THE ČECH COHOMOLOGY AND THE SPECTRUM FOR 1-DIMENSIONAL TILING SYSTEMS

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Abstract. This paper shows that there is a close relationship between three groups: the dynamical cohomology of a 0-dimensional substitution dynamical system (X,T), the first integer Čech cohomology group $\check{H}^1(X_{\mathbf{g}})$ of the corresponding 1-dimensional tiling space $X_{\mathbf{g}}$, for a given heigh vector $\mathbf{g}$, and the point-spectrum $\mathcal{E}_{T_{\mathbf{g}}}$ of the tiling flow $(X_{\mathbf{g}},T^t_{\mathbf{g}})$. In particular, the group of eigenfunctions of $(X_{\mathbf{g}},T^t_{\mathbf{g}})$ can be embedded as a subgroup of $\check{H}^1(X_{\mathbf{g}})$. There is a real-valued functional W on $\check{H}^1(X_{\mathbf{g}})$, called the winding number, that assigns each eigenfunction its eigenvalue. The paper gives conditions for W to be injective and for the image of W to equal $\mathcal{E}_{T_{\mathbf{g}}}$. In the injective case, the spectrum does not depend on the height $\mathbf{g}$, but is completely determined by $\check{H}^1(X_{\mathbf{g}})$.

1. Introduction

In this paper we consider the 1-dimensional tiling substitution $S_{\mathbf{g}}$ and the corresponding tiling flow $(X_{\mathbf{g}},T^t_{\mathbf{g}})$ that comes from a discrete substiution S and a positive tile length vector $\mathbf{g}$. In particular, $T_{\mathbf{g}}$ is the suspension flow for the substitution shift T corresponding to a piecewise constant function with heights $\mathbf{g}$. Our purpose is to relate two properties of these dynamical systems that have each received a great deal of study, but have seldom been discussed together: their point-spectrum (see for example [Ho-86, Ra-90, FMN-96, So-97, AI-01, CSg-01, SiSo-02, HS-03, FiHR-03, Sg-03, CS-03, R-04, BK-06, BBK-06]), and their cohomolgy (see for example [AP-98, BD-01, FoHK-02, CS-06, BD-08, S-08, BD-09]). We show that, contrary to a common belief, these two invariants are closely related. Every eigenfunction for $(X_{\mathbf{g}},T^t_{\mathbf{g}})$ corresponds to a unique element of the first integer Čech cohomology $\check{H}^1(X_{\mathbf{g}})$. Moreover, there is a real-valued functional W on $\check{H}^1(X_{\mathbf{g}})$ that assigns to each eigenfunction its corresponding eigenvalue. The image of W, which is a subgroup of $\mathbb{R}$, is the largest possible group of eigenvalues among the different choices of $\mathbf{g}$. Of particular interest is the case when W is injective and the image of W is completely made up of eigenvalues. We say, in this case, that $T^t_{\mathbf{g}}$ has *full spectrum*.

For a finite *alphabet* $\mathcal{A}=\{0,1,\ldots,d-1\}$ let $\mathcal{A}^*$ denote the finite words in $\mathcal{A}$, including the empty word ε. A *discrete substitution* is a mapping $S:\mathcal{A}\to\mathcal{A}^*\backslash\{\varepsilon\}$. For example, the *golden mean* substitution S on $\mathcal{A}=\{0,1\}$ is given by $0\to 01,\ 1\to 0.$. The *substitution shift* $X\subseteq\mathcal{A}^{\mathbb{Z}}$ corresponding to S consists of all sequences $x\in\mathcal{A}^{\mathbb{Z}}$ such that every finite subword is a subword of some $S^k a$, $a\in\mathcal{A}$

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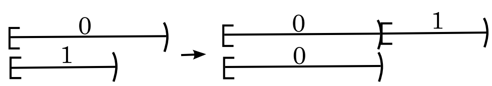
and $k \geq 1$. Assuming S satisfies a few standard hypotheses (is *primitive* and *shift-aperiodic*), X is a Cantor set, and the shift T restricted to X is a strictly ergodic homeomorphism. The dynamical system (X, T) is called a *substitution shift*.

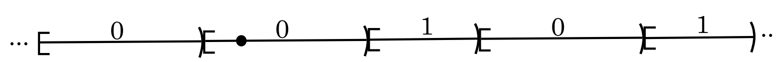
Now let $\mathbf{g} = (g_0, g_1, \ldots, g_{d-1}) \in \mathbb{R}^d$ be positive, and let $\mathcal{T} = \{I_0, I_1, \ldots, I_{d-1}\}$ be a set of labeled, half-closed intervals $I_a = [0, g_a)_a$, called *prototiles*. For any $u = u_0 u_2 \ldots u_{n-1} \in \mathcal{A}^*$, let $t_0 = 0$ and for $j \geq 1$, $t_j = g_{u_0} + \cdots + g_{u_{j-1}}$, and define a *tiling*

$$I_u = [t_0, t_1)_{u_0} [t_1, t_2)_{u_1} \ldots [t_{n-1}, t_n)_{u_n}$$

of $[0, t_n)$. Let of $\mathcal{T}^*$ denote the set of all tilings of the intervals $[0, t)$. Define the *tiling substitution* $S_{\mathbf{g}} : \mathcal{T} \to \mathcal{T}^*$ by $S_{\mathbf{g}}(I_a) = I_{S(a)}$. For example, when S is the golden mean substitution, and $\mathbf{g} = (\lambda + 1)^{-1}(1, \lambda)$, $\lambda = (1 + \sqrt{5})/2$, we have $S_{\mathbf{g}}$:



Let $X_{\mathbf{g}}$ be the set of all tilings y of $\mathbb{R}$ with the property that every finite subtiling of y is a translate of a subtiling of some $S_{\mathbf{g}}^k(I_a)$, $a \in \mathcal{A}$, and $k \geq 0$. The *tiling space* $X_{\mathbf{g}}$ has a natural compact metric topology (see e.g., [R-04]), and the *tiling flow* $T_{\mathbf{g}}^t$ on $X_{\mathbf{g}}$, which is defined as translation by $\mathbb{R}$, is strictly ergodic. A portion of y for the Fibonacci tiling substitution $S_{\mathbf{g}}$ is shown below (where the dot indicates the position of $0 \in \mathbb{R}$):



This paper studies the 1-dimensional integer Čech cohomology $\check{H}^1(X_{\mathbf{g}})$ of $X_{\mathbf{g}}$. Using Bruslinski's Theorem ([Br-34], see also [PT-82]) we identify $\check{H}^1(X_{\mathbf{g}})$ with the *Bruslinski group* $Br(X_{\mathbf{g}})$ of circle-valued continuous functions $f : X_{\mathbf{g}} \to \mathbb{T}$, modulo homotopy. There is a real valued functional W on $\check{H}^1(X_{\mathbf{g}})$, called the *Scwhartzman winding number* [Sc-57] (see also [PT-82]) with the property that $W(f) = \omega$ when f is an eigenfunction with eigenvalue ω. We define the *winding number group* $\mathcal{W}_{T_{\mathbf{g}}} = \mathrm{im}(W) \subseteq \mathbb{R}$. If $\mathcal{E}_{T_{\mathbf{g}}}$ denotes the group of all eigenvalues, then $\mathcal{E}_{T_{\mathbf{g}}} \subseteq \mathcal{W}_{T_{\mathbf{g}}}$. If W is injective on $\check{H}^1(X_{\mathbf{g}})$, we say $(T_{\mathbf{g}}^t, X_{\mathbf{g}})$ is *saturated*. In the saturated case the winding number group gives a faithful numerical representation of $\check{H}^1(X_{\mathbf{g}})$. We find sufficient conditions for $T_{\mathbf{g}}^t$ to be saturated, and in many cases, we completely determine $\mathcal{W}_{T_{\mathbf{g}}}$. When $\mathcal{W}_{T_{\mathbf{g}}} = \mathcal{E}_{T_{\mathbf{g}}}$, we say $T_{\mathbf{g}}^t$ has *full spectrum*.

Our proofs depend on the fact that $(X_{\mathbf{g}}, T_{\mathbf{g}}^t)$ is a suspension of (X, T). Thus almost all of our results have analogues for the substitution shift (X, T). As a Cantor set, X has a trivial Čech cohomology, but we replace it with the dynamical cohomology $H(T)$ (i.e., the continuous integer functions mod coboundaries). The winding number group is replaced by $\mathcal{M}_T$, the subgroup of $\mathbb{R}$ generated by the measures of clopen sets. The concept of saturation in this context is due to [BK-06].

This paper is organized as follows. In the first section we consider Cantor dynamical systems in general, and substitutions in particular, and we define $\mathcal{M}_T$. Then we give sufficient conditions for a substitution shift T to be saturated. In the second section we study suspensions of Cantor dynamical systems, their Čech cohomology. We define winding numbers, and apply these results to substitution tilings. In the third section, we consider examples.

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2. Cantor Dynamical Systems

2.1. **Basic definitions.** A *Cantor dynamical system* (X,T) is a homeomorphism $T : X \to X$ of a Cantor set X. Every Cantor dynamical system has at least one T-invariant Borel probability measure m, and if m is unique, T is called *uniquely ergodic*. If the orbit $O_T(x) := \{T^n x : n \in \mathbb{Z}\}$ of every $x \in X$ is dense, then T is called *minimal*. We usually assume T is *strictly ergodic*, which means it is both minimal and uniquely ergodic.

Let $C(X,\mathbb{Z})$ denote the additive group of integer-valued continuous functions on X. The group $B(T)$ of *coboundaries* is the subgroup of $n \in C(X,\mathbb{Z})$ satisfying $n(x) = p(Tx) - p(x)$ for some $p \in C(X,\mathbb{Z})$, and the quotient $H(T) := C(X,\mathbb{Z})/B(T)$ is called the (dynamical) *cohomology group* of T. The group $Inf(T)$ of *infinitesimals* is defined to be those $n \in C(X,\mathbb{Z})$ with $\int n\,dm = 0$. Note that $B(T) \subseteq Inf(T)$ because $\int_X ((p(Tx) - p(x))\,dm = 0$. The group $G(T) = C(X,\mathbb{Z})/Inf(T)$ is called the *dimension group* of T.

A homeomorphism U of X is said to be *orbit equivalent* to T if $O_T(x) = O_U(x)$ for all $x \in X$. The set of all U orbit equivalent to T is denoted $[T]$, and called the *full group* of T. Since T is strictly ergodic, the same is true for any $U \in [T]$. Minimality implies that for $U \in [T]$, there is a unique $n : X \to \mathbb{Z}$ such that $Ux = T^{n(x)}x$. It n has at most one discontinuity, then U is said to be *strongly orbit* to T. If $n(x)$ is continuous, U is said to be in the *topological full group* of T, denoted $U \in [[T]]$.

Remark 1. *Boyle* [Bo-83] *shows that $U \in [[T]]$ implies $U = V^{-1}TV$ or $U = V^{-1}T^{-1}V$ for some homeomorphism V of X. Giordano, Putnam and Skau* [GPS-95] *show that for minimal Cantor dynamical systems, complete invariants for orbit equivalence and strong orbit equivalence are given by isomorphisms of $H(T)$ and $G(T)$ that preserve positive functions and constants.*

A set $E \subseteq X$ is called *clopen* if it is both open and closed. Assume T is uniquely ergodic with unique invariant measure m, and define the *measure group* $\mathcal{M}_T$ of (X,T) to be the additive subgroup of $\mathbb{R}$ generated by the measures $m(E)$ of the clopen sets $E \subseteq X$. For $n(x) \in H(T)$, we define $I(n) = \int_X n(x)\,dm$. It is clear $I : H(T) \to \mathcal{M}_T$ is a surjective group homomorphism.

Two clopen sets $E_1, E_2 \subseteq X$ are called *T-equivalent* if there exists $U \in [[T]]$ so that $E_2 = UE_1$. Clearly if E_1 and E_2 are T-equivalent, then $m(E_1) = m(E_2)$. We call a uniquely ergodic Cantor dynamical system T *saturated* if any two clopen sets with $m(E_1) = m(E_2)$ are T-equivalent. This definition is due to Bezuglyi and Kwiatkowski, [BzK-00], who also prove the following:

Theorem 2. (Bezuglyi, Kwiatkowski, [BzK-00]) *A uniquely ergodic Cantor dynamical system T is saturated if and only if $Inf(T) = B(T)$.*

Clearly the homomorphism $I : H(T) \to \mathcal{M}_T$ is an isomorphism (we write $H(T) \cong \mathcal{M}_T$) if and only if T is saturated.

Remark 3. *Bezuglyi and Kwiatkowski* [BK-06] *show that Chacon's transformation T (as a homeomorphism of a Cantor set X) is not saturated. Chacon's transformation is topologically conjugate to the substitution shift (X,T) corresponding to the primitive, aperiodic substitution $0 \to 0012, 1 \to 12, 2 \to 012$ (see* [Fg-02]*).*

2.2. **Subshifts.** For $d \geq 1$, let $\mathcal{A} = \{0, \ldots, d-1\}$, $d > 1$, be an *alphabet*, and consider $\mathcal{A}^{\mathbb{Z}}$, which is a Cantor set with the product topology. Let T be the left-shift homeomorphism. A *subshift* is a closed T-invariant subset $X \subseteq \mathcal{A}^{\mathbb{Z}}$. An uncountable subshift X is a Cantor set, and thus (X,T) is a Cantor dynamical system. We call $u = u_0 u_1 \ldots u_{n-1} \in \mathcal{A}^n$ a *word* with *length* $|u| = n$, and let $\mathcal{A}^* = \bigcup_{n \geq 0} \mathcal{A}^n$, where $\mathcal{A}^0 = \{\varepsilon\}$, with ε the empty word. Any $\mathcal{L} \subseteq \mathcal{A}^*$ is called a *language*. If $x \in \mathcal{A}^{\mathbb{Z}}$ and $[p,q] \subseteq \mathbb{Z}$, is an interval, let $x_{[p,q]} \in \mathcal{A}^{q-p+1}$ be the word u with $u_i = x_{p+i}$ for $i = 0, \ldots, q-p$. A subshift (X,T) is completely determined by its language $\mathcal{L}$. In particular, $\mathcal{L}$ determines whether or not T is strictly ergodic, and determines $\mathcal{M}_T$. For $u \in \mathcal{L}$, $|u| = n$ define the *cylinder* $\lfloor u \rfloor := \{x \in X : x_{[0,|u|-1]} = w\} \subseteq X$. More generally, we also call $T^k \lfloor u \rfloor$ a cylinder. Cylinders are nonempty clopen sets that form a sub-base for the topology on X. Any clopen set is a finite disjoint union of cylinders.

3. Substitutions

3.1. **Basic properties.** Let $\mathcal{A} = \{0, \ldots, d-1\}$. A *substitution* on $\mathcal{A}$ (or on d letters) is a mapping $S : \mathcal{A} \to \mathcal{A}^* \backslash \{\varepsilon\}$. A substitution S is called *primitive* if there exists $k \geq 0$ so that for each $a, b \in \mathcal{A}$, b appears in $S^k(a)$. The *structure matrix* A of a substitution S is $d \times d$ matrix with entries $e_{a,b}$ equal to the number of times b appears in $S(a)$. A $d \times d$ non-negative integer matrix A is called *primitive* if $A^k > 0$ for some $k \geq 1$. Thus a substitution is primitive if and only if it has a primitive structure matrix.

Given a primitive substitution S, let $\mathcal{L}' = \{S^k(a) : a \in \mathcal{A}, k \geq 0\}$ be the words obtained by iterating S on each letter in $\mathcal{A}$. The set $\mathcal{L}$ of all subwords of $\mathcal{L}'$ is thus the language of a subshift (X,T) that we call the *substitution shift* corresponding to S (see [Qu-87] or [Fg-02]). A substitution S is called *shift-aperiodic* if the corresponding subshift (X,T) has no periodic points. It is well known (see [Qu-87] or [Fg-02]) that if S is a primitive shift-aperiodic substitution, then (X,T) is a strictly ergodic Cantor dynamical system. For a primitive matrix A, the *Perron-Frobenius Theorem* (see [LM-95]), says that there exists a maximal real eigenvalue $\lambda > 0$, called the *Perron-Frobenius eigenvalue*, and positive (left and right) eigenvectors $\mathbf{m}$ and $\mathbf{h}$ (i.e., $A\mathbf{m} = \lambda\mathbf{m}$ and $A^t\mathbf{h} = \lambda\mathbf{h}$). We normalize $\mathbf{m}$ and $\mathbf{h}$ so that $\mathbf{m} \cdot \mathbf{1} = 1$ and $\mathbf{h} \cdot \mathbf{m} = 1$. We call $\mathbf{m}$ and $\mathbf{h}$ the *normalized* right and left *Perron-Frobenius eigenvectors*. for A (or S). We call S *irreducible* if the characteristic polynomial $p(z)$ of A is irreducible over $\mathbb{Z}$, and *unimodular* if $\det(A) = \pm 1$. Note that λ is always an algebraic integer. Irreducibility means it is degree d, and unimodularity means it is a unit.

For $\mathbf{v} \in \mathbb{R}^d$, write $\mathbb{Z}[\mathbf{v}] = \{\mathbf{v} \cdot \mathbf{n} : \mathbf{n} \in \mathbb{Z}^d\}$. For $\lambda \in \mathbb{R}$ let $\mathbb{Z}[\lambda]$ denote the $\mathbb{Z}$-module generated by the numbers λ^k, $k = 0, 1, 2, \ldots$. We define the *Perron-Frobenius group* of a primitive irreducible matrix A by $\mathcal{G}_A = \bigcup_{k=0}^{\infty} \mathbb{Z}[\lambda^{-k}\mathbf{m}]$.

Lemma 4. *If S is primitive and irreducible, then*

(1) *S is shift-aperiodic,*
(2) $\mathbb{Z}[\mathbf{m}] \cong \mathbb{Z}^d$,
(3) $\mathbb{Z}[\mathbf{m}] = \alpha\mathbb{Z}[\lambda] \cong \mathbb{Z}^d$ *for some $\alpha \in \mathbb{Q}(\lambda)$, and*
(4) $\mathbb{Z}[\lambda^{-k}\mathbf{m}] \subseteq \mathbb{Z}[\lambda^{-(k+1)}\mathbf{m}]$.

If S is also unimodular, then $\mathbb{Z}[\lambda^{-k}\mathbf{m}] = \mathbb{Z}[\lambda^{-(k+1)}\mathbf{m}]$, and $\mathcal{G}_A = \mathbb{Z}[\mathbf{m}]$. But if S is not unimodular then $\mathcal{G}_A$ is not finitely generated.

Proof. If $p(z)$ is irreducible then $\lambda \in \mathbb{R}\backslash\mathbb{Q}$. Shift-aperiodicity follows. Let C be the companion matrix for $p(z)$. Then $C\mathbf{v} = \lambda\mathbf{v}$ for $\mathbf{v} = (\lambda^{d-1}, \ldots, \lambda, 1)^t$. By Corollary 18 of [Du-99], there exists $Q \in Sl(\mathbb{Q}, d)$ so that $A = Q^{-1}CQ$. Thus $\mathbf{m}' = Q^{-1}\mathbf{v}$ satisfies $A\mathbf{m}' = \lambda\mathbf{m}'$. Clearly $\mathbf{m}' \in \frac{1}{k}\mathbb{Z}[\lambda]$ for some $k \in \mathbb{N}$. Also $\mathbf{m}\cdot\mathbf{1} = 1$, so $\alpha' = (\mathbf{m}'\cdot\mathbf{1})^{-1} \in \mathbb{Q}(\lambda)$. It follows that $\mathbf{m} \in \alpha\mathbb{Z}[\lambda]$ where $\alpha = \alpha'/k$. A similar argument applies to $\mathbf{h}$.

Let $\lambda^{-k}\mathbf{m}\cdot\mathbf{n} \in \mathbb{Z}[\lambda^{-k}\mathbf{m}]$. Since $\lambda^{-(k+1)}A\mathbf{m} = \lambda^{-k}\mathbf{m}$, we have $\lambda^{-k}\mathbf{m}\cdot\mathbf{n} = \lambda^{-(k+1)}A\mathbf{m}\cdot\mathbf{n} = \lambda^{-(k+1)}\mathbf{m}\cdot A^t\mathbf{n} \in \mathbb{Z}[\lambda^{-(k+1)}\mathbf{m}]$. In the unimodular case, A is invertible, λ is a unit, and $\mathbb{Z}[\lambda^{-k}\mathbf{m}] = \mathbb{Z}[\mathbf{m}]$ for all k. If S is not unimodular, then all the inclusions

$$\mathbb{Z}[\lambda^{-k}\mathbf{m}] \subseteq \mathbb{Z}[\lambda^{-(k+1)}\mathbf{m}] \tag{1}$$

are proper. This shows $\mathcal{G}_A$ is not finitely generated. □

If S is irreducible then the inclusions (1) induce a directed system

$$\mathbb{Z}^d \xrightarrow{A} \mathbb{Z}^d \xrightarrow{A} \mathbb{Z}^d \xrightarrow{A} \mathbb{Z}^d \ldots.$$

Given such a directed system, the direct limit $\mathcal{D}_A = \varinjlim(\mathbb{Z}^d, A)$ is called the *dimension group* of A (see [LM-95]). A more concrete presentation of $\mathcal{D}_A$ is given by $\mathcal{D}_A = \{\mathbf{g} \in \mathbb{Q}^d : (A^t)^n\mathbf{g} \in \mathbb{Z}^d \text{ for some } n \in \mathbb{Z}\}$, (see [LM-95]). It is easy to see with this latter presentation that (in the irreducible case) $\mathcal{G}_A \cong \mathcal{D}_A \subseteq \mathbb{Q}^d$ via $I : \mathcal{D}_A \to \mathcal{G}_A$, defined $I(\mathbf{g}) = \mathbf{g}\cdot\mathbf{m}$.

3.2. **Kakutani-Rohlin partitions.** Let (X, T) be a Cantor dynamical system. A *semi-partition* on (X, T) is a collection $\mathcal{P} = \{P_0, \ldots, P_{n-1}\}$ of pairwise disjoint clopen sets in X. Two semi-partitions $\mathcal{P}$ and $\mathcal{Q}$ are *disjoint* if any $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ satisfy $P \cap Q = \emptyset$, which implies $\mathcal{P} \cup \mathcal{Q}$ is a semi-partition. A *partition* is a semi-partition such that $\cup_{P\in\mathcal{P}} P = X$. A partition $\mathcal{Q}$ *refines* partition $\mathcal{P}$, denoted $\mathcal{Q} \geq \mathcal{P}$, if for each $Q \in \mathcal{Q}$ there is a $P \in \mathcal{P}$ with $Q \subseteq P$. A sequence $\mathcal{P}_k$ of partitions is *refining* if $\mathcal{P}_{k+1} \geq \mathcal{P}_k$ for all k. A semi-partition of the form $\mathcal{P} = \{B, TB, T^2B, \ldots, T^{h-1}B\}$, for some clopen set B is called a *Rohlin tower*. The *base* is B, and the *height* $h \geq 1$. A *Kakutani-Rohlin partition* is a partition $\mathcal{P}$ that is a union of d disjoint Rohlin towers $\{\mathcal{P}^0, \mathcal{P}^1, \ldots, \mathcal{P}^{d-1}\}$. A Kakutani-Rohlin partition is specified by its bases $B_0, B_1, \ldots, B_{d-1}$ and its heights $h_0, h_1, \ldots, h_{d-1}$.

We say $p \in \mathcal{A}$ is a *common prefix* for a substitution S if for all $a \in \mathcal{A}$, there is a $u_a \in \mathcal{A}^*$, so that $S(a) = pu_a$. Similarly, we say $s \in \mathcal{A}$ is a *common suffix* if for all $a \in \mathcal{A}$ there is a $v_a \in \mathcal{A}^*$ so that $S(a) = v_a s$. A substitution S is called *proper* (see [DHS-99]) if it has both a common prefix and a common suffix. Any primitive aperiodic substitution shift (X, T) is topologically conjugate to a substitution shift for a proper substitution, but generally one with a larger alphabet (see [DHS-99]).

Proposition 5. (Durand, Host, Skau [DHS-99]) *Let (X, T) be the substitution shift corresponding to a primitive aperiodic substitution S on an alphabet $\mathcal{A}$ with d symbols. For each $k \geq 0$, there is a Kakutani-Rohlin partition $\mathcal{P}_k$ with bases $B_a^{(k)} = S^k(\lfloor a\rfloor)$, $a \in \mathcal{A}$, and heights $h_a = |S^k(a)|$. These partitions satisfy $\mathcal{P}_{k+1} \geq \mathcal{P}_k$.*

We say that a sequence $\mathcal{P}_k$ of partitions on a Cantor set X *generates* (the topology) if for all k sufficiently large, any clopen set E is the union of sets in $\mathcal{P}_k$. Alternatively, the sequence $\mathcal{P}_k$ generates the topology on X if and only if for each $n \geq 1$, there is a $K \in \mathbb{N}$, so that for $k \geq K$, the function $x \mapsto x_{[-n,n]} : X \to \mathcal{A}^{2n+1}$

is constant on each $P \in \mathcal{P}^k$. Durand, Host and Skau [DHS-99] show that if, in addition to the hypotheses of Proposition 5, S is proper, then the sequence $\mathcal{P}_k$ generates the topology on X. The next proposition extends this result.

Proposition 6. *Let S be a primitive aperiodic substitution on d symbols, let T be the corresponding susbtitution shift, and let r_k be a sequence of integers. Then there is a sequence $\mathcal{P}_k$ of Kakutani-Rohlin partition with bases $B_a^{(k)} = T^{r_k}S^k(\lfloor a\rfloor)$, $a \in \mathcal{A}$, and heights $\ell_a = |S^k(a)|$. If in addition S has a common prefix (or suffix), then there exists a sequence r_k so that the sequence $\mathcal{P}_k$ of partitions generates the topology on X.*

Proof. It is easy to see (using Proposition 5) that $\mathcal{P}_k$ is always a Kakutani-Rohlin partition. Now suppose S has a common prefix (the common suffix case is similar). Then for $a \in \mathcal{A}$, $S(a) = pu_a$, and for any $ab \in \mathcal{L}$ (including, possibly aa), $S(ab) = pu_apu_b$. Thus for $k > 1$,

$$S^k(ab) = S^{k-1}(p)S^{k-1}(u_a)S^{k-1}(p)S^{k-1}(u_b). \tag{2}$$

Let $e_k = |S^{k-1}(p)|$ and let $d_k = \lfloor e_k/2\rfloor$ (the integer part of $e_k/2$), and let $c_{a,k} = |S^{k-1}(u_a)|$.

Fix $n \geq 0$ and take k large enough that $e_k \geq n$. This is possible because S is primitive. Since $S^k(a) = S^{k-1}(p)S^{k-1}(u_a)$, it follows that $\ell_{a,k} = |S^k(a)| = |S^{k-1}(p)| + |S^{k-1}(u_a)| = e_k + c_{a,k}$, which implies $[0, 2e_k + c_{a,k}] = [0, e_k + \ell_{a,k}]$. Thus by (2), any $x \in \lfloor S^k(a)\rfloor$ satisfies

$$x_{[0,e_k+\ell_{a,k}]} = S^{k-1}(p)S^{k-1}(u_a)S^{k-1}(p) \tag{3}$$

Now $S^k(\lfloor a\rfloor) \subseteq \lfloor S^k(a)\rfloor$, so (3) remains true for $x \in S^k(\lfloor a\rfloor)$.

Let $Q_{\mathbf{g}} \in \mathcal{Q}_k$. Then there exists $a \in \mathcal{A}$ and $0 \leq j < \ell_{a,k}$ so that $Q = T^{j+d_k}S^k(\lfloor a\rfloor)$. Any $x \in Q$ satisfies $T^{-j+d_k}x \in S^k(\lfloor a\rfloor)$, so by (3)

$$x_{[-j-d_k,\ell_{a,k}-j]} = (T^{-j-d_k}x)_{[0,d_k+\ell_{a,k}]} = S^{k-1}(p)S^{k-1}(u_a)S^{k-1}(p). \tag{4}$$

But $n < e_k \leq d_k/2 < d_k + j$, and since $j \leq \ell_{a,k}$, $n < e_k \leq d_k/2 < \ell$ Thus $[-n, n] \subseteq [-j - d_k, d_k + c_{a,k} - j]$, so $x_{[-n,n]}$ is constant on $S_{\mathbf{g}}$. □

Let S be primitive shift-aperiodic substitution, and let $\mathcal{P}_k$ be the sequence of Kakutain-Rohlin partitions from Proposition 6, corresponding to an arbitrary sequence r_k. Let $\mathbf{m}_a^{(k)} = m(B_a^{(k)}) = m(S^k(\lfloor a\rfloor))$ be the measures of the bases, and let $\mathbf{m}^{(k)} = (m_0^{(k)}, m_1^{(k)}, \ldots, m_{d-1}^{(k)})$. We call the vectors $\mathbf{m}^{(k)}$ *tower base vectors.*

Lemma 7. *The tower base vectors satisfy $\mathbf{m}^{(k)} = \lambda^{-k}\mathbf{m}$, where $\mathbf{m}$ is the normalized right Perron-Frobenius eigenvalue. The subgroup of $\mathbb{R}$ generated by the measures of the tower bases is the Perron-Frobenius group $\mathcal{G}_A$.*

Proof. First note that $\mathbf{m}^{(0)} = \mathbf{m}$. Also, $A\mathbf{m}^{(k)} = \mathbf{m}^{(k-1)}$ for all k. Assume $\mathbf{m}^{(k-1)} = \lambda^{k-1}\mathbf{m}$. Then $\mathbf{m}^{(k)} = A\mathbf{m}^{(k-1)} = \lambda^{k-1}A\mathbf{m} = \lambda^k\mathbf{m}$. The measures of the bases generate $\cup\mathbb{Z}[\mathbf{m}^{(k)}] = \cup\mathbb{Z}[\lambda^{-k}\mathbf{m}] = \mathcal{G}_A$. □

The next theorem is the main result of this section:

Theorem 8. *If S is a primitive, shift-aperiodic substitution with a common prefix (or suffix), then $\mathcal{M}_T = \mathcal{G}_A$. If S is also irreducible, then (X, T) is saturated, and $H(T) \cong \mathcal{M}_T = \mathcal{G}_A$.*

Proof. Let $\mathcal{P}_k$ be the sequence of Kakutani-Rohlin towers from Proposition 6. It follows from Proposition 6 that for any clopen set E, there exists k, so that E is a union of levels of $\mathcal{P}_k$. Thus by Lemma 7, $m(E) \in \mathbb{Z}[\lambda^{-k}\mathbf{m}] \subseteq \mathcal{G}_A$.

Now suppose S is irreducible, and let E and F be clopen sets such that $m(E) = m(F)$. By Proposition 6, there exists k so that E and F are both unions of the levels of $\in \mathcal{P}_k$. The entries of $\mathbf{m}^{(k)}$ give the measures of the bases of $\mathcal{P}_k$. Since T preserves m all the levels in each tower have the same measure as the base. By Lemma 4, $\mathbf{m}^{(k)}$ has rationally independent entries, since by Lemma 7, $\mathbf{m}^{(k)} = \lambda^{-k}\mathbf{m}$. Thus E and F must consist of the same number of levels from each tower in $\mathcal{P}_k$. Clearly we can define a homeomorphism $U \in [[T]]$ that matches up components of E with components of F within each tower, and that is constant on the rest of X. It follows that (X, T) is saturated. The last assertion follows from Theorem 2. □

4. One dimensional tiling systems

4.1. **Suspensions.** Throughout this section, we fix a strictly ergodic Cantor dynamical system (X, T), with unique T-invariant probability measure m. For a positive continuous function, $g : X \to \mathbb{R}$ (called a *height function*), define $R : X \times \mathbb{R} \to X \times \mathbb{R}$ by $R(x, s) = (Tx, g(x) + s)$. Put $(x, s) \sim (x', s')$ if $(x', s') = R^n(x, s)$ for some $n \in \mathbb{Z}$. The *suspension space* for (X, T) and g is defined $X_g = (X \times \mathbb{R})/\sim$. Note that $X_{\mathbf{g}}$ is a 1-dimensional compact metric space such that each $(x, t) \in X_g$ has a neighborhood of that is a product of an interval and a Cantor set. A space of this type is called a 1-*dimensional lamination.* In particular, X_g has a well defined 1-dimensional integer Čech cohomology $\check{H}^1(X_g)$ (see [PT-82]). Define the *suspension flow* (or *flow under a function*) T_g^t on X_g to be the $\sim$ quotient of the flow $(x, s) \mapsto (x, t + s)$. The minimality of (X, T) implies that (X_g, T_g^t) is minimal.

Let $\gamma = \int g\, dm$. Call a suspension *even* if $\gamma = 1$, and *uneven* otherwise. A probability measure μ on X_g is given by $\mu(A \times [a, b]) = \gamma^{-1}(b - a)m(A)$, where $A \subseteq X_g$ and $0 \leq a < b < g(x)$. The unique ergodicity of (X, T) implies (T_g^t, X_g) is uniquely ergodic for μ. Given an even suspension T_g^t, let $T_{\gamma g}^t$ be the uneven suspension corresponding to γg for a constant $\gamma \neq 1$. It is easy to see that $T_{\gamma g}^t$ is topologically conjugate to a time change $T_g^{\gamma^{-1}t}$ of T_g.

4.2. **The Brushlinski group.** We now describe a way to compute the 1-dimensional integer Čech cohomology $\check{H}^1(X_g)$ for a suspension space X_g. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle group in $\mathbb{C}$, and let $C(X_g, \mathbb{T})$ be the group of circle valued continuous functions on X_g, with pointwise multiplication. Recall that $f_0, f_1 \in C(X_g, \mathbb{T})$ are *homotopic* if there exists continuous $F : X_g \times [0, 1] \to \mathbb{T}$ so that $f_0(y) = F(y, 0)$ and $f_1(y) = F(y, 1)$. The *Brushlinski group* of X_g, denoted $Br(X_g)$, is defined to be $C(Y, \mathbb{T})$ modulo homotopy. According to Brushlinski's Theorem ([Br-34], see [PT-82]), $\check{H}^1(X_g)$ is isomorphic to the Brushlinski group $Br(X_g)$. From now on we will identify these two groups, writing both as $\check{H}^1(X_g)$ (this follows [PT-82]).

For $n(x) \in C(X, \mathbb{Z})$ define

$$f_n(x, t) = \exp\left(2\pi i\, t \frac{n(x)}{g(x)}\right). \tag{5}$$

Clearly $f_n \in C(Y, \mathbb{T})$ since $t\, n(x)/g(x) \in \mathbb{Z}$ for $t = 0$ and $t = g(x)$.

Proposition 9. *Suppose (X_g, T_g^t) is a suspension of a uniquely ergodic Cantor dynamical system (X, T). Then $n(x)$ is a dynamical coboundary if and only if f_n is homotopic to a constant. Moreover, for any $f \in C(Y, \mathbb{T})$ there is an $n \in C(X, \mathbb{Z})$ so that f_n is homotopic to f.*

Proof. Suppose $n(x) = p(Tx) - p(x)$ for $p \in C(X, \mathbb{Z})$. Then

$$f_n(x,t) = \exp(2\pi i t p(Tx)/g(x)) \exp(2\pi i t p(x)/g(x)),$$

Define a continuous real valued function $k(x,t) = (p(x)(1-t) + p(Tx)t)/g(x)$ on X_g. Since X_g is compact, $k(x,t)$ is continuous, uniformly bounded, and it follows that k is homotopic to 0. But $k(x,t) = (p(x) + tn(x))/g(x)$, and $p(x) \in \mathbb{Z}$, so $f_n(x,t) = \exp(2\pi i k(x,t))$, which implies f_n is homotopic to 1.

Conversely, if $f_n(x,t) = \exp(2\pi i t n(x)/g(x))$ is homotopic to 1, then there is a lift $\tilde{f}_n : X_g \to \mathbb{R}$ so that $\exp(2\pi i\, t\, n(x)/g(x)) = \exp(2\pi i\, \tilde{f}_n(x,t))$, and

$$t\,\frac{n(x)}{g(x)} - \tilde{f}_n(x,t) = p(x,t), \tag{6}$$

for some $p \in C(Y, \mathbb{Z})$. But p can not depend on t, so if we define $p(x) = p(x,0)$, then we have

$$t\,\frac{n(x)}{g(x)} - \tilde{f}_n(x,t) = p(x). \tag{7}$$

By the continuity of the left hand side of (7) for $t \to g(x)$ we have $n(x) - \tilde{f}_n(x, g(x)) = p(x)$. On the other hand $\tilde{f}_n(x, g(x)) = \tilde{f}_n(Tx, 0)$. For $t = 0$ we have $-\tilde{f}_n(Tx, 0) = p(Tx)$, and therefore $n(x) = p(Tx) - p(x)$. The final statement was proved in the case $g(x) = 1$ by Parry and Tuncel [PT-82]. The proof in this case is almost the same. □

Corollary 10. *If (X_g, T_g^t) is a suspension of a Cantor dynamical system (X, T), then the dynamical cohomology group $H(T)$ is isomorphic to the Čech cohomology group $\check{H}^1(X_g)$.*

4.3. **Winding numbers.** Let T^t be a flow on a compact metric space Y. We say that $f \in C(Y, \mathbb{T})$ is *differentiable* at $y \in Y$ if the limit

$$f'(y) = \lim_{h \to 0} \frac{1}{h}(f(T^t y) - f(y)) \tag{8}$$

exists. If f is continuously differentiable, we define the *winding number* of f by

$$W(f) = \frac{1}{2\pi i} \int_Y \frac{f'(y)}{f(y)}\, d\mu(y). \tag{9}$$

If we want to note the dependence on T^t we write $W(T^t, f)$. In particular, for a time change $F^{\gamma t}$ we have $W(T^{\gamma t}, f) = \gamma^{-1} W(T^t, f)$.

The winding number was defined by Schwartzman [Sc-57], who proved the following.

Lemma 11. (Schwartzman, [Sc-57]) *Every homotopy class in $C(Y, \mathbb{T})$ contains a continuously differentiable function. Moreover, if two continuously differentiable f_1 and f_2 are homotopic then $W(f_0) = W(f_1)$. Thus W is a well defined real-valued functional on $\check{H}^1(Y)$.*

For a flow (Y, T^t), we define the *winding number group* $\mathcal{W}_{T^t} \subseteq \mathbb{R}$ to be the image $\mathrm{im}(W)$ of the winding number operator W. By the comment following the definition of W, we have $\mathcal{W}_{T^{\gamma t}} = \gamma^{-1}\mathcal{W}_{T^t}$ for a time change.

Proposition 12. *Suppose (X_g, T_g^t) is an even suspension of a uniquely ergodic Cantor dynamical system (X, T). If $n \in C(X, \mathbb{Z})$, and $f_n \in C(X_g, \mathbb{T})$ is defined by (5), then $W(f_n) = \int_X n dm$. Thus $\mathcal{W}_{T_g} = \mathcal{M}_T$.*

Proof. Since $f_n'(x,t) = 2\pi i n(x)/g(x) f_n(x,t)$, $f_n'/f_n = 2\pi i n(x)/g(x)$ Thus $W(f_f) = \int_Y n(x)/g(x)\, d\mu = \int_X n\, dm$. $\square$

Remark 13. *The conclusion of Proposition* 12 *does not depend on the height function g, except for the assumption that g is even. If we replace T_g^t with the uneven suspension $T_{\gamma g}^t$, then $\mathcal{W}_{T_{\gamma g}} = \gamma^{-1}\mathcal{W}_{T_g} = \gamma^{-1}\mathcal{M}_T$.*

Remark 14. *When T is not saturated,* $\ker(W) \neq \emptyset$*. We call $f \in$* $\ker(W)$ *an* invisible cocycle*. These are the elements of $\check{H}^1(X_g)$, that have winding number zero, and correspond to the dynamical cocycles in $Inf(T)$.*

Remark 15. *Conversely, when T is saturated (i.e., when* $\ker(W) = \{0\}$*), any $f \in C(X_g, \mathbb{T})$ with $W(f) = 0$ is (homotopic to) a constant. We think of this as a strong form of unique ergodicity for T_g^t.*

4.4. **Eigenvalues.** Suppose (T^t, Y) is a minimal and uniquely ergodic continuous flow on a compact metric space Y for the invariant probability measure μ. We say that $\omega \in \mathbb{R}$ is an *eigenvalue* corresponding to the *eigenfunction* $f \in L^2(X, \mu)$ such that

$$f(T^t y) = \exp(2\pi i \omega t) \cdot f(y), \tag{10}$$

for μ a.e., $y \in Y$. One can think of the eigenvalue ω as the angular velocity of the eigenfunction f along the flow. We will assume (T^t, Y) is *strictly ergodic*, and also *homogeneous*, which means every eigenfunction f can be chosen to be continuous. We let $\mathcal{E}_{T^t} \subseteq \mathbb{R}$ denote the set of all eigenvalues. Fixing $y_0 \in Y$, we may assume that the eigefunctions f are normalized so that $f(y_0) = 1$, so that the mapping $\omega \mapsto f_\omega$ from eigenvalues to corresponding eigenfunction is well defined. Letting $E(Y) = \{f_\omega : \omega \in \mathcal{E}_{T^t}\}$, we have that $E(Y) \subseteq C(Y, \mathbb{T})$, and $\mathcal{E}_{T^t}$ and $E(Y)$ are both groups.

Lemma 16. *Let $\omega, \omega_1, \omega_2 \in \mathcal{E}_{T^t}$*

1. *$W(f_\omega) = \omega$, and*
2. *If f_{ω_1} is homotopic to f_{ω_2} then $\omega_1 = \omega_2$.*

Proof. For (1), $f_\omega'(y) = 2\pi i\omega\, f_\omega(y)$. Thus $f_\omega'/f_\omega = 2\pi i \omega$. Part (2) follows from part (1) and Lemma 11. $\square$

The next theorem applies these results to the case of even suspensions.

Theorem 17. *If (X_g, T_g^t) is an even suspension of a strictly ergodic Cantor system (X, T), then $\mathcal{E}_{T_g} \subseteq \mathcal{W}_{T_g} = \mathcal{M}_T$ and $E(X_g) \subseteq \check{H}^1(X_g)$.*

Proof. Lemma 16 shows that $E(X_g) \subseteq \check{H}^1(X_g)$. Since $E(X_g) \subseteq C(X_g, \mathbb{T})$, Proposition 12 shows that $\mathcal{E}_{T_g} = W(E(X_g)) \subseteq W(C(X_g, \mathbb{T})) = \mathcal{W}_{T_g}$. $\square$

Remark 18. *While $\mathcal{M}_T$ does not depend on the (even) height function $g(x)$, the eigenvalues $\mathcal{E}_{T_g}$ do. Thus $\mathcal{M}_T$ provides an upper bound on the possible sets $\mathcal{E}_{T_g}$ of eigenvalues for different even suspension flows (X_g, T_g^t).*

4.5. **Substitution tilings.** Let S be a primitive shift-aperiodic substitution on $\mathcal{A} = \{0, \ldots, d-1\}$ and let (X, T) be the corresponding substitution shift. Let $\mathbf{m}$ and $\mathbf{h}$ be the right and left Perron-Frobenius eigenvectors, normalized so that $\mathbf{m} \cdot \mathbf{1} = 1$ and $\mathbf{h} \cdot \mathbf{m} = 1$. We call a vector $\mathbf{g}$ an *even height vector* if $\mathbf{g} \cdot \mathbf{m} = 1$. Two particular even height vectors are $\mathbf{g} = \mathbf{1}$, called the *unit* height vector, and $\mathbf{g} = \mathbf{h}$, the normalized left Perron-Frobenius eigenvector. Given an even height vector $\mathbf{g} = (g_0, g_1, \ldots, g_{d-1})$, define

$$g(x) = \sum_{a=1}^{d} g_a \chi_{\lfloor a \rfloor}(x),$$

and let $(X_\mathbf{g}, T_\mathbf{g}^t)$ be the corresponding even suspension flow. We interpret this flow as the *substitution tiling flow* for the tiling substitution $S_\mathbf{g}$ obtained from S and $\mathbf{g}$ (see the Introduction or [RaS-01]).

Lemma 19. *For a primitive aperiodic tiling substitution $S_\mathbf{g}$ the substitution tiling flow $(X_\mathbf{g}, T_\mathbf{g}^t)$ is strictly ergodic and homogeneous. Moreover $\mathcal{E}_{T_\mathbf{g}} \subseteq \mathcal{W}_{T_\mathbf{g}} = \mathcal{M}_T$.*

For strict ergodicity and homogeneity, see [CS-03]. The second statement follows from Theorem 17. The next theorem is our second main result.

Theorem 20. *Let $S_\mathbf{g}$ be a tiling substitution $S_\mathbf{g}$ corresponding to a primitive, irreducible discrete substitution S with a common prefix. Then $\ker(W)$ is trivial, $\mathcal{E}_{T_\mathbf{g}} \subseteq \mathcal{W}_{T_\mathbf{g}} = \mathcal{M}_T = \mathcal{G}_A \cong \check{H}^1(X_\mathbf{g})$. If, in addition, S is unimodular, then $\mathcal{W}_{T_\mathbf{g}} = \mathcal{G}_A = \mathbb{Z}[\mathbf{m}] \cong \mathbb{Z}^d$.*

Proof. This follows from Theorem 8 and Lemma 19. □

4.6. **The spectrum of a substitution.** Theorem 20 gives conditions for $\mathcal{E}_{T_\mathbf{g}} \subseteq \mathcal{W}_{T_\mathbf{g}} = \mathcal{G}_A$. In this section, we give some conditions for equality.

Definition 21. *If a saturated strictly ergodic flow (Y, T^t) on a compact metric space has $\mathcal{E}_{T^t} = \mathcal{W}_{T^t} \cong \check{H}^1(Y)$, then we say T^t has* full spectrum.

A primitive irreducible integer matrix A is called an *Pisot matrix* if all the eigenvalues λ', except the Perron-Frobenius eigenvalue $\lambda > 1$, satisfy $|\lambda'| < 1$. In particular, the Perron Frobenius eigenvalue λ of a Pisot matrix is a Pisot number (a real algebraic integer $\lambda > 1$ whose Galois conjugates all satisfy $|\lambda'| < 1$; see e.g., [Me-75]). We call a (primitive irreducible) substitution S a *Pisot substitution* if its structure matrix is a Pisot matrix. Similarly we call a (primitive aperiodic) tiling substitution $S_\mathbf{g}$ a Pisot substitution if the underlying discrete substitution S is a Pisot substitution.

Our main result in this section is the following.

Theorem 22. *The tiling flow $T_\mathbf{g}^t$ corresponding to an even Pisot tiling substitution $S_\mathbf{g}$, with a common prefix, has full spectrum: $\mathcal{E}_{T_\mathbf{g}} = \mathcal{W}_{T_\mathbf{g}} = \mathcal{G}_A$.*

For a word $u = u_0 u_1 \ldots u_{n-1} \in \mathcal{L}$, the *population vector* is defined $\mathbf{p}_u = (p_0, p_1, \ldots, p_{d-1})^t \in \mathbb{Z}^d$, where $p_a = |\{j = 0, \ldots, n-1 : u_j = a\}|$. We say $u \in \mathcal{L}$ is a *recurrence word* if $u = x_{[r,s]}$, $r \leq s$, for some $x \in X$ such that $x_r = x_{s+1}$. For a real number t define $\{t\} = t - \lfloor t \rfloor$. We begin with a general lemma from Clark and Sadun [CS-03] (see also Host [Ho-86] and Solomyak [So-98]).

Lemma 23. *Let $T_{\mathbf{g}}^t$ be be a tiling flow corresponding to a primitive aperiodic tiling substitution $S_{\mathbf{g}}$. A real number satisfies $\omega \in \mathcal{E}_{T_{\mathbf{g}}}$ if and only if for every recurrence word u,*

$$\lim_{n\to\infty} \{\omega\, \mathbf{g} \cdot A^n \mathbf{p}_u\} = 0. \tag{11}$$

In the case of an even suspension, where $\mathbf{g} = \mathbf{h}$ is the normalized left Perron-Frobenius eigenvector, we have the following:

Lemma 24. *Let $T_{\mathbf{h}}^t$ be the even suspension of a primitive aperiodic substitution shift T, corresponding to the normalized left Perron-Frobenius eigenvector $\mathbf{h}$ (i.e, the Perron-Frobenuis suspension). If $\omega \in \mathbb{R}$ satisfies*

$$\lim_{n\to\infty} \{\omega\, h_a\, \lambda^n\} = 0 \tag{12}$$

for each $a \in \mathcal{A}$, where h_a is the height of the ath Kakutani-Rokhlin tower for T, then $\omega \in \mathcal{E}_{T_{\mathbf{h}}}$.

Proof. To prove ω satisfies (11) for each return word, it suffices to establish (11) for $\mathbf{p}_a$ for each $a \in \mathcal{A}$. This is because every return word is made up of symbols. But in this case $\omega\mathbf{h}\cdot\, A^n\mathbf{p}_a = \omega(A^t)^n\mathbf{h}\cdot\mathbf{p}_a = \omega\lambda^n\mathbf{h}\cdot\mathbf{p}_a = \omega\lambda^n h_a$, since $\mathbf{p}_a$ is the ath standard basis vector. ☐

A Pisot matrix A has 1-dimensional expanding subspace $H^u(A)$ and a $d-1$ dimensional contracting subspace $H^s(A)$.

Lemma 25. *We have $H^u(A) \perp H^s(A^t)$ and $H^u(A^t) \perp H^s(A)$. Moreover, if we define $P\mathbf{v} = \mathbf{v} - (\mathbf{h}\cdot\mathbf{v})\mathbf{m}$ for $\mathbf{v} \in \mathbb{R}^d$, then $P\mathbf{v} \in H^s(A)$.*

Proof. For $j = 0, \dots, d-1$, let $A\mathbf{m}_j = \lambda_j\mathbf{m}_j$ and $A^t\mathbf{h}_j = \lambda_j\mathbf{h}_j$, where $\mathbf{m}_1 = \mathbf{m}$, $\mathbf{h}_1 = \mathbf{h}$. Then $\mathbf{h}_i\cdot\mathbf{m}_j = 0$, for $i \neq j$. This is because $i \neq j$ implies $\lambda_j \neq \lambda_j$, but $\lambda_i\mathbf{h}_i\cdot\mathbf{m}_j = (A^t\mathbf{h}_i)\cdot\mathbf{m}_j = \mathbf{h}_i\cdot(A\mathbf{h}_j) = \lambda_j\mathbf{h}_i\cdot\mathbf{m}_j$. Note that $H^u(A)$ and $H^u(A^t)$ are the spans of $\mathbf{m}$ and $\mathbf{h}$, and that $H^s(A)$ and $H^s(A^t)$ are the real parts of the spans of $\mathbf{m}_2, \dots, \mathbf{m}_d$ and $\mathbf{h}_2, \dots, \mathbf{h}_d$. For the second assertion, $\mathbf{h}\cdot P\mathbf{v} = \mathbf{h}\cdot(\mathbf{v} - (\mathbf{h}\cdot\mathbf{v})\mathbf{m}) = \mathbf{h}\cdot\mathbf{v} - (\mathbf{h}\cdot\mathbf{v})(\mathbf{h}\cdot\mathbf{m}) = 0$ since $\mathbf{h}\cdot\mathbf{m} = 1$. ☐

The next result is one direction of Theorem 22 in the case of $\mathbf{g} = \mathbf{h}$.

Proposition 26. *The the tiling flow $T_{\mathbf{h}}^t$ corresponding to the Perron-Frobenious Pisot tiling substitution satisfies $\mathbb{Z}[\mathbf{m}] \subseteq \mathcal{E}_{T_{\mathbf{h}}}$.*

Proof. For $\mathbf{v}, \mathbf{k} \in \mathbb{R}^n$, let $t = (\mathbf{v}\cdot\mathbf{h})$, and $\alpha = (\mathbf{k}\cdot\mathbf{m})$. Then

$$\begin{aligned} t\alpha\lambda^n &= \lambda^n(\mathbf{v}\cdot\mathbf{h})(\mathbf{k}\cdot\mathbf{m}) = \big(\mathbf{k}\cdot(\mathbf{v}\cdot\mathbf{h})\lambda^n\mathbf{m}\big) \\ &= \big(\mathbf{k}\cdot(\mathbf{v}\cdot\mathbf{h})A^n\mathbf{m}\big) = \big((A^t)^n\mathbf{k}\cdot(\mathbf{v}\cdot\mathbf{h})\mathbf{m}\big) \\ &= (A^t)^n\mathbf{k}\cdot(\mathbf{v} - P\mathbf{v}\big) = \mathbf{k}\cdot A^n\mathbf{v} - \mathbf{k}\cdot A^n P\mathbf{v}. \end{aligned}$$

If $\mathbf{k} = \mathbf{p}_a$ and $\mathbf{v} = \mathbf{p}_b$, then

$$\{m_b h_a \lambda^n\} = \{\mathbf{p}_a\cdot A^n\mathbf{p}_b - \mathbf{p}_a\cdot A^n P\mathbf{p}_b\} = \{-\mathbf{p}_a\cdot A^n P\mathbf{p}_b\}, \tag{13}$$

since $\mathbf{p}_a\cdot A^n\mathbf{p}_b \in \mathbb{Z}$, and thus $\{m_b h_a\lambda^n\} \to 0$ since $P\mathbf{p}_b \in H^s(A)$. ☐

Remark 27. *In* [BK-06]*, this result is said to be well known and attributed to* [BT-86]*. The proof in* [BK-06] *constructs a torus rotation factor of $T^t_{\mathbf{h}}$ that has eigenvalues* $\mathbb{Z}[\mathbf{m}]$ *(but see Remark* 28 *regarding the scaling). The authors go on to prove the opposite inclusion $\mathcal{E}_{T_{\mathbf{h}}} \subseteq \mathbb{Z}[\mathbf{m}]$, which is their main result. For us, this opposite inclusion is Corollary* 29*, which will follow from Theorem* 20*, once we assume S has a common prefix. Unfortunately, the common prefix assumption seems necessary for our approach.*

Remark 28. *In* [BK-06] *and* [BBK-06] *the Perron-Frobenius eigenvectors are normalized $\mathbf{m}\cdot\mathbf{m} = 1$ and $\mathbf{h}\cdot\mathbf{m} = 1$, so their tiling flow $T^t_{\mathbf{h}}$ is not, in our terminology, an even suspension. However, the conclusion of Proposition* 26 *holds verbatim, since it differs from an even suspension by a time change.*

Corollary 29. *If S is a Pisot substitution with structure matrix A, and $(X_{\mathbf{h}}, T^t_{\mathbf{h}})$ is the Perron Frobenuis suspension, then $\mathcal{G}_A \subseteq \mathcal{E}_{T_{\mathbf{g}}}$.*

Proof. It suffices to show that $\mathbb{Z}[\lambda^{-k}\mathbf{m}] \subseteq \mathcal{E}_{T_{\mathbf{h}}}$. But clearly $\{(\lambda^{-k}m_b)h_a\lambda^n\} = \{m_b h_a \lambda^{n-k}\} \to 0$ as $n \to \infty$. □

The next result, which completes the proof of Theorem 22, shows that for Pisot tiling substitutions any even height vector $\mathbf{g}$ can be substituted for the Perron-Frobenius eigenvector $\mathbf{h}$.

Proposition 30. *For a Pisot substitution S, let $\mathbf{m}$ and $\mathbf{h}$ be the normalized Perron-Frobenius eigenvectors, and let $\mathbf{g} \geq 0$ be a normalized height vector* (*so that* $\mathbf{g}\cdot\mathbf{m} = 1$)*. Then $\mathcal{E}_{T_{\mathbf{g}}} = \mathcal{E}_{T_{\mathbf{h}}}$.*

Proof. Let $\omega \in \mathcal{E}_{T_{\mathbf{h}}}$, so that by Lemma 23, $\{\omega\mathbf{h}\cdot A^n\mathbf{p}_w\} \to 0$ for every recurrence vector $\mathbf{p}_w$. Then

$$\begin{aligned}\{\omega\mathbf{h}\cdot A^n\mathbf{p}_w\} &= \{\omega(\mathbf{g} - (\mathbf{g}-\mathbf{h}))\cdot A^n\mathbf{p}_w\} \\ &= \{\omega\mathbf{g}\cdot A^n\mathbf{p}_w - \omega(A^t)^n(\mathbf{g}-\mathbf{h})\cdot\mathbf{p}_w\}.\end{aligned}$$

Since $(\mathbf{g}-\mathbf{h})\cdot\mathbf{m} = \mathbf{g}\cdot\mathbf{m} - \mathbf{h}\cdot\mathbf{m} = 1 - 1 = 0$ it follows that $\mathbf{g}-\mathbf{h} \in H^s(A^t)$ and $(A^t)^n(\mathbf{g}-\mathbf{h}) \to \mathbf{0}$. Thus $\{\omega\mathbf{h}\cdot A^n\mathbf{p}_w\} \to 0$ and $\omega \in \mathcal{E}_{T_{\mathbf{g}}}$. □

5. Examples

5.1. **Metallic and alloy substitutions.** Let us consider two families of substitutions on $\mathcal{A} = \{0,1\}$. Define the substitutions S_n, for $n \geq 1$ (left), and S'_n for $n \geq 3$ (right), by

$$\begin{array}{ccc} 0 & \to & 0^n1 \\ 1 & \to & 0 \end{array} \quad \text{and} \quad \begin{array}{ccc} 0 & \to & 0^{n-1}1^{n-2} \\ 1 & \to & 01. \end{array}$$

We call S_n the nth metallic substitution and S'_n the nth alloy substitution.

Proposition 31. *The substitutions S_n and S'_n are primitive, shift-aperiodic, irreducible, unimodular, Pisot, and have a common prefix. For each of them, the substitution shift (X,T) is strictly ergodic and saturated, with $H(T) \cong \mathcal{M}_T = \mathbb{Z}[\mathbf{m}]$. For $\mathbf{g}$ an even height vector the flow $T^t_{\mathbf{g}}$ has $\mathcal{W}_{T_{\mathbf{g}}} = \mathbb{Z}[\mathbf{m}] \cong H^1(X_{\mathbf{g}})$.*

Proof of Proposition 31. The primitive unimodular structure matrices A_n and A'_n for S_n, and S'_n are given by

$$A_n = \begin{bmatrix} n & 1 \\ 1 & 0 \end{bmatrix}, \text{ and } \quad A'_n = \begin{bmatrix} n-1 & 1 \\ n-2 & 1 \end{bmatrix}, \tag{14}$$

($A_n^2 > 0$, $\det(A_n) = -1$, and $\det(A_n') = 1$). The characteristic polynomials $p_n(z) = z^2 - nz - 1$, and $q_n(z) = z^2 - nz + 1$, are irreducible, and have roots

$$\lambda_n, \lambda_n' = \frac{n \pm \sqrt{n^2+4}}{2}, \quad \text{and} \quad \beta_n, \beta_n' = \frac{n \pm \sqrt{n^2-4}}{2},$$

so the substitutions are Pisot. All the other stated properties follow. □

The numbers λ_n are sometimes called the *metallic numbers*. The number $\lambda_1 = (1/2)(1+\sqrt{5})$ is called the *golden mean*, $\lambda_2 = 1+\sqrt{2}$ is called the *silver mean*, etc. We call the numbers β_n *alloy numbers* because $A_{n+2}' = A_1 A_n$. Any monic quadratic polynomial over $\mathbb{Z}$ with constant term ± 1 is either $p_n(z)$, $p_n(-z)$, $q_n(z)$, $q_n(-z)$ or is $r(z) = z^2 \pm 1$. Thus the metallic and alloy numbers make up all the quadratic units, and all quadratic units are Pisot. Any product of the matrices of types A_n and A_k has a Pisot quadratic unit as its Perron-Frobenius eigenvalue.

Theorem 32. *A metallic tiling substitution flow* $(X_{\mathbf{g}}, T_{\mathbf{g}}^t)$ *for an even height* $\mathbf{g}$ *has full spectrum with*

$$\mathcal{W}_{T_{\mathbf{g}}} = \mathcal{E}_{T_{\mathbf{g}}} = \frac{1}{\lambda+1}\mathbb{Z}[\lambda] \cong \mathbb{Z}^2, \tag{15}$$

where λ *is the Perron-Frobenius eigenvalue. Similarly, a metallic tiling substitution flow* $(X_{\mathbf{g}}, T_{\mathbf{g}}^t)$ *for an even height* $\mathbf{g}$ *has full spectrum with*

$$\mathcal{W}_{T_{\mathbf{g}}} = \mathcal{E}_{T_{\mathbf{g}}} = \frac{1}{\beta-(n-2)}\mathbb{Z}[\beta] \cong \mathbb{Z}^2, \tag{16}$$

where β *is the Perron-Frobenis eigenvalue.*

Remark 33. *In the golden mean case* $n=1$, $\lambda = (1+\sqrt{5})/2$, $\lambda+1 = \lambda^2$ *is a unit, and* $\mathcal{W}_{T_{\mathbf{g}}} = \mathcal{E}_{T_{\mathbf{g}}} = \mathbb{Z}[\lambda]$.

Proof. The normalized right Perron-Frobenius eigenvectors of the matrices A_n (which is symmetric) are $\mathbf{m} = (\lambda+1)^{-1}(1,\lambda)$. Thus $\mathcal{E}_{T_{\mathbf{g}}} = \mathbb{Z}[\mathbf{m}] = (\lambda+1)^{-1}\mathbb{Z}[\lambda]$. Similarly, fixing $n \geq 3$, the normalized right Perron-Frobenius eigenvectors of B_n are $\mathbf{m} = (\beta - n + 2)^{-1}(1, \beta - n + 1)$, and $\mathbb{Z}[\beta - n + 1] = \mathbb{Z}[\beta]$. □

Remark 34. *For any substitution* S_n *or* S_n'*, the spectrum of the substitution shift* (X,T) *is* $\exp(2\pi i\, \mathcal{E}_{T_{\mathbf{g}}})$*. It is well known that* T *has* pure point spectrum.

5.2. **A unimodular non-Pisot substitution.** A *non-Piot* substitution is primitive substitution S with a non-Pisot Perron-Frobenius eigenvalue. The 4-letter substitution $S\; 0 \to 0313, 1 \to 031313, 2 \to 03223, 3 \to 0323$ studied in [FiHR-03] is non-Pisot because the Perron-Frobenius eigenvalue, $\lambda = \frac{1}{4}(7+\sqrt{5}+\sqrt{2}\sqrt{19+7\sqrt{5}}) \approx 4.39026$, and has a conjugate $\lambda' \approx 1.83785$ (the other two conjugates are reciprocals of these). It is primitive, shift-aperiodic, has a common prefix, is irreducible, and unimodular. Let $(X_{\mathbf{g}}, T_{\mathbf{h}}^t)$ be strictly ergodic even Perron-Frobenius substitution tiling flow. It is shown in [FiHR-03] that $T_{\mathbf{h}}^t$ is weakly mixing, which means $\mathcal{E}_{T_{\mathbf{h}}} = \{0\}$. On the other hand, $\check{H}^1(X_{\mathbf{h}}) \cong \mathcal{W}_{T_{\mathbf{h}}}$ which is equal to $\mathbb{Z}[\mathbf{m}] \cong \mathbb{Z}^4$. Thus the spectrum is not full.

Remark 35. *In* [FiHR-03] *it is proved that* $(X_{\mathbf{h}}, T_{\mathbf{h}}^t)$ *is homeomorphic to an almost 1:1 extension of a flow* (G_2, F^t) *that is a suspension of a four-interval interval exchange transformation. The space* G_2 *is a surface of genus 2, and the substitution* $S_{\mathbf{h}}$ *induces a pseudo-Anosov diffeomorphism on* G_2*. The space* $X_{\mathbf{h}}$

can be obtained as an inverse limit $\lim_{n\leftarrow\infty} G_2^n$*, where* G_2^n *is the surface* G_2 *with a six pointed asterisk of radius* n *(part of a singular orbit) removed. For each* n*,* G_2^n *is homeomorphic to* $G_2\backslash\{p\}$*, which is* G_2 *with a single puncture. Thus* $\check{H}^1(G_2^n) = H^1(G_2\{p\}) \cong \mathbb{Z}^4$*, and thus* $\check{H}^1(X_{\mathbf{g}}) = \lim_{n\to\infty} \check{H}^1(G_2^n) \cong \mathbb{Z}^4$*. This alternate way to compute* $\check{H}^1(X_{\mathbf{g}})$*, suggests an explanation for why the cohomology is not related to the spectrum for this substitution* S*. However, a similar calculation for the golden mean substitution* S_1*, unzipping a single orbit from an irrational flow in* $\mathbb{T}^2$*, shows that* $\check{H}^1(X_{\mathbf{g}}) = H^1(\mathbb{T}^2\backslash\{p\}) \cong \mathbb{Z}^2$*, even though* $T_{\mathbf{g}}^t$ *has full spectrum in this case.*

5.3. **The completely non-Pisot case.** We call a d-letter substitution S *completely non-Pisot* if all the eigenvalues λ satisfy $|\lambda| \geq 1$. Such a substitution is never unimodular. Clark and Sadun [CS-03] study the case of primitive, aperiodic, completely non-Pisot substitution S, with the additional assumption that there is a *full* recurrence word $u \in \mathcal{L}$: the vectors $\mathbf{p}_u, A\mathbf{p}_u, \ldots, A^{d-1}\mathbf{p}_u$ are linearly independent. They show that if a height vector $\mathbf{g} = (g_0, g_1, \ldots, d_{d-1})$ (not necessarily even) has rationally independent entries, then the tiling flow $T_{\mathbf{g}}^t$ is weakly mixing. This provides us with many examples of substitutions that do not have full spectrum since $\mathcal{E}_{T_{\mathbf{g}}} = \{\mathbf{0}\}$ is a proper subgroup of $\mathcal{W}_{T_{\mathbf{g}}}$. Consider, for example the substitution $0 \to 0111, 1 \to 0$ from [FR-08]. It is also shown in [CS-03] that if $g_i/g_j \in \mathbb{Q}$ $\forall i, j$, for a primitive, aperiodic, completely non-Pisot substitution then $\mathcal{E}_{T_{\mathbf{g}}} \subseteq g_0\mathbb{Q}$.

In the two-letter primitive, aperiodic, completely non-Pisot case, Kenyon, Sadun, and Solomyak [KSS-05] show that for a substitutions S, the tiling flow $T_{\mathbf{g}}^t$ is topologically mixing if and only if $\mathbf{g}$ has rationally independent entries (i.e., $g_0/g_1 \notin \mathbb{Q}$). This implies weak mixing. In fact, whenever the second (non-Perron-Frobenius) eigenvalue satisfies $|\lambda'| \neq 1$, weak mixing for $T_{\mathbf{g}}^t$ is equivalent to topological mixing (but never measure theoretic mixing; see [CS-03]).

Sometimes it is possible to say something about the spectrum even if $g_0/g_1 \in \mathbb{Q}$. Let us fix a a primitive, irreducible, shift-aperiodic two-letter substitution S that is completely non-Pisot. We assume, in addition, that S has a common prefix, and that $00, 11 \in \mathcal{L}$. For a vector $\mathbf{k} = (k_0, k_1) \in \mathbb{Z}^2$, let $\gcd(\mathbf{k}) = \gcd(k_0, k_)$. We say $\mathbf{k}$ satisfies the *gcd condition* if $\gcd((A^t)^n\mathbf{k}) = 1$ for all $n \geq 0$. It is shown in [KSS-05] that if S is a primitive, shift-aperiodic completely non-Pisot substitution, and if $\mathbf{1}$ satisfies the gcd condition, then (the discrete substitution shift) T is topologically mixing, which implies weak mixing. Conversely, if $\mathbf{1}$ does not satisfy the gcd condition then T is not weak mixing.

Theorem 36. *Let* S *be a two-letter primitive irreducible completely non-Pisot substitution with a common prefix. Then for any even height vector* $\mathbf{g}$*, the tiling flow* $T_{\mathbf{g}}$ *is saturated and* $\check{H}^1(X_{\mathbf{g}}) \cong \mathcal{W}_{T_{\mathbf{g}}} = \bigcup_{k\geq 0} \mathbb{Z}[\lambda^{-k}\mathbf{m}]$*, which is not finitely generated. Suppose, in addition, that* $00, 11 \in \mathcal{L}$*. Let* $\mathbf{g} = \gamma\mathbf{k}$ *be an even height vector where* $\mathbf{k} \in \mathbb{Z}^2$ *satisfies the gcd condition. Then* $\gamma^{-1} = \mathbf{m}\cdot\mathbf{k}$ *and* $\mathcal{E}_{T_{\mathbf{g}}} = \gamma^{-1}\mathbb{Z}$*. In particular,* $T_{\mathbf{g}}^t$ *does not have full spectrum.*

In [KSS-05] an algorithm is given to test the gcd condition. It is easy to see the substitution $0 \to 00011, 1 \to 0111$, satisfies all the hypotheses of Theorem 36), and the vector $\mathbf{1}$ satisfies the gcd condition.

Proof of Theorem 36. Consider $T_{\mathbf{k}}^t$ where $\mathbf{k}$ satisfies the gcd condition. Since $00, 11 \in \mathcal{L}$, and λ is irrational, both 0 and 1 are full return words. Thus Lemma 23 implies that $\omega \in \mathcal{E}_{T_{\mathbf{k}}}$ if and only if $\lim_{n\to\infty}\{\omega\,\mathbf{k}\cdot A^n\mathbf{p}_u\} = 0$ for both $u = 0$ and $u = 1$. The argument in [CS-03] shows that this is the case if and only if $\omega(A^t)^n\mathbf{k}\cdot\mathbf{p}_u = \omega\,\mathbf{k}\cdot A^n\mathbf{p}_u \in \mathbb{Z}$ for $u = 0, 1$ and all n sufficiently large, or equivalently, $\omega(A^t)^n\mathbf{k} \in \mathbb{Z}^2$, for n sufficiently large. Since $(A^t)^n\mathbf{k} \in \mathbb{Z}^2$ and $\gcd((A^t)^n\mathbf{k}) = 1$, it follows that $\mathcal{E}_{T_{\mathbf{k}}} = \mathbb{Z}$. Thus $\mathcal{E}_{T_{\mathbf{g}}} = \gamma^{-1}\mathbb{Z}$. □

5.4. **The case of T.** Let (X, T) be the substitution shift for a primitive aperiodic substitution S. We say a measurable, complex valued function f on X is an eigenfunction for eigenvalue ν if $f(Tx) = \nu f(x)$. Because T is strictly ergodic and homogeneous, we may assume $|f(x)| = 1$, $f \in C(X, \mathbb{T})$, and $|\nu| = 1$. The set $\mathcal{E}'_T$ of all eigenvalues is then a countable subgroup of $\mathbb{T}$ and we let $\mathcal{E}_T = \{\omega \in \mathbb{R} : e^{2\pi i\omega} \in \mathcal{E}'_T\}$. It is easy to see that $\mathcal{E}_T = \mathcal{E}_{T_{\mathbf{1}}}$ (T is isomorphic to $T^1_{\mathbf{1}}$ on $\{(x, 0) \in X_{\mathbf{1}}\}$).

It makes no sense to study $\check{H}^1(X)$ (since X is a Cantor set) but we can study $H(T)$ instead, keeping in mind that $H(T) \cong \check{H}^1(X_{\mathbf{1}})$. We also know that $\mathcal{M}_T = \mathcal{W}_{T^t_{\mathbf{1}}}$ and $\mathcal{M}_T = \mathcal{W}_{T^t_{\mathbf{1}}}$, so that $\mathcal{E}_T \subseteq \mathcal{M}_T$. If T is saturated (e.g., if S is irreducible and has a common prefix) then $H(T) \cong \mathcal{M}_T$. In a weak mixing case, like (**??**), for example, $\mathcal{M}_T = \mathcal{G}_A$, whereas $\mathcal{E}_T = \mathbb{Z}$ and $\mathcal{E}'_T = \{1\}$. In a unimodular Pisot case, on the other hand, $\mathcal{M}_T = \mathcal{E}_T = \mathbb{Z}[\mathbf{m}]$ and $\mathcal{E}'_T = \exp(2\pi i\mathbb{Z}[\mathbf{m}])$.

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