

ON THE SPECTRA OF THREE-DIMENSIONAL LAMELLAR SOLUTIONS OF THE DIBLOCK COPOLYMER PROBLEM*

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Abstract. One-dimensional free energy local minimizers are viewed as three-dimensional lamellar-type critical points in a box. To determine whether they model the lamellar phase of diblock copolymers in the strong segregation region, we analyze their spectra. We obtain the asymptotic expansions of their eigenvalues and eigenfunctions. Consequently we find that they are stable, i.e., are local minimizers in space, only if they have sufficiently many interfaces. Interestingly the one-dimensional global minimizer is near the borderline of three-dimensional stability.

Key words. spectrum, three-dimensional stability, lamellar solution, diblock copolymer

AMS subject classifications. 35J55, 34D15, 45J05, 82D60

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1. Introduction. In a diblock copolymer melt a molecule is a linear chain consisting of two subchains grafted covalently to each other. The first subchain has N_A type A monomer units, and the second subchain has N_B type B monomer units. In polymer systems even a weak repulsion between unlike monomers A and B induces a strong repulsion between subchains. With many chain molecules in a polymer melt the different type subchains tend to segregate below some critical temperature, but as they are chemically bonded in chain molecules, even a complete segregation of subchains cannot lead to a macroscopic phase separation. Only a local microphase separation occurs: microdomains rich in A and B are formed. These microdomains form morphological patterns/phases in a larger scale. The commonly observed phases include the spherical, cylindrical, and lamellar, depicted in Figure 1.

We consider a scenario in which a diblock copolymer melt is placed in a domain D and maintained at fixed temperature. D is scaled to have unit volume in space. Let $a = N_A/(N_A + N_B) \in (0, 1)$ be the relative number of the A monomers in a chain molecule. Similarly $b = N_B/(N_A + N_B)$, so $a + b = 1$. The relative A monomer density field u is an order parameter. $u \approx 1$ stands for high concentration of A monomers. The melt is incompressible so the relative B monomer density is $1 - u$, and $u \approx 0$ stands for high concentration of B monomers.

Ohta and Kawasaki [10] introduced an equilibrium theory in which the free energy of the system is a functional of the relative A monomer density:

$$(1.1) \quad I(u) = \int_D \left\{ \frac{\epsilon^2}{2} |\nabla u|^2 + \frac{\sigma}{2} |(-\Delta)^{-1/2}(u - a)|^2 + W(u) \right\},$$

defined in $X_a = \{u \in W^{1,2}(D) : \bar{u} = a\}$, where $\bar{u} := \frac{1}{|D|} \int_D u$ is the average of u on D . The original formula in [10] is given for the whole space. The expression here on a bounded domain D first appeared in Nishiura and Ohnishi [8].

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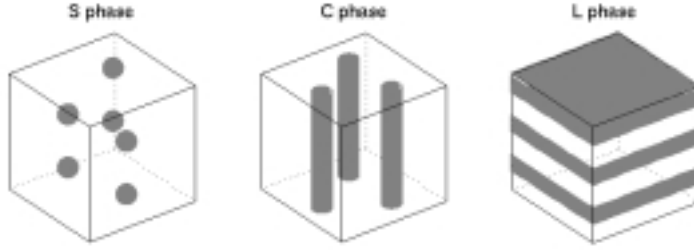


FIG. 1. The spherical, cylindrical, and lamellar morphology phases commonly observed in diblock copolymer melts. The dark color indicates the concentration of type A monomer, and the white color indicates the concentration of type B monomer.

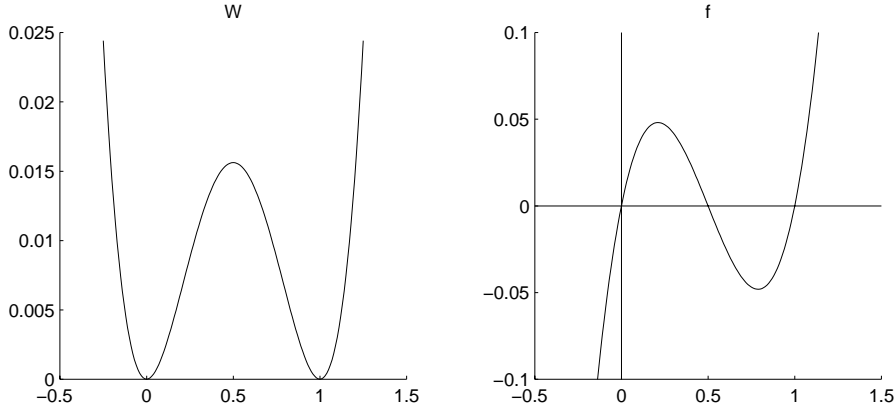


FIG. 2. The graphs of W and $f = W'$.

ϵ and σ are positive dimensionless parameters that depend on various physical quantities such as N_A , N_B , the average distance between two adjacent monomers in a chain, the interaction between monomers, the temperature, and the size of the sample. In the strong segregation region where morphology patterns form, ϵ is very small. The size of σ in this paper is chosen to be of order ϵ ; i.e., there is a fixed positive constant γ so that $\sigma = \epsilon\gamma$. This particular parameter range is realized if we take the sample size to be of the $(N_A + N_B)^{2/3}$ order.¹

The local function W is smooth and has the shape of a double well, as depicted in Figure 2. It has the global minimum value 0 at two numbers: 0 and 1. To avoid unnecessary technical difficulties we assume that $W(p) = W(1 - p)$. The two global minimum points are nondegenerate, i.e., $W''(0) = W''(1) \neq 0$. A simple example is $W(u) = \frac{1}{4}((u - \frac{1}{2})^2 - \frac{1}{4})^2$.

The most mathematically interesting part in equation (1.1) is the nonlocal term $(-\Delta)^{-1/2}(u - a)$ in the integrand. Let $(-\Delta)^{-1}(u - a)$ be the solution v of

$$-\Delta v = u - a \text{ in } D, \quad \partial_\nu v = 0 \text{ on } \partial D, \quad \bar{v} = 0,$$

where $\partial_\nu v$ is the outward normal derivative of v . $(-\Delta)^{-1/2}$ in I is the square root of the positive operator $(-\Delta)^{-1}$ from $\{w \in L^2(D) : \bar{w} = 0\}$ to itself. If we let

¹See Choksi and Ren [3] for more on these parameters.

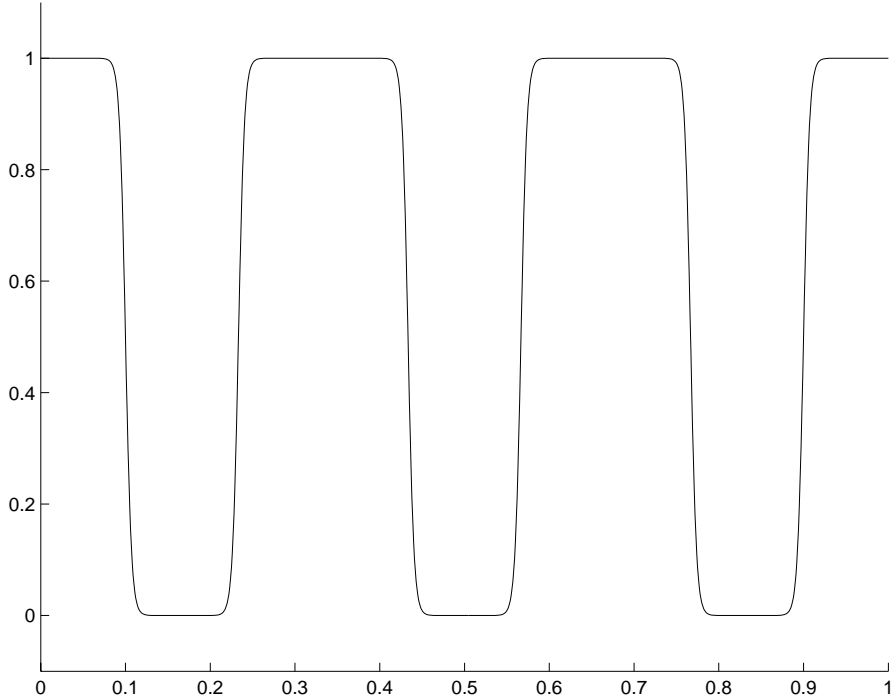


FIG. 3. A one-dimensional local minimizer u with $K = 6$. The regions where u is close to 1 are microdomains with high concentration of A monomers, and the regions where u is close to 0 are microdomains with high concentration of B monomers.

$v = (-\Delta)^{-1}(u - a)$, then an often more useful formula is

$$I(u) = \int_D \left\{ \frac{\epsilon^2}{2} |\nabla u|^2 + \frac{\epsilon\gamma}{2} |\nabla v|^2 + W(u) \right\}.$$

Let $f(u) = W'(u)$ as in Figure 2. For the particular $W(u) = \frac{1}{4}((u - \frac{1}{2})^2 - \frac{1}{4})^2$, $f(u) = u(u - \frac{1}{2})(u - 1)$. The Euler-Lagrange equation of I is

$$(1.2) \quad -\epsilon^2 \Delta u + \epsilon\gamma(-\Delta)^{-1}(u - a) + f(u) - \overline{f(u)} = 0, \quad \partial_\nu u = 0 \text{ on } \partial D.$$

The term $\overline{f(u)}$ is equal to the Lagrange multiplier corresponding to the constraint $\overline{u} = a$.

It is proved in Ren and Wei [13] using the Γ -limit theory that when $D = (0, 1)$ for any positive integer K there exists a local minimizer u with K interfaces and $K + 1$ microdomains if ϵ is small enough.² An example of u with $K = 6$ is shown in Figure 3. u is close to 0 in three regions and close to 1 in four regions. These regions are separated by sharp interfaces. That such u is energetically favored is not too difficult to explain. Note that the W term in I likes to have $u \approx 0$ or $u \approx 1$. The gradient term penalizes oscillation of u , but since it has a small coefficient it tolerates a number of sharp interfaces. The best profile for the nonlocal term is $u \approx a$. But this is impossible due to the presence of the W term and the fact $0 < a < 1$. The second

²See Theorem 2.1.

best profile for the nonlocal term is for u to have wild oscillation about a . When all the three terms are present in I , a compromise must be reached, and u as in Figure 3 emerges as a local minimizer.

Now we place such a one-dimensional (1-D) local minimizer in a three-dimensional (3-D) box through trivial extension. The extended u becomes a 3-D critical point of I , i.e., a solution of (1.2). We ask whether this u is a good model of the lamellar phase depicted in plot 3 of Figure 1. In general a morphology phase must be at least *metastable* in the sense that it is described by a local minimizer of I in space. Such a 3-D local minimizer is also called a stable solution of (1.2). We take $D = (0, 1) \times (0, 1) \times (0, 1)$ and study the spectrum of u , i.e., the second variation of I at u . The linearized operator at u is

$$(1.3) \quad \begin{aligned} L(\phi) &:= -\epsilon^2 \Delta \phi + \epsilon \gamma (-\Delta)^{-1} \phi + f'(u)\phi - \overline{f'(u)\phi}, \\ \partial_\nu \phi &= 0 \text{ on } \partial D, \quad \phi \in W^{2,2}(D), \quad \bar{\phi} = 0. \end{aligned}$$

This is an unbounded self-adjoint operator defined densely on $\{\phi \in L^2(D) : \bar{\phi} = 0\}$ whose spectrum consists of eigenvalues only.

We will obtain detailed information on the spectrum of u when ϵ is small. In particular we will find the asymptotic expansions of the important eigenvalues of small absolute values in terms of ϵ . We will also derive asymptotic expansions of the corresponding eigenfunctions. The analysis in this paper culminates in the following theorem.

THEOREM 1.1. *The eigenvalues λ of L are classified into λ_m by $m = (m_1, m_2)$, which is a pair of nonnegative integers. The following three statements hold when ϵ is sufficiently small:*

1. *There exists $M(K)$, depending on K but not ϵ , so that when $|m| := \sqrt{m_1^2 + m_2^2} \geq M(K)$, $\lambda_m \geq C\epsilon^2$ for some $C > 0$ independent of ϵ .*
2. *When $m = (0, 0)$, there are K small positive $\lambda_{(0,0)}$'s. One of them is of order ϵ whose only eigenfunction is approximately $\sum_j (h_j(x) - \bar{h}_j)$. The other $K - 1$ $\lambda_{(0,0)}$'s are of order ϵ^2 . Their only eigenfunctions are approximately $\sum_j c_j^0 h_j(x)$ for some vectors c^0 satisfying $\sum_j c_j^0 = 0$. The remaining $\lambda_{(0,0)}$'s are positive and bounded below by a positive constant independent of ϵ .*
3. *When $m \neq (0, 0)$ and $|m| < M(K)$, there are K λ_m 's of order ϵ^2 , which are not necessarily positive, whose only eigenfunctions are approximately $\sum_j c_j^0 h_j(x) \cos(m_1 \pi y_1) \cos(m_2 \pi y_2)$. The remaining λ_m 's are positive and bounded below by a positive constant independent of ϵ . Only when K is sufficiently large or γ is sufficiently small are all the eigenvalues of L positive and u stable.*

Here a point in D is denoted by (x, y_1, y_2) , where x is in the direction perpendicular to the interfaces of a lamellar phase, the up direction in plot 3, Figure 1. The functions h_j are defined in (3.5), and the c^0 vectors are given in sections 5 and 7. The $\lambda_{(0,0)}$ eigenvalues are just the eigenvalues in the 1-D problem. That they are positive, as noted in statement 2, is consistent with the fact that u is a 1-D local minimizer.

The most exciting discovery is apparently statement 3. The presence of the λ_m 's there is a 3-D phenomenon. A 1-D local minimizer is *not necessarily* a local minimizer in three dimensions. Not all 1-D local minimizers may be used to model the lamellar phase of diblock copolymers. Only the ones with sufficiently many interfaces, or in other words with sufficiently thin microdomains, are suitable candidates.

Of particular interest is the 1-D global minimizer, which is one of the 1-D local minimizers with $K \approx (\frac{a^2 b^2 \gamma}{3\tau})^{1/3}$, where τ is a positive number specified in (2.7). Since

its energy is lower than that of any other 1-D local minimizer, it is thermodynamically more preferred. But if it were unstable in three dimensions, then the lamellar phase would only be a transient metastable phase. Thermal fluctuation would eventually destroy any metastable lamellar phase. It turns out that the 1-D global minimizer has a delicate spectral property. It actually lies near the *borderline* of the stability of lamellar solutions.³

The stability of a solution of (1.2) may also be defined by a dynamic problem. As observed in [8] one may consider negative gradient flows of I in various function spaces. The simplest one is probably

$$(1.4) \quad u_t = \epsilon^2 \Delta u - \epsilon \gamma (-\Delta)^{-1} (u - a) - f(u) + \overline{f(u)}, \quad \partial_\nu u = 0 \text{ on } \partial D \times (0, \infty).$$

A physically more realistic dynamic model is the Cahn–Hilliard-like [1] fourth order problem:

$$(1.5) \quad u_t = \Delta(-\epsilon^2 \Delta u + \epsilon \gamma (-\Delta)^{-1} (u - a) + f(u)), \quad \partial_\nu \Delta u = \partial_\nu u = 0 \text{ on } \partial D \times (0, \infty).$$

The stability of steady states of (1.4) or (1.5) agrees with our static stability definition that a stable solution of (1.2) is a local minimizer of I .

Some preliminary work is done in section 2. We derive inner and outer asymptotic expansions of the lamellar solution u in a rigorous way. The first statement of the theorem is proved in section 3, the second in sections 4 and 5, and the third in sections 6 and 7. In the last section we discuss the spectrum of the 1-D global minimizer.

To avoid clumsy notation a quantity's dependence on ϵ is usually suppressed. For example, we write u , the lamellar solution, instead of u_ϵ . On the other hand we often emphasize a quantity's independence of ϵ with a superscript 0. For example, the limit of a lamellar solution u as $\epsilon \rightarrow 0$ is denoted by u^0 . In estimates, C is always a positive constant independent of ϵ . Its value may vary from line to line. The shorthand *e.s.* stands for a quantity that is exponentially small, i.e., equals $O(e^{-C/\epsilon})$. The L^2 inner product is denoted by $\langle \cdot, \cdot \rangle$ and the L^p norm by $\| \cdot \|_p$.

References on the mathematical aspects of the block copolymer theory include, in addition to the ones cited already, Ohnishi et al. [9], Choksi [2], Fife and Hillhorst [4], Henry [6], and Ren and Wei [11, 14] on diblock copolymers. On triblock copolymers we refer to Ren and Wei [15, 16].

2. The lamellar solution u . The lamellar solutions we consider in this paper were constructed in [13] by the Γ -limit theory.

THEOREM 2.1 (Ren and Wei [13]). *In 1-D for each positive integer K the functional*

$$(2.1) \quad I_1(u) := \int_0^1 \left\{ \frac{\epsilon^2}{2} \left(\frac{du}{dx} \right)^2 + \frac{\epsilon \gamma}{2} \left| \left(-\frac{d^2}{dx^2} \right)^{-1/2} (u - a) \right|^2 + W(u) \right\} dx$$

in $\{u \in W^{1,2}(0,1) : \bar{u} = a\}$ has a local minimizer u near u^0 , under the L^2 norm, when ϵ is sufficiently small. It satisfies the Euler–Lagrange equation

$$-\epsilon^2 u'' + f(u) - \overline{f(u)} + \epsilon \gamma G_0 [u - a] = 0, \quad u'(0) = u'(1) = 0$$

³This phenomenon compares well with the *marginal* stability, observed in Muratov [7], of the corresponding 1-D global minimizer in the Γ -limit.

and the properties

$$\lim_{\epsilon \rightarrow 0} \|u - u^0\|_2 = 0 \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \epsilon^{-1} I_1(u) = \tau K + \frac{\gamma}{2} \int_0^1 |(v^0)'| dx.$$

Here u^0 is a step function defined to be

$$u^0(x) = 1 \text{ on } (0, x_1^0), \quad 0 \text{ on } (x_1^0, x_2^0), \quad 1 \text{ on } (x_2^0, x_3^0), \quad 0 \text{ on } (x_3^0, x_4^0), \quad 1 \text{ on } (x_4^0, x_5^0), \dots$$

with (recall $b = 1 - a$)

$$x_1^0 = \frac{a}{K}, \quad x_2^0 = \frac{1+b}{K}, \quad x_3^0 = \frac{2+a}{K}, \quad x_4^0 = \frac{3+b}{K}, \quad x_5^0 = \frac{4+a}{K}, \dots,$$

$v^0 = G_0[u^0 - a]$. G_0 is the solution operator of $-v'' = g$, $v'(0) = v'(1) = 0$, $\bar{v} = 0$. The constant τ is positive and defined in (2.7).

There is another K -interface lamellar solution whose limiting value as $\epsilon \rightarrow 0$ is instead of 1 on the first interval $(0, b/K)$. This solution has the same properties as u does, so we focus on u , the solution of the first type, only.

Remark 2.2. This second solution is just $1 - \tilde{u}$, where \tilde{u} is a solution of the first type, but with $\tilde{u} = 1 - a$.

In this section we learn more about u . In particular u is periodic.

THEOREM 2.3. *When ϵ is small, for every $x \in (0, 1/K)$,*

$$\begin{aligned} u(x) &= u\left(\frac{2}{K} - x\right) = u\left(x + \frac{2}{K}\right) = u\left(\frac{4}{K} - x\right) = u\left(x + \frac{4}{K}\right) = \dots \\ &= \begin{cases} u(1-x) & \text{if } K \text{ is even,} \\ u(x + \frac{K-1}{K}) & \text{if } K \text{ is odd.} \end{cases} \end{aligned}$$

Moreover when ϵ is small, u is the unique local minimizer of I_1 in an L^2 neighborhood of u^0 . If u on $((j-1)/K, j/K)$ for some $j = 1, 2, \dots, K$ is scaled to a function on $(0, 1)$, then it is exactly a one-layer local minimizer of (2.1) with ϵ and γ replaced by $\tilde{\epsilon} = \epsilon K$ and $\tilde{\gamma} = \gamma/K^3$.

The nuts and bolts needed to prove this theorem are available in [11]. We give the proof in Appendix A, so the reader may skip it first in order to focus on the spectral properties of u in the following sections.

For that purpose we need asymptotic expansions of u in terms of ϵ . By Lemma A.1 in Appendix A there exist exactly K points x_j , $j = 1, 2, \dots, K$, in $(0, 1)$ so that $u(x_j) = 1/2$. These K points identify the interfaces of u . Theorem 2.3 implies that $x_2 = \frac{2}{K} - x_1$, $x_3 = \frac{4}{K} - x_2$, $x_4 = \frac{6}{K} - x_3$, etc. The first approximation of u is

$$\begin{aligned} w(x) &= H\left(-\frac{x-x_1}{\epsilon}\right) + H\left(\frac{x-x_2}{\epsilon}\right) + H\left(-\frac{x-x_3}{\epsilon}\right) - 1 + \dots \\ (2.2) \quad &+ \begin{cases} H\left(\frac{x-x_K}{\epsilon}\right) & \text{if } K \text{ is even,} \\ H\left(-\frac{x-x_K}{\epsilon}\right) - 1 & \text{if } K \text{ is odd.} \end{cases} \end{aligned}$$

Here H is the heteroclinic solution of

$$-H'' + f(H) = 0, \quad H(-\infty) = 0, \quad H(\infty) = 1, \quad H(0) = 1/2.$$

In the case $W(u) = \frac{1}{4}((u - \frac{1}{2})^2 - \frac{1}{4})^2$, it is explicitly known that $H(t) = (1/2)(\tanh \frac{t}{2\sqrt{2}} + 1)$. $H(t)$ converges to 1 as $t \rightarrow \infty$ (and to 0 as $t \rightarrow -\infty$) exponentially fast. Also $H'(t)$ and $H''(t)$ decay to 0 exponentially fast as $t \rightarrow \pm\infty$. H ,

or $H(\cdot)$, gives the profile of interfaces between the microdomains of u . At every $x \neq x_j^0$, $j = 1, 2, \dots, K$, $\lim_{\epsilon \rightarrow 0} w(x) = u^0(x)$.

Next we define

$$(2.3) \quad z^0(x) = -\frac{\gamma(v^0(x) - v^0(x_j^0))}{f'(0)}.$$

Let us compute

$$(2.4) \quad (v^0)'(x) = \begin{cases} (a-1)x & \text{on } (0, x_1^0), \\ a(x - \frac{1}{K}) & \text{on } (x_1^0, x_2^0), \\ (a-1)(x - \frac{2}{K}) & \text{on } (x_2^0, x_3^0), \\ a(x - \frac{3}{K}) & \text{on } (x_3^0, x_4^0), \\ (a-1)(x - \frac{4}{K}) & \text{on } (x_4^0, x_5^0), \\ \dots & \end{cases}$$

If we integrate $(v^0)'$ over an interval (x_{j-1}^0, x_j^0) , we get 0. So $v^0(x_j^0)$ is independent of j , and the definition of z^0 makes sense. Note that z^0 is independent of ϵ .

LEMMA 2.4. *Let z be defined by $u = w + \epsilon z$. Then $\|z - z^0\|_\infty = O(\epsilon)$.*

Proof. Combine Lemma A.3 in Appendix A and Theorem 2.3. \square

LEMMA 2.5. *There exists a constant $C > 0$ independent of ϵ so that $|\epsilon^{-1}z(x_j + \epsilon t)| \leq C(1 + |t|)$ for all $t \in (-\frac{x_j}{\epsilon}, \frac{1-x_j}{\epsilon})$. $\epsilon^{-1}z(x_j + \epsilon \cdot)$ converges to P in $C_{loc}^2(-\infty, \infty)$, where $P(t)$ is the solution of*

$$-P'' + f'(H)P = -\gamma(v^0)'(x_j^0)t, \quad P \perp H'$$

in $(-\infty, \infty)$.

There are two different P 's depending on whether j is odd or even. But they just differ by a sign, and it is always easy to tell from the context which one is referred to. Once j is given, there exists a unique P since the right side of its equation is perpendicular to the kernel H' .

Proof. Without the loss of generality we assume that j is even. Define $Z(t) = z(x_j + \epsilon t)$. Lemma 2.4 implies $Z = O(1)$ and hence, with the help of Lemma A.4, $f(u) = O(\epsilon)$. From the 1-D Euler-Lagrange equation in Theorem 2.1, which u satisfies, we find the equation for Z :

$$-Z'' + f'(H)Z + O(\epsilon)Z^2 + \gamma G_0[u - a] - \epsilon^{-1}\overline{f(u)} = 0.$$

From this equation we also have $Z'' = O(1)$ and $Z' = O(1)$. Multiply the equation by H' and integrate. Set $v(x) = G_0[u - a](x)$. Then

$$\begin{aligned} \text{e.s.} &= \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (-Z''H' + f'(H)ZH') dt \\ &= \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (-O(\epsilon)Z^2 - \gamma v(x_j + \epsilon t) + \epsilon^{-1}\overline{f(u)})H' dt \\ &= -\gamma v(x_j) + \epsilon^{-1}\overline{f(u)} + O(\epsilon). \end{aligned}$$

Hence $\gamma v(x_j) - \epsilon^{-1}\overline{f(u)} = O(\epsilon)$ and $\gamma v(x) - \epsilon^{-1}\overline{f(u)} = O(\epsilon) + O(\epsilon)t$. The equation for Z is now simplified to

$$-Z'' + f'(H)Z + O(\epsilon) + O(\epsilon)t = 0.$$

As $\epsilon \rightarrow 0$, $Z \rightarrow cH'$ in $C_{loc}^2(-\infty, \infty)$ for some c . But $Z(0) = \text{e.s.}$ implies $c = 0$ since $H'(0) \neq 0$. Therefore $Z \rightarrow 0$ in $C_{loc}^2(-\infty, \infty)$.

Next we study $\epsilon^{-1}Z$, whose equation is written as

$$-(\epsilon^{-1}Z)'' + f'(H)(\epsilon^{-1}Z) + O(1)Z^2 + \gamma\epsilon^{-1}v - \epsilon^{-2}\overline{f(u)} = 0.$$

We again multiply by H' and integrate:

$$\begin{aligned} \text{e.s.} &= \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (-\epsilon^{-1}Z''H' + f'(H)\epsilon^{-1}ZH') dt \\ &= \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (-O(1)Z^2 - \gamma\epsilon^{-1}v(x_j + \epsilon t) + \epsilon^{-2}\overline{f(u)})H' dt \\ &= \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (-O(1)Z^2 - \gamma\epsilon^{-1}v(x_j) - \gamma v'(x_j)t + O(\epsilon)t^2 + \epsilon^{-2}\overline{f(u)})H' dt \\ &= -\gamma\epsilon^{-1}v(x_j) + \epsilon^{-2}\overline{f(u)} + o(1), \end{aligned}$$

where we have used the facts that $Z \rightarrow 0$ locally and $\int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} tH' dt = \text{e.s.}$ Hence $\gamma\epsilon^{-1}v(x_j) - \epsilon^{-2}\overline{f(u)} = o(1)$, which simplifies the equation for $\epsilon^{-1}Z$ to

$$(2.5) \quad -(\epsilon^{-1}Z)'' + f'(H)(\epsilon^{-1}Z) + O(1)Z^2 + \gamma v'(x_j)t + O(\epsilon)t^2 + o(1) = 0.$$

Next we show that $|\epsilon^{-1}Z(t)| \leq C(1 + |t|)$. Without the loss of generality we consider $t > 0$. Let $\epsilon^{-1}Z(t) = (1 + t)R(t)$, where R satisfies

$$-R'' - \frac{2R'}{1+t} + f'(H)R + O(1) = 0 \quad \text{in} \quad \left(0, \frac{1-x_j}{\epsilon}\right), \quad R(0) = \text{e.s.}, \quad R\left(\frac{1-x_j}{\epsilon}\right) = O(1).$$

Suppose that $R = O(1)$ is invalid. We let $\hat{R} = R/\|R\|_{L^\infty}$, which satisfies

$$-\hat{R}'' - \frac{2\hat{R}'}{1+t} + f'(H)\hat{R} + o(1) = 0, \quad \hat{R}(0) = \text{e.s.}, \quad \hat{R}\left(\frac{1-x_j}{\epsilon}\right) = o(1).$$

From this equation we see that $|\hat{R}|$ must attain its maximum value 1 in a bounded region around 0. In the limit \hat{R} approaches in $C_{loc}^2[0, \infty)$ to a nonzero, bounded solution of

$$-\hat{R}''_\infty - \frac{2\hat{R}'_\infty}{1+t} + f'(H)\hat{R}_\infty = 0 \quad \text{in} \quad (0, \infty), \quad \hat{R}_\infty(0) = 0.$$

Then $(1+t)\hat{R}_\infty$ satisfies

$$-((1+t)\hat{R}_\infty)'' + f'(H)(1+t)\hat{R}_\infty = 0 \quad \text{in} \quad (0, \infty).$$

Thus, $(1+t)\hat{R}_\infty(t) = cH'(t)$ for some c . This is because $|(1+t)\hat{R}_\infty(t)|$ grows at most like t , and any other solution, independent of H' , of the last equation grows exponentially fast. Since $(1+0)\hat{R}_\infty(0) = 0$ and $H'(0) \neq 0$, we derive $c = 0$ and $\hat{R}_\infty = 0$, a contradiction.

Since $|\epsilon^{-1}Z(t)| \leq C(1 + |t|)$, we may send $\epsilon \rightarrow 0$ in (2.5) and find that $\epsilon^{-1}Z$ approaches in $C_{loc}^2(-\infty, \infty)$ to a solution of

$$-P'' + f'(H)P = -\gamma(v^0)'(x_j^0)t \quad \text{in} \quad (-\infty, \infty).$$

We write the solution family as $P + cH'$ with $P \perp H'$. Here $P(0) = 0$, and $P(0) + cH'(0) = cH'(0)$, where $H'(0) \neq 0$. Since $\epsilon^{-1}Z(0) = \text{e.s.}$, we must have $c = 0$, and $\epsilon^{-1}Z \rightarrow P$ in $C_{loc}^2(-\infty, \infty)$. \square

In the language of singular perturbation theory, the last two lemmas assert that the outer expansion of u is $u^0 + \epsilon z^0 + \dots$ and the inner expansion at x_j (when j is even) is $H + \epsilon^2 P + \dots$. The fact that $z^0(x_j^0) = 0$ matches the absence of the ϵ order term in the inner expansion. The function w defined in (2.2) is the 0th order uniform approximation of u .

We close this section by defining two frequently used constants. The first one is

$$(2.6) \quad s := \int_{-\infty}^{\infty} f''(H(t))(H'(t))^2 P(t) dt = -\frac{\gamma ab}{K}.$$

Here P is associated with an even j . When P is associated with an odd j in this paper, $f''(H(t))$ will always be changed to $f''(H(-t)) = -f''(H(t))$, so s remains the same. To verify the equality in (2.6) we differentiate the equation for P , multiply by H' , and integrate. The right side becomes $-\gamma(v^0)'(x_2^0)$. The left side becomes

$$\int_{-\infty}^{\infty} (-P'''H' + f'(H)P'H' + f''(H)(H')^2 P) dt = \int_{-\infty}^{\infty} f''(H)(H')^2 P dt,$$

where the first two terms on the left side cancel after integration by parts and using $-H''' + f'(H)H' = 0$, which follows after differentiating the equation for H . From (2.4) we find $(v^0)'(x_2^0) = ab/K$ and $s = -\gamma ab/K$.

The second constant is

$$(2.7) \quad \tau := \int_{-\infty}^{\infty} (H'(t))^2 dt > 0.$$

Because the equation for H has a first integral $-\frac{(H')^2}{2} + W(H) = 0$, then $\tau = \int_{-\infty}^{\infty} \sqrt{2W(H(t))} H'(t) dt = \int_0^1 \sqrt{2W(p)} dp$.⁴ In the special case $W(u) = \frac{1}{4}((u - \frac{1}{2})^2 - \frac{1}{4})^2$, $\tau = \frac{\sqrt{2}}{12}$.

3. Linearization at u . The 1-D local minimizer u of I_1 is now viewed as a function on D through extension to the second and third dimensions trivially, so $u(x, y_1, y_2) = u(x)$. It is a solution of (1.2) and $I_1(u) = I(u)$.

For an eigenpair (λ, ϕ) of (1.3) we separate variables so that

$$(3.1) \quad \phi(x, y_1, y_2) = \sum_{m_1, m_2=0}^{\infty} \phi_m(x) \cos(m_1 \pi y_1) \cos(m_2 \pi y_2).$$

We set $m = (m_1, m_2)$ and let $m^2 = m_1^2 + m_2^2$. Note that

$$(-\Delta)^{-1} \{\phi_m(x) \cos(m_1 \pi y_1) \cos(m_2 \pi y_2)\} = X(x) \cos(m_1 \pi y_1) \cos(m_2 \pi y_2),$$

where X is the solution of

$$-X'' = \phi_{(0,0)}, \quad X'(0) = X'(1) = 0, \quad \bar{X} = 0 \quad \text{if } m = (0,0)$$

or

$$-X'' + m^2 \pi^2 X = \phi_m, \quad X'(0) = X'(1) = 0 \quad \text{if } m \neq (0,0).$$

⁴In [13, 14, 16] this constant is defined by the last integral.

The solution operator of the first equation is G_0 , already defined. Let $G_m[\cdot]$ be the solution operator of the second equation. They are identified with the Green functions $G_m(\cdot, \cdot)$ in this paper. Therefore $X = G_m[\phi_m]$. The eigenvalue problem $L\phi = \lambda\phi$ now becomes

$$\begin{aligned} \sum_m \{ -\epsilon^2(\phi_m'' - m^2\pi^2\phi_m) + \epsilon\gamma G_m[\phi_m] + f'(u)\phi_m \} \cos(m_1\pi y_1) \cos(m_2\pi y_2) - \overline{f'(u)\phi_0} \\ = \lambda \sum_m \phi_m(x) \cos(m_1\pi y_1) \cos(m_2\pi y_2). \end{aligned}$$

Here we have used the fact that $\overline{f'(u)\phi_m(x) \cos(m_1\pi y_1) \cos(m_2\pi y_2)} = 0$ if $m \neq (0, 0)$.

Multiplying the equation by $\cos(m_1\pi y_1) \cos(m_2\pi y_2)$ and integrating with respect to y_1 and y_2 , we find two cases:

1. When $m = (0, 0)$,

$$\begin{aligned} (3.2) \quad -\epsilon^2\phi_{(0,0)}'' + \epsilon\gamma G_0[\phi_{(0,0)}] + f'(u)\phi_{(0,0)} - \overline{f'(u)\phi_{(0,0)}} = \lambda\phi_{(0,0)}, \\ \phi'_{(0,0)}(0) = \phi'_{(0,0)}(1) = \overline{\phi_{(0,0)}} = 0. \end{aligned}$$

2. When $m \neq (0, 0)$,

$$\begin{aligned} (3.3) \quad -\epsilon^2(\phi_m'' - m^2\pi^2\phi_m) + \epsilon\gamma G_m[\phi_m] + f'(u)\phi_m = \lambda\phi_m, \\ \phi_m'(0) = \phi_m'(1) = 0. \end{aligned}$$

Because the λ 's are classified by m , we use λ_m to denote an eigenvalue that is associated with m . The corresponding eigenfunction is $\phi_m(x) \cos(m_1\pi y_1) \cos(m_2\pi y_2)$.

Proof of Theorem 1.1, statement 1. We first consider the local eigenvalue problem

$$(3.4) \quad E(\phi) := -\epsilon^2\phi'' + f'(u)\phi = \nu\phi, \quad \phi'(0) = \phi'(1) = 0.$$

In this proof an eigenpair of (3.4) is denoted by (ν, ϕ) . We will prove that $\nu \geq -C\epsilon^2$ for some $C > 0$.

Claim 1. If $\nu \rightarrow \nu^0$ as $\epsilon \rightarrow 0$, then $\nu^0 \geq 0$.

Suppose on the contrary that $\nu^0 < 0$. Let $y \in [0, 1]$ so that $\phi(y) = \max|\phi| = 1$. Then $y - x_j = O(\epsilon)$ for some j . Otherwise $-\epsilon^2\phi''(y) \geq 0$, $f'(u(y))\phi(y) > 0$, $\nu\phi(y) < 0$, and hence (3.4) is not satisfied. Then we consider $\Phi(t) = \phi(x_j + \epsilon t)$, which satisfies $-\Phi'' + f'(u)\Phi = \nu\Phi$ in $(-x_j/\epsilon, (1 - x_j)/\epsilon)$. As $\epsilon \rightarrow 0$, Φ approaches $\Phi_\infty \neq 0$ in $C_{loc}^2(-\infty, \infty)$, which satisfies $-\Phi_\infty'' + f'(H)\Phi_\infty = \nu^0\Phi_\infty$ in $(-\infty, \infty)$. But this is impossible since the last equation has no negative eigenvalues. This proves the claim.

The case $\nu^0 > 0$ does not concern us, so we assume $\nu \rightarrow 0$. We introduce, for $j = 1, 2, \dots, K$,

$$(3.5) \quad h_j(x) = H' \left(\frac{x - x_j}{\epsilon} \right) + \text{e.s.}$$

Here e.s. is an exponentially small correction term. It is chosen so that $h_j(0) = h_j(1) = h_j'(0) = h_j'(1) = 0$, $\|h_j' - \epsilon^{-1}H''(\frac{\cdot - x_j}{\epsilon})\|_\infty = \text{e.s.}$, and $\|h_j'' - \epsilon^{-2}H'''(\frac{\cdot - x_j}{\epsilon})\|_\infty = \text{e.s.}$

Remark 3.1. Should we weaken the condition $W(p) = W(1 - p)$, H' would no longer be even and we would set

$$h_j(x) = \begin{cases} H'(\frac{x - x_j}{\epsilon}) + \text{e.s.} & \text{if } j \text{ is even,} \\ H'(-\frac{x - x_j}{\epsilon}) + \text{e.s.} & \text{if } j \text{ is odd.} \end{cases}$$

Consider the subspace of $L^2(0, 1)$ generated by h_j . Decompose $\phi = \sum_j c_j h_j + \psi$, so that $h_j \perp \psi$ for each $j = 1, 2, \dots, K$. Note that

$$E(h_j) = (f'(u) - f'(H))h_j + \text{e.s.},$$

and by Lemma 2.5,

$$(3.6) \quad \begin{aligned} |(f'(u) - f'(H))h_j| &= |(f'(w(x_j + \epsilon t) + \epsilon z(x_j + \epsilon t)) - f'(H(t)))H'(t)| + \text{e.s.} \\ &= |f''(H(t))\epsilon z(x_j + \epsilon t)H'(t)| + O(\epsilon^4) = O(\epsilon^2). \end{aligned}$$

Hence we deduce

$$(3.7) \quad E(h_j) = O(\epsilon^2).$$

We write (3.4) as

$$(3.8) \quad \sum_{j=1}^K c_j E(h_j) + E(\psi) = \nu \sum_j c_j h_j + \nu \psi.$$

Claim 2. $\langle E(\psi), \psi \rangle \geq C \|\psi\|_2^2$ for some $C > 0$ independent of ϵ .

When we minimize the quotient $\frac{\langle E(\tilde{\psi}), \tilde{\psi} \rangle}{\|\tilde{\psi}\|_2^2}$ among nonzero $\tilde{\psi}$ subject to $\tilde{\psi} \perp h_j$ for every j , the minimizer, denoted by $\tilde{\psi}$ in this paragraph, satisfies

$$(3.9) \quad -\epsilon^2 \tilde{\psi}'' + f'(u)\tilde{\psi} = \iota \tilde{\psi} + \sum_j d_j h_j.$$

The constant $\iota = \frac{\langle E(\tilde{\psi}), \tilde{\psi} \rangle}{\|\tilde{\psi}\|_2^2}$. Suppose that Claim 2 is false. Then $\lim_{\epsilon \rightarrow 0} \iota = \iota^0 \leq 0$. We multiply $\tilde{\psi}$ by a proper constant so there exists $y \in [0, 1]$ such that $\tilde{\psi}(y) = \max |\tilde{\psi}| = 1$. Now we multiply (3.9) by h_k and integrate.

$$\langle E(h_k), \tilde{\psi} \rangle = \sum_j d_j \langle h_j, h_k \rangle.$$

The left side is $O(\epsilon^2)$ by (3.7). The right side is

$$\int_0^1 \sum_j d_j h_j h_k = \sum_j \epsilon d_j \tau \delta_{jk} + \text{e.s.} |d| = \epsilon \tau d_k + \text{e.s.} |d|,$$

where $\delta_{jk} = 1$ if $j = k$ and 0 otherwise, and $|d| = \sqrt{d_1^2 + d_2^2 + \dots + d_K^2}$ is the norm of the vector d . Therefore $d_k = O(\epsilon)$. As in the proof of Claim 1, $y - x_j = O(\epsilon)$ for some j . Moreover we consider $\Psi(t) = \tilde{\psi}(x_j + \epsilon t)$, which satisfies $-\Psi'' + f'(u)\Psi = \iota \Psi + o(1)$. Passing to the limit we find a nonzero Ψ_∞ which satisfies $-\Psi_\infty'' + f'(H)\Psi_\infty = \iota^0 \Psi_\infty$ in $(-\infty, \infty)$. Therefore $\iota^0 = 0$ and Ψ_∞ is proportional to H' . But on the other hand $\tilde{\psi} \perp h_j$ implies $\Psi_\infty \perp H'$. Hence $\Psi_\infty = 0$, contradicting the fact that Ψ_∞ is nonzero. This proves Claim 2.

We now return to (3.8). Multiply it by ψ and integrate. Use (3.7) to deduce

$$|c|O(\epsilon^2)\|\psi\|_2 + \langle E(\psi), \psi \rangle = \nu \int_0^1 \psi^2.$$

Then Claim 2 implies

$$(3.10) \quad \|\psi\|_2 = O(\epsilon^2)|c|.$$

Next we multiply (3.8) by h_k and integrate. The left side is

$$(3.11) \quad \begin{aligned} & \int_0^1 \left\{ E(h_k)\psi + \sum_j c_j E(h_j)h_k \right\} \\ &= \int_0^1 \left\{ (f'(u) - f'(H))h_k\psi + \text{e.s.}\psi + \sum_j c_j ((f'(u) - f'(H))h_j h_k + \text{e.s.}) \right\}, \end{aligned}$$

in which

$$\left| \int_0^1 (f'(u) - f'(H))h_k\psi \right| \leq \| (f'(u) - f'(H))h_k \|_\infty \|\psi\|_2 = O(\epsilon^4)|c|$$

by (3.6) and (3.10), and

$$(3.12) \quad \begin{aligned} & \int_0^1 (f'(u) - f'(H))h_j h_k \\ &= \epsilon \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} \{f'(w(x_j + \epsilon t) + \epsilon z(x_j + \epsilon t)) - f'(H(t))\} H'(t) H'(t + (x_j - x_k)/\epsilon) dt + \text{e.s.} \\ &= \epsilon \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} \{f''(H(t))\epsilon z(x_j + \epsilon t) + O(\epsilon^2)z^2(x_j + \epsilon t)\} H'(t) H'(t + (x_j - x_k)/\epsilon) dt + \text{e.s.} \\ &= \epsilon^3 \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} f''(H(t))P(t)H'(t)H'(t + (x_j - x_k)/\epsilon) dt + o(\epsilon^3) \\ &= \epsilon^3 s\delta_{jk} + o(\epsilon^3) \end{aligned}$$

by Lemma 2.5. The above argument applies to the case when j is even. When j is odd, $f''(H(t))$ becomes $f''(H(-t)) = -f''(H(t))$ and $P(t)$ has a different sign, but the final result remains unchanged. Hence (3.11) becomes $\epsilon^3 s c_k + o(\epsilon^3)|c|$. The right side of (3.8) multiplied by h_k and integrated is $\nu \epsilon \tau c_k + \text{e.s.}|c|$. Equating the last two quantities, we find that for every k

$$s c_k + o(1)|c| = \frac{\nu \tau}{\epsilon^2} c_k.$$

Therefore $\nu \geq -C\epsilon^2$ for some $C > 0$ independent of ϵ .

Since G_m is a bounded, positive operator in the eigenvalue problem

$$(3.13) \quad -\epsilon^2 \phi'' + \epsilon \gamma G_m[\phi] + f'(u)\phi = \nu \phi, \quad \phi'(0) = \phi'(1) = 0,$$

we again have $\nu \geq -C\epsilon^2$ for some $C > 0$ independent of ϵ . This can be seen easily by comparing the variational characterization of the principle eigenvalue of (3.13),

$$\inf \left\{ \int_0^1 \{ \epsilon^2 (\phi')^2 + \epsilon \gamma G_m[\phi]\phi + f'(u)\phi^2 \} dx : \phi \in W^{1,2}(0,1), \|\phi\|_2 = 1 \right\},$$

to a similar one without the $\epsilon \gamma G_m[\phi]\phi$ term for (3.4). Finally, in (3.3), by setting m^2 large enough, we find $\lambda_m \geq C\epsilon^2$. \square

4. $m = (0, 0)$ eigenvalues. Here we study the $m_1 = m_2 = 0$ problem (3.2). Denote the linear operator there by L_0 . An eigenpair of (3.2) is denoted by (λ, ϕ) in this section. Since (3.2) is precisely the linearized operator of the 1-D problem I_1 defined in (2.1) at a 1-D local minimizer u , we have $\lambda \geq 0$. The case $\lambda \rightarrow \lambda^0 > 0$ as $\epsilon \rightarrow 0$ does not concern us. So we assume $\lambda \rightarrow 0$ along a subsequence throughout the rest of this section.

We decompose $\phi = \sum_j c_j(h_j - \bar{h}_j) + \psi$, where $\psi \perp h_j - \bar{h}_j$ for every j . Note that $L_0(h_j - \bar{h}_j) = (f'(u) - f'(H))h_j + \epsilon\gamma G_0[h_j - \bar{h}_j] + (\overline{f'(u)} - f'(u))\bar{h}_j - \overline{f'(u)h_j} + \text{e.s.}$ A few terms on the right side are estimated once and for all.

$$(4.1) \quad \epsilon\gamma G_0[h_j - \bar{h}_j](x) = \gamma\epsilon^2 G_0[(h_j - \bar{h}_j)/\epsilon](x) = \gamma\epsilon^2 G_0(x, x_j) + O(\epsilon^3).$$

$$\begin{aligned} \overline{f'(u)h_j} &= \epsilon \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} f'(w(x_j + \epsilon t) + \epsilon z(x_j + \epsilon t))H'(t) dt + \text{e.s.} \\ &= \epsilon \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (f'(H(t)) + f''(H(t))\epsilon z(x_j + t) + O(\epsilon^4))H'(t) dt + \text{e.s.} \\ (4.2) \quad &= \epsilon^2 \int_{-\infty}^{\infty} f''(H(t))z(x_j + \epsilon t)H'(t) dt + O(\epsilon^5) = O(\epsilon^3), \end{aligned}$$

where the last line follows from Lemma 2.5. The next estimate is not the sharpest.

$$(4.3) \quad |(\overline{f'(u)} - f'(u))\bar{h}_j| = |(\overline{f'(u)} - f'(u))|(\epsilon + \text{e.s.}) = O(\epsilon).$$

So based on the last three estimates and (3.6) we find

$$(4.4) \quad L_0(h_j - \bar{h}_j) = O(\epsilon).$$

We also need an L^1 version of (4.3):

$$\|(\overline{f'(u)} - f'(u))\bar{h}_j\|_1 = O(\epsilon)\|\overline{f'(u)} - f'(u)\|_1 = O(\epsilon)\|\overline{f'(w)} - f'(w) + O(\epsilon)\|_1 = O(\epsilon^2),$$

so we obtain

$$(4.5) \quad \|L_0(h_j - \bar{h}_j)\|_1 = O(\epsilon^2).$$

Rewrite (3.2) as

$$(4.6) \quad \sum_j c_j L_0(h_j - \bar{h}_j) + L_0\psi = \lambda \sum_j c_j(h_j - \bar{h}_j) + \lambda\psi.$$

LEMMA 4.1. $\langle L_0(\psi), \psi \rangle \geq C\|\psi\|_2^2$ for some $C > 0$ independent of ϵ .

Proof. When we minimize the quotient $\frac{\langle L_0(\tilde{\psi}), \tilde{\psi} \rangle}{\|\tilde{\psi}\|_2^2}$ among nonzero $\tilde{\psi}$ of zero average subject to $\tilde{\psi} \perp h_j - \bar{h}_j$ for every j , the minimizer, denoted by $\tilde{\psi}$ in this proof, satisfies

$$-\epsilon^2 \tilde{\psi}'' + \epsilon\gamma G_0[\tilde{\psi}] + f'(u)\tilde{\psi} - \overline{f'(u)\tilde{\psi}} = \iota\tilde{\psi} + \sum_j d_j(h_j - \bar{h}_j).$$

The constant $\iota = \frac{\langle L_0(\tilde{\psi}), \tilde{\psi} \rangle}{\|\tilde{\psi}\|_2^2}$. Suppose the lemma is false. Then $\lim_{\epsilon \rightarrow 0} \iota = \iota^0 \leq 0$. We multiply $\tilde{\psi}$ by a proper constant so there exists $y \in [0, 1]$ such that $\tilde{\psi}(y) = \max |\tilde{\psi}| = 1$.

Now we multiply the last equation by $h_k - \overline{h_k}$ and integrate: $\langle L_0(h_k - \overline{h_k}), \tilde{\psi} \rangle = \sum_j d_j \langle h_j - \overline{h_j}, h_k - \overline{h_k} \rangle$. The left side is $O(\epsilon^2)$ by (4.5). The right side is

$$\int_0^1 \sum_j d_j (h_j - \overline{h_j})(h_k - \overline{h_k}) = \epsilon \tau d_k + O(\epsilon^2)|d|.$$

Therefore $d_k = O(\epsilon)$. The rest of the proof is the same as that of Claim 2 in section 3, since the additional terms in the equation satisfy

$$\epsilon \gamma G_0[\tilde{\psi}] = O(\epsilon),$$

$$\overline{f'(u)\tilde{\psi}} = \overline{(f'(u) - f'(0))\tilde{\psi}} = O(1)\|f'(u) - f'(0)\|_1 = O(\epsilon).$$

A minor difference is that $\tilde{\psi} \perp h_j$ here is a consequence of $\tilde{\psi} \perp h_j - \overline{h_j}$ and $\overline{\tilde{\psi}} = 0$. \square

Multiply (4.6) by ψ and integrate. Using (4.4) we find

$$|c|O(\epsilon)\|\psi\|_2 + \langle L_0(\psi), \psi \rangle = \lambda\|\psi\|_2^2.$$

Lemma 4.1 implies that

$$(4.7) \quad \|\psi\|_2 = O(\epsilon)|c|.$$

Remark 4.2. As a comparison we compute

$$\left\| \sum_j c_j (h_j - \overline{h_j}) \right\|_2 = \left\{ \sum_j c_j^2 \int_0^1 h_j^2 + O(\epsilon^2)|c|^2 \right\}^{1/2} = \{\epsilon \tau |c|^2 + O(\epsilon^2)|c|^2\}^{1/2} \sim \epsilon^{1/2}|c|.$$

So in the decomposition of ϕ , $\sum_j c_j (h_j - \overline{h_j})$ is more prominent than ψ .

Multiply (4.6) by $h_k - \overline{h_k}$ and integrate:

$$(4.8) \quad \int_0^1 L_0 \psi (h_k - \overline{h_k}) + \sum_j c_j \int_0^1 L_0 (h_j - \overline{h_j})(h_k - \overline{h_k}) = \lambda \sum_j c_j \int_0^1 (h_j - \overline{h_j})(h_k - \overline{h_k}).$$

The first term on the left side is written as

$$(4.9) \quad \begin{aligned} \int_0^1 L_0(\psi)(h_k - \overline{h_k}) &= \int_0^1 L_0(h_k - \overline{h_k})\psi \\ &= \int_0^1 \{(f'(u) - f'(H))h_k \psi + \epsilon \gamma G_0[h_k - \overline{h_k}]\psi + (\overline{f'(u)} - f'(u))\overline{h_k} \psi - \overline{f'(u)h_k} \psi + \text{e.s.} \psi\} \\ &= \int_0^1 \{(f'(u) - f'(H))h_k \psi + \epsilon \gamma G_0[\psi]h_k + (f'(0) - f'(u))\overline{h_k} \psi + \text{e.s.} \psi\}. \end{aligned}$$

The four terms are estimated as follows:

$$\begin{aligned} \left| \int_0^1 (f'(u) - f'(H))h_k \psi \right| &\leq \| (f'(u) - f'(H))h_k \|_\infty \|\psi\|_2 = O(\epsilon^2)\|\psi\|_2 = O(\epsilon^3)|c|, \\ \int_0^1 \epsilon \gamma G_0[\psi]h_k &= O(\epsilon)\|G_0[\psi]\|_\infty \|h_k\|_1 = O(\epsilon^2)\|\psi\|_2 = O(\epsilon^3)|c|, \\ \int_0^1 (f'(0) - f'(u))\overline{h_k} \psi &= \|f'(u) - f'(0)\|_2 O(\epsilon)\|\psi\|_2 = O(\epsilon^{2.5})|c|, \\ \int_0^1 \text{e.s.} \psi &= \text{e.s.}\|\psi\|_2 = \text{e.s.}|c|. \end{aligned}$$

Note that the first estimate follows from (3.6). The second term on the left of (4.8) is, for each j , by (4.1) and (4.2),

$$\begin{aligned}
\int_0^1 L_0(h_j - \bar{h}_j)(h_k - \bar{h}_k) &= \int_0^1 L_0(h_j - \bar{h}_j)h_k \\
&= \int_0^1 \{(f'(u) - f'(H))h_j h_k + \epsilon \gamma G_0[h_j - \bar{h}_j]h_k + (\overline{f'(u)} - f'(u))\bar{h}_j h_k - \overline{f'(u)}\bar{h}_j h_k + \text{e.s.}\} \\
&= \epsilon^3 s \delta_{jk} + \gamma \epsilon^3 G_0(x_j, x_k) + \epsilon^2 \overline{f'(u)} + o(\epsilon^3).
\end{aligned}
\tag{4.10}$$

The last line follows from the estimates (3.12), (4.1), (4.2), and

$$\begin{aligned}
\int_0^1 (\overline{f'(u)} - f'(u))\bar{h}_j h_k &= (\epsilon + \text{e.s.}) \left\{ \overline{f'(u)}(\epsilon + \text{e.s.}) - \int_0^1 f'(u)h_k \right\} \\
&= \epsilon^2 \overline{f'(u)} - \epsilon \int_0^1 f'(u)h_k + \text{e.s.} \\
&= \epsilon^2 \overline{f'(u)} - \epsilon^2 \int_{-x_k/\epsilon}^{(1-x_k)/\epsilon} (f'(H) + O(\epsilon^2))H'(t) dt + \text{e.s.} = \epsilon^2 \overline{f'(u)} + O(\epsilon^4).
\end{aligned}$$

The right side of (4.8) is

$$\lambda \sum_j c_j \int_0^1 (h_j - \bar{h}_j)(h_k - \bar{h}_k) = \lambda(\epsilon \tau c_k + O(\epsilon^2)|c|).
\tag{4.11}$$

In summary, for every k

$$\epsilon^3 s c_k + \sum_j \{\gamma \epsilon^3 G(x_j, x_k) + \epsilon^2 \overline{f'(u)} + o(\epsilon^3)\} c_j + O(\epsilon^{2.5})|c| = \lambda(\tau \epsilon c_k + O(\epsilon^2)|c|).
\tag{4.12}$$

If we consider the c_k of the largest absolute value, since $\epsilon^2 \overline{f'(u)} \sim \epsilon^2$, $\lambda = O(\epsilon)$. On the left side of (4.12), $\epsilon^2 \overline{f'(u)}$ is the largest term. Because $\epsilon^2 \overline{f'(u)}$ is multiplied by $\sum_j c_j$, a dichotomy appears at this point, unless $K = 1$.

Case 1. $\frac{\sum_k c_k}{|c|} \not\rightarrow 0$. Note that when $K = 1$, this is the only case. We rewrite (4.12) as

$$\overline{f'(u)} \sum_j c_j + O(\epsilon^{1/2})|c| = \frac{\lambda \tau}{\epsilon} c_k.
\tag{4.13}$$

In the limit we have

$$f'(0) \sum_j c_j^0 = \eta \tau c_k^0, \quad \sum_j c_j^0 \neq 0,
\tag{4.14}$$

since $\lim_{\epsilon \rightarrow 0} \overline{f'(u)} = f'(0)$. Here $\eta = \lim_{\epsilon \rightarrow 0} \lambda/\epsilon$ and $\lim_{\epsilon \rightarrow 0} c_j = c_j^0$. Solving (4.14) we find

$$\eta = \frac{f'(0)K}{\tau}, \quad c_1^0 = c_2^0 = \dots = c_K^0.
\tag{4.15}$$

Thus we obtain the asymptotic expansions for one eigenpair $\lambda_{(0,0)} = \lambda$ and $\phi_{(0,0)} = \phi$ of (3.2):

$$(4.16) \quad \lambda_{(0,0)} = \frac{\epsilon f'(0)K}{\tau} + o(\epsilon), \quad \phi_{(0,0)} \approx \sum_j (h_j - \bar{h}_j).$$

Note that this $\lambda_{(0,0)}$ is positive.

Case 2. $\frac{\sum_k c_k}{|c|} \rightarrow 0$. This occurs when $K \geq 2$. To study this case, we rewrite $L_0(\phi) = \lambda\phi$ as

$$L_0(\psi) = - \sum_j c_j L_0(h_j - \bar{h}_j) + \lambda \sum_j c_j (h_j - \bar{h}_j) + \lambda\psi.$$

Note that

$$\sum_j c_j L_0(h_j - \bar{h}_j) = O(\epsilon^2)|c| + \sum_j c_j (\overline{f'(u)} - f'(u))\bar{h}_j = O(\epsilon^2)|c| + \left(\sum_j c_j \right) O(\epsilon) = o(\epsilon)|c|$$

by the assumption and by (4.1), (4.2), (3.6), and (4.3). Hence

$$(4.17) \quad L_0(\psi) = o(\epsilon)|c| + |\lambda|O(1)|c| + \lambda\psi.$$

LEMMA 4.3. $\|\psi\|_\infty = o(\epsilon)|c| + |\lambda|O(1)|c|$.

Proof. Suppose that the lemma is false. Replacing ψ by $\pm \frac{\psi}{\|\psi\|_\infty}$ in (4.17) we obtain $L_0(\psi) = o(1) + \lambda\psi$, where at some $y \in [0, 1]$, $\psi(y) = \|\psi\|_\infty = 1$. We show that $y - x_j = O(\epsilon)$ for some j . Otherwise $-\epsilon^2\psi''(y) \geq 0$, $\epsilon\gamma G_0[\psi] = O(\epsilon)$, $f'(u)\psi(y) \rightarrow f'(0) > 0$, $-\overline{f'(u)}\bar{\psi} = O(\epsilon)$, and $\lambda\psi(y) = O(\epsilon)$. Hence the equation $L_0(\psi) = o(1) + \lambda\psi$ is not satisfied. Then we set $\Psi(t) = \psi(x_j + \epsilon t)$, which satisfies $-\Psi'' + f'(u)\Psi = o(1)$ in $(-x_j/\epsilon, (1 - x_j)/\epsilon)$. As $\epsilon \rightarrow 0$, $\Psi \rightarrow \Psi_\infty \not\equiv 0$ in $C_{loc}^2(-\infty, \infty)$ and Ψ_∞ satisfies $-\Psi_\infty'' + f'(H)\Psi_\infty = 0$. Hence Ψ_∞ is proportional to H' . On the other hand $\psi \perp h_j$ implies $\int_{-\infty}^\infty \Psi_\infty H' = 0$. Thus $\Psi_\infty = 0$, contradicting the fact that $\Psi_\infty \not\equiv 0$. \square

With this lemma we return to (4.8) and recall

$$\int_0^1 (f'(u) - f'(H))h_k\psi = O(\epsilon^2)\|\psi\|_2, \quad \int_0^1 \epsilon\gamma G_0[\psi]h_k = O(\epsilon^2)\|\psi\|_2.$$

Rederive

$$\int_0^1 (f'(0) - f'(u))\bar{h}_k\psi = \|f'(u) - f'(0)\|_1 O(\epsilon)\|\psi\|_\infty = O(\epsilon^2)\|\psi\|_\infty.$$

Therefore

$$(4.18) \quad \int_0^1 L_0(\psi)(h_k - \bar{h}_k) = o(\epsilon^3)|c| + |\lambda|O(\epsilon^2)|c|,$$

and from (4.10)

$$\begin{aligned} \sum_j c_j \int_0^1 L_0(h_j - \bar{h}_j)(h_k - \bar{h}_k) &= \epsilon^3 s c_k + \sum_j (\gamma\epsilon^3 G_0(x_j, x_k) + \epsilon^2 \overline{f'(u)})c_j + o(\epsilon^3)|c| \\ &= O(\epsilon^3)|c| + \epsilon^2 \overline{f'(u)} \left(\sum_j c_j \right) = o(\epsilon^2)|c|. \end{aligned}$$

This estimate and (4.18), (4.11) turn (4.8) into

$$|\lambda|O(\epsilon^2)|c| + o(\epsilon^2)|c| = \lambda(\epsilon\tau c_k + O(\epsilon^2)|c|).$$

From the c_k of the largest absolute value, we find, with the help of Lemma 4.3,

$$(4.19) \quad \lambda = o(\epsilon), \quad \|\psi\|_\infty = o(\epsilon)|c|.$$

So (4.18) is improved to $o(\epsilon^3)|c|$ and (4.8) reads

$$(4.20) \quad \epsilon^3 s c_k + \sum_j (\gamma \epsilon^3 G_0(x_j, x_k) + \epsilon^2 \overline{f'(u)}) c_j + o(\epsilon^3)|c| = \lambda \epsilon \tau c_k.$$

We sum over k . $\sum_k G_0(x_j^0, x_k^0)$ is independent of j , an issue further addressed in the next section, so we denote it by g . Then after dividing by $\epsilon^2|c|$ we obtain, using (4.19),

$$\frac{\sum_j c_j}{|c|} (\epsilon s + \gamma \epsilon g + \overline{f'(u)} K) + o(\epsilon) = o(1) \frac{\sum_j c_j}{|c|}.$$

Since $\overline{f'(u)} \sim 1$, $\frac{\sum_j c_j}{|c|} = o(\epsilon)$. Return to (4.20). Divide by ϵ^3 . Since $\overline{f'(u)} \sum_j c_j = o(\epsilon)|c|$,

$$(4.21) \quad s c_k + \gamma \sum_j G_0(x_j, x_k) c_j + o(1)|c| = \frac{\lambda}{\epsilon^2} \tau c_k$$

for all k . In the limit we have

$$(4.22) \quad s c_k^0 + \gamma \sum_j G_0(x_j^0, x_k^0) c_j^0 = \eta \tau c_k^0, \quad \sum_j c_j^0 = 0.$$

Here $\eta = \lim_{\epsilon \rightarrow 0} \lambda/\epsilon^2$ and $c_j^0 = \lim_{\epsilon \rightarrow 0} c_j$. In the next section we will solve (4.22) to find $K-1$ pairs of η and c^0 . Once they are determined we obtain the asymptotic expansions of $K-1$ eigenpairs $\lambda_{(0,0)} = \lambda$ and $\phi_{(0,0)} = \phi$ of (3.2):

$$(4.23) \quad \lambda_{(0,0)} = \epsilon^2 \eta + o(\epsilon^2), \quad \phi_{(0,0)} \approx \sum_j c_j^0 h_j.$$

Here the $\overline{h_j}$ terms drop out in $\phi_{(0,0)}$ since $\sum_j c_j^0 \overline{h_j} = \sum_j (1 + \text{e.s.}) c_j^0 = \text{e.s.}|c|$ is negligible.

5. The spectrum of $[G_0(x_j^0, x_k^0)]$. To understand (4.22) we must find the spectrum of the K by K matrix $G_0(x_j^0, x_k^0)$. Suppose for every k that

$$\sum_j G_0(x_j^0, x_k^0) b_j = \Lambda b_k.$$

Note that from (4.22) $s + \gamma \Lambda = \tau \eta$. From the formula

$$G_0(x, y) = \begin{cases} \frac{x^2}{2} + \frac{(1-y)^2}{2} - \frac{1}{6} & \text{if } x < y, \\ \frac{(1-x)^2}{2} + \frac{y^2}{2} - \frac{1}{6} & \text{if } x > y \end{cases}$$

we see by straight computation that $\sum_k G_0(x_j^0, x_k^0)$ is independent of j . This number is an eigenvalue whose associated eigenvector is $(1, 1, \dots, 1)^T$, where the superscript T denotes the transpose of a vector. However, this eigenpair is discarded since by (4.22) we require that $\sum_j b_j = 0$.

To find other eigenpairs we let $\zeta = \sum_j G_0(x_j^0, \cdot) b_j$. Then ζ satisfies $-\zeta'' = \sum_j (\delta(\cdot - x_j^0) - 1) b_j = \sum_j \delta(\cdot - x_j^0)$, $\zeta'(0) = \zeta'(1) = 0$. Moreover $\zeta(x_k) = \Lambda b_k$ and $[-\zeta']_{x_j^0} = b_j$. Then for every k , $[-\zeta']_{x_j^0} = (1/\Lambda)\zeta(x_k^0)$. We need to express $[-\zeta']_{x_j^0}$ in terms of $\zeta(x_k^0)$. In other words we find a K by K matrix T so that $(T\vec{\zeta})_j = [-\zeta']_{x_j^0}$, where $\vec{\zeta} = (\zeta(x_1^0), \zeta(x_2^0), \dots, \zeta(x_K^0))^T$. This way the original eigenvalue problem is converted to

$$(5.1) \quad T\vec{\zeta} = \frac{1}{\Lambda}\vec{\zeta} \quad \text{with } b = \frac{1}{\Lambda}\vec{\zeta}.$$

To find T note that $-\zeta$ is affine between the x_j^0 's. From $(0, x_1^0)$ we deduce

$$-\zeta'(x_1^0-) = \frac{-\zeta(x_1^0) + \zeta(x_0^0)}{a/K} = 0$$

since $\zeta'(0) = 0$. From (x_1^0, x_2^0) we obtain

$$-\zeta'(x_1^0+) = \frac{-\zeta(x_2^0) + \zeta(x_1^0)}{2b/K}.$$

Hence

$$[-\zeta']_{x_1^0} = \frac{K}{2b}\zeta(x_1^0) - \frac{K}{2b}\zeta(x_2^0).$$

On the other intervals we find

$$[-\zeta']_{x_j^0} = \left(\frac{K}{2a} + \frac{K}{2b}\right)\zeta(x_j^0) - \begin{cases} \frac{K}{2b}\zeta(x_{j-1}^0) - \frac{K}{2a}\zeta(x_{j+1}^0) & \text{if } j \text{ is even,} \\ \frac{K}{2a}\zeta(x_{j-1}^0) - \frac{K}{2b}\zeta(x_{j+1}^0) & \text{if } j \text{ is odd.} \end{cases}$$

When $K = 2$ we have

$$T = \begin{bmatrix} K/(2b) & -K/(2b) \\ -K/(2b) & K/(2b) \end{bmatrix}.$$

Therefore after discarding the 0 eigenvalue of the matrix, we find $\Lambda = \frac{b}{K}$. And for (4.22)

$$(5.2) \quad \eta = \frac{1}{\tau} \left(-\frac{\gamma ab}{K} + \frac{\gamma b}{K} \right), \quad c^0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Note that $\eta > 0$. When $K \geq 3$ we have $T = (\alpha + \beta)I_{K \times K} - Q$, where $I_{K \times K}$ is the K by K identity matrix, $\alpha = K/(2a)$, $\beta = K/(2b)$, and

$$Q = \begin{bmatrix} \alpha & \beta & & & \\ \beta & 0 & \alpha & & \\ & \alpha & 0 & \beta & \\ & & \beta & 0 & \alpha \\ & & & & \dots \end{bmatrix}.$$

The K distinct eigenvalues of Q are found in (B.5) of Appendix B. One of them, $\alpha + \beta$, is discarded, for its eigenvector is $(1, 1, \dots, 1)^T$. If we denote the rest of them by q_1, q_2, \dots, q_{K-1} , we have $K - 1$ Λ 's:

$$\Lambda = \frac{1}{\alpha + \beta - q_j}, \quad j = 1, 2, \dots, K - 1.$$

Therefore $K - 1$ pairs of

$$(5.3) \quad \eta = \frac{1}{\tau} \left(-\frac{\gamma ab}{K} + \frac{\gamma}{\alpha + \beta - q_j} \right), \quad c^0 = \vec{\zeta}$$

for (4.22) are found.

When concerned with the positivity of η , we consider the smallest Λ , which is associated with the smallest q_j . According to equation (B.5), the smallest q_j is $-\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta}$, where $\theta = 2\pi/K$. Hence the smallest Λ is

$$\Lambda = \frac{1}{\alpha + \beta + \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta}} > \frac{1}{2(\alpha + \beta)} = \frac{ab}{K}.$$

Therefore the smallest η of (4.22) is

$$\eta = \frac{s + \gamma\Lambda}{\tau} > \frac{1}{\tau} \left(-\frac{\gamma ab}{K} + \frac{\gamma ab}{K} \right) = 0.$$

Thus the η 's in both (5.2) and (5.3) are positive.

Finally, we show that L_0 has exactly K simple eigenpairs with the asymptotic expansions (4.16) and (4.23). Let F be the linear subspace generated by small eigenvalues. It is defined nonambiguously by $F = \text{span}\{\phi \in L^2(0, 1) : \bar{\phi} = 0, L_0(\phi) = \lambda\phi, |\lambda| < \epsilon^{1/2}\}$. Since the small eigenvalues of L_0 are of order ϵ^2 or ϵ , F addresses all the small eigenvalues when ϵ is small enough.

First $\dim F$, the dimensional of F , is at most K . Suppose that this is not the case. There exist two distinct eigenpairs (λ, ϕ) and (λ', ϕ') with the same asymptotic behavior. That is,

$$\lambda = \epsilon^2 \eta + o(\epsilon^2), \quad \lambda' = \epsilon^2 \eta + o(\epsilon^2), \quad \text{or} \quad \lambda = \epsilon \eta + o(\epsilon), \quad \lambda' = \epsilon \eta + o(\epsilon),$$

$$\phi = \sum_j c_j (h_j - \bar{h}_i) + \psi, \quad \phi' = \sum_j c'_j (h_j - \bar{h}_i) + \psi', \quad \lim_{\epsilon \rightarrow 0} c_j = \lim_{\epsilon \rightarrow 0} c'_j = c_j^0.$$

However, the two eigenfunctions must be orthogonal, so

$$\begin{aligned} 0 &= \langle \phi, \phi' \rangle \\ &= \sum_{j,k} c_j c'_k \langle h_j - \bar{h}_i, h_k - \bar{h}_k \rangle + O(|c|) \|\psi\|_2 \|h_j\|_2 + O(|c|) \|\psi'\|_2 \|h_j\|_2 + \|\psi\|_2 \|\psi'\|_2 \\ &= \sum_j c_j^2 \int_0^1 h_j^2 dx + o(\epsilon) |c|^2 = \epsilon |c^0|^2 \int_{-\infty}^{\infty} (H'(t))^2 dt + o(\epsilon) |c^0|^2 \end{aligned}$$

by Remark 4.2. This is obviously impossible when ϵ is sufficiently small.

Next $\dim F$ is at least K . Suppose instead that $\dim F < K$. Define a subspace of $\{\phi \in L^2(0, 1) : \bar{\phi} = 0\}$: $S = \text{span}\{\sum_j c_j^0 (h_j - \bar{h}_j) : \text{all } c^0 \text{ found in (5.3)}\}$. We use

a perturbation argument. The asymmetric distance between the closed subspaces S and F is

$$d(S, F) = \sup\{d(\varphi, F) : \varphi \in S, \|\varphi\|_2 = 1\},$$

where $d(x, F) = \inf\{\|x - y\|_2 : y \in F\}$. Since $\dim F < \dim S$, there exists $\sum_j c_j^0(h_j - \bar{h}_j) \in S$ such that for every eigenvector in F which may be written as $\sum_j c'_j(h_j - \bar{h}_j) + \psi$ with $\|\psi\|_2 = O(\epsilon)|c'|$ according to (4.7), $\sum_j \frac{c'_j}{|c'|} \frac{c_j^0}{|c^0|} = o(1)$. Then a straight computation shows

$$\left\langle \frac{\sum_j c'_j(h_j - \bar{h}_j) + \psi}{\|\sum_j c'_j(h_j - \bar{h}_j) + \psi\|_2}, \frac{\sum_j c_j^0(h_j - \bar{h}_j)}{\|\sum_j c_j^0(h_j - \bar{h}_j)\|_2} \right\rangle = o(1).$$

So if we use

$$\varphi = \frac{\sum_j c_j^0(h_j - \bar{h}_j)}{\|\sum_j c_j^0(h_j - \bar{h}_j)\|_2},$$

$d(\varphi, F) = 1 - o(1)$ and $d(S, F) = 1 - o(1)$. The following lemma due to Helffer and Sjöstrand [5] will give us a contradiction.

LEMMA 5.1. *Let L be a self-adjoint operator on a Hilbert space H , let R be a compact interval in $(-\infty, \infty)$, and let e_1, e_2, \dots, e_K be normalized linearly independent elements in the domain of L . Assume that the following are true:*

1. $L(e_k) = p_k e_k + r_k$, $\|r_k\| \leq \epsilon'$, and $p_j \in R$, $k = 1, 2, \dots, K$.
2. There is $\omega > 0$ so that R is ω -isolated in the spectrum of L , i.e., $(\sigma(L) \setminus R) \cap (R + (-\omega, \omega)) = \emptyset$.

Then $d(S, F) \leq \frac{K^{1/2}\epsilon'}{\omega\kappa^{1/2}}$, where $S = \text{span}\{e_1, \dots, e_K\}$, $F =$ the closed subspace associated with $\sigma(L) \cap R$, and κ equals the smallest eigenvalue of the matrix $[\langle e_j, e_k \rangle]$.

Here we take $L = L_0$, each $e_k \propto \sum_j c_j^0(h_j - \bar{h}_j)$ for each one of the K vectors c^0 , and S, F as before. ω and κ are positive and bounded away from 0 as $\epsilon \rightarrow 0$. Set $p_k = \eta\epsilon^2$ or $\eta\epsilon$ depending on c^0 and $R = [-\epsilon^{1/2}, \epsilon^{1/2}]$. From (4.4) we find

$$L_0 \left(\sum_j c_j^0(h_j - \bar{h}_j) \right) - p_k \sum_j c_j^0(h_j - \bar{h}_j) = O(\epsilon)|c^0|,$$

and on the other hand $\|\sum_j c_j^0(h_j - \bar{h}_j)\|_2 \sim \epsilon^{1/2}|c^0|$, as discussed in Remark 4.2. Therefore $\|r_k\|_2 = O(\epsilon^{1/2})$. Consequently $d(S, F) = o(1)$, a contradiction. Statement 2 of Theorem 1.1 is proved.

6. $m \neq (0, 0)$ eigenvalues. Rewrite (3.3) as

$$(6.1) \quad L_m(\phi) := -\epsilon^2 \phi'' + \epsilon \gamma G_m[\phi] + f'(u)\phi = \mu\phi,$$

where $\mu = \lambda_m - \epsilon^2 m^2 \pi^2$. In this section an eigenpair of (6.1) is denoted by (μ, ϕ) .

LEMMA 6.1. *If $\mu \rightarrow \mu^0$ as $\epsilon \rightarrow 0$, then $\mu^0 \geq 0$.*

The proof of this lemma is almost identical to that of Claim 1 in section 3, and we skip it, because the extra term $\epsilon \gamma G_m[\phi]$ is of order $O(\epsilon)$. The case $\mu^0 > 0$ does not concern us, so we assume $\mu \rightarrow 0$. Decompose $\phi = \sum_j c_j h_j + \psi$, where $\psi \perp h_j$, $j = 1, 2, \dots, K$. Note that

$$L_m(h_j) = (f'(u) - f'(H))h_j + \epsilon \gamma G_m[h_j] + \text{e.s.}$$

Because of (3.6) and

$$\epsilon\gamma G_m[h_j](x) = \gamma\epsilon^2 G_m\left[\frac{h_j}{\epsilon}\right](x) = \gamma\epsilon^2 G_m(x, x_j) + O(\epsilon^3),$$

we deduce

$$(6.2) \quad L_m(h_j) = O(\epsilon^2).$$

We write (6.1) as

$$(6.3) \quad \sum_{j=1}^K c_j L_m(h_j) + L_m(\psi) = \mu \sum_j c_j h_j + \mu\psi.$$

LEMMA 6.2. $\langle L_m(\psi), \psi \rangle \geq C\|\psi\|_2^2$ for some $C > 0$ independent of ϵ .

We skip the proof of this lemma since it is similar to that of Claim 2 in section 3. Multiply (6.3) by ψ and integrate. Use (6.2) to deduce

$$|c|O(\epsilon^2)\|\psi\|_2 + \langle L_m(\psi), \psi \rangle = \mu\|\psi\|_2^2.$$

Then Lemma 6.2 implies

$$(6.4) \quad \|\psi\|_2 = O(\epsilon^2)|c|.$$

Next we multiply (6.3) by h_k and integrate. The left side is

$$(6.5) \quad \begin{aligned} & \int_0^1 \left\{ L_m(\psi)h_k + \sum_j c_j L_m(h_j)h_k \right\} = \int_0^1 \left\{ L_m(h_k)\psi + \sum_j c_j L_m(h_j)h_k \right\} \\ & = \int_0^1 \{(f'(u) - f'(H))h_k\psi + \epsilon\gamma G_m[h_k]\psi + \text{e.s. } \psi\} \\ & + \sum_j c_j \int_0^1 \{(f'(u) - f'(H))h_j h_k + G_m[h_j]h_k + \text{e.s. } h_k\}. \end{aligned}$$

All terms in (6.5) are estimated.

$$\left| \int_0^1 (f'(u) - f'(H))h_k\psi \right| \leq \|(f'(u) - f'(H))h_k\|_\infty \|\psi\|_2 = O(\epsilon^4)|c|$$

by (3.6) and (6.4).

$$\int_0^1 \epsilon\gamma G_m[h_k]\psi = O(\epsilon) \int_0^1 G_m[\psi]h_k = O(\epsilon)\|G_m[\psi]\|_\infty \|h_k\|_1 = O(\epsilon^2)\|\psi\|_2 = O(\epsilon^4)|c|$$

by (6.4). The rest of (6.5) are estimated as in section 4:

$$\int_0^1 (f'(u) - f'(H))h_j h_k = \epsilon^3 s\delta_{jk} + O(\epsilon^4), \quad \int_0^1 \epsilon\gamma G_m[h_j]h_k = \gamma\epsilon^3 G_m(x_j, x_k) + o(\epsilon^3).$$

Hence (6.5) becomes

$$\epsilon^3 s c_k + \sum_j c_j \gamma \epsilon^3 G_m(x_j, x_k) + o(\epsilon^3)|c|.$$

The right side of (6.3) multiplied by h_k and integrated is

$$\int_0^1 \mu \sum_j c_j h_j h_k = \sum_j \mu \epsilon c_j \tau \delta_{jk} + \text{e.s.}|c| = \mu \epsilon \tau c_k + \text{e.s.}|c|.$$

Equating the last two quantities, we find that $\mu = O(\epsilon^2)$ and, for every k ,

$$(6.6) \quad s c_k + \gamma \sum_j G_m(x_j, x_k) c_j + o(1)|c| = \frac{\mu \tau}{\epsilon^2} c_k.$$

So in the limit

$$(6.7) \quad s c_k^0 + \gamma \sum_j G_m(x_j^0, x_k^0) c_j^0 = \eta \tau c_k^0.$$

Here $\eta = \lim_{\epsilon \rightarrow 0} \mu / \epsilon^2 =$ and $c_j^0 = \lim_{\epsilon \rightarrow 0} c_j$. In the next section we will solve (6.7) to determine η and c^0 . Once they are found we obtain the asymptotic expansions for the eigenpair $\lambda_m = \mu + \epsilon^2 m^2 \pi^2$ and $\phi_m = \phi$:

$$(6.8) \quad \lambda_m = \epsilon^2 (\eta + m^2 \pi^2) + o(\epsilon^2), \quad \phi_m \approx \sum_j c_j^0 h_j.$$

7. The spectrum of $[G_m(x_j^0, x_k^0)]$. When dealing with

$$\sum_j G_m(x_j^0, x_k^0) b_j = \Lambda b_k,$$

we first consider the simplest case $K = 1$. Then $\Lambda = G_m(x_1^0, x_1^0)$. On $(0, x_1^0)$

$$G_m(x, x_1^0) = \frac{G_m(x_1^0, x_1^0)}{\cosh \tilde{m} x_1^0} \cosh \tilde{m} x,$$

where $\tilde{m} = \pi \sqrt{m_1^2 + m_2^2}$, and on $(x_1^0, 1)$

$$G_m(x, x_1^0) = \frac{G_m(x_1^0, x_1^0)}{\cosh \tilde{m}(1 - x_1^0)} \cosh \tilde{m}(1 - x).$$

Then

$$1 = [-G'_m(\cdot, x_1^0)]_{x_1^0} = \left\{ \frac{\tilde{m} \sinh \tilde{m} x_1^0}{\cosh \tilde{m} x_1^0} + \frac{\tilde{m} \sinh \tilde{m}(1 - x_1^0)}{\cosh \tilde{m}(1 - x_1^0)} \right\} G_m(x_1^0, x_1^0).$$

Therefore

$$\Lambda = \frac{1}{\tilde{m}(\tanh \tilde{m} a + \tanh \tilde{m} b)},$$

and in (6.7)

$$(7.1) \quad \eta = \frac{1}{\tau} \left(-\gamma ab + \frac{\gamma}{\tilde{m}(\tanh \tilde{m} a + \tanh \tilde{m} b)} \right), \quad c^0 = 1.$$

To see the sign of λ_m , we recall

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_m}{\epsilon^2} = \eta + m^2 \pi^2 = \frac{1}{\tau} \left(-\gamma ab + \frac{\gamma}{\tilde{m}(\tanh \tilde{m} a + \tanh \tilde{m} b)} \right) + m^2 \pi^2.$$

The right side is positive for all $m \neq (0,0)$ if γ is small enough, because $m^2\pi^2$ dominates the negative term. However, when γ is sufficiently large, we may find some large \tilde{m} that makes the right side negative. To see this we first take \tilde{m} large enough so that sum of the two terms in the parentheses is negative. Then we take γ large enough so the entire right side is negative.

When $K \geq 2$, $\sum_j G_m(x_j^0, x_k^0)b_j = \Lambda b_k$ is a more complex problem. Let ζ be the solution of $-\zeta'' + m^2\pi^2\zeta = \sum_j \delta(\cdot - x_j^0)b_j$, $\zeta'(0) = \zeta'(1) = 0$. Hence $[-\zeta']_{x_k^0} = b_k$ and $\zeta(x_k^0) = \Lambda b_k$. Then for every k , $[-\zeta']_{x_k^0} = \frac{1}{\Lambda}\zeta(x_k^0)$. As in section 5 we express $[-\zeta']_{x_k^0} = (T\vec{\zeta})_k$ in order to convert to the new eigenvalue problem $T\vec{\zeta} = (1/\Lambda)\vec{\zeta}$. Away from x_j^0 , $\zeta = g_1 \cosh \tilde{m}x + g_2 \sinh \tilde{m}x$. From here we write, in the matrix notation,

$$\begin{bmatrix} \zeta(x_{j-1}^0) \\ \zeta(x_j^0) \end{bmatrix} = \begin{bmatrix} \cosh \tilde{m}x_{j-1}^0 & \sinh \tilde{m}x_{j-1}^0 \\ \cosh \tilde{m}x_j^0 & \sinh \tilde{m}x_j^0 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

We denote the 2 by 2 matrix by A_L for the left of x_j^0 . To the right we have similarly

$$\begin{bmatrix} \zeta(x_j^0) \\ \zeta(x_{j+1}^0) \end{bmatrix} = \begin{bmatrix} \cosh \tilde{m}x_j^0 & \sinh \tilde{m}x_j^0 \\ \cosh \tilde{m}x_{j+1}^0 & \sinh \tilde{m}x_{j+1}^0 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix},$$

with the 2 by 2 matrix denoted by A_R . Hence

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = A_L^{-1} \begin{bmatrix} \zeta(x_{j-1}^0) \\ \zeta(x_j^0) \end{bmatrix} \text{ on } (x_{j-1}^0, x_j^0), \quad \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = A_R^{-1} \begin{bmatrix} \zeta(x_j^0) \\ \zeta(x_{j+1}^0) \end{bmatrix} \text{ on } (x_j^0, x_{j+1}^0).$$

Then

$$-\zeta'(x_j^0-) = -\tilde{m}[\sinh \tilde{m}x_j^0, \cosh \tilde{m}x_j^0]A_L^{-1} \begin{bmatrix} \zeta(x_{j-1}^0) \\ \zeta(x_j^0) \end{bmatrix},$$

$$-\zeta'(x_j^0+) = -\tilde{m}[\sinh \tilde{m}x_j^0, \cosh \tilde{m}x_j^0]A_R^{-1} \begin{bmatrix} \zeta(x_j^0) \\ \zeta(x_{j+1}^0) \end{bmatrix},$$

and

$$[-\zeta']_{x_j^0} = \tilde{m}[\sinh \tilde{m}x_j^0, \cosh \tilde{m}x_j^0] \left\{ A_L^{-1} \begin{bmatrix} \zeta(x_{j-1}^0) \\ \zeta(x_j^0) \end{bmatrix} - A_R^{-1} \begin{bmatrix} \zeta(x_j^0) \\ \zeta(x_{j+1}^0) \end{bmatrix} \right\}.$$

We also compute

$$A_L^{-1} = \frac{1}{\sinh \tilde{m}(x_j^0 - x_{j-1}^0)} \begin{bmatrix} \sinh \tilde{m}x_j^0 & -\sinh \tilde{m}x_{j-1}^0 \\ -\cosh \tilde{m}x_j^0 & \cosh \tilde{m}x_{j-1}^0 \end{bmatrix},$$

$$A_R^{-1} = \frac{1}{\sinh \tilde{m}(x_{j+1}^0 - x_j^0)} \begin{bmatrix} \sinh \tilde{m}x_{j+1}^0 & -\sinh \tilde{m}x_j^0 \\ -\cosh \tilde{m}x_{j+1}^0 & \cosh \tilde{m}x_j^0 \end{bmatrix}.$$

Thus T is a triangular matrix. The three entries of the j th row where $j \neq 1, K$ are

$$\begin{aligned} & -\tilde{m}\text{csch } \tilde{m}(x_j^0 - x_{j-1}^0), \quad \tilde{m} \coth \tilde{m}(x_j^0 - x_{j-1}^0) + \tilde{m} \coth \tilde{m}(x_{j+1}^0 - x_j^0), \\ & -\tilde{m}\text{csch } \tilde{m}(x_{j+1}^0 - x_j^0). \end{aligned}$$

For the first row,

$$\zeta(x) = \frac{\zeta(x_1^0)}{\cosh \tilde{m}x_1^0} \cosh \tilde{m}x$$

and

$$[-\zeta']_{x_1^0} = \tilde{m}(\tanh \tilde{m}x_1^0 + \coth \tilde{m}(x_2^0 - x_1^0))\zeta(x_1^0) - \tilde{m}\operatorname{csch} \tilde{m}(x_2^0 - x_1^0)\zeta(x_2^0).$$

When $K = 2$ the matrix T is

$$\begin{aligned} & \tilde{m} \begin{bmatrix} \tanh \tilde{m}a/2 + \coth \tilde{m}b & -\operatorname{csch} \tilde{m}b \\ -\operatorname{csch} \tilde{m}b & \tanh \tilde{m}b/2 + \coth \tilde{m}a \end{bmatrix} \\ &= \tilde{m}(\coth \tilde{m}a + \coth \tilde{m}b)I_{K \times K} - \tilde{m} \begin{bmatrix} \operatorname{csch} \tilde{m}a & \operatorname{csch} \tilde{m}b \\ \operatorname{csch} \tilde{m}b & \operatorname{csch} \tilde{m}a \end{bmatrix}. \end{aligned}$$

The two $(1/\Lambda)$'s are

$$\begin{aligned} & \tilde{m}(\coth \tilde{m}a + \coth \tilde{m}b - \operatorname{csch} \tilde{m}a - \operatorname{csch} \tilde{m}b), \\ & \tilde{m}(\coth \tilde{m}a + \coth \tilde{m}b - \operatorname{csch} \tilde{m}a + \operatorname{csch} \tilde{m}b), \end{aligned}$$

which again lead to η for (6.7). To see the sign of λ_m , we take the smaller Λ so that the smaller λ_m satisfies

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_m}{\epsilon^2} = \eta + m^2\pi^2 = \frac{1}{\tau} \left(-\frac{\gamma ab}{2} + \frac{\gamma}{\tilde{m}(\coth \tilde{m}a + \coth \tilde{m}b - \operatorname{csch} \tilde{m}a + \operatorname{csch} \tilde{m}b)} \right) + m^2\pi^2.$$

As in the $K = 1$ case the right side is positive for all $m \neq (0, 0)$ if γ is small, and is negative for some m if γ is large.

When $K \geq 3$ we write $T = dI_{K \times K} - Q$ with

$$Q = \begin{bmatrix} \alpha & \beta & & & \\ \beta & 0 & \alpha & & \\ & \alpha & 0 & \beta & \\ & & \beta & 0 & \alpha \\ & & & & \dots \end{bmatrix},$$

where

$$\alpha = \tilde{m}\operatorname{csch} \frac{2\tilde{m}a}{K}, \quad \beta = \tilde{m}\operatorname{csch} \frac{2\tilde{m}b}{K}, \quad d = \tilde{m} \coth \frac{2\tilde{m}a}{K} + \tilde{m} \coth \frac{2\tilde{m}b}{K}.$$

Because of diagonal domination the matrix $dI_{K \times K} - Q$ is positive definite. The K eigenvalues of Q are found in (B.5) of Appendix B. We again denote them by q_j , $j = 1, 2, \dots, K$. Then $\Lambda = \frac{1}{d - q_j}$, and for (6.7), K eigenpairs

$$(7.2) \quad \eta = \frac{1}{\tau} \left(-\frac{\gamma ab}{K} + \frac{\gamma}{d - q_j} \right), \quad c^0 = \vec{\zeta}$$

are found. To see the sign of λ_m , we focus on the smallest η , which is associated with $q_j = -\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta}$, where $\theta = 2\pi/K$. For this q_j

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{\lambda_m}{\epsilon^2} &= \eta + m^2\pi^2 = \frac{1}{\tau} \left(-\frac{\gamma ab}{K} + \frac{\gamma}{d - q_j} \right) + m^2\pi^2 \\ &= \frac{1}{\tau} \left(-\frac{\gamma ab}{K} + \frac{\gamma}{d + \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta}} \right) + m^2\pi^2. \end{aligned}$$

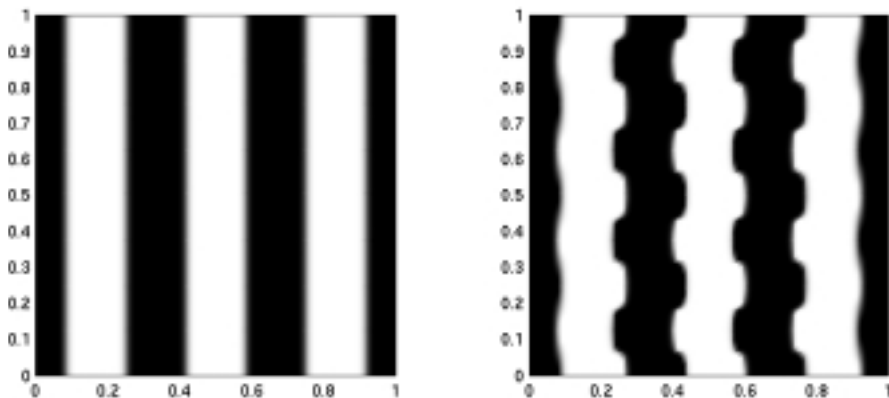


FIG. 4. A lamellar solution u and its deformation in a cross section perpendicular to y_2 .

The dependence of the positivity of the right side on γ is still the same; i.e., the right side is positive for all m if γ is small and negative for some m if γ is large. The dependence of the positivity on K is also clear. When K is large, $m^2\pi^2$ dominates the negative term, so the whole quantity is positive for all m .

Remark 7.1. If γ and K are held fixed, then the last line is positive if $|m|$ is sufficiently large. This is consistent with statement 1 of Theorem 1.1.

We omit the proof that L_m has exactly K simple eigenpairs with small eigenvalues, because it is similar to that for L_0 . This concludes the proof of statement 3, Theorem 1.1.

To visualize statement 3, Theorem 1.1, consider the example $a = 0.4$ and $K = 6$. We study $m = (8, 0)$ and find the c^0 associated with the smallest $\lambda_{(8,0)}$ by numerically diagonalizing Q :

$$c^0 = (0.0424, -0.4774, 0.5199, -0.5199, 0.4774, -0.0424)^T.$$

The eigenfunction of L associated with this $\lambda_{(8,0)}$ and with c^0 is approximately $\sum_j c_j^0 h_j(x) \cos(8\pi y_1)$. When γ is sufficiently large, we have $\lambda_{(8,0)} < 0$. Then the unstable lamellar solution u may easily be deformed in the direction of this eigenfunction. In Figure 4 we make a cross section of D , perpendicular to the y_2 direction. The first plot shows u on this cross section, where the black color indicates $u \approx 1$ and the white color indicates $u \approx 0$. The second plot shows u deformed by the eigenfunction. Note that under this deformation the straight interfaces in u become wiggled curves. See [7] for a heuristic argument for this change of shape.

8. The 1-D global minimizer. The integral $\int_0^1 |(v^0)'|^2 dx$ in the conclusion of Theorem 2.1 may be calculated as

$$\begin{aligned} \int_0^1 |(v^0)'|^2 dx &= K \int_0^{a/K} |(v^0)'|^2 dx + K \int_{a/K}^{1/K} |(v^0)'|^2 dx \\ &= K \int_0^{a/K} (1-a)^2 x^2 dx + K \int_{a/K}^{1/K} a^2 \left(x - \frac{1}{K}\right)^2 dx \\ &= \frac{a^3 b^2}{3K^2} + \frac{a^2 b^3}{3K^2} = \frac{a^2 b^2}{3K^2}. \end{aligned}$$

Hence

$$(8.1) \quad \lim_{\epsilon \rightarrow 0} \epsilon^{-1} I_1(u) = \tau K + \frac{\gamma a^2 b^2}{6K^2}.$$

It was shown in [13] that the 1-D global minimizer is a 1-D local minimizer whose number of interfaces K_* minimizes the right side of (8.1). Note that in some less likely cases two integers K_* and $K_* + 1$ may both minimize the right side of (8.1). Then we may have two global minimizers with K_* and $K_* + 1$ interfaces, respectively.⁵ If we pretend that K is a positive real number and minimize the right side with respect to K , then the minimum is achieved at

$$(8.2) \quad K_* = \left(\frac{a^2 b^2 \gamma}{3\tau} \right)^{1/3}.$$

We set $t = \tilde{m}/K$. Consider the eigenvalue λ_m that is associated with the smallest η of section 7.

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{\lambda_m}{\epsilon^2} \frac{\tau \tilde{m}}{\gamma} = \frac{\tau \tilde{m}(\eta + m^2 \pi^2)}{\gamma} \\ = & -abt + \frac{1}{\coth 2at + \coth 2bt + \sqrt{(\operatorname{csch} 2at)^2 + (\operatorname{csch} 2bt)^2 + 2\operatorname{csch} 2at \operatorname{csch} 2bt \cos \theta}} + \frac{\tau K^3 t^3}{\gamma}. \end{aligned}$$

For the 1-D global minimizer, we use K_* in (8.2) for K to find

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \frac{\lambda_m}{\epsilon^2} \frac{\tau \tilde{m}}{\gamma} = \frac{\tau \tilde{m}(\eta + m^2 \pi^2)}{\gamma} \\ = & -abt + \frac{1}{\coth 2at + \coth 2bt + \sqrt{(\operatorname{csch} 2at)^2 + (\operatorname{csch} 2bt)^2 + 2\operatorname{csch} 2at \operatorname{csch} 2bt \cos \theta}} + \frac{a^2 b^2 t^3}{3}. \end{aligned} \quad (8.3)$$

Note that a natural lower bound for the second term in (8.3) is

$$\begin{aligned} & \frac{1}{\coth 2at + \coth 2bt + \operatorname{csch} 2at + \operatorname{csch} 2bt} \\ = & \frac{\sinh at \sinh bt}{\sinh t} = abt - \frac{a^2 b^2 t^3}{3} + \frac{(a^2 b^2 + 2a^3 b^3) t^5}{45} + \dots \end{aligned}$$

by replacing $\cos \theta$ by 1. This lower bound is sharp if K_* is large, i.e., γ is large. The first three terms of the Taylor expansion are given. We observe that the first two terms in the Taylor expansion are *exactly* canceled by the first and third terms in (8.3). This is certainly no coincidence. The fifth order term is positive. Our numerical tests confirm that all of (8.3) remains positive. The particular K_* of (8.2) is barely large enough to overcome the negative third order term in $\frac{\sinh at \sinh bt}{\sinh t}$.

To contemplate the physical significance of the shaky stability property of the 1-D global minimizer, we first note that the value (8.2) for K_* is only approximate. But the 1-D global minimizer is very close to the borderline of 3-D stability. 1-D local minimizers with larger K are likely to be stable in three dimensions, and 1-D local minimizers with smaller K are likely to be unstable in three dimensions. In the real

⁵Actually there are four global minimizers because of Remark 2.2 if we include solutions of both types.

physical system only the 3-D global minimizer, which is unlikely to be lamellar, is the thermal equilibrium. Other stable solutions of (1.2) are only transient, metastable states. In general lamellar phases, including the 1-D global minimizer, are transient. They are vulnerable to perturbations of the form $\sum_j c_j^0 h_j(x) \cos(m_1 \pi y_1) \cos(m_2 \pi y_2)$ found in statement 3 of Theorem 1.1, which push the straight interfaces in a lamellar state to a wriggled shape; see Figure 4. In a forthcoming paper [12] we will actually prove, using bifurcation analysis, that (1.2) admits wriggled solutions for some values of γ .

Appendix A. Proof of Theorem 2.3.

LEMMA A.1. *u has exactly K transition layers in the sense that there are exactly K points, x_1, x_2, \dots, x_K , in $(0, 1)$, so that $u(x_j) = 1/2$, $j = 1, 2, \dots, K$, and $\lim_{\epsilon \rightarrow 0} x_j = x_j^0$.*

The proof of this lemma is similar to that of [11, Proposition 8.2].

LEMMA A.2. *The derivative of $v = G_0[u - a]$ has exactly $K - 1$ zeros, denoted by y_1, y_2, \dots, y_{K-1} , in $(0, 1)$, such that $\lim_{\epsilon \rightarrow 0} y_j = j/K$.*

Proof. The derivative of $v^0 = G_0[u^0 - a]$ has zeros at $1/K, 2/K, \dots, (K-1)/K$. The convergence of v' to $(v^0)'$ implies that v' has exactly $K - 1$ zeros y_j with the property $\lim_{\epsilon \rightarrow 0} y_j = j/K$. \square

We set $y_0 = 0$ and $y_K = 1$. Let $l_i = y_i - y_{i-1}$, $i = 1, \dots, K$. Between two zeros of v' we integrate the equation $-v'' = u - a$ and find $\frac{1}{l_i} \int_{y_{i-1}}^{y_i} u dx = a$. This allows us to localize the energy of u on (y_{i-1}, y_i) . If we set $l_i \xi + y_{i-1} = x$, $\mathcal{U}_i(\xi) = u(x)$, and $\mathcal{V}_i(\xi) = l_i^{-2} v(x) = l_i^{-2} G_0[u - a](x)$, then $\int_0^1 \mathcal{U}_i dz = a$, $-\mathcal{V}_i'' = \mathcal{U}_i - a$, $\mathcal{V}_i'(0) = \mathcal{V}_i'(1) = 0$. More importantly,

$$\begin{aligned} I_1(u) &= \sum_{i=1}^K \int_{y_{i-1}}^{y_i} \left\{ \frac{\epsilon^2}{2} |u|^2 + \frac{\gamma \epsilon}{2} |v'|^2 + W(u) \right\} dx \\ (A.1) \quad &= \sum_{i=1}^K l_i \int_0^1 \left\{ \frac{\epsilon^2}{2l_i^2} |\mathcal{U}_i'|^2 + \frac{l_i^2 \gamma \epsilon}{2} |\mathcal{V}_i'|^2 + W(\mathcal{U}_i) \right\} d\xi = \sum_{i=1}^K l_i J_{l_i}(\mathcal{U}_i) \end{aligned}$$

if we define a new variational problem:

$$(A.2) \quad J_l(\mathcal{U}) = \int_0^1 \left\{ \frac{\epsilon^2}{2l^2} |\mathcal{U}'|^2 + \frac{l^3 \gamma \epsilon}{2l} \left| \left(-\frac{d^2}{d\xi^2} \right)^{-1/2} (\mathcal{U} - a) \right|^2 + W(\mathcal{U}) \right\} d\xi, \quad \mathcal{U} \in X_a.$$

This new J_l is similar to the original I_1 . l lies in a compact subinterval of $(0, 1)$, so we take $l \sim 1$. We consider a one-layer local minimizer \mathcal{U} that is close to \mathcal{U}^0 , which is 0 on $(0, 1-a)$ and 1 on $(1-a, 0)$. The dependence of \mathcal{U} on l and ϵ is suppressed in the notation. It is proved in Proposition 9.2 of [11] that this local minimizer is unique in an L^2 ball centered at \mathcal{U}^0 of radius δ . δ is small but independent of ϵ . Denote the transition point of \mathcal{U} by χ , i.e., $\mathcal{U}(\chi) = 1/2$. This one-layer local minimizer has the following asymptotic expansion.

LEMMA A.3. *Let $\tilde{\epsilon} = \frac{\epsilon}{l}$ and $\tilde{\gamma} = l^3 \gamma$. Then $\mathcal{U} = H(\frac{\cdot - \chi}{\tilde{\epsilon}}) + \tilde{\epsilon} \mathcal{Z}$ with $\|\mathcal{Z} - \mathcal{Z}^0\|_\infty = O(\tilde{\epsilon})$. Here $\mathcal{Z}^0 = -\frac{\tilde{\gamma}(\mathcal{V}^0 - \mathcal{V}^0(1-a))}{f'(0)}$, $\mathcal{V}^0 = G_0[\mathcal{U}^0 - a]$. Note that $\mathcal{Z}^0(1-a) = 0$.*

Proof. See Proposition 8.3 in [11]. \square

LEMMA A.4. *Let $F \in C^2(-\infty, \infty)$ be such that $F(0) = F(1) = 0$. Then*

$$\int_0^1 F(\mathcal{U}) d\xi = \tilde{\epsilon} \int_{-\infty}^{\infty} F(H) dt + \tilde{\epsilon} \int_0^{1-a} F'(0) \mathcal{Z}^0 d\xi + \tilde{\epsilon} \int_{1-a}^1 F'(1) \mathcal{Z}^0 d\xi + O(\tilde{\epsilon}^2).$$

Proof. See Lemma 8.4 in [11]. \square

LEMMA A.5. Let $\mathcal{W} = \frac{\partial \mathcal{U}}{\partial l}$. Then

$$\mathcal{W}(\xi) = H' \left(\frac{l(\xi - \chi)}{\epsilon} \right) \frac{\xi - \chi}{\epsilon} - \overline{H' \left(\frac{l(\xi - \chi)}{\epsilon} \right) \frac{\xi - \chi}{\epsilon}} + \varphi,$$

with $\|\varphi\|_2 = O(1)$. And $\varphi = c(h - \bar{h}) + \psi$, with $h = H'(\frac{l(\xi - \chi)}{\epsilon})$, $h - \bar{h} \perp \psi$, $c = O(\epsilon^{-1/2})$, and $\|\psi\|_2 = O(\epsilon)$.

Proof. The brief argument here summarizes the more elaborate proof of the similar Proposition 9.3 in [11]. Differentiate the Euler–Lagrange equation of (A.2) with respect to l to obtain

$$(A.3) \quad - \left(\frac{\epsilon}{l} \right)^2 \mathcal{W}'' + \gamma \epsilon l^2 G_0[\mathcal{W}] + f'(\mathcal{U})\mathcal{W} + 4\gamma \epsilon l G_0[u - a] + \frac{2}{l} f(\mathcal{U}) - \frac{2}{l} \overline{f(\mathcal{U})} = \frac{df(\mathcal{U})}{dl}$$

for \mathcal{W} . Define $g(\xi) = H'(\frac{l(\xi - \chi)}{\epsilon}) \frac{\xi - \chi}{\epsilon} + \text{e.s.}$ so that g and its derivative vanish at 0 and 1, and $\varphi = \mathcal{W} - (g - \bar{g})$. Here g satisfies the equation

$$- \left(\frac{\epsilon}{l} \right)^2 g'' + f'(H)g + \frac{2}{l} f(H) = \text{e.s.}$$

Subtract this from (A.3) and use the facts $\|g\|_2 = O(\epsilon^{1/2})$, $\bar{g} = O(\epsilon)$, and $(f'(\mathcal{U}) - f'(H))(g - \bar{g}) = O(\epsilon^2)$, where the last one follows from Lemma A.3, to deduce the equation for φ :

$$(A.4) \quad - \left(\frac{\epsilon}{l} \right)^2 \varphi'' + \epsilon \gamma l^2 G_0[\varphi] + f'(\mathcal{U})\varphi + O(\epsilon) = \text{Const.},$$

where we simply write Const. for a constant since its exact value is not needed in this proof. We multiply this equation by φ and integrate:

$$\int_0^1 \left\{ \left(\frac{\epsilon}{l} \right)^2 |\varphi'|^2 + \epsilon \gamma l^2 G_0[\varphi]\varphi + f'(\mathcal{U})\varphi^2 \right\} dz = O(\epsilon) \|\varphi\|_2.$$

By Proposition 9.1 in [11] we find $\|\varphi\|_2 = O(1)$.

Decompose $\varphi = c(h - \bar{h}) + \psi$, where $h = H'(\frac{l(\xi - \chi)}{\epsilon}) + \text{e.s.}$ and $h - \bar{h} \perp \psi$. The exponentially small correction term e.s. is added so that h and h' vanish at 0 and 1. Then

$$c = \frac{\int_0^1 \varphi(h - \bar{h}) dz}{\|h - \bar{h}\|_2^2} \leq \frac{\|\varphi\|_2}{\|h - \bar{h}\|_2} = O(\epsilon^{-1/2}).$$

The equation satisfied by ψ is

$$- \left(\frac{\epsilon}{l} \right)^2 \psi'' + f'(\mathcal{U})\psi + O(\epsilon) = \text{Const.},$$

where we have used the fact $(f(\mathcal{U}) - f(H))h = O(\epsilon^2)$, again a consequence of Lemma A.3. Argue as in Lemma 4.1 to deduce $\int_0^1 \{ -(\frac{\epsilon}{l})^2 \psi'' + f'(\mathcal{U})\psi \} \psi d\xi \geq C \|\psi\|_2^2$, which implies $\|\psi\|_2 = O(\epsilon)$. \square

LEMMA A.6. Let $E(l) = lJ_l(\mathcal{U})$. Then $E(l)$ is strictly convex in l in any compact subset of $(0, 1)$.

Proof. This lemma is similar to Proposition 10.1 in [11]. Differentiating E with respect to l yields

$$(A.5) \quad \frac{\partial E}{\partial l} = \int_0^1 \left\{ -\frac{\epsilon^2}{2l^2} |\mathcal{U}'|^2 + W(\mathcal{U}) + \frac{3\epsilon\gamma l^2}{2} |\mathcal{V}'|^2 \right\} d\xi,$$

where $\mathcal{V} = G_0[\mathcal{U} - a]$. We have used the fact that \mathcal{U} is a critical point of J_l . Differentiate (A.5) with respect to l :

$$\frac{\partial^2 E}{\partial l^2} = \int_0^1 \left\{ \frac{\epsilon^2}{l^3} |\mathcal{U}'|^2 + 3\epsilon\gamma l |\mathcal{V}'|^2 \right\} d\xi + \int_0^1 \{2f(\mathcal{U})\mathcal{W} + 4\epsilon\gamma l^2 \mathcal{V}\mathcal{W}\} d\xi.$$

Call the first integral on the right side T_1 and the second integral T_2 . Multiplying the Euler–Lagrange equation of \mathcal{U} by $\mathcal{U} - a$ and integrating by parts, we find the useful integral identity

$$\int_0^1 \left\{ \left(\frac{\epsilon}{l}\right)^2 |\mathcal{U}'|^2 + f(\mathcal{U})(\mathcal{U} - a) + \epsilon\gamma l^2 |\mathcal{V}'|^2 \right\} d\xi = 0.$$

Using this identity and Lemma A.4, we obtain

$$\begin{aligned} T_1 &= \frac{1}{l} \int_0^1 \{-f(\mathcal{U})(\mathcal{U} - a) + 2\epsilon\gamma l^2 |\mathcal{V}'|^2\} d\xi \\ &= \frac{\epsilon}{l^2} \int_{-\infty}^{\infty} -f(H)(H - a) dt + \frac{\epsilon\gamma l a^2 b^2}{3} + \frac{2\epsilon\gamma l a^2 b^2}{3} + O(\epsilon^2) \\ &= \frac{\epsilon}{l^2} \int_{-\infty}^{\infty} -f(H)H dt + \epsilon\gamma l a^2 b^2 + O(\epsilon^2). \end{aligned}$$

Here we have used

$$\int_0^1 |\mathcal{V}'|^2 d\xi = \int_0^1 |\mathcal{V}'_0|^2 d\xi + O(\epsilon) = \frac{a^2 b^2}{3} + O(\epsilon),$$

which follows from (8.17) in [11]. By Lemmas A.3 and A.5

$$\begin{aligned} T_2 &= \int_0^1 (2f(H) + O(\epsilon)) \left(\frac{\xi - \chi}{\epsilon} H' + cH' - c\overline{H}' + \psi \right) d\xi \\ &= \frac{\epsilon}{l^2} \int_{-\chi/\epsilon}^{l(1-\chi)/\epsilon} 2f(H(t))H'(t)t dt + O(\epsilon^{1.5}) = \frac{\epsilon}{l^2} \int_{-\infty}^{\infty} -2W(H) dt + O(\epsilon^{1.5}). \end{aligned}$$

We have used the estimates

$$\begin{aligned} \int_0^1 \left| \frac{\xi - \chi}{\epsilon} H' \right| d\xi &= \frac{\epsilon}{l^2} \int_{-\chi/\epsilon}^{l(1-\chi)/\epsilon} |H'(t)t| dt = O(\epsilon), \\ \int_0^1 |f(H)| d\xi &= \frac{\epsilon}{l} \int_{-\chi/\epsilon}^{l(1-\chi)/\epsilon} |f(H(t))| dt = O(\epsilon), \\ \|2f(H) + O(\epsilon)\|_2 &= O(\epsilon^{1/2}), \\ \int_0^1 f(H)H' d\xi &= \text{e.s.} \end{aligned}$$

Adding T_1 and T_2 , since $\int_{-\infty}^{\infty} (f(H)H + 2W(H)) dt = 0$ (a consequence of the integral identity $\int_{-\infty}^{\infty} \{(H')^2 + f(H)H\} dt = 0$ and the first integral of H), we arrive at

$$(A.6) \quad \frac{\partial^2 E}{\partial l^2} = \epsilon l \gamma a^2 b^2 + O(\epsilon^{1.5}),$$

proving the lemma. \square

Proof of Theorem 2.3. We construct a particular periodic solution u_* with K transition layers and show that $u = u_*$. Let \mathcal{U} be the unique minimum of J_l in a δ neighborhood of \mathcal{U}^0 , with $l = 1/K$, and let $\mathcal{U}^R = \mathcal{U}(1 - \cdot)$ be its reversal. Set $u_*(x) = \mathcal{U}^R(Kx)$ for $x \in (0, 1/K)$. Extend u_* antiperiodically to $(0, 1)$, i.e., $u_*(x) = \mathcal{U}(Kx - 1)$ for $x \in (1/K, 2/K)$, $u_*(x) = \mathcal{U}^R(Kx - 2)$ for $x \in (2/K, 3/K)$, \dots . Clearly u_* is periodic with $K/2$ periods.

For small ϵ , u and u_* belong to the same small L^2 neighborhood in which u is a minimizer. Using the strict convexity of E in Lemma A.6 and (A.1), we find

$$I(u_*) \geq I(u) = \sum_{i=1}^K l_i J_{l_i}(u(l_i \cdot + y_{i-1})) \geq \sum_{i=1}^K E(l_i) \geq KE \left(\frac{1}{K} \right) = I(u_*).$$

All the inequalities above must be equalities. Therefore $l_i = 1/K$, $y_i = i/K$ for all i , and $l_i J_{l_i}(u(l_i \cdot + y_{i-1})) = E(l_i)$. Moreover $u((1/K) \cdot + y_{i-1}) = \mathcal{U}$ when i is even or $= \mathcal{U}^R$ when i is odd by the local uniqueness of \mathcal{U} and \mathcal{U}^R [11, Proposition 9.2]. Thus $u = u_*$. \square

Appendix B. The matrix Q . Consider a matrix Q like those in sections 5 and 7 with $\alpha, \beta > 0$. In this appendix Q , whose size is at least 3 by 3, acts on the complex vector space \mathbf{C}^K . Let $\vec{q} = (z, tz^2, z^3, tz^4, \dots)^T$, where $z, t \in \mathbf{C}$ and $|z| = 1$. Suppose the eigenvalue problem $Q\vec{q} = q\vec{q}$ holds for the second through the next-to-last equations, excluding the first and the last. In these $K - 2$ equations

$$(B.1) \quad \begin{cases} \alpha tz^{l-1} + \beta tz^{l+1} = qz^l & \text{if } l \text{ is odd,} \\ \beta z^{l-1} + \alpha z^{l+1} = qtz^l & \text{if } l \text{ is even.} \end{cases}$$

They imply

$$(B.2) \quad t = \pm \frac{\alpha z + \beta \bar{z}}{|\alpha z + \beta \bar{z}|}.$$

In particular $|t| = 1$. In order to have the first and the last equations satisfied, we let $\vec{h} = A\vec{q} + B\vec{q}$ and study $Q\vec{h} = q\vec{h}$.

If the vector \vec{q} is extended by t as the 0th entry and by tz^{K+1} as the $(K + 1)$ th entry if K is odd, or by z^{K+1} as the $(K + 1)$ th entry if K is even, then the first and the last equations of $Q\vec{h} = q\vec{h}$ are satisfied if the 0th entry is equal to the first entry and the K th entry is equal to the $(K + 1)$ th entry. That is,

$$(B.3) \quad \begin{cases} Az + B\bar{z} = At + B\bar{t} \\ Az^K + B\bar{z}^K = Atz^{K+1} + B\bar{t}\bar{z}^{K+1} \end{cases} \quad \text{if } K \text{ is odd;}$$

$$(B.4) \quad \begin{cases} Az + B\bar{z} = At + B\bar{t} \\ Atz^K + B\bar{t}\bar{z}^K = Az^{K+1} + B\bar{z}^{K+1} \end{cases} \quad \text{if } K \text{ is even.}$$

They should have nontrivial solutions for A and B . In the case of (B.3) this means

$$(z - t)(1 - \bar{t}\bar{z})\bar{z}^K = (\bar{z} - \bar{t})(1 - tz)z^K$$

or, since $|z| = |t| = 1$, $z^{2K} = 1$. The case (B.4) gives the same condition. Define $\theta = \frac{2\pi(j-1)}{K}$, $j = 1, 2, \dots, 2K$. Then $z = e^{i\theta/2}$. From (B.1) we find

$$q = \alpha t\bar{z} + \beta tz = \pm\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta}.$$

Here θ ranges from 0 to $4\pi - (2\pi/K)$, which is too wide a range. We restrict j to $1, 2, \dots, (K+1)/2$ if K is odd and $j = 1, 2, \dots, K/2 + 1$ if K is even. Even then we have some extra values. When $z = 1$ and $t = -1$, which occur if $\theta = 0$ and $q = -(\alpha + \beta)$, we find $A + B = 0$ and $\vec{h} = \vec{0}$, which is not an eigenvector. Also when K is even, $z = i$, and $t = -i$, which occur if $\theta = \pi$, and $q = \beta - \alpha$, we find $A - B = 0$ and again $\vec{h} = \vec{0}$. In summary the K distinct eigenvalues of Q are

$$\alpha + \beta, \pm\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta} \quad \left(\theta = \frac{2\pi(j-1)}{K}, j = 2, 3, \dots, \frac{K+1}{2} \right) \quad \text{if } K \text{ is odd;}$$

(B.5)

$$\alpha + \beta, \pm\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta} \quad \left(\theta = \frac{2\pi(j-1)}{K}, j = 2, 3, \dots, \frac{K}{2} \right), \alpha - \beta \quad \text{if } K \text{ is even.}$$

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