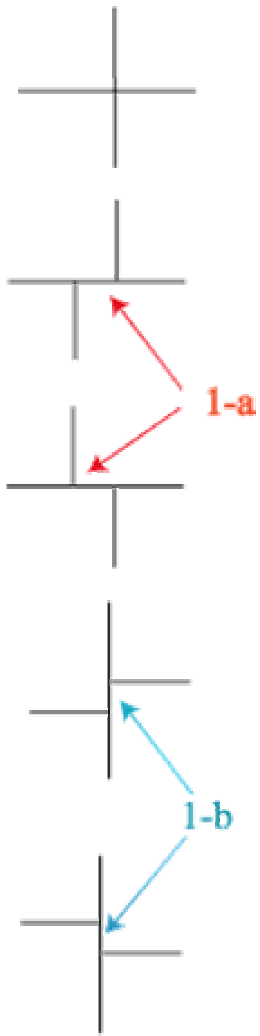
Problem: "Einstein" problem.

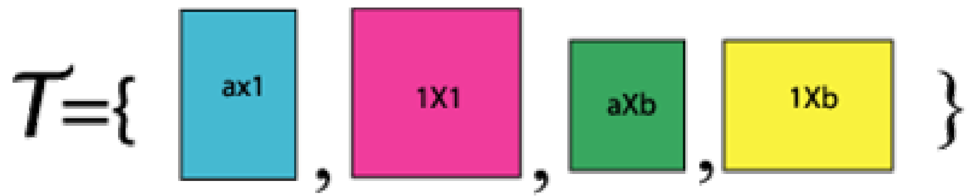


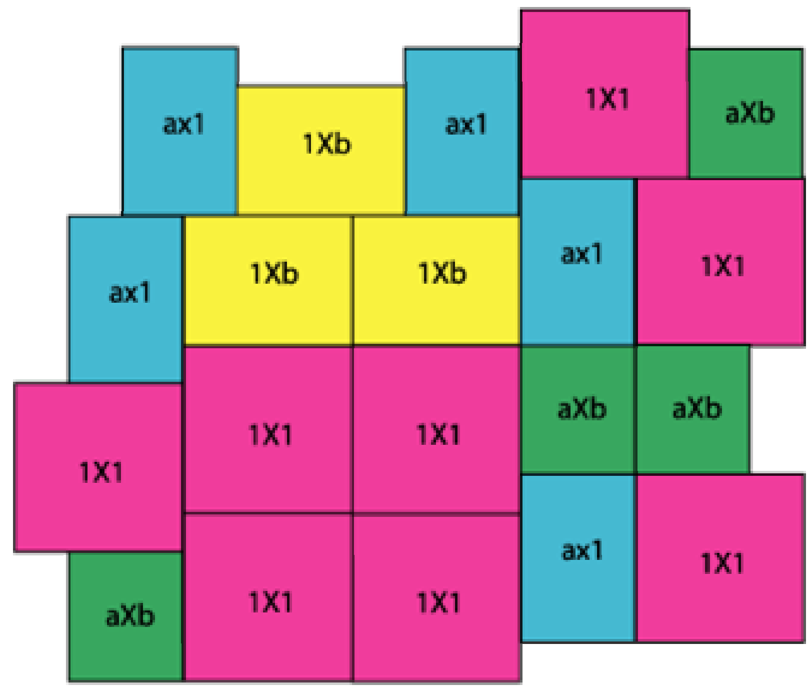
Positive entropy:

Rudolph tilings

$\mathcal{T}^{(4)} =$







Rudolph tilings have a universal modeling property.

Positive entropy:

- $\exists$ positive entropy finite type tiling spaces with *no periodic points*.

Topological mixing does not imply positive entropy (a result in $\mathbb{Z}^d$).

Even with lots of topological mixing, measures of maximal entropy need not be unique and need not be weakly mixing (a result in $\mathbb{Z}^d$).

= known true in $\mathbb{Z}^d$, expected true in $\mathbb{R}^d$

Problem. Find a finite type tiling system that is topologically mixing (as an $\mathbb{R}^d$ action).

Problem. Find other sets T that have Rudolph's *universal modeling property*:

For any ergodic m.p. $\mathbb{Z}^d$ action (Y,S,ν) there is a T-invariant measure on (X_T, T) so that (X_T, T, μ) is a *metric factor* of (Y,S,ν).

If the factor is isomorphism, call it strong universal modeling.

Problem. Do there exist *strong* universal models for $d > 1$?

• Substitution tilings

- $L \in \mathrm{Gl}(d, \mathbb{R})$ a linear expansion (an *Affinity*).
- *Similarity*: the case $L = \lambda M$, where M =isometry.

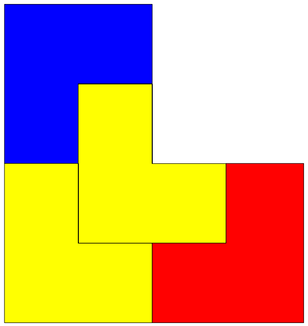
- *Decomposition*: a mapping $C : \mathcal{T} \to L^{-1}\mathcal{T}^{*}$

- **Assume:** *perfect overlap* condition

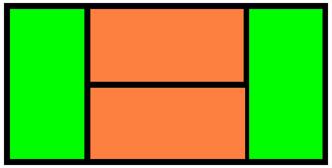
$$\mathrm{supp}(C(D)) = \mathrm{supp}(D)$$

Some decompositions:

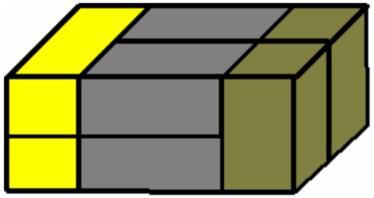
- A few self-similar “polyomino” decompositions



The chair



The table

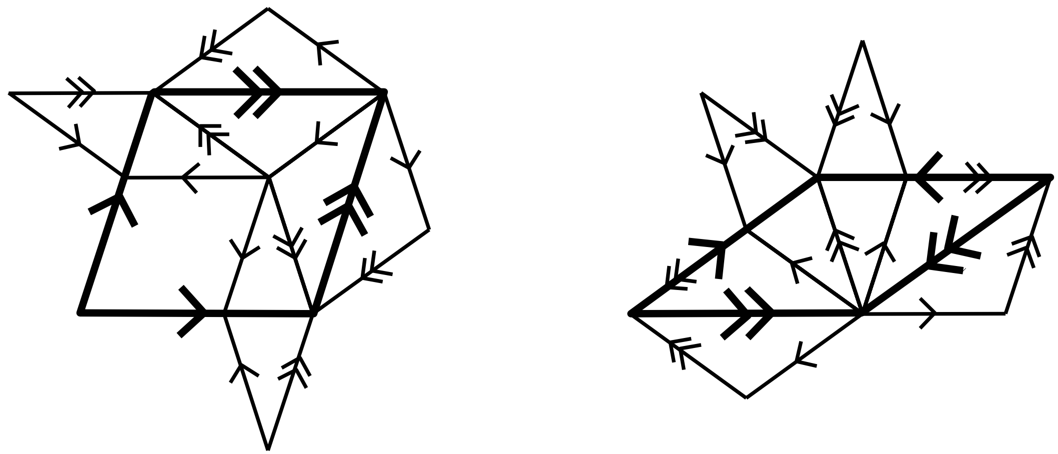


3-dimensional table

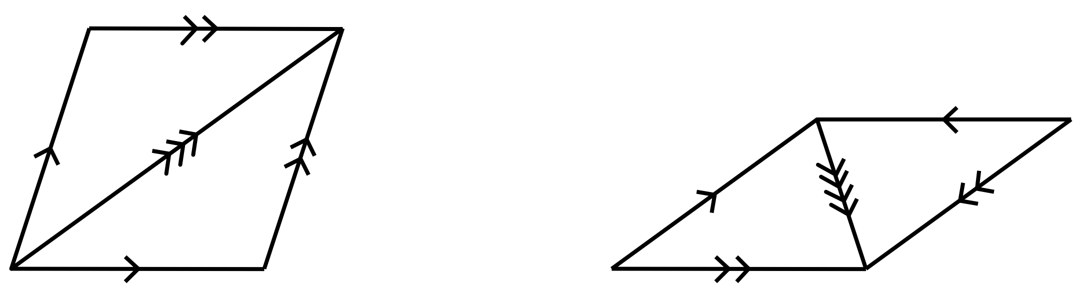
A mapping $S = LC$ is called a *tiling substitution*.

Penrose decompositions:

- The rhombic Penrose decomposition ("imperfect")

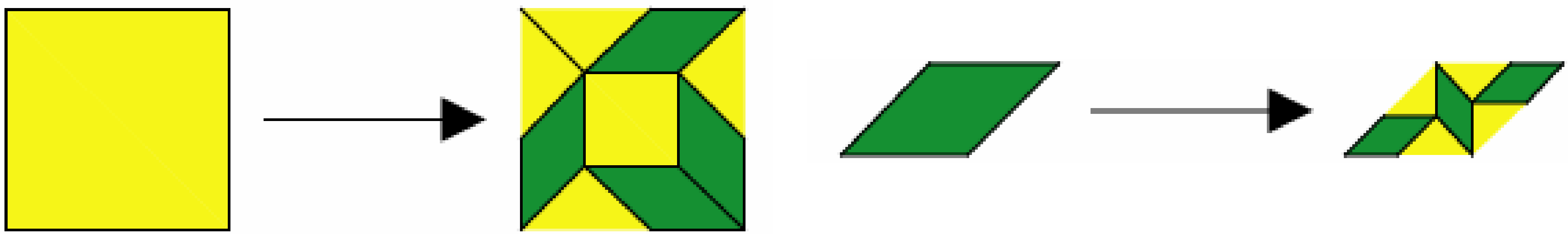


- Can be "perfected" by cutting tiles in half.

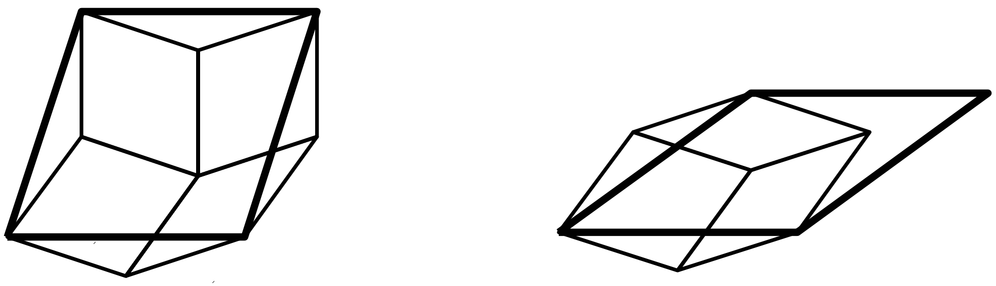


More “imperfect” self-similar substitutions

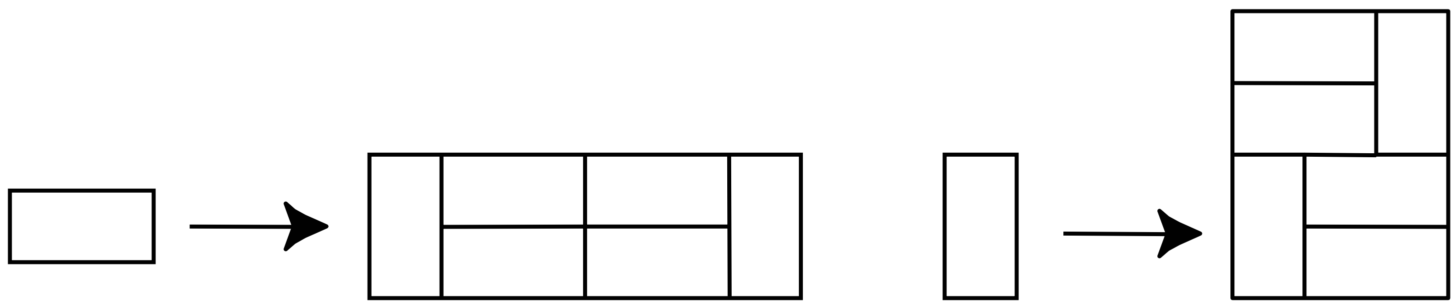
- The *octagonal* or *Ammann-Beenker* tiling



- The binary decomposition (note the required 2p/20 rotation)



A self-affine polyomino tiling substitution.



In this example $L = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ (i.e. it's non self-similar)

How to make a substitution tiling space...

- Put $x_1 = \{D\}$ for some De$\mathcal{T}$
- Let $x_k = Sx_{k\text{-}1.}$
- Define

  $X = \{x \in X_{\mathcal{T}}$: each patch in x is in some $x_k\}$.

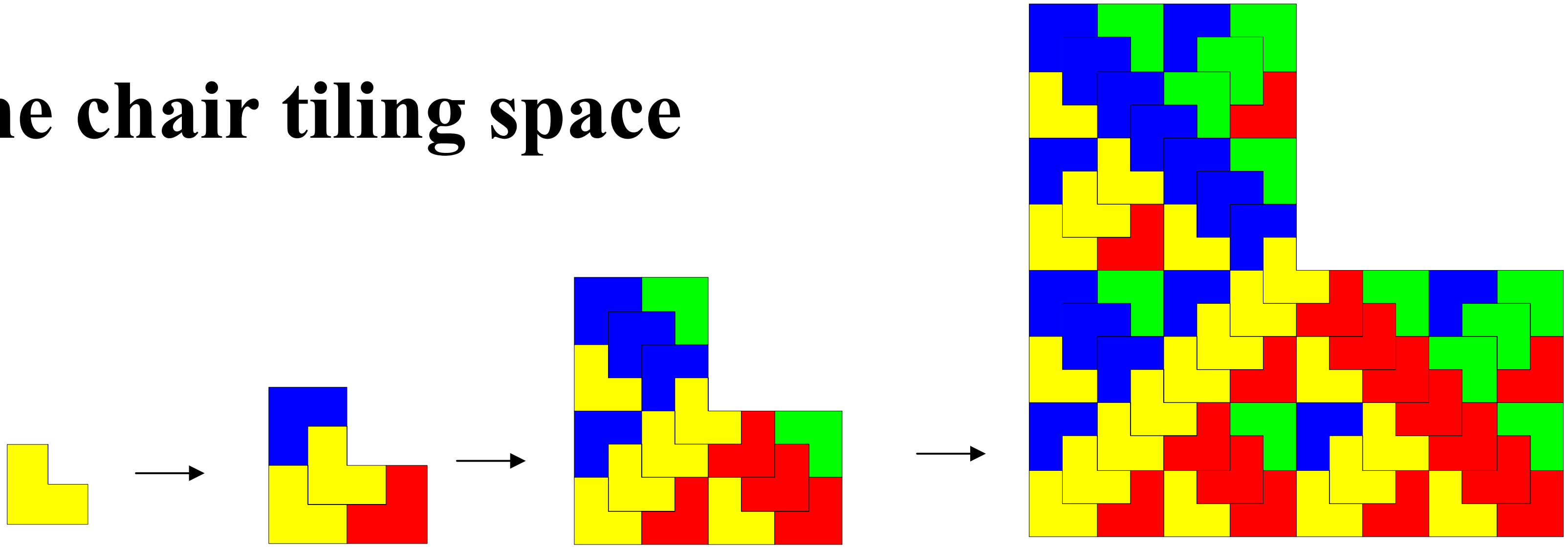
- (X,T) is a tiling dynamical system.

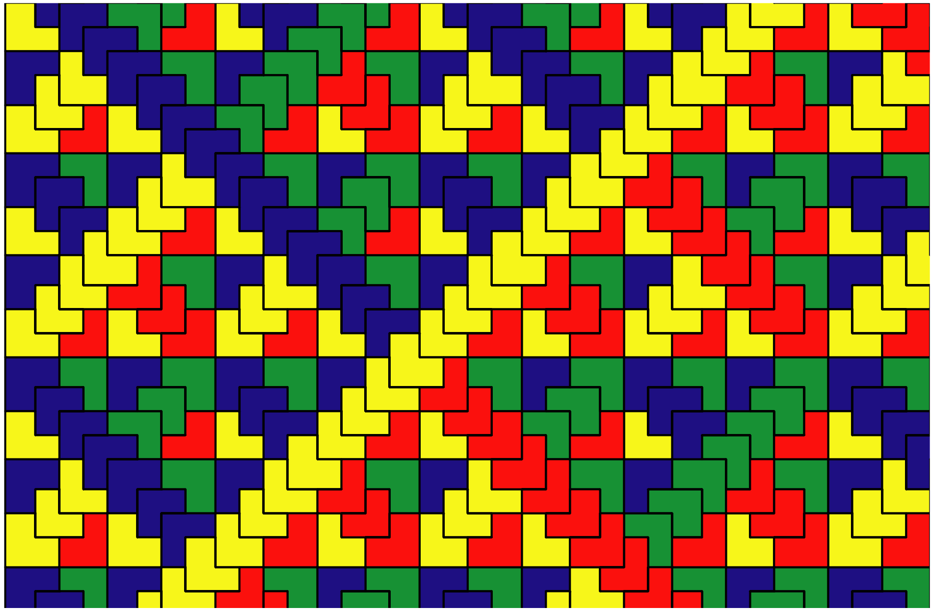
Usual Assumptions:

•S is *primitive*:

•S is *primitive*:

The chair tiling space





Basic Properties

Theorem. If S is *primitive* then $X \neq \emptyset$ and (X,T) *strictly ergodic* (minimal and uniquely ergodic) (Pragastis, Solomyak).

Penrose tilings do exist!

Theorem. S is *invertible* if and only if all $x \in X$ are *aperiodic* (Solomyak).

Basic Properties, continued

Examples are known with (metric) pure point spectrum, mixed spectrum and weak mixing.

Theorem. (X,T) has (topological) entropy zero (R. & Hansen).

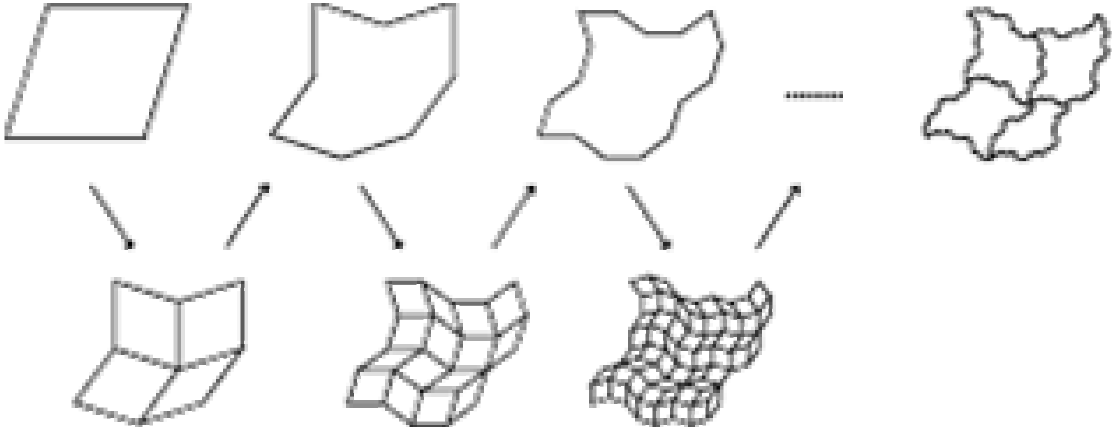
Theorem. (X,T) is not (metrically) mixing (Solomyak)

Problem. Is topological mixing possible?

An interesting fact.

Theorem. Imperfect substitutions can always be "perfected" in the case d=2. (Priebe/Frank, Solomyak)

This means the corresponding tilings systems are MLD (i.e.,topologically conjugate: see later)



Problem. Is this result true for $d > 2$?

Question. What linear maps L can be the expansion for a tiling substitution?

Definition. Diagonalizable L satisfies the *Perron condition* if for each e.v. λ of L, if λ' is a Galois conjugate of λ with $|\lambda'| > |\lambda|$, then λ' must occur more multiplicity than λ (complex conjugates count in pairs).

Theorem. L is an expansion for a self-affine tiling if and only if it satisfies the *Perron condition*. (Thurston, Kenyon)

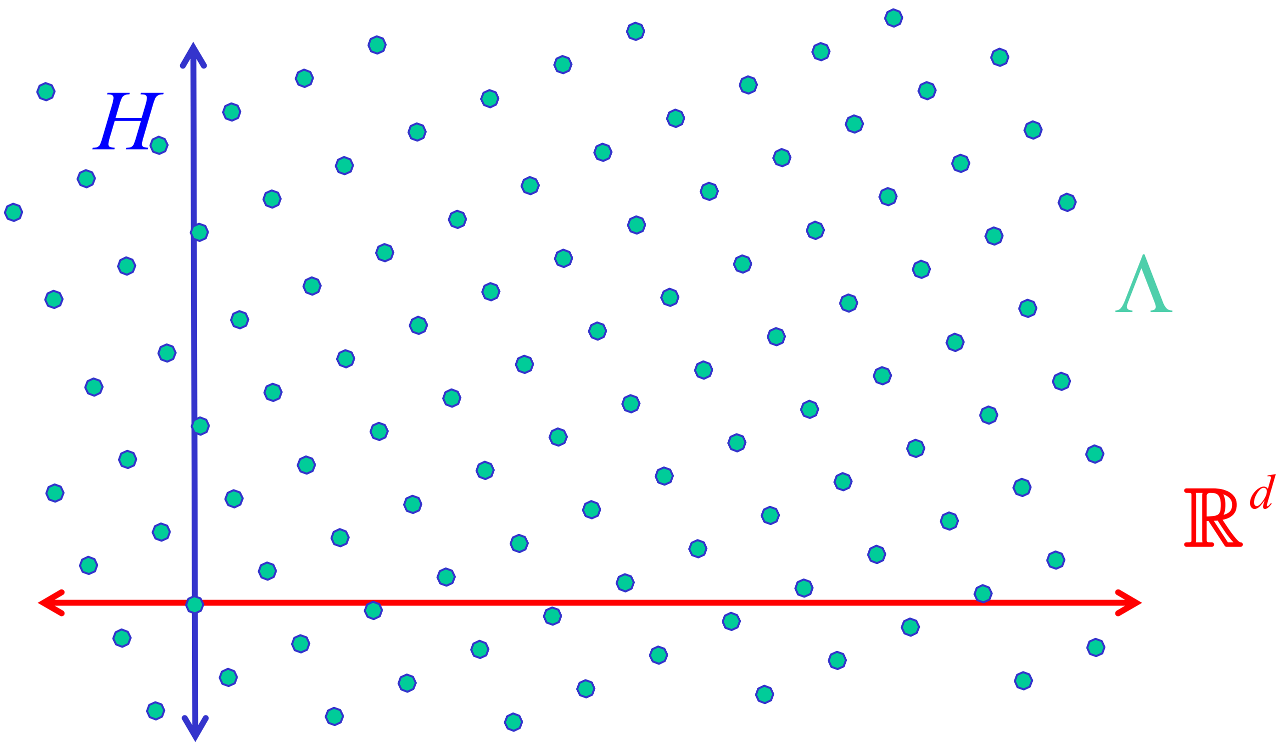
•Quasiperiodic tilings

$\mathbb{H}$ = Locally compact, abelian

σ-compact (the "internal space").

$\Lambda \subseteq \mathbb{R}^d \times \mathbb{H}$ a *lattice*:

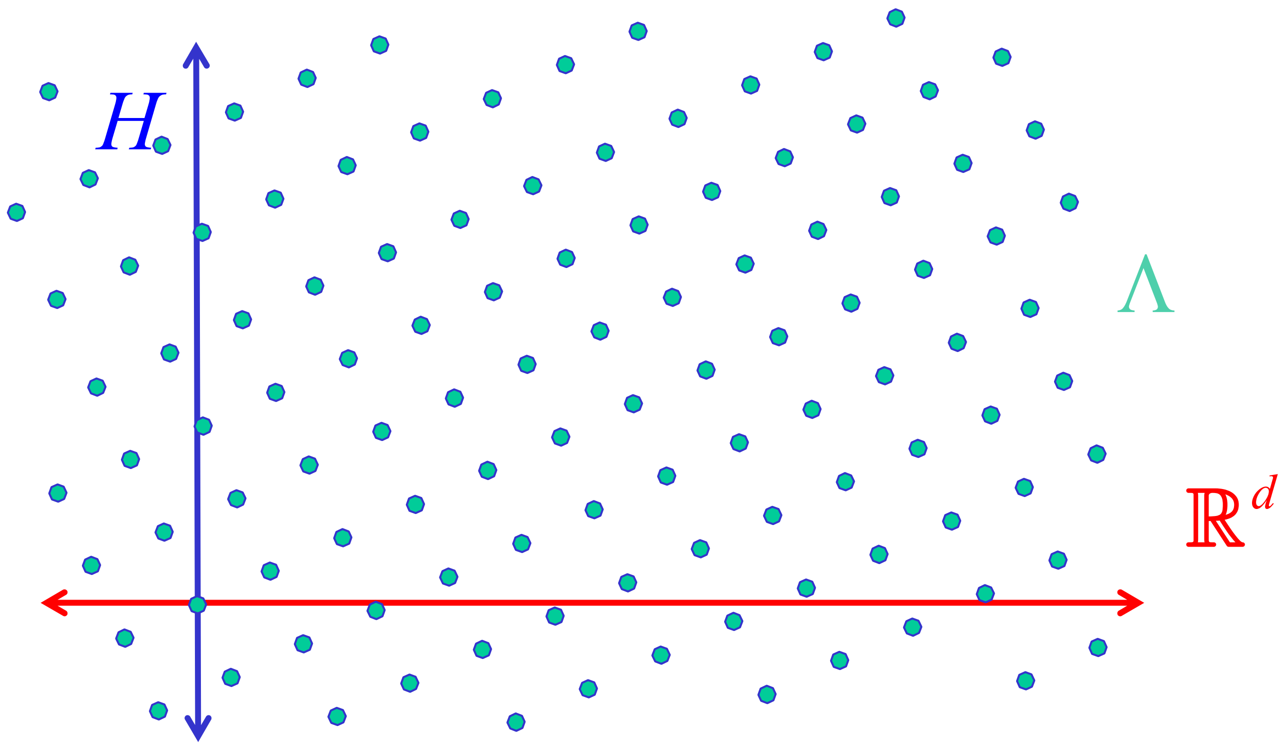
i.e., $\mathbb{G} := (\mathbb{R}^d \times \mathbb{H}) / \Lambda$ compact.



Assume: $\mathbb{R}^d \cap \Lambda = \{0\}$.

Define $\mathrm{K}^{\mathbf{t}}$: $\mathbb{G} \rightarrow \mathbb{G}$ by $K^{\mathbf{t}} g = (0,\mathbf{t}) + g$.

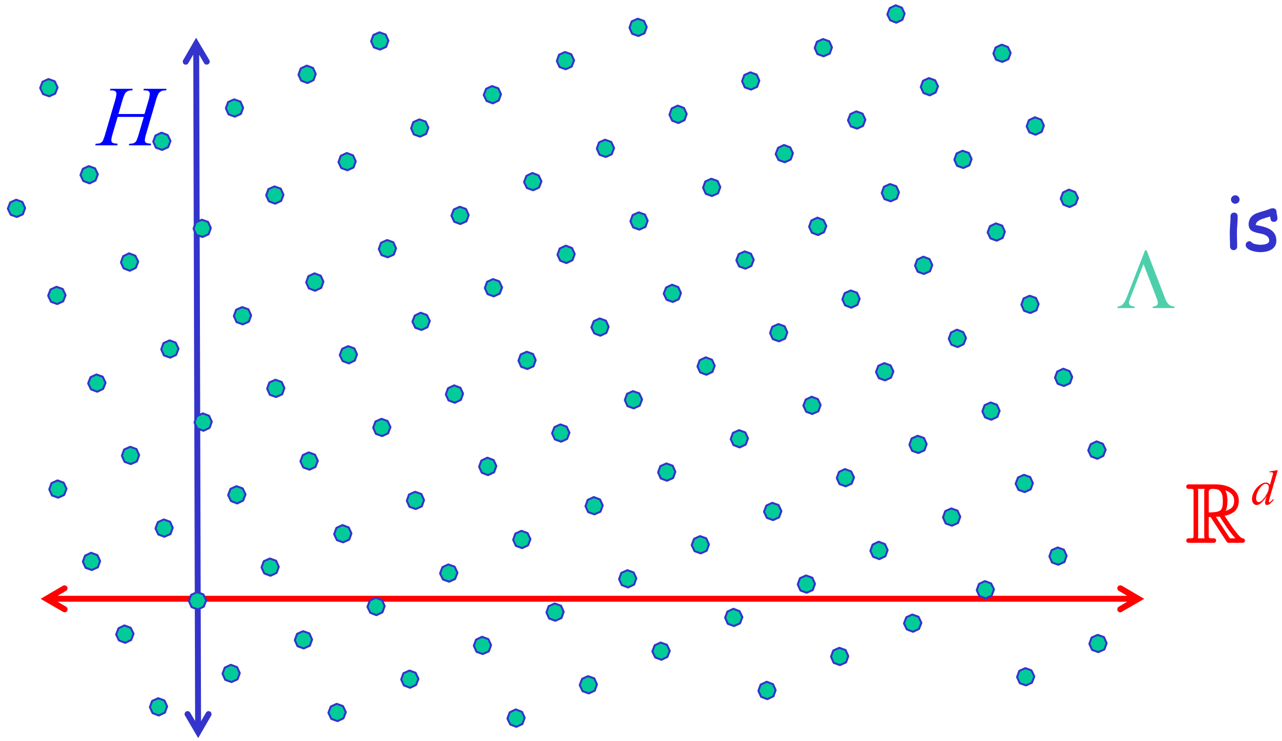
- The dynamical system $(\mathbb{G}, K)$ is piecewise almost periodic. (every "point" almost periodic)



Let $\mathbb{G}_0$ = closure $\{K^{\mathbf{t}}\, 0 \colon \mathbf{t} \in \mathbb{R}^d\}$ (a closed subgroup).

$(\mathbb{G}_0, T)$ is *strictly ergodic* (minimal & uniquely ergodic) and *Kronecker* (pure point spectrum).

$m \in \mathbb{M} = \mathbb{G}/\mathbb{G}_0$ indexes other minimal sets $\mathbb{G}_m = \mathbb{G}_0 + m$

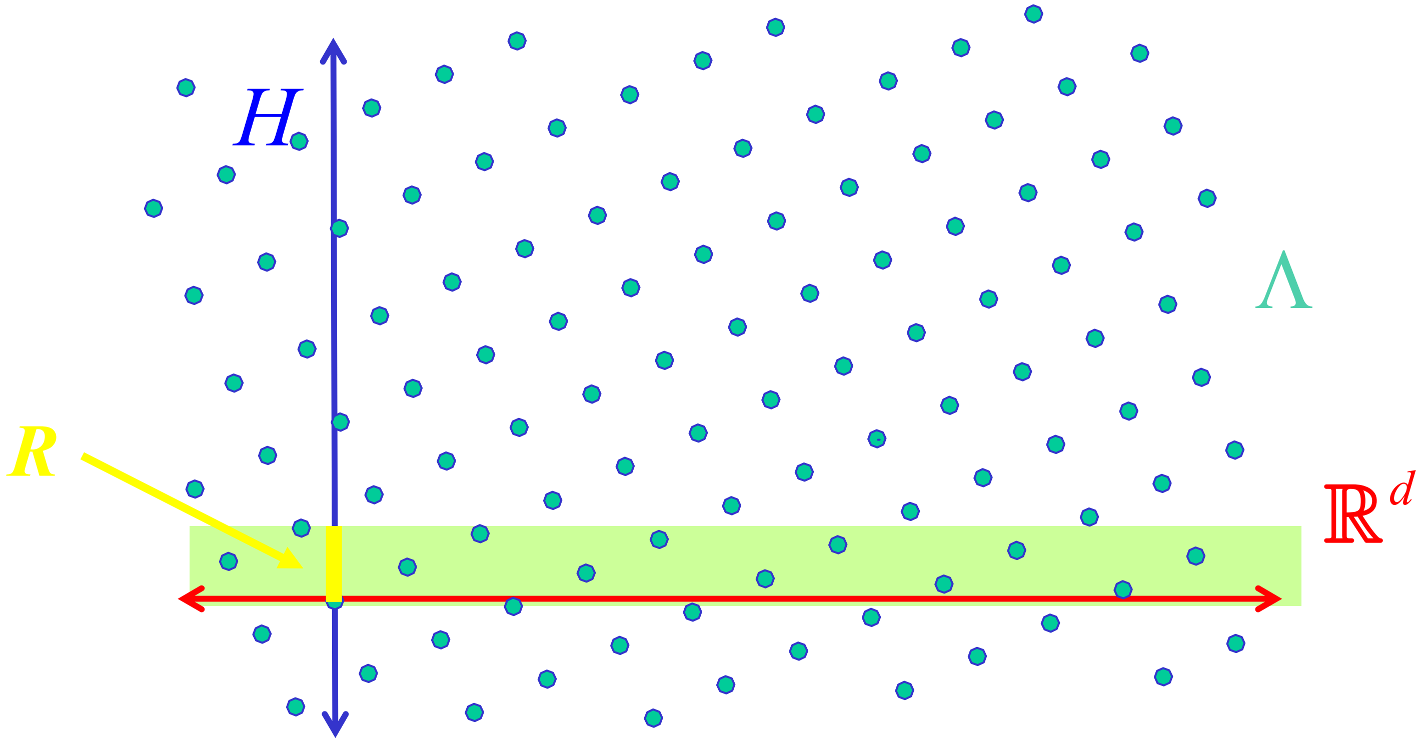


...all isomorphic

Choose $R \subseteq H$ *compact* ("window", "atomic surface")

For $g \in \mathbb{G}$ let

$$y_g = \{\pi_{\mathbb{R}^d}(h) : h \in \Lambda + g, \pi_{\mathbb{H}}(h) \in R\}$$



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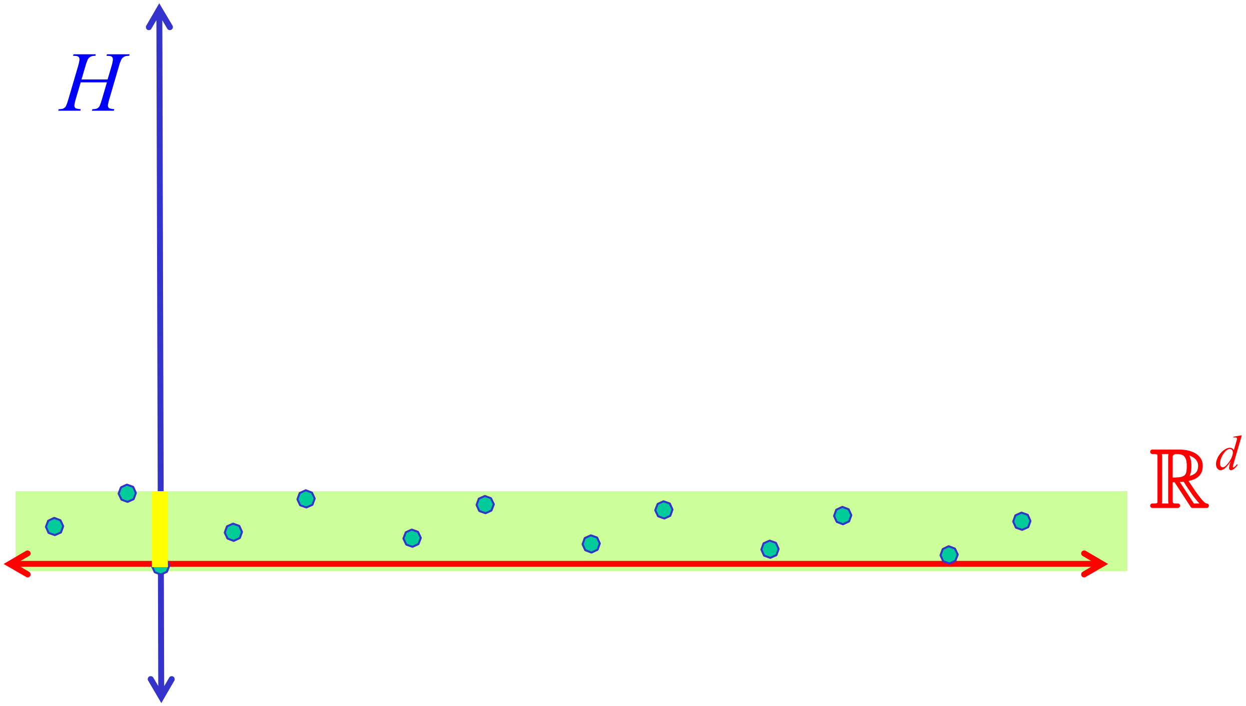

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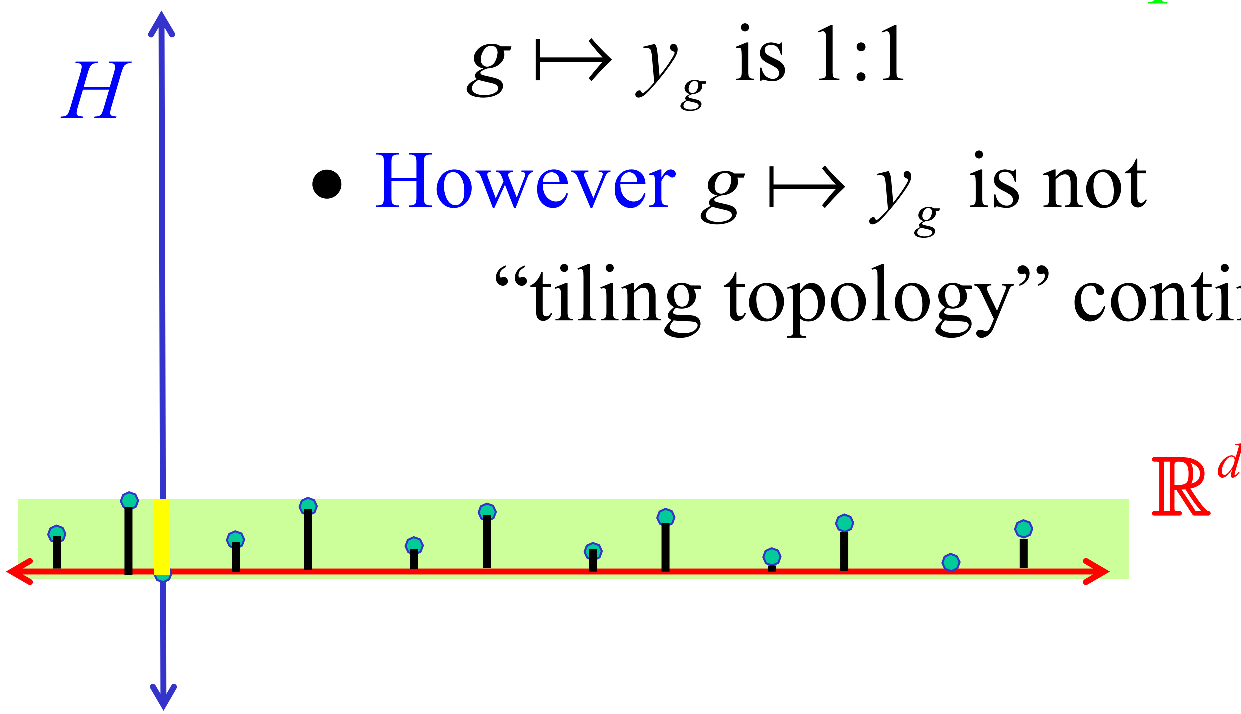
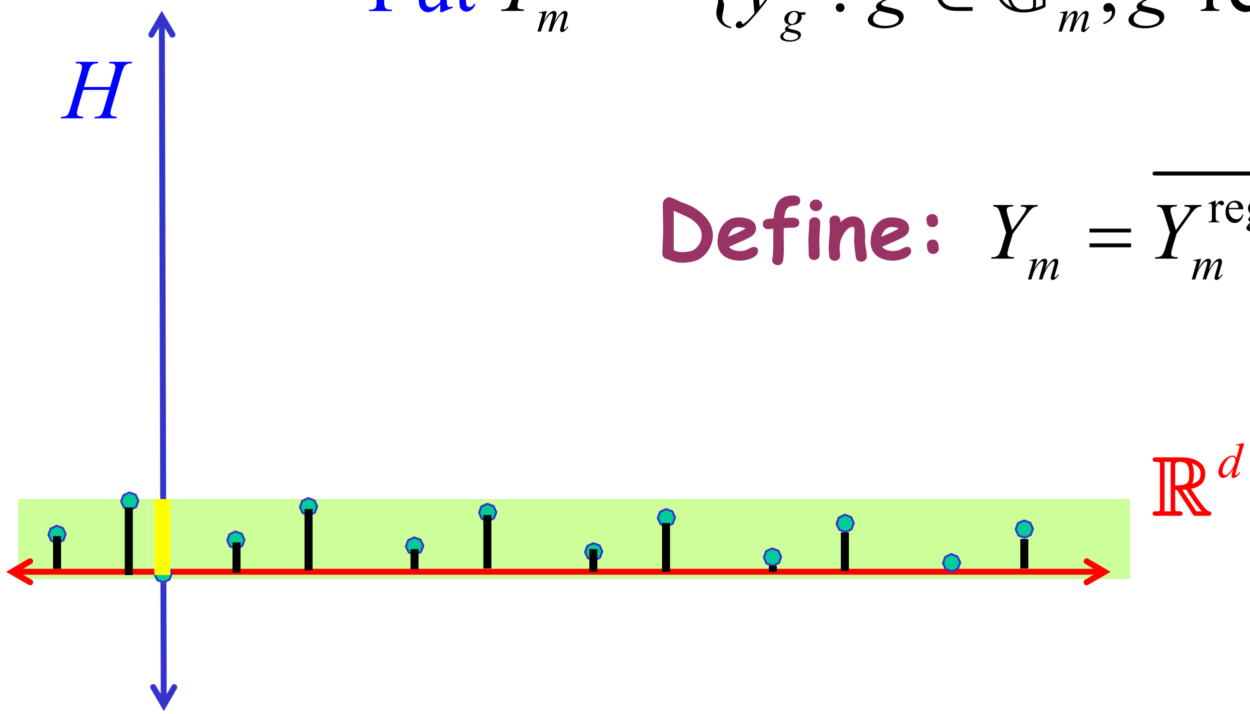
- $y_{K^{\mathbf{t}}g} = T^{\mathbf{t}} y_g$
- Under additional assumptions: $g \mapsto y_g$ is 1:1
- However $g \mapsto y_g$ is not “tiling topology” continuous!



Discontinuities occur when $h \in \Lambda + g$ lies on the boundary of $R \times \mathbb{R}^d$.

Define: $g \in \mathbb{G}$ *regular* if $\pi_{\mathbb{H}}(\Lambda + g) \cap \partial(R) = \emptyset$

Put $Y_m^{\mathrm{reg}} = \{ y_g : g \in \mathbb{G}_m, g \text{ regular} \}$

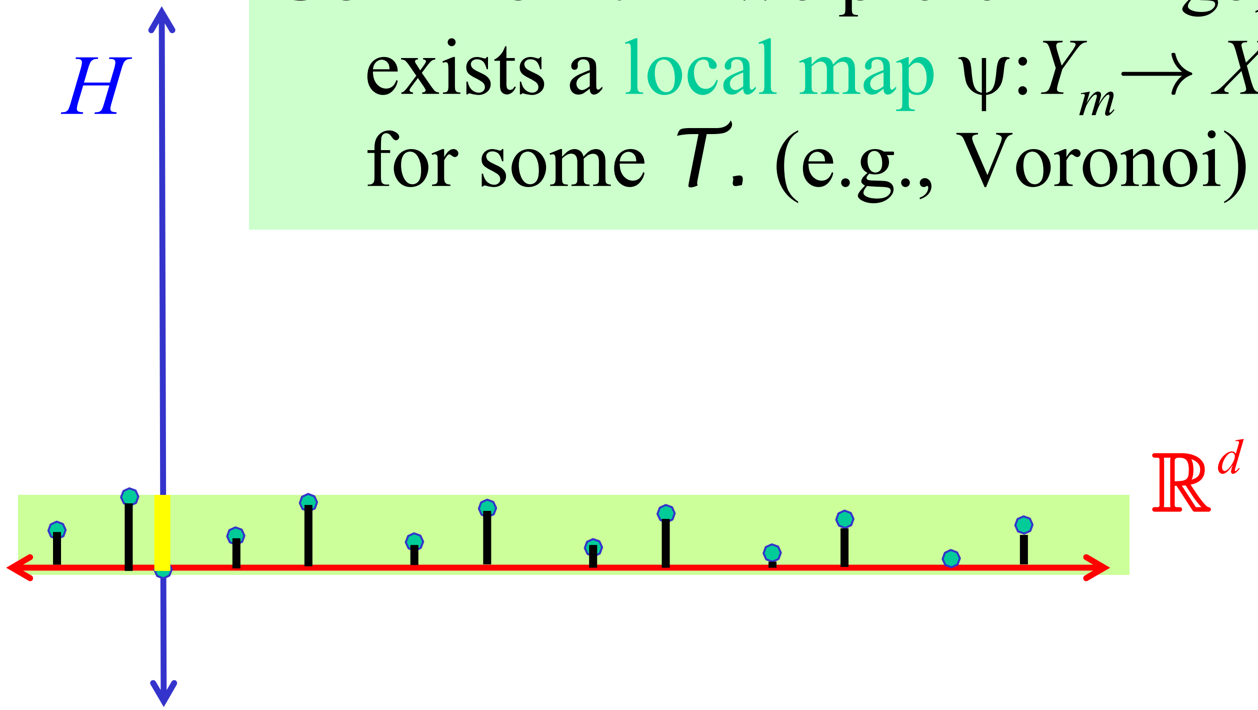


Define: $Y_m = \overline{Y_m^{\mathrm{reg}}}$.

- Y_m *is* a T-invariant space of locally finite *Delone sets*.

- (Y_m, T) is a dynamical system (of Delone sets).

Comment: If we prefer tilings, there exists a local map $\psi: Y_m \to X_m \subseteq X_{\mathcal{T}}$ for some $\mathcal{T}$. (e.g., Voronoi)



Names of related constructions

- Cut method
- Generalized grid method
- Projection method
- Atomic surface theory

Short list of people who have worked in this area

Y. Meyer, N. G. de Bruijn, G. Rauzy

A. Katz, M. Duneau, C. Oguey, K. Niizeki

L. Levitov, P. Steinhardt, T.Q.T. Le

P. Arnoux, V. Berthé, S. Ito, J. Lagarias

R. Moody, J. Lee M. Baake, M. Schlottman,

Properties of "Quasiperiodic" tiling systems:

- $\exists$ *continuous* $\psi: Y_m \to \mathbb{G}_m$ s.t. $\psi(y_g)=g$.
- $\psi \circ T^{\mathbf{t}} = K^{\mathbf{t}} \circ \psi$
- Each (Y_m, T) is an ***almost* 1:1 *extension*** of $(\mathbb{G}_m, K)$.
- Each (Y_m, T) is *strictly ergodic* and has metric *pure point spectrum* (the same for all m).
- They **do not** have topological *pure point spectrum* and are **not** topologically conjugate.

Points that have the same *non-regular* ψ-images are homoclinic or proximal.

Comments:

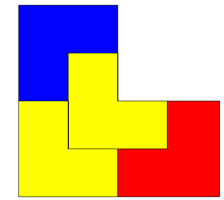
- The point spectrum – the set of "eigenvalues" Σ of (X_m, T) – is given by an easy algebraic formula (involving Λ and $\mathbb{H}$)

- The point spectrum of a Delone set dynamical system $(Y_m, T)$is essentially the *diffraction spectrum* of any of of the contained Delone sets $y \in Y_{m.}$ (Dworkin, Hof)

- Virtually any point "quasisymmetry" can be achieved: **no** crystalographic restriction.

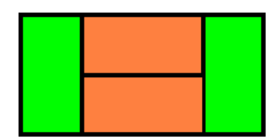
Two interesting special cases.

First… The chair tiling corresponds to the case where $\mathbb{H} = \widehat{\mathbb{Z}_2} \times \widehat{\mathbb{Z}_2}$ (Baake, Moody, Solomyak, R.)



This result extends to any lattice polynomial substitution satisfying a ***coincidence condition***

- Without coincidence (e.g., the table) we get a *finite extension* of K^t that has mixed spectrum (Solomyak, R.)



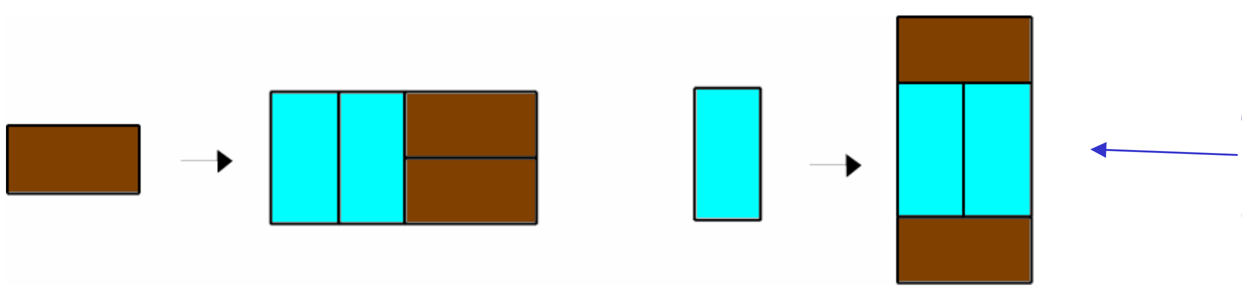


Table with coincidence

Second... Rauzy Fractals

$$\theta := \begin{cases} 1 \to 12 \\ 2 \to 13 \\ 3 \to 1 \end{cases}$$

"Rauzy substitution"

$$\boxed{\theta(u) = u}$$

Abelianize:

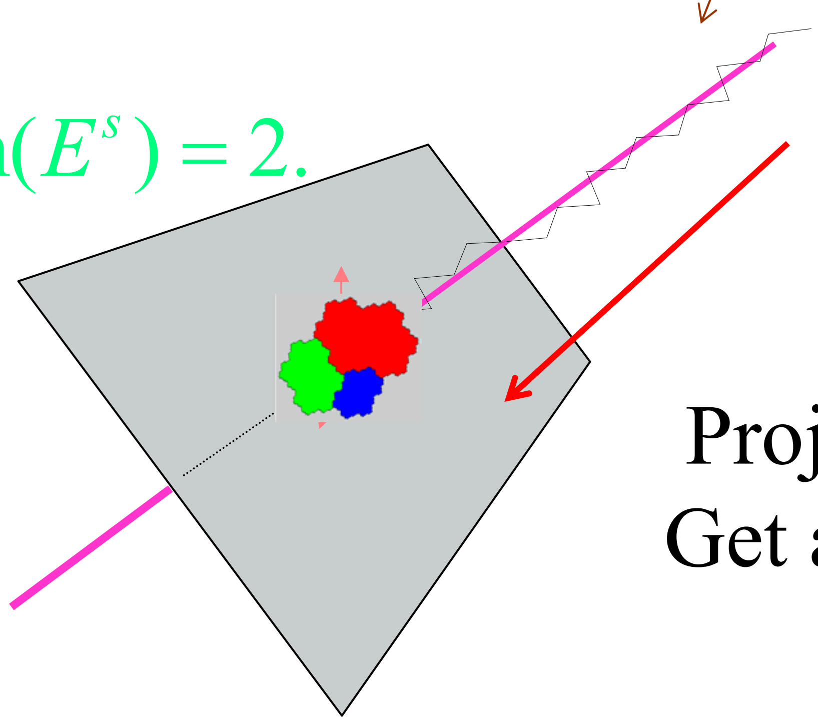
$$\begin{array}{ccc} \mathcal{F}(\mathcal{A}) & \xrightarrow{\theta} & \mathcal{F}(\mathcal{A}) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p}\downarrow \\ \mathbb{Z}^2 & \xrightarrow{A} & \mathbb{Z}^2 \end{array}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

u stays near E^u by the Pisot property,

$\dim(E^u)=1$

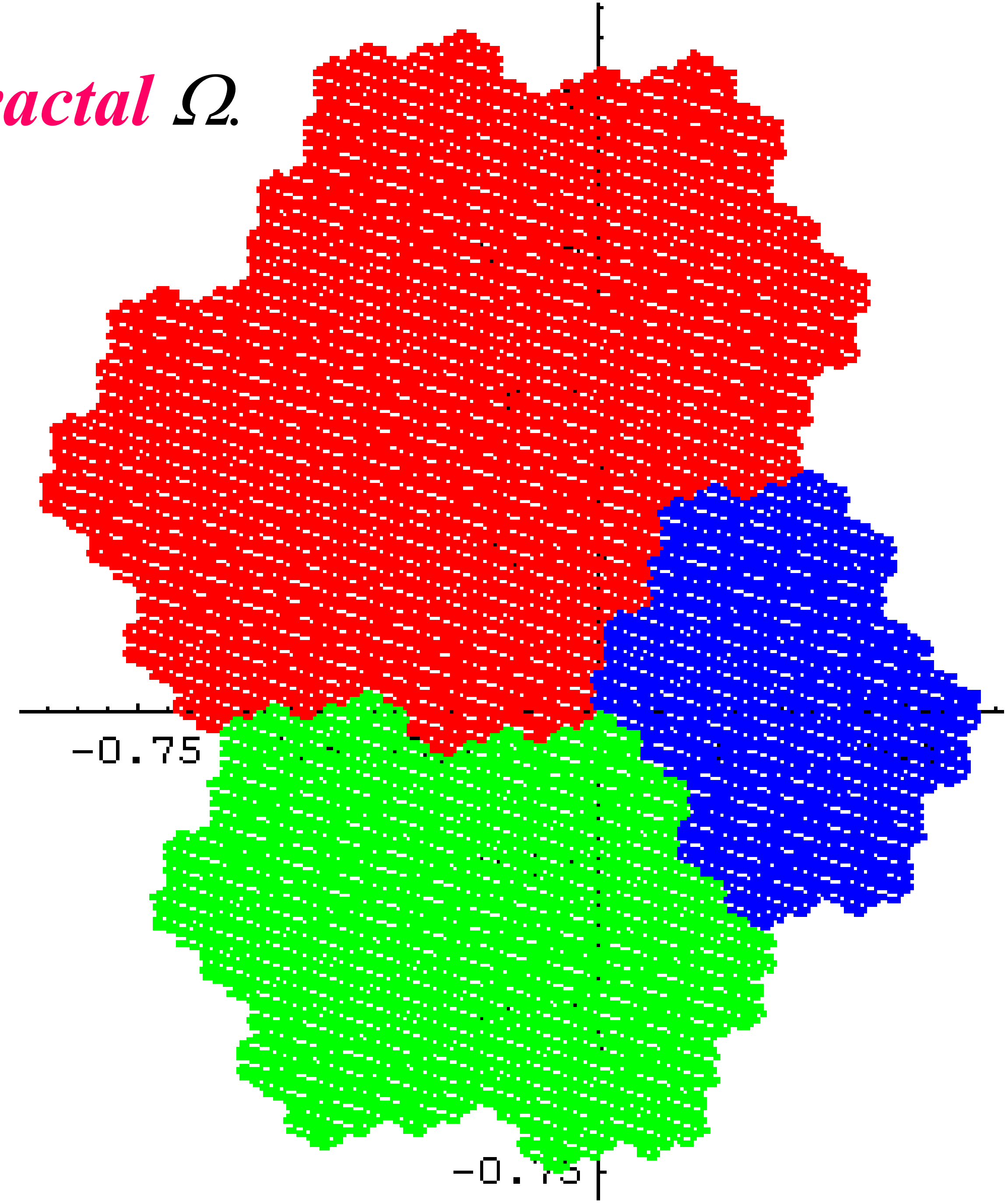
$\dim(E^s) = 2.$



Project to E^{u}. Get Rauzy quasicrystal

Project to E_{s} .
Get atomic surface

...the ***Rauzy Fractal*** Ω.



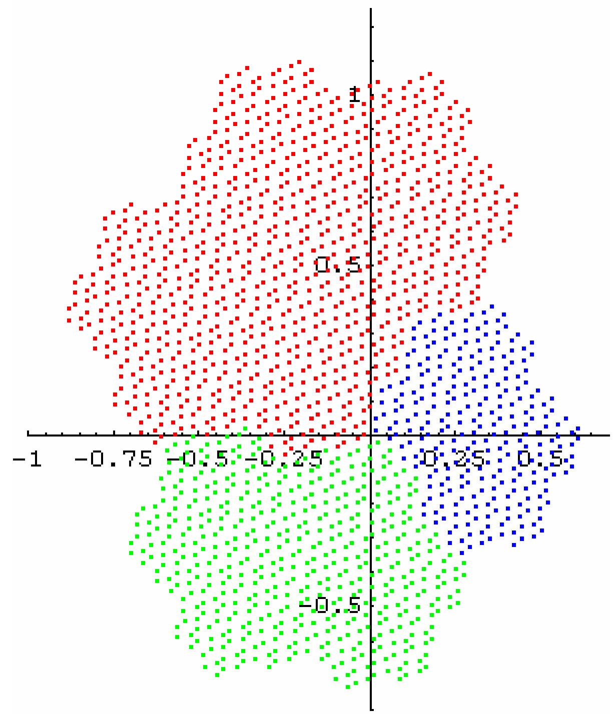
ρ-integers:

0.
1.
10.
11.
100.
101.
110.
1000.
...

ρ -van der Corput points:

0.0
0.1
0.01
0.11
0.001
0.101
0.011
0.0001
...

$$x_{i-1}x_ix_{i+1} = 0$$

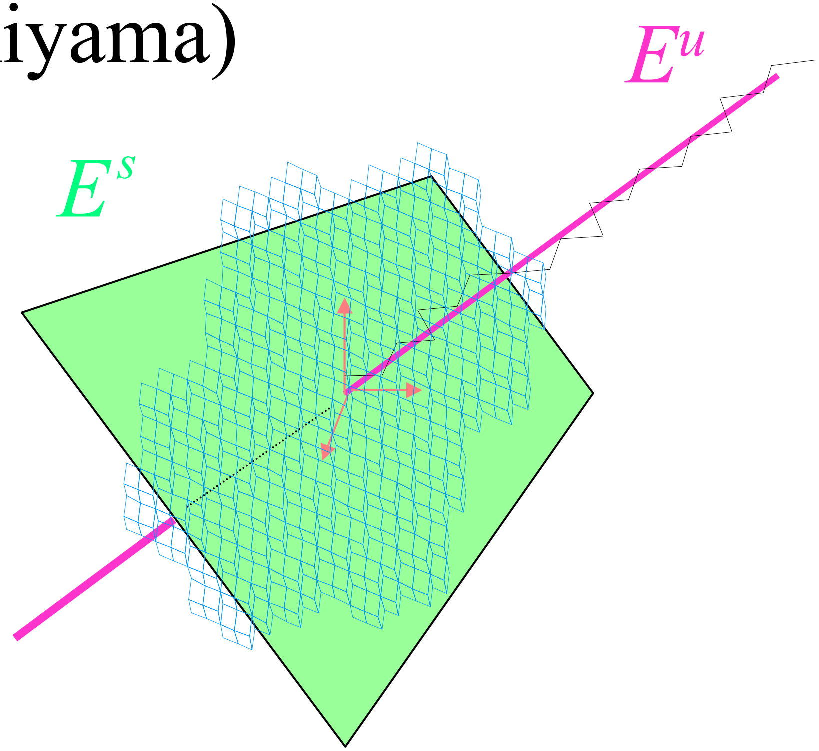


Rauzy quasicrystal:
u projected to E^u

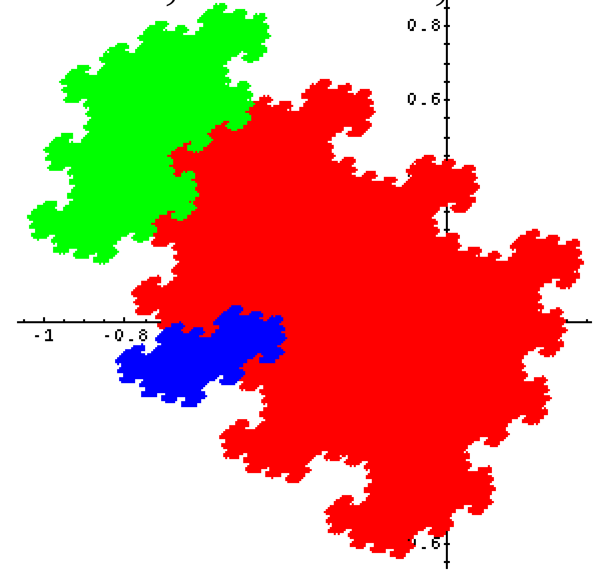
Rauzy fractal
u projected to E^s

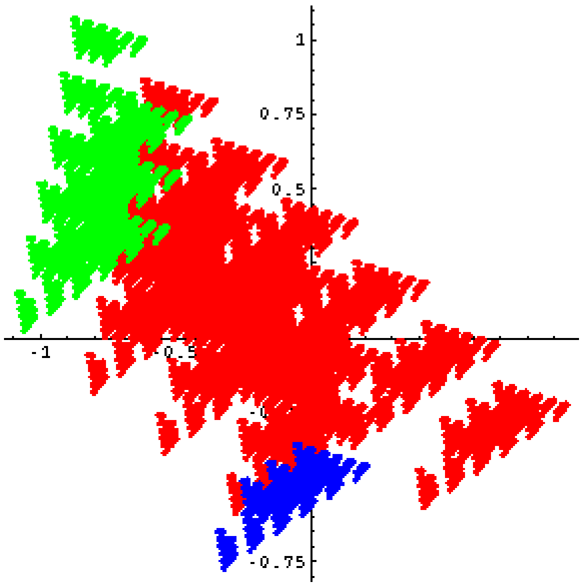
This is the basis for the now nearly complete theory of *Pisot* substitutions. Related to β expansions. Related to Markov partitions for toral automorphisms.

(Arnoux, Ito, Siegel, Cantereni, Solomyak, Sirvient, Akiyama)

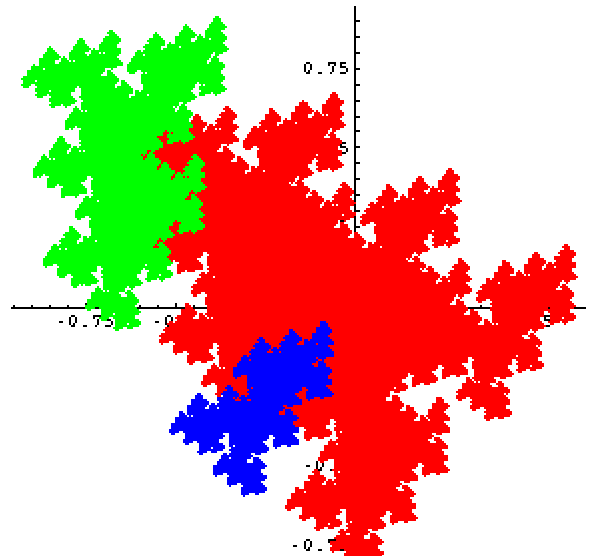


$1 \to 1112, \quad 2 \to 113, \quad 3 \to 1$





$1 \to 1112, \quad 2 \to 311, \quad 3 \to 1$



$1 \to 1112, \quad 2 \to 131, \quad 3 \to 1$

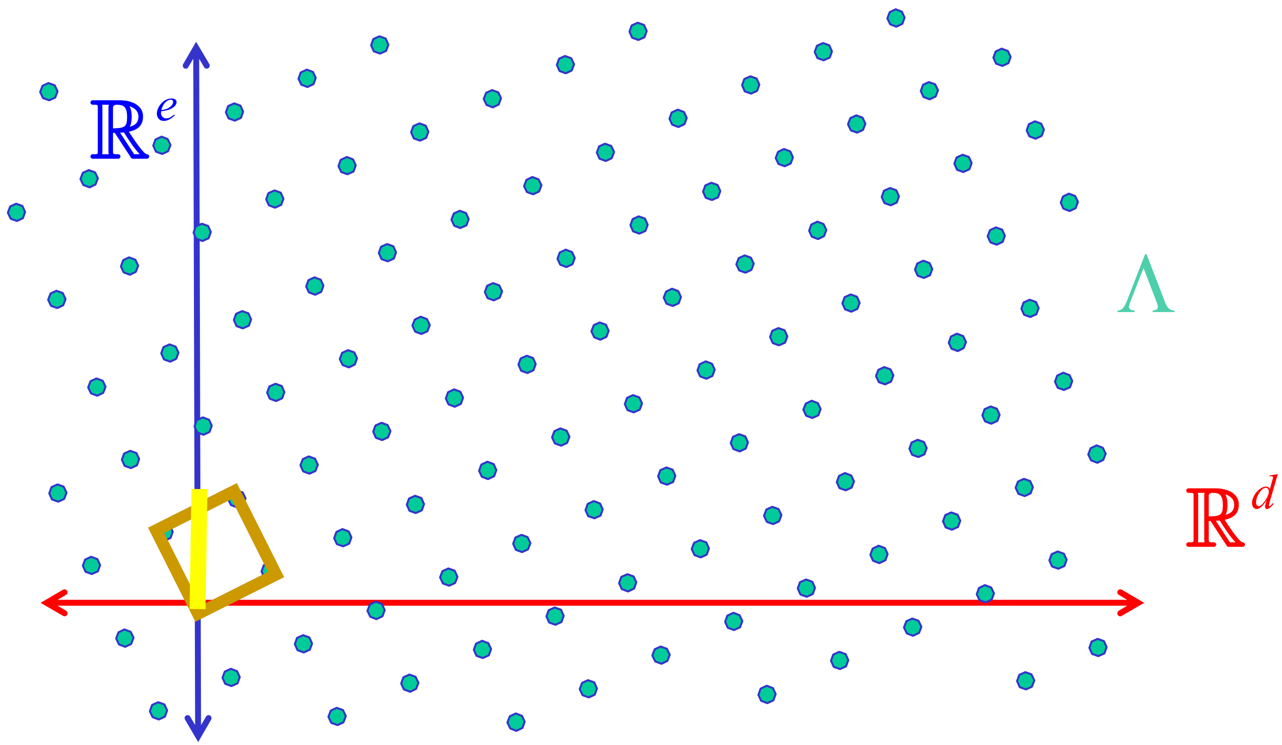
http://www.ma.usb.ve/~vsirvent/gallery/rauzy.html (Victor Sirvent)

•The classical quasiperiodic tilings

Put $\mathbb{H}=\mathbb{R}^e$.

Let $\Lambda=D(\mathbb{Z}^{d+e})$ where $D\in \mathrm{O}(d+e)$.

Take $R=\pi_{\mathbb{R}^e}(D(Q))$ where Q=unit cube.



• Penrose tilings (unmarked) arise this way for d=2, e=3 for a "symmetric" choice of D. Here $\mathbb{M}=\mathbb{T}$ and Penrose tilings correspond to the case m=0.

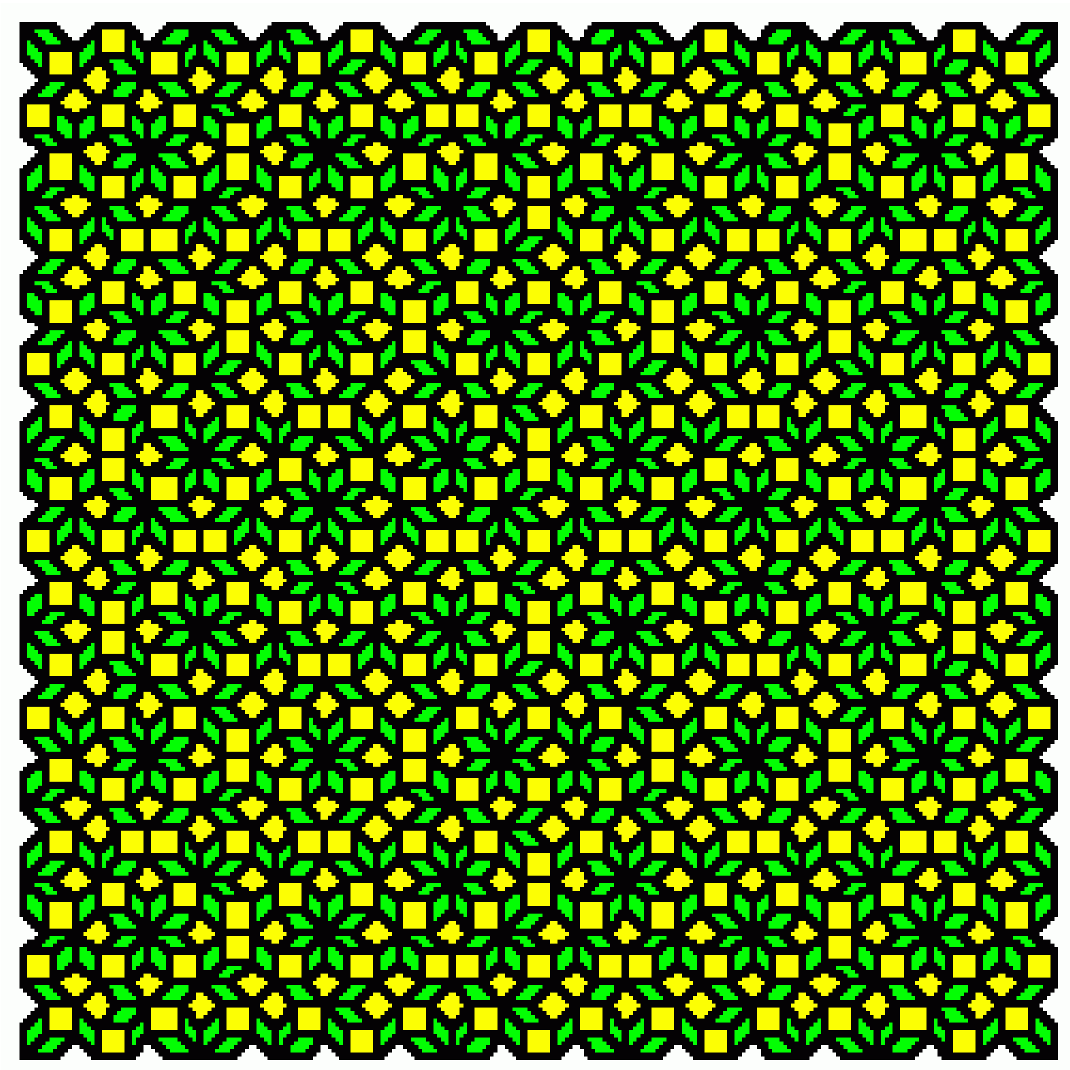
de Bruijn showed that the markings can be restored uniquely by a "local map".

• The case m=1/2 is sometimes called the *anti-Penrose tiling*

Different values of m result in different local isomorphism classes, different point symmetries and different configurations of non-regular points.

While generally not (never?) topologically conjugate, all resulting tiling systems are metrically isomorphic.

• The case d=2, e=2, for a similar choice of D, yields the octagonal or Ammann-Beekner tilings. Here $\mathbb{M}=\{0\}$.

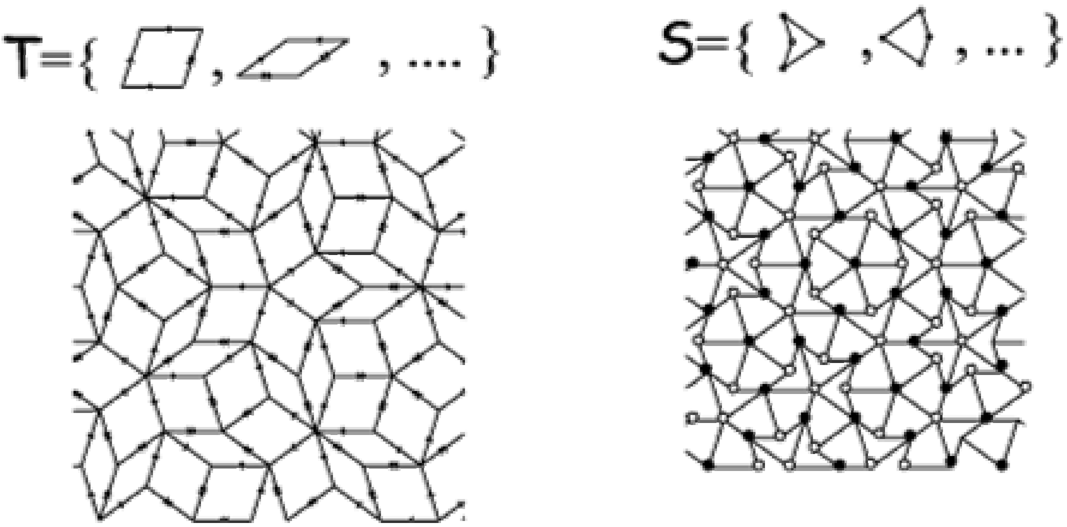


There is also a tiling substitution and a "finite type" version of this tiling.

III. Properties

- Types of isomorphism
- Issues surrounding finite type
- Entropy and complexity
- Spectral theory

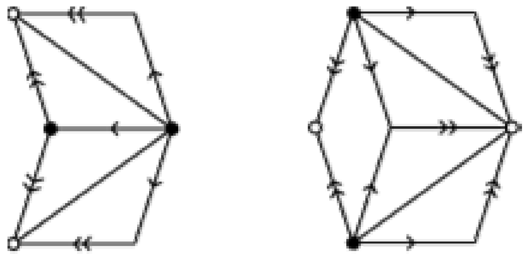
•Types of isomorphism



Two kinds Penrose tilings:

- Rhombs
- Kites & darts

- Mutual Local derivability (MLD)



Types of dynamical isomorphism on (X,T):

- MLD: $\phi: X_T \to X_S$

implies

- Topological conjugacy

implies

- Almost topological conjugacy

Almost 1:1 extensions arise naturally throughout the theory of tiling dynamical systems.

Theorem. Topological conjugacy does not imply MLD (Petersen, Radin-Sadun, 1999).

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i.e., No Curtis-Lyndon-Hedlund Theorem.

Assuming (X,T) has been given a T-invariant measure μ,

- Metric isomorphism

Note: (X,T) frequently uniquely ergodic.

- Tiling space homeomorphism

Due to the nature of the tiling topology, any homeomorphism is automatically an orbit equivalence.

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In any case, topoloical conjugacy requires homeomorphism.

C*-algebras, K-theory and cohomology

(Anderson, Putnam, Forrest, Kellendonk, Gähler) :

Tiling	H^0	H^1	H^2	H^3
Penrose	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^8$	0
Anti-Penrose	$\mathbb{Z}$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{34}$	0
Octagonal	$\mathbb{Z}$	$\mathbb{Z}^5$	$\mathbb{Z}^9$	0
FT-Octagonal	$\mathbb{Z}$	$\mathbb{Z}^8$	$\mathbb{Z}^{23}$	0

• Issues surrounding finite type

Theorem. The finite type property is an invariant of topological conjugacy (Radin, Sadun).

However, finite type does not pass to factors. We call a factor of a finite type tiling system *sofic*.

Example. One can show that the unmarked octagonal tilings are not finite type. The forgetful map is almost 1:1.

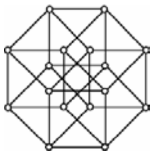
Problem. Do all sofic tilings have almost 1:1 finite type covers?

•What is the relation between substitution tiling spaces and finite type tiling spaces?

Theorem. Suppose (X,T) is substitution system $X \subseteq X_T$. Then there exists a marking $T_\#$ of T, and a $T_\#^{(2)}$ such that the forgetful mapping

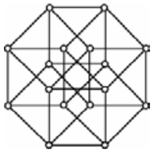
$$F{:}X_{T\#} \;\circ\; X_T$$

is almost 1:1, and satisfies $F(X_{T\#}) = X$. (Goodman-Strauss, Radin, Mozes)



- *In general sofic,*
- Occasionally (e.g., Penrose) finite type.

Problem. Find a fundamentally different (i.e., less combinatorial) proof of the previous theorem (cohomology, etc.?)



Call a tiling that comes from a substitution or similar iterative construction *hierarchical*.

Problem. Are there, in fact, any zero entropy aperiodic finite type tiling systems that are not hierarchical?

- Kari-Culic tilings?
- Quasiperiodic tilings?

- What is the relation between quasiperiodc tilings and the finite type property?

Studied in great detail by T. Q. T. Le (and others)

Necessary (all cases). The orientation D of the lattice Λ must be ***algebraic***.

Theorem. Let $\tau = \frac{1+\sqrt{5}}{2}$. For the generalized Penrose tilings X_m:

- X_m is finite type iff $m \in \mathbb{Z}[\tau]$
- X_m is sofic iff $\mathrm{m} \in \mathbb{Q}(\tau)$. (Le)

•Entropy and complexity

For $x \in X$, let $x[Q_n]$ denote the patch of tiles $D \in x$ that meet Q_n, the n^d cube centered at the origin.

Definition. The *complexity* $c(n)$ of (X,T) is the number of different $x[Q_n]$, up to translation, in all $x \in X$.

Topological entropy: $h(X,T) = \lim_{n\to\infty} \frac{\log(c(n))}{n^d}$

Complexity itself is of interest in the case $h(X,T)=0$.

Theorem. Any *uniquely ergodic*, *finite type* tiling system has (X,T) has $h(X,T)=0$ (Radin, Shieh)

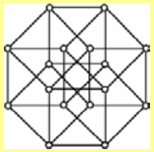
- The proof actually shows $c(n) \approx C|\mathcal{T}|^{n-1}$.
- One can also show *directional entropy* is zero in every direction.

Weakly mixing substitutions give weakly mixing finite type systems, but cannot be strongly mixing.

Problem. Can a uniquely ergodic, finite type tiling system be strongly mixing?

Theorem. Let (X,T) be a primitive invertible substitution system for $S=LC$. Suppose L is diagonalizable and has eigenvalues $\lambda_1 \ldots \lambda_d$ where $|\lambda_d| \le |\lambda_j|$, $j=1\ldots d$.

Let $$c = \frac{\log|\det(L)|}{\log|\lambda_d|} = \frac{\log(|\lambda_1|\ldots|\lambda_d|)}{\log|\lambda_d|}$$



Then $c(n) \le K \cdot n^c$.

- This implies $h=0$ and all directional entropies 0.
- For self-similar tilings c=d.
- So far we have no example with c>d.

What about quasiperiodic tilings?

- They all have h=0.

Classical quasiperiodic tilings have been studied by Berthé and Vuillon in the codimension-1 (i.e., e=1) case.

This is inspired by 1-dim Sturmian sequences, which satisfy $c(n)=n+1$. Any smaller complexity, $c(n) \leq n$, results in a periodic sequence.

- For d=2 and e=1 they get $c(n)=n^2+2n$. However, they have a discrete example with $c(n)=n^2+n$.

- **Problem.** Prove the conjecture is $c(n)\leq n^2$ implies periodic.

Note that all substitution and quasiperiodic tilings have $c(n) \leq Kb^c$ for some K, c.

The Radin-Shieh result for uniquely ergodic finite type systems is $c(n) = |T|^{n-1}$

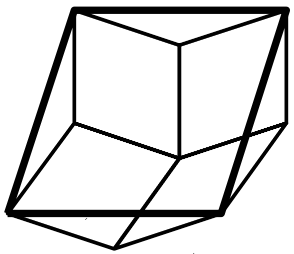
Problem. Is there a uniquely ergodic, finite type tiling system with $c(n)$ super-polynomial?

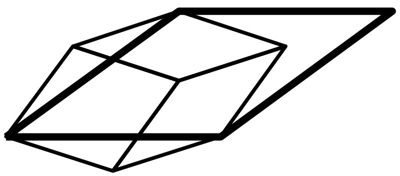
Defintion. A real algebraic integer $\lambda>1$is *Pisot* if all its conjugates satisfy $|\lambda'| <1$. A complex algebraic integer λ is complex Pisot if the above is true except for its complex conjugate. Any other real or complex number is called *non-Pisot*.

• There is a similar definition for a *totally non-Pisot* linear mapping L (based on Mauduit's idea of a Pisot family). It includes the case where $L=\lambda\, I$ for λ a non-Piost number (assumes diagonalizable).

Theorem. If the expansion L for a substitution tiling system is totally non-Pisot then (X,T) is weakly mixing.

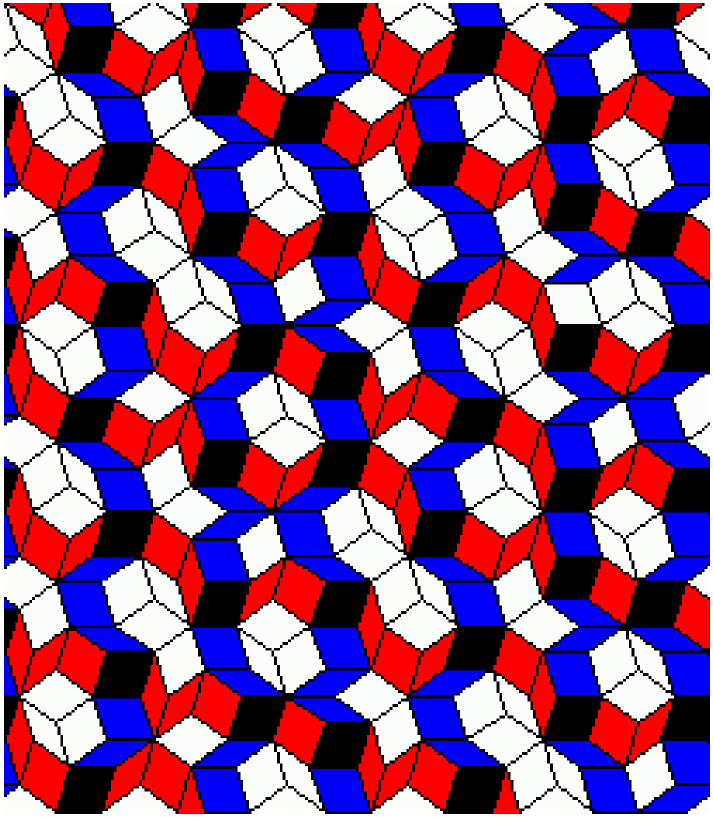
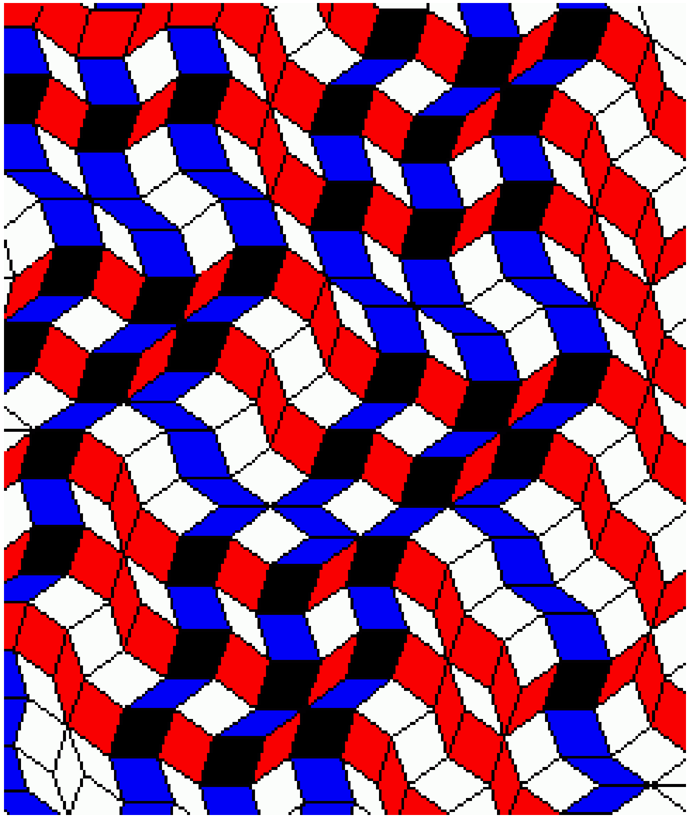
Example. The binary tiling:





$$A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} \text{ and } p_A(\theta) = \theta^2 - 5\theta + 5.$$

Tilings in a weakly mixing system look more “disordered” than in one with pure point spectrum:

Penrose tiling v.s. binary tiling

Example. There is a 1-dimensional weakly mixing substitution tiling system that is a suspension of a 4-interval exchange.

• The tiling space is is an almost 1:1 extension of a surface of genus 2.

• The substitution mapping S is a pseudo-Anosov diffeomorphism. (Fitzkee, Hockett, R)

Example. There is a 2-dimensional substitution tiling system with mixed Lebesgue spectrum with multiplicity >1. (Frank)

The End