

# On the Spectra of 3-D Lamellar Solutions of the Diblock Copolymer Problem \*

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## Abstract

One-dimensional free energy local minimizers are viewed as three-dimensional lamellar type critical points in a box. To determine whether they model the lamellar phase of diblock copolymers in the strong segregation region, we analyze their spectra. We obtain the asymptotic expansions of their eigenvalues and eigenfunctions. Consequently we find that they are stable, i.e. are local minimizers in space, only if they have sufficiently many interfaces. Interestingly the 1-D global minimizer is near the borderline of 3-D stability.

Key words. spectrum, 3-D stability, lamellar solution, diblock copolymer  
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## 1 Introduction

In a di-block copolymer melt a molecule is a linear-chain consisting of two sub-chains grafted covalently to each other. The first sub-chain has  $N_A$  type  $A$  monomer units and the second sub-chain has  $N_B$  type  $B$  monomer units. In polymer systems even a weak repulsion between unlike monomers  $A$  and  $B$  induces a strong repulsion between sub-chains. With many chain molecules in a polymer melt the different type sub-chains tend to segregate below some critical temperature, but as they are chemically bonded in chain molecules, even a complete segregation of sub-chains cannot lead to a macroscopic phase separation. Only a local micro-phase separation occurs: micro-domains rich in  $A$  and  $B$  are formed. These micro-domains form morphological patterns/phases in a larger scale. The commonly observed phases include the spherical, cylindrical and lamellar, depicted in Figure 1.

We consider a scenario that a diblock copolymer melt is placed in a domain  $D$  and maintained at fixed temperature.  $D$  is scaled to have unit volume in space. Let  $a = N_A/(N_A + N_B) \in (0, 1)$  be the relative number of the  $A$  monomers in a chain molecule. Similarly  $b = N_B/(N_A + N_B)$ , so  $a + b = 1$ . The relative  $A$  monomer density field  $u$  is an order parameter.  $u \approx 1$  stands for high concentration

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\*Abbreviated title. Spectra of Lamellar Solutions

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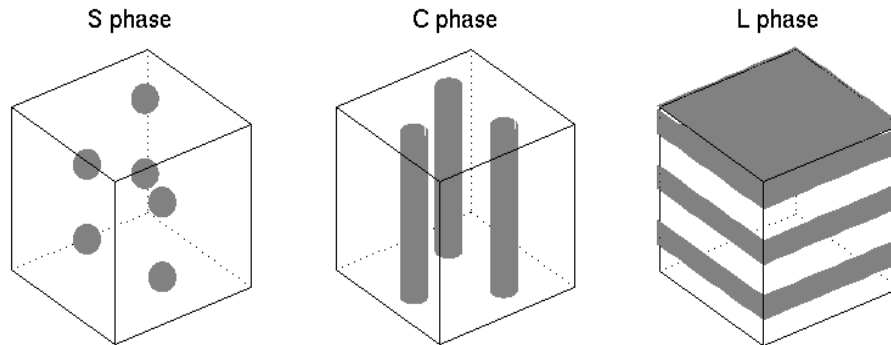


Figure 1: The spherical, cylindrical, and lamellar morphology phases commonly observed in diblock copolymer melts. The dark color indicates the concentration of type  $A$  monomer, and the white color indicates the concentration of type  $B$  monomer.

of  $A$  monomers. The melt is incompressible so the relative  $B$  monomer density is  $1 - u$  and  $u \approx 0$  stands for high concentration of  $B$  monomers.

Ohta and Kawasaki [10] introduced an equilibrium theory, in which the free energy of the system is a functional of the relative  $A$  monomer density:

$$I(u) = \int_D \left\{ \frac{\epsilon^2}{2} |\nabla u|^2 + \frac{\sigma}{2} |(-\Delta)^{-1/2}(u - a)|^2 + W(u) \right\}, \quad (1.1)$$

defined in  $X_a = \{u \in W^{1,2}(D) : \bar{u} = a\}$ , where  $\bar{u} := \frac{1}{|D|} \int_D u$  is the average of  $u$  on  $D$ . The original formula in [10] is given for the whole space. The expression here on a bounded domain  $D$  first appeared in Nishiura and Ohnishi [8].

$\epsilon$  and  $\sigma$  are positive dimensionless parameters that depend on various physical quantities such as  $N_A$ ,  $N_B$ , the average distance between two adjacent monomers in a chain, the interaction between monomers, the temperature, the size of the sample. In the strong segregation region where morphology patterns form,  $\epsilon$  is very small. The size of  $\sigma$  in this paper is chosen to be of order  $\epsilon$ , i.e. there is a fixed positive constant  $\gamma$  so that  $\sigma = \epsilon\gamma$ . This particular parameter range is realized if we take the sample size to be of the  $(N_A + N_B)^{2/3}$  order<sup>1</sup>.

The local function  $W$  is smooth and has the shape of a double well, as depicted in Figure 2. It has the global minimum value 0 at two numbers: 0 and 1. To avoid unnecessary technical difficulties we assume that  $W(p) = W(1 - p)$ . The two global minimum points are non-degenerate, i.e.  $W''(0) = W''(1) \neq 0$ . A simple example is  $W(u) = \frac{1}{4}((u - \frac{1}{2})^2 - \frac{1}{4})^2$ .

The most mathematically interesting part in (1.1) is the nonlocal term  $(-\Delta)^{-1/2}(u - a)$  in the integrand. Let  $(-\Delta)^{-1}(u - a)$  be the solution  $v$  of

$$-\Delta v = u - a \text{ in } D, \quad \partial_\nu v = 0 \text{ on } \partial D, \quad \bar{v} = 0$$

<sup>1</sup>See Choksi and Ren [3] for more on these parameters.

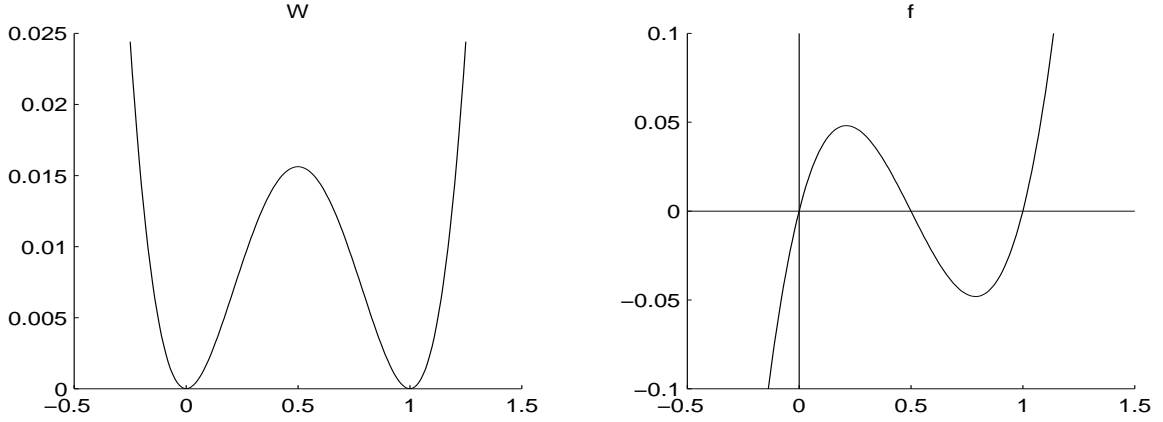


Figure 2: The graphs of  $W$  and  $f = W'$ .

where  $\partial_\nu v$  is the outward normal derivative of  $v$ .  $(-\Delta)^{-1/2}$  in  $I$  is the square root of the positive operator  $(-\Delta)^{-1}$  from  $\{w \in L^2(D) : \bar{w} = 0\}$  to itself. If we let  $v = (-\Delta)^{-1}(u - a)$ , then an often more useful formula is

$$I(u) = \int_D \left\{ \frac{\epsilon^2}{2} |\nabla u|^2 + \frac{\epsilon\gamma}{2} |\nabla v|^2 + W(u) \right\}.$$

Let  $f(u) = W'(u)$  as in Figure 2. For the particular  $W(u) = \frac{1}{4}((u - \frac{1}{2})^2 - \frac{1}{4})^2$ ,  $f(u) = u(u - \frac{1}{2})(u - 1)$ . The Euler-Lagrange equation of  $I$  is

$$-\epsilon^2 \Delta u + \epsilon\gamma(-\Delta)^{-1}(u - a) + f(u) - \overline{f(u)} = 0, \quad \partial_\nu u = 0 \text{ on } \partial D. \quad (1.2)$$

The term  $\overline{f(u)}$  is equal to the Lagrange multiplier corresponding to the constraint  $\bar{u} = a$ .

It is proved in Ren and Wei [12] using the  $\Gamma$ -limit theory that when  $D = (0, 1)$  for any positive integer  $K$  there exists a local minimizer  $u$  with  $K$  interfaces and  $K + 1$  micro-domains if  $\epsilon$  is small enough<sup>2</sup>. An example of  $u$  with  $K = 6$  is shown in Figure 3.  $u$  is close to 0 in 3 regions and close to 1 in 4 regions. These regions are separated by sharp interfaces. That  $u$  is energetically favored is not too difficult to explain. Note that the  $W$  term in  $I$  likes to have  $u \approx 0$  or  $u \approx 1$ . The gradient term penalizes oscillation of  $u$ , but since it has a small coefficient it tolerates a number of sharp interfaces. The best profile for the nonlocal term is  $u \approx a$ . But this is impossible due to the presence of the  $W$  term and the fact  $0 < a < 1$ . The second best profile for the nonlocal term is for  $u$  to have wild oscillation about  $a$ . When all the three terms are present in  $I$ , a compromise must be reached and  $u$  as in Figure 3 emerges as a local minimizer.

Now we place such a 1-D local minimizer in a 3-D box through trivial extension. The extended  $u$  becomes a 3-D critical point of  $I$ , i.e. a solution of (1.2). We ask whether this  $u$  is a good model of the lamellar phase depicted in plot 3 of Figure 1. In general a morphology phase must be at least *meta-stable* in the sense that it is described by a local minimizer of  $I$  in space. Such a 3-D local minimizer is also called a stable solution of (1.2). We take  $D = (0, 1) \times (0, 1) \times (0, 1)$  and study the spectrum of  $u$ , i.e. the second variation of  $I$  at  $u$ . The linearized operator at  $u$  is

$$L(\phi) := -\epsilon^2 \Delta \phi + \epsilon\gamma(-\Delta)^{-1}\phi + f'(u)\phi - \overline{f'(u)\phi}, \quad \partial_\nu \phi = 0 \text{ on } \partial D, \quad \phi \in W^{2,2}(D), \quad \bar{\phi} = 0. \quad (1.3)$$

<sup>2</sup>See Theorem 2.1.

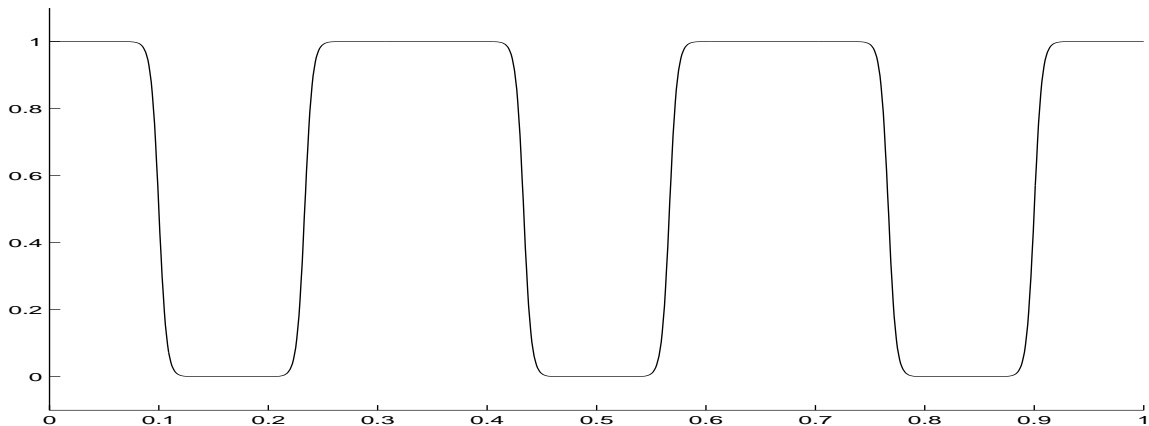


Figure 3: A 1-D local minimizer  $u$  with  $K = 6$ . The regions where  $u$  is close to 1 are micro-domains with high concentration of  $A$  monomers and the regions where  $u$  is close to 0 are micro-domains with high concentration of  $B$  monomers.

This is an unbounded self-adjoint operator defined densely on  $\{\phi \in L^2(D) : \bar{\phi} = 0\}$  whose spectrum consists of eigenvalues only.

We will obtain detailed information on the spectrum of  $u$  when  $\epsilon$  is small. In particular we will find the asymptotic expansions of the important eigenvalues of small absolute values in terms of  $\epsilon$ . We will also derive asymptotic expansions of the corresponding eigenfunctions. The analysis in this paper culminates in the following theorem.

**Theorem 1.1** *The eigenvalues  $\lambda$  of  $L$  are classified into  $\lambda_m$  by  $m = (m_1, m_2)$  which is a pair of non-negative integers. The following 3 statements hold when  $\epsilon$  is sufficiently small.*

1. *There exists  $M(K)$  depending on  $K$  but not  $\epsilon$  so that when  $|m| := \sqrt{m_1^2 + m_2^2} \geq M(K)$ ,  $\lambda_m \geq C\epsilon^2$  for some  $C > 0$  independent of  $\epsilon$ .*
2. *When  $m = (0, 0)$ , there are  $K$  small positive  $\lambda_{(0,0)}$ 's. One of them is of order  $\epsilon$  whose only eigenfunction is approximately  $\sum_j (h_j(x) - \bar{h}_j)$ . The other  $K - 1$   $\lambda_{(0,0)}$ 's are of order  $\epsilon^2$ . Their only eigenfunctions are approximately  $\sum_j c_j^0 h_j(x)$  for some vectors  $c^0$  satisfying  $\sum_j c_j^0 = 0$ . The remaining  $\lambda_{(0,0)}$ 's are positive and bounded below by a positive constant independent of  $\epsilon$ .*
3. *When  $m \neq (0, 0)$  and  $|m| < M(K)$ , there are  $K$   $\lambda_m$ 's of order  $\epsilon^2$ , which are not necessarily positive, whose only eigenfunctions are approximately  $\sum_j c_j^0 h_j(x) \cos(m_1 \pi y_1) \cos(m_2 \pi y_2)$ . The remaining  $\lambda_m$ 's are positive and bounded below by a positive constant independent of  $\epsilon$ . Only when  $K$  is sufficiently large or  $\gamma$  is sufficiently small, all the eigenvalues of  $L$  are positive and  $u$  is stable.*

Here a point in  $D$  is denoted by  $(x, y_1, y_2)$ , where  $x$  is in the direction perpendicular to the interfaces of a lamellar phase, the up direction in Plot 3, Figure 1. The functions  $h_j$  are defined in (3.17), and the  $c^0$  vectors are given in Sections 5 and 7. The  $\lambda_{(0,0)}$  eigenvalues are just the

eigenvalues in the 1-D problem. That they are positive, stated in Part 2, is consistent with the fact that  $u$  is a 1-D local minimizer.

The most exciting discovery is apparently Part 3. The presence of the  $\lambda_m$ 's there is a 3-D phenomenon. A 1-D local minimizer is *not necessarily* a local minimizer in 3-D. Not all 1-D local minimizers may be used to model the lamellar phase of diblock copolymers. Only the ones with sufficiently many interfaces, or in other words with sufficiently thin micro-domains, are suitable candidates.

Of particular interest is the 1-D global minimizer, which is one of the 1-D local minimizers with  $K \approx (\frac{a^2 b^2 \gamma}{3\tau})^{1/3}$ , where  $\tau$  is a positive number specified in (2.12). Since its energy is lower than that of any other 1-D local minimizer, it is thermodynamically more preferred. But if it were unstable in 3-D then the lamellar phase would only be a transient meta-stable phase. Thermal fluctuation would eventually destroy any meta-stable lamellar phase. It turns out that the 1-D global minimizer has a delicate spectral property. It actually lies near the *borderline* of the stability of lamellar solutions<sup>3</sup>.

The stability of a solution of (1.2) may also be defined by a dynamic problem. As observed in [8] one may consider negative gradient flows of  $I$  in various function spaces. The simplest one is probably

$$u_t = \epsilon^2 \Delta u - \epsilon \gamma (-\Delta)^{-1} (u - a) - f(u) + \overline{f(u)}, \quad \partial_\nu u = 0 \text{ on } \partial D \times (0, \infty). \quad (1.4)$$

A physically more realistic dynamic model is the Cahn-Hilliard [1] like fourth order problem:

$$u_t = \Delta(-\epsilon^2 \Delta u + \epsilon \gamma (-\Delta)^{-1} (u - a) + f(u)), \quad \partial_\nu \Delta u = \partial_\nu u = 0 \text{ on } \partial D \times (0, \infty). \quad (1.5)$$

The stability of steady states of (1.4) or (1.5) agrees with our static stability definition that a stable solution of (1.2) is a local minimizer of  $I$ .

Some preliminary work is done in Section 2. We derive inner and outer asymptotic expansions of the lamellar solution  $u$  in a rigorous way. The first statement of the theorem is proved in Section 3, the second in Sections 4 and 5, and the third in Sections 6 and 7. In the last section we discuss the spectrum of the 1-D global minimizer.

To avoid clumsy notations a quantity's dependence on  $\epsilon$  is usually suppressed. For example we write  $u$ , the lamellar solution, instead of  $u_\epsilon$ . On the other hand we often emphasize a quantity's independence of  $\epsilon$  with a superscript 0. For example the limit of a lamellar solution  $u$  as  $\epsilon \rightarrow 0$  is denoted by  $u^0$ . In estimates  $C$  is always a positive constant independent of  $\epsilon$ . Its value may vary from line to line. The short hand *e.s.* stands for a quantity that is exponentially small, i.e.  $= O(e^{-C/\epsilon})$ . The  $L^2$  inner product is denoted by  $\langle \cdot, \cdot \rangle$  and the  $L^p$  norm by  $\| \cdot \|_p$ .

References on the mathematical aspects of the block copolymer theory include, in addition to the ones cited already, Ohnishi *et al* [9], Choksi [2], Fife and Hilhorst [4], Henry [6], Ren and Wei [15, 13], on diblock copolymers. On triblock copolymers we refer to Ren and Wei [14, 16].

## 2 The lamellar solution $u$

The lamellar solutions we consider in this paper are constructed in [12] by the  $\Gamma$ -limit theory.

**Theorem 2.1 (Ren and Wei [12])** *In 1-D for each positive integer  $K$  the functional*

$$I_1(u) := \int_0^1 \left\{ \frac{\epsilon^2}{2} \left( \frac{du}{dx} \right)^2 + \frac{\epsilon \gamma}{2} \left| \left( -\frac{d^2}{dx^2} \right)^{-1/2} (u - a) \right|^2 + W(u) \right\} dx, \quad (2.6)$$

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<sup>3</sup>This phenomenon compares well with the *marginal* stability, observed in Muratov [7], of the corresponding 1-D global minimizer in the  $\Gamma$ -limit.

in  $\{u \in W^{1,2}(0,1) : \bar{u} = a\}$ , has a local minimizer  $u$  near  $u^0$ , under the  $L^2$  norm, when  $\epsilon$  is sufficiently small. It satisfies the Euler-Lagrange equation

$$-\epsilon^2 u'' + f(u) - \overline{f(u)} + \epsilon \gamma G_0[u - a] = 0, \quad u'(0) = u'(1) = 0,$$

and the properties

$$\lim_{\epsilon \rightarrow 0} \|u - u^0\|_2 = 0, \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \epsilon^{-1} I_1(u) = \tau K + \frac{\gamma}{2} \int_0^1 |(v^0)'| dx.$$

Here  $u^0$  is a step function defined to be

$$u^0(x) = 1 \text{ on } (0, x_1^0), 0 \text{ on } (x_1^0, x_2^0), 1 \text{ on } (x_2^0, x_3^0), 0 \text{ on } (x_3^0, x_4^0), 1 \text{ on } (x_4^0, x_5^0), \dots$$

with (recall  $b = 1 - a$ )

$$x_1^0 = \frac{a}{K}, \quad x_2^0 = \frac{1+b}{K}, \quad x_3^0 = \frac{2+a}{K}, \quad x_4^0 = \frac{3+b}{K}, \quad x_5^0 = \frac{4+a}{K}, \dots$$

$v^0 = G_0[u^0 - a]$ .  $G_0$  is the solution operator of  $-v'' = g$ ,  $v'(0) = v'(1) = 0$ ,  $\bar{v} = 0$ . The constant  $\tau$  is positive and defined in (2.12).

There is another  $K$ -interface lamellar solution whose limiting value as  $\epsilon \rightarrow 0$  is 0 instead of 1 on the first interval  $(0, b/K)$ . This solution has the same properties as  $u$  does, so we focus on  $u$ , the solution of the first type, only.

**Remark 2.2** *This second solution is just  $1 - \tilde{u}$  where  $\tilde{u}$  is a solution of the first type, but with  $\bar{\tilde{u}} = 1 - a$ .*

In this section we learn more about  $u$ . In particular  $u$  is periodic.

**Theorem 2.3** *When  $\epsilon$  is small, for every  $x \in (0, 1/K)$ ,*

$$u(x) = u\left(\frac{2}{K} - x\right) = u\left(x + \frac{2}{K}\right) = u\left(\frac{4}{K} - x\right) = u\left(x + \frac{4}{K}\right) = \dots = \begin{cases} u(1-x) & \text{if } K \text{ is even} \\ u\left(x + \frac{K-1}{K}\right) & \text{if } K \text{ is odd} \end{cases} .$$

Moreover when  $\epsilon$  is small,  $u$  is the unique local minimizer of  $I_1$  in an  $L^2$  neighborhood of  $u^0$ . If  $u$  on  $((j-1)/K, j/K)$  for some  $j = 1, 2, \dots, K$  is scaled to a function on  $(0, 1)$ , then it is exactly a one-layer local minimizer of (2.6) with  $\epsilon$  and  $\gamma$  replaced by  $\tilde{\epsilon} = \epsilon K$  and  $\tilde{\gamma} = \gamma/K^3$ .

The nuts and bolts needed to prove this theorem are available in [15]. We leave the proof in Appendix A, so the reader may skip it first in order to focus on the spectral properties of  $u$  in the following sections.

For that purpose we need asymptotic expansions of  $u$  in terms of  $\epsilon$ . By Lemma A.1 in Appendix A there exist exactly  $K$  points  $x_j$ ,  $j = 1, 2, \dots, K$ , in  $(0, 1)$  so that  $u(x_j) = 1/2$ . These  $K$  points identify the interfaces of  $u$ . Theorem 2.3 implies that  $x_2 = \frac{2}{K} - x_1$ ,  $x_3 = \frac{4}{K} - x_2$ ,  $x_4 = \frac{6}{K} - x_3$ , etc. The first approximation of  $u$  is

$$w(x) = H\left(-\frac{x-x_1}{\epsilon}\right) + H\left(\frac{x-x_2}{\epsilon}\right) + H\left(-\frac{x-x_3}{\epsilon}\right) - 1 + \dots + \begin{cases} H\left(\frac{x-x_K}{\epsilon}\right) & \text{if } K \text{ is even} \\ H\left(-\frac{x-x_K}{\epsilon}\right) - 1 & \text{if } K \text{ is odd} \end{cases} . \quad (2.7)$$

Here  $H$  is the heteroclinic solution of

$$-H'' + f(H) = 0, \quad H(-\infty) = 0, \quad H(\infty) = 1, \quad H(0) = 1/2.$$

In the case  $W(u) = \frac{1}{4}((u - \frac{1}{2})^2 - \frac{1}{4})^2$ , it is explicitly known:  $H(t) = (1/2)(\tanh \frac{t}{2\sqrt{2}} + 1)$ .  $H(t)$  converges to 1 as  $t \rightarrow \infty$  (and to 0 as  $t \rightarrow -\infty$ ) exponentially fast. Also  $H'(t)$  and  $H''(t)$  decay to 0 exponentially fast as  $t \rightarrow \pm\infty$ .  $H$ , or  $H(\cdot)$ , gives the profile of interfaces between the micro-domains of  $u$ . At every  $x \neq x_j^0$ ,  $j = 1, 2, \dots, K$ ,  $\lim_{\epsilon \rightarrow 0} w(x) = u^0(x)$ .

Next we define

$$z^0(x) = -\frac{\gamma(v^0(x) - v^0(x_j^0))}{f'(0)}. \quad (2.8)$$

Let us compute

$$(v^0)'(x) = \begin{cases} (a-1)x & \text{on } (0, x_1^0) \\ a(x - \frac{1}{K}) & \text{on } (x_1^0, x_2^0) \\ (a-1)(x - \frac{2}{K}) & \text{on } (x_2^0, x_3^0) \\ a(x - \frac{3}{K}) & \text{on } (x_3^0, x_4^0) \\ (a-1)(x - \frac{4}{K}) & \text{on } (x_4^0, x_5^0) \\ \dots & \dots \end{cases}. \quad (2.9)$$

If we integrate  $(v^0)'$  over an interval  $(x_{j-1}^0, x_j^0)$ , we get 0. So  $v^0(x_j^0)$  is independent of  $j$ , and the definition of  $z^0$  makes sense. Note that  $z^0$  is independent of  $\epsilon$ .

**Lemma 2.4** *Let  $z$  be defined by  $u = w + \epsilon z$ . Then  $\|z - z^0\|_\infty = O(\epsilon)$ .*

*Proof.* Combine Lemma A.3 in Appendix A and Theorem 2.3.  $\square$

**Lemma 2.5** *There exists a constant  $C > 0$  independent of  $\epsilon$  so that  $|\epsilon^{-1}z(x_j + \epsilon t)| \leq C(1 + |t|)$  for all  $t \in (-\frac{x_j}{\epsilon}, \frac{1-x_j}{\epsilon})$ .  $\epsilon^{-1}z(x_j + \epsilon \cdot)$  converges to  $P$  in  $C_{loc}^2(-\infty, \infty)$ , where  $P(t)$  is the solution of*

$$-P'' + f'(H)P = -\gamma(v^0)'(x_j^0)t, \quad P \perp H'$$

in  $(-\infty, \infty)$ .

There are two different  $P$ 's depending on whether  $j$  is odd or even. But they just differ by a sign, and it is always easy to tell from the context which one is referred to. Once  $j$  is given, there exists a unique  $P$  since the right side of its equation is perpendicular to the kernel  $H'$ .

*Proof.* Without the loss of generality we assume that  $j$  is even. Define  $Z(t) = z(x_j + \epsilon t)$ . Lemma 2.4 implies  $Z = O(1)$  and hence, with the help of Lemma A.4,  $\overline{f(u)} = O(\epsilon)$ . From the 1-D Euler-Lagrange equation in Theorem 2.1, which  $u$  satisfies, we find the equation for  $Z$ :

$$-Z'' + f'(H)Z + O(\epsilon)Z^2 + \gamma G_0[u - a] - \epsilon^{-1}\overline{f(u)} = 0.$$

From this equation we also have  $Z'' = O(1)$  and  $Z' = O(1)$ . Multiply the equation by  $H'$  and integrate. Set  $v(x) = G_0[u - a](x)$ . Then

$$\begin{aligned} e.s. &= \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (-Z''H' + f'(H)ZH') dt \\ &= \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (-O(\epsilon)Z^2 - \gamma v(x_j + \epsilon t) + \epsilon^{-1}\overline{f(u)})H' dt \\ &= -\gamma v(x_j) + \epsilon^{-1}\overline{f(u)} + O(\epsilon). \end{aligned}$$

Hence  $\gamma v(x_j) - \epsilon^{-1} \overline{f(u)} = O(\epsilon)$  and  $\gamma v(x) - \epsilon^{-1} \overline{f(u)} = O(\epsilon) + O(\epsilon)t$ . The equation for  $Z$  is now simplified to

$$-Z'' + f'(H)Z + O(\epsilon) + O(\epsilon)t = 0.$$

As  $\epsilon \rightarrow 0$ ,  $Z \rightarrow cH'$  in  $C_{loc}^2(-\infty, \infty)$  for some  $c$ . But  $Z(0) = e.s.$  implies  $c = 0$  since  $H'(0) \neq 0$ . Therefore  $Z \rightarrow 0$  in  $C_{loc}^2(-\infty, \infty)$ .

Next we study  $\epsilon^{-1}Z$  whose equation is written as

$$-(\epsilon^{-1}Z)'' + f'(H)(\epsilon^{-1}Z) + O(1)Z^2 + \gamma\epsilon^{-1}v - \epsilon^{-2}\overline{f(u)} = 0.$$

We again multiply by  $H'$  and integrate:

$$\begin{aligned} e.s. &= \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (-\epsilon^{-1}Z''H' + f'(H)\epsilon^{-1}ZH') dt \\ &= \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (-O(1)Z^2 - \gamma\epsilon^{-1}v(x_j + \epsilon t) + \epsilon^{-2}\overline{f(u)})H' dt \\ &= \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (-O(1)Z^2 - \gamma\epsilon^{-1}v(x_j) - \gamma v'(x_j)t + O(\epsilon)t^2 + \epsilon^{-2}\overline{f(u)})H' dt \\ &= -\gamma\epsilon^{-1}v(x_j) + \epsilon^{-2}\overline{f(u)} + o(1), \end{aligned}$$

where we have used the facts  $Z \rightarrow 0$  locally, and  $\int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} tH' dt = e.s..$  Hence  $\gamma\epsilon^{-1}v(x_j) - \epsilon^{-2}\overline{f(u)} = o(1)$ , which simplifies the equation for  $\epsilon^{-1}Z$  to

$$-(\epsilon^{-1}Z)'' + f'(H)(\epsilon^{-1}Z) + O(1)Z^2 + \gamma v'(x_j)t + O(\epsilon)t^2 + o(1) = 0. \quad (2.10)$$

Next we show that  $|\epsilon^{-1}Z(t)| \leq C(1 + |t|)$ . Without the loss of generality we consider  $t > 0$ . Let  $\epsilon^{-1}Z(t) = (1 + t)R(t)$  where  $R$  satisfies

$$-R'' - \frac{2R'}{1+t} + f'(H)R + O(1) = 0 \quad \text{in } (0, \frac{1-x_j}{\epsilon}), \quad R(0) = e.s., \quad R(\frac{1-x_j}{\epsilon}) = O(1).$$

Suppose that  $R = O(1)$  is invalid. We let  $\hat{R} = R/\|R\|_{L^\infty}$ , which satisfies

$$-\hat{R}'' - \frac{2\hat{R}'}{1+t} + f'(H)\hat{R} + o(1) = 0, \quad \hat{R}(0) = e.s., \quad \hat{R}(\frac{1-x_j}{\epsilon}) = o(1).$$

From this equation we see that  $|\hat{R}|$  must attain its maximum value 1 in a bounded region around 0. In the limit  $\hat{R}$  approaches in  $C_{loc}^2[0, \infty)$  to a nonzero, bounded solution of

$$-\hat{R}'' - \frac{2\hat{R}}{1+t} + f'(H)\hat{R} = 0 \quad \text{in } (0, \infty), \quad \hat{R}(0) = 0.$$

Then  $(1+t)\hat{R}_\infty$  satisfies

$$-((1+t)\hat{R}_\infty)'' + f'(H)(1+t)\hat{R}_\infty = 0 \quad \text{in } (0, \infty).$$

Thus,  $(1+t)\hat{R}_\infty(t) = cH'(t)$  for some  $c$ . This is because  $|(1+t)\hat{R}_\infty(t)|$  grows at most like  $t$ , and any other solution, independent of  $H'$ , of the last equation grows exponentially fast. Since  $(1+0)\hat{R}_\infty(0) = 0$  and  $H'(0) \neq 0$ , we derive  $c = 0$  and  $\hat{R}_\infty = 0$ . A contradiction.

Since  $|\epsilon^{-1}Z(t)| \leq C(1+|t|)$ , we may send  $\epsilon \rightarrow 0$  in (2.10), and find that  $\epsilon^{-1}Z$  approaches in  $C_{loc}^2(-\infty, \infty)$  to a solution of

$$-P'' + f'(H)P = -\gamma(v^0)'(x_j^0)t \quad \text{in } (-\infty, \infty).$$

We write the solution family as  $P + cH'$  with  $P \perp H'$ . Here  $P(0) = 0$ , and  $P(0) + cH'(0) = cH'(0)$  where  $H'(0) \neq 0$ . Since  $\epsilon^{-1}Z(0) = e.s.$ , we must have  $c = 0$ , and  $\epsilon^{-1}Z \rightarrow P$  in  $C_{loc}^2(-\infty, \infty)$ .  $\square$

In the language of singular perturbation theory, the last two lemmas assert that the outer expansion of  $u$  is  $u^0 + \epsilon z^0 + \dots$  and the inner expansion at  $x_j$  (when  $j$  is even) is  $H + \epsilon^2 P + \dots$ . The fact  $z^0(x_j^0) = 0$  matches the absence of the  $\epsilon$  order term in the inner expansion. The function  $w$  defined in (2.7) is the 0-th order uniform approximation of  $u$ .

We close this section by defining two frequently used constants. The first one is

$$s := \int_{-\infty}^{\infty} f''(H(t))(H'(t))^2 P(t) dt = -\frac{\gamma ab}{K}. \quad (2.11)$$

Here  $P$  is associated with an even  $j$ . When  $P$  is associated with an odd  $j$  in this paper,  $f''(H(t))$  will always be changed to  $f''(H(-t)) = -f''(H(t))$ , so  $s$  remains the same. To verify the equality in (2.11) we differentiate the equation for  $P$ , multiply by  $H'$  and integrate. The right side becomes  $-\gamma(v^0)'(x_j^0)$ . The left side becomes

$$\int_{-\infty}^{\infty} (-P'''H' + f'(H)P'H' + f''(H)(H')^2P) dt = \int_{-\infty}^{\infty} f''(H)(H')^2P dt,$$

where the first two terms on the left side cancel after integration by parts, and using  $-H''' + f'(H)H' = 0$ , which follows after differentiating the equation for  $H$ . From (2.9) we find  $(v^0)'(x_2^0) = ab/K$  and  $s = -\gamma ab/K$ .

The second constant is

$$\tau := \int_{-\infty}^{\infty} (H'(t))^2 dt > 0. \quad (2.12)$$

Because the equation for  $H$  has a first integral  $-\frac{(H')^2}{2} + W(H) = 0$ ,  $\tau = \int_{-\infty}^{\infty} \sqrt{2W(H(t))}H'(t) dt = \int_0^1 \sqrt{2W(p)} dp$ <sup>4</sup>. In the special case  $W(u) = \frac{1}{4}((u - \frac{1}{2})^2 - \frac{1}{4})^2$ ,  $\tau = \frac{\sqrt{2}}{12}$ .

### 3 Linearization at $u$

The 1-D local minimizer  $u$  of  $I_1$  is now viewed as a function on  $D$ , through extension to the second and third dimensions trivially, so  $u(x, y_1, y_2) = u(x)$ . It is a solution of (1.2) and  $I_1(u) = I(u)$ .

For an eigen pair  $(\lambda, \phi)$  of (1.3) we separate variables so that

$$\phi(x, y_1, y_2) = \sum_{m_1, m_2=0}^{\infty} \phi_m(x) \cos(m_1 \pi y_1) \cos(m_2 \pi y_2). \quad (3.13)$$

<sup>4</sup>In [12, 13, 16] this constant is defined by the last integral.

We set  $m = (m_1, m_2)$  and let  $m^2 = m_1^2 + m_2^2$ . Note

$$(-\Delta)^{-1}\{\phi_m(x) \cos(m_1\pi y_1) \cos(m_2\pi y_2)\} = X(x) \cos(m_1\pi y_1) \cos(m_2\pi y_2)$$

where  $X$  is the solution of

$$\begin{aligned} -X'' &= \phi_{(0,0)}, \quad X'(0) = X'(1) = 0, \quad \overline{X} = 0, \quad \text{if } m = (0,0) \text{ or} \\ -X'' + m^2\pi^2 X &= \phi_m, \quad X'(0) = X'(1) = 0, \quad \text{if } m \neq (0,0). \end{aligned}$$

The solution operator of the first equation is  $G_0$ , already defined. Let  $G_m[\cdot]$  be the solution operator of the second equation. They are identified with the Green functions  $G_m(\cdot, \cdot)$  in this paper. Therefore  $X = G_m[\phi_m]$ . The eigenvalue problem  $L\phi = \lambda\phi$  now becomes

$$\begin{aligned} \sum_m \{-\epsilon^2(\phi_m'' - m^2\pi^2\phi_m) + \epsilon\gamma G_m[\phi_m] + f'(u)\phi_m\} \cos(m_1\pi y_1) \cos(m_2\pi y_2) - \overline{f'(u)\phi_0} \\ = \lambda \sum_m \phi_m(x) \cos(m_1\pi y_1) \cos(m_2\pi y_2). \end{aligned}$$

Here we have used the fact  $\overline{f'(u)\phi_m(x) \cos(m_1\pi y_1) \cos(m_2\pi y_2)} = 0$  if  $m \neq (0,0)$ .

Multiplying the equation by  $\cos(m_1\pi y_1) \cos(m_2\pi y_2)$  and integrating with respect to  $y_1$  and  $y_2$ , we find two cases:

1. When  $m = (0,0)$ ,

$$-\epsilon^2\phi_{(0,0)}'' + \epsilon\gamma G_0[\phi_{(0,0)}] + f'(u)\phi_{(0,0)} - \overline{f'(u)\phi_{(0,0)}} = \lambda\phi_{(0,0)}, \quad \phi_{(0,0)}'(0) = \phi_{(0,0)}'(1) = \overline{\phi_{(0,0)}} = 0. \quad (3.14)$$

2. When  $m \neq (0,0)$ ,

$$-\epsilon^2(\phi_m'' - m^2\pi^2\phi_m) + \epsilon\gamma G_m[\phi_m] + f'(u)\phi_m = \lambda\phi_m, \quad \phi_m'(0) = \phi_m'(1) = 0. \quad (3.15)$$

Because the  $\lambda$ 's are classified by  $m$ , we use  $\lambda_m$  to denote an eigenvalue that is associated with  $m$ . The corresponding eigenfunction is  $\phi_m(x) \cos(m_1\pi y_1) \cos(m_2\pi y_2)$ .

*Proof of Theorem 1.1, Part 1.* We first consider the local eigenvalue problem

$$E(\phi) := -\epsilon^2\phi'' + f'(u)\phi = \nu\phi, \quad \phi'(0) = \phi'(1) = 0. \quad (3.16)$$

In this proof an eigen pair of (3.16) is denoted by  $(\nu, \phi)$ . We will prove that  $\nu \geq -C\epsilon^2$  for some  $C > 0$ .

**Claim 1.** If  $\nu \rightarrow \nu^0$  as  $\epsilon \rightarrow 0$ , then  $\nu^0 \geq 0$ .

Suppose on the contrary that  $\nu^0 < 0$ . Let  $y \in [0, 1]$  so that  $\phi(y) = \max|\phi| = 1$ . Then  $y - x_j = O(\epsilon)$  for some  $j$ . Otherwise  $-\epsilon^2\phi''(y) \geq 0$ ,  $f'(u(y))\phi(y) > 0$ ,  $\nu\phi(y) < 0$  and hence (3.16) is not satisfied. Then we consider  $\Phi(t) = \phi(x_j + \epsilon t)$  which satisfies  $-\Phi'' + f'(u)\Phi = \nu\Phi$  in  $(-x_j/\epsilon, (1-x_j)/\epsilon)$ . As  $\epsilon \rightarrow 0$ ,  $\Phi$  approaches  $\Phi_\infty \neq 0$  in  $C_{loc}^2(-\infty, \infty)$  which satisfies  $-\Phi_\infty'' + f'(H)\Phi_\infty = \nu^0\Phi_\infty$  in  $(-\infty, \infty)$ . But this is impossible since the last equation has no negative eigenvalues. This proves the claim.

The case  $\nu^0 > 0$  does not concern us, so we assume  $\nu \rightarrow 0$ . We introduce, for  $j = 1, 2, \dots, K$ ,

$$h_j(x) = H'\left(\frac{x - x_j}{\epsilon}\right) + e.s.. \quad (3.17)$$

Here  $e.s.$  is an exponentially small correction term. It is chosen so that  $h_j(0) = h_j(1) = h'_j(0) = h'_j(1) = 0$ ,  $\|h'_j - \epsilon^{-1}H''(\frac{\cdot - x_j}{\epsilon})\|_\infty = e.s.$ , and  $\|h''_j - \epsilon^{-2}H'''(\frac{\cdot - x_j}{\epsilon})\|_\infty = e.s..$

**Remark 3.1** *Should we weaken the condition  $W(p) = W(1 - p)$ ,  $H'$  would no longer be even and we would set*

$$h_j(x) = \begin{cases} H'\left(\frac{x - x_j}{\epsilon}\right) + e.s. & \text{if } j \text{ is even} \\ H'\left(-\frac{x - x_j}{\epsilon}\right) + e.s. & \text{if } j \text{ is odd} \end{cases}.$$

Consider the subspace of  $L^2(0, 1)$  generated by  $h_j$ . Decompose  $\phi = \sum_j c_j h_j + \psi$ , so that  $h_j \perp \psi$  for each  $j = 1, 2, \dots, K$ . Note

$$E(h_j) = (f'(u) - f'(H))h_j + e.s.$$

and by Lemma 2.5,

$$\begin{aligned} |(f'(u) - f'(H))h_j| &= |(f'(w(x_j + \epsilon t) + \epsilon z(x_j + \epsilon t)) - f'(H(t)))H'(t)| + e.s. \\ &= |f''(H(t))\epsilon z(x_j + \epsilon t)H'(t)| + O(\epsilon^4) = O(\epsilon^2). \end{aligned} \quad (3.18)$$

Hence we deduce

$$E(h_j) = O(\epsilon^2). \quad (3.19)$$

We write (3.16) as

$$\sum_{j=1}^K c_j E(h_j) + E(\psi) = \nu \sum_j c_j h_j + \nu \psi. \quad (3.20)$$

**Claim 2.**  $\langle E(\psi), \psi \rangle \geq C\|\psi\|_2^2$  for some  $C > 0$  independent of  $\epsilon$ .

When we minimize the quotient  $\frac{\langle E(\tilde{\psi}), \tilde{\psi} \rangle}{\|\tilde{\psi}\|_2^2}$  among nonzero  $\tilde{\psi}$  subject to  $\tilde{\psi} \perp h_j$  for every  $j$ , the minimizer, denoted by  $\tilde{\psi}$  in this paragraph, satisfies

$$-\epsilon^2 \tilde{\psi}'' + f'(u)\tilde{\psi} = \iota \tilde{\psi} + \sum_j d_j h_j. \quad (3.21)$$

The constant  $\iota = \frac{\langle E(\tilde{\psi}), \tilde{\psi} \rangle}{\|\tilde{\psi}\|_2^2}$ . Suppose Claim 2 is false. Then  $\lim_{\epsilon \rightarrow 0} \iota = \iota^0 \leq 0$ . We multiply  $\tilde{\psi}$  by a proper constant so there exists  $y \in [0, 1]$  such that  $\tilde{\psi}(y) = \max |\tilde{\psi}| = 1$ . Now we multiply (3.21) by  $h_k$  and integrate.

$$\langle E(h_k), \tilde{\psi} \rangle = \sum_j d_j \langle h_j, h_k \rangle.$$

The left side is  $O(\epsilon^2)$  by (3.19). The right side is

$$\int_0^1 \sum_j d_j h_j h_k = \sum_j \epsilon d_j \tau \delta_{jk} + e.s. |d| = \epsilon \tau d_k + e.s. |d|,$$

where  $\delta_{jk} = 1$  if  $j = k$  and 0 otherwise, and  $|d| = \sqrt{d_1^2 + d_2^2 + \dots + d_K^2}$  is the norm of the vector  $d$ . Therefore  $d_k = O(\epsilon)$ . As in the proof of Claim 1,  $y - x_j = O(\epsilon)$  for some  $j$ . Moreover we consider  $\Psi(t) = \tilde{\psi}(x_j + \epsilon t)$ , which satisfies  $-\Psi'' + f'(u)\Psi = \iota\Psi + o(1)$ . Passing to the limit we find a nonzero  $\Psi_\infty$  which satisfies  $-\Psi_\infty'' + f'(H)\Psi_\infty = \iota^0\Psi_\infty$  in  $(-\infty, \infty)$ . Therefore  $\iota^0 = 0$  and  $\Psi_\infty$  is proportional to  $H'$ . But on the other hand  $\tilde{\psi} \perp h_j$  implies  $\Psi_\infty \perp H'$ . Hence  $\Psi_\infty = 0$ , contradicting the fact that  $\Psi_\infty$  is nonzero. This proves Claim 2.

We now return to (3.20). Multiply it by  $\psi$  and integrate. Use (3.19) to deduce

$$|c|O(\epsilon^2)\|\psi\|_2 + \langle E(\psi), \psi \rangle = \nu \int_0^1 \psi^2.$$

Then Claim 2 implies

$$\|\psi\|_2 = O(\epsilon^2)|c|. \quad (3.22)$$

Next we multiply (3.20) by  $h_k$  and integrate. The left side is

$$\int_0^1 \{E(h_k)\psi + \sum_j c_j E(h_j)h_k\} = \int_0^1 \{(f'(u) - f'(H))h_k\psi + e.s.\psi + \sum_j c_j((f'(u) - f'(H))h_j h_k + e.s.)\} \quad (3.23)$$

in which

$$\left| \int_0^1 (f'(u) - f'(H))h_k\psi \right| \leq \|(f'(u) - f'(H))h_k\|_\infty \|\psi\|_2 = O(\epsilon^4)|c|$$

by (3.18) and (3.22), and

$$\begin{aligned} & \int_0^1 (f'(u) - f'(H))h_j h_k \\ &= \epsilon \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} \{f'(w(x_j + \epsilon t) + \epsilon z(x_j + \epsilon t)) - f'(H(t))\} H'(t) H'(t + (x_j - x_k)/\epsilon) dt + e.s. \\ &= \epsilon \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} \{f''(H(t))\epsilon z(x_j + \epsilon t) + O(\epsilon^2)z^2(x_j + \epsilon t)\} H'(t) H'(t + (x_j - x_k)/\epsilon) dt + e.s. \\ &= \epsilon^3 \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} f''(H(t))P(t)H'(t)H'(t + (x_j - x_k)/\epsilon) dt + o(\epsilon^3) \\ &= \epsilon^3 s \delta_{jk} + o(\epsilon^3) \end{aligned} \quad (3.24)$$

by Lemma 2.5. The above argument applies to the case when  $j$  is even. When  $j$  is odd  $f''(H(t))$  becomes to  $f''(H(-t)) = -f''(H(t))$  and  $P(t)$  has a different sign, but the final result remains unchanged. Hence (3.23) becomes  $\epsilon^3 s c_k + o(\epsilon^3)|c|$ . The right side of (3.20) multiplied by  $h_k$  and integrated is  $\nu \epsilon \tau c_k + e.s. |c|$ . Equating the last two quantities we find that for every  $k$

$$s c_k + o(1)|c| = \frac{\nu \tau}{\epsilon^2} c_k.$$

Therefore  $\nu \geq -C\epsilon^2$  for some  $C > 0$  independent of  $\epsilon$ .

Since  $G_m$  is a bounded, positive operator in the eigenvalue problem

$$-\epsilon^2\phi'' + \epsilon\gamma G_m[\phi] + f'(u)\phi = \nu\phi, \quad \phi'(0) = \phi'(1) = 0, \quad (3.25)$$

we again have  $\nu \geq -C\epsilon^2$  for some  $C > 0$  independent of  $\epsilon$ . This can be seen easily by comparing the variational characterization of the principle eigenvalue of (3.25):

$$\inf\left\{\int_0^1 \{\epsilon^2(\phi')^2 + \epsilon\gamma G_m[\phi]\phi + f'(u)\phi^2\} dx : \phi \in W^{1,2}(0,1), \|\phi\|_2 = 1\right\},$$

to a similar one without the  $\epsilon\gamma G_m[\phi]\phi$  term for (3.16). Finally in (3.15) by setting  $m^2$  large enough, we find  $\lambda_m \geq C\epsilon^2$ .  $\square$

## 4 $m = (0, 0)$ eigenvalues

Here we study the  $m_1 = m_2 = 0$  problem (3.14). Denote the linear operator there by  $L_0$ . An eigen pair of (3.14) is denoted by  $(\lambda, \phi)$  in this section. Since (3.14) is precisely the linearized operator of the 1-D problem  $I_1$  defined in (2.6) at a 1-D local minimizer  $u$ , we have  $\lambda \geq 0$ . The case  $\lambda \rightarrow \lambda^0 > 0$  as  $\epsilon \rightarrow 0$  does not concern us. So we assume  $\lambda \rightarrow 0$  along a subsequence throughout the rest of this section.

We decompose  $\phi = \sum_j c_j(h_j - \bar{h}_j) + \psi$  where  $\psi \perp h_j - \bar{h}_j$  for every  $j$ . Note

$$L_0(h_j - \bar{h}_j) = (f'(u) - f'(H))h_j + \epsilon\gamma G_0[h_j - \bar{h}_j] + (\overline{f'(u)} - f'(u))\bar{h}_j - \overline{f'(u)h_j} + e.s..$$

A few terms on the right side are estimated once and for all.

$$\epsilon\gamma G_0[h_j - \bar{h}_j](x) = \gamma\epsilon^2 G_0[(h_j - \bar{h}_j)/\epsilon](x) = \gamma\epsilon^2 G_0(x, x_j) + O(\epsilon^3). \quad (4.26)$$

$$\begin{aligned} \overline{f'(u)h_j} &= \epsilon \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} f'(w(x_j + \epsilon t) + \epsilon z(x_j + \epsilon t))H'(t) dt + e.s. \\ &= \epsilon \int_{-x_j/\epsilon}^{(1-x_j)/\epsilon} (f'(H(t)) + f''(H(t))\epsilon z(x_j + t) + O(\epsilon^4))H'(t) dt + e.s. \\ &= \epsilon^2 \int_{-\infty}^{\infty} f''(H(t))z(x_j + \epsilon t)H'(t) dt + O(\epsilon^5) = O(\epsilon^3) \end{aligned} \quad (4.27)$$

where the last line follows from Lemma 2.5. The next estimate is not the sharpest.

$$|(\overline{f'(u)} - f'(u))\bar{h}_j| = |(\overline{f'(u)} - f'(u))|(\epsilon + e.s.) = O(\epsilon). \quad (4.28)$$

So based on the last three estimates and (3.18) we find

$$L_0(h_j - \bar{h}_j) = O(\epsilon). \quad (4.29)$$

We also need an  $L^1$  version of (4.28):

$$\|(\overline{f'(u)} - f'(u))\bar{h}_j\|_1 = O(\epsilon)\|\overline{f'(u)} - f'(u)\|_1 = O(\epsilon)\|\overline{f'(w)} - f'(w) + O(\epsilon)\|_1 = O(\epsilon^2),$$

so we obtain

$$\|L_0(h_j - \bar{h}_j)\|_1 = O(\epsilon^2). \quad (4.30)$$

Rewrite (3.14) as

$$\sum_j c_j L_0(h_j - \bar{h}_j) + L_0\psi = \lambda \sum_j c_j (h_j - \bar{h}_j) + \lambda\psi. \quad (4.31)$$

**Lemma 4.1**  $\langle L_0(\psi), \psi \rangle \geq C\|\psi\|_2^2$  for some  $C > 0$  independent of  $\epsilon$ .

*Proof.* When we minimize the quotient  $\frac{\langle L_0(\tilde{\psi}), \tilde{\psi} \rangle}{\|\tilde{\psi}\|_2^2}$  among nonzero  $\tilde{\psi}$  of zero average subject to  $\tilde{\psi} \perp h_j - \bar{h}_j$  for every  $j$ , the minimizer, denoted by  $\tilde{\psi}$  in this proof, satisfies

$$-\epsilon^2 \tilde{\psi}'' + \epsilon\gamma G_0[\tilde{\psi}] + f'(u)\tilde{\psi} - \overline{f'(u)\tilde{\psi}} = \iota\tilde{\psi} + \sum_j d_j (h_j - \bar{h}_j).$$

The constant  $\iota = \frac{\langle L_0(\tilde{\psi}), \tilde{\psi} \rangle}{\|\tilde{\psi}\|_2^2}$ . Suppose that the lemma is false. Then  $\lim_{\epsilon \rightarrow 0} \iota = \iota^0 \leq 0$ . We multiply  $\tilde{\psi}$  by a proper constant so there exists  $y \in [0, 1]$  such that  $\tilde{\psi}(y) = \max |\tilde{\psi}| = 1$ . Now we multiply the last equation by  $h_k - \bar{h}_k$  and integrate:  $\langle L_0(h_k - \bar{h}_k), \tilde{\psi} \rangle = \sum_j d_j \langle h_j - \bar{h}_j, h_k - \bar{h}_k \rangle$ . The left side is  $O(\epsilon^2)$  by (4.30). The right side is

$$\int_0^1 \sum_j d_j (h_j - \bar{h}_j)(h_k - \bar{h}_k) = \epsilon\tau d_k + O(\epsilon^2)|d|.$$

Therefore  $d_k = O(\epsilon)$ . The rest of the proof is the same as that of Claim 2 in Section 3, since the additional terms in the equation satisfies

$$\begin{aligned} \epsilon\gamma G_0[\tilde{\psi}] &= O(\epsilon) \\ \overline{f'(u)\tilde{\psi}} &= \overline{(f'(u) - f'(0))\tilde{\psi}} = O(1)\|f'(u) - f'(0)\|_1 = O(\epsilon). \end{aligned}$$

A minor difference is that  $\tilde{\psi} \perp h_j$  here is a consequence of  $\tilde{\psi} \perp h_j - \bar{h}_j$  and  $\bar{\tilde{\psi}} = 0$ .  $\square$

Multiply (4.31) by  $\psi$  and integrate. Using (4.29) we find

$$|c|O(\epsilon)\|\psi\|_2 + \langle L_0(\psi), \psi \rangle = \lambda\|\psi\|_2^2.$$

Lemma 4.1 implies that

$$\|\psi\|_2 = O(\epsilon)|c|. \quad (4.32)$$

**Remark 4.2** As a comparison we compute

$$\left\| \sum_j c_j (h_j - \bar{h}_j) \right\|_2 = \left\{ \sum_j c_j^2 \int_0^1 h_j^2 + O(\epsilon^2)|c|^2 \right\}^{1/2} = \{ \epsilon\tau|c|^2 + O(\epsilon^2)|c|^2 \}^{1/2} \sim \epsilon^{1/2}|c|.$$

So in the decomposition of  $\phi$ ,  $\sum_j c_j (h_j - \bar{h}_j)$  is more prominent than  $\psi$  is.

Multiply (4.31) by  $h_k - \bar{h}_k$  and integrate:

$$\int_0^1 L_0 \psi (h_k - \bar{h}_k) + \sum_j c_j \int_0^1 L_0 (h_j - \bar{h}_j) (h_k - \bar{h}_k) = \lambda \sum_j c_j \int_0^1 (h_j - \bar{h}_j) (h_k - \bar{h}_k). \quad (4.33)$$

The first term on the left side is written as

$$\begin{aligned} \int_0^1 L_0(\psi)(h_k - \bar{h}_k) &= \int_0^1 L_0(h_k - \bar{h}_k)\psi \\ &= \int_0^1 \{(f'(u) - f'(H))h_k\psi + \epsilon\gamma G_0[h_k - \bar{h}_k]\psi + (\overline{f'(u)} - f'(u))\bar{h}_k\psi - \overline{f'(u)\bar{h}_k}\psi + e.s.\psi\} \\ &= \int_0^1 \{(f'(u) - f'(H))h_k\psi + \epsilon\gamma G_0[\psi]h_k + (f'(0) - f'(u))\bar{h}_k\psi + e.s.\psi\}. \end{aligned} \quad (4.34)$$

The four terms are estimated as follows.

$$\begin{aligned} \left| \int_0^1 (f'(u) - f'(H))h_k\psi \right| &\leq \| (f'(u) - f'(H))h_k \|_\infty \|\psi\|_2 = O(\epsilon^2)\|\psi\|_2 = O(\epsilon^3)|c| \\ \int_0^1 \epsilon\gamma G_0[\psi]h_k &= O(\epsilon)\|G_0[\psi]\|_\infty \|h_k\|_1 = O(\epsilon^2)\|\psi\|_2 = O(\epsilon^3)|c| \\ \int_0^1 (f'(0) - f'(u))\bar{h}_k\psi &= \|f'(u) - f'(0)\|_2 O(\epsilon)\|\psi\|_2 = O(\epsilon^{2.5})|c| \\ \int_0^1 e.s.\psi &= e.s.\|\psi\|_2 = e.s.|c|. \end{aligned}$$

Note that the first estimate follows from (3.18). The second term on the left of (4.33) is, for each  $j$ , by (4.26) and (4.27),

$$\begin{aligned} \int_0^1 L_0(h_j - \bar{h}_j)(h_k - \bar{h}_k) &= \int_0^1 L_0(h_j - \bar{h}_j)h_k \\ &= \int_0^1 \{(f'(u) - f'(H))h_j h_k + \epsilon\gamma G_0[h_j - \bar{h}_j]h_k + (\overline{f'(u)} - f'(u))\bar{h}_j h_k - \overline{f'(u)\bar{h}_j}h_k + e.s.\} \\ &= \epsilon^3 s \delta_{jk} + \gamma \epsilon^3 G_0(x_j, x_k) + \epsilon^2 \overline{f'(u)} + o(\epsilon^3). \end{aligned} \quad (4.35)$$

The last line follows from the estimates (3.24), (4.26), (4.27), and

$$\begin{aligned} \int_0^1 (\overline{f'(u)} - f'(u))\bar{h}_j h_k &= (\epsilon + e.s.)\{\overline{f'(u)}(\epsilon + e.s.) - \int_0^1 f'(u)h_k\} = \epsilon^2 \overline{f'(u)} - \epsilon \int_0^1 f'(u)h_k + e.s. \\ &= \epsilon^2 \overline{f'(u)} - \epsilon^2 \int_{-x_k/\epsilon}^{(1-x_k)/\epsilon} (f'(H) + O(\epsilon^2))H'(t) dt + e.s. = \epsilon^2 \overline{f'(u)} + O(\epsilon^4). \end{aligned}$$

The right side of (4.33) is

$$\lambda \sum_j c_j \int_0^1 (h_j - \bar{h}_j)(h_k - \bar{h}_k) = \lambda(\epsilon\tau c_k + O(\epsilon^2)|c|). \quad (4.36)$$

In summary for every  $k$

$$\epsilon^3 s c_k + \sum_j \{\gamma \epsilon^3 G(x_j, x_k) + \epsilon^2 \overline{f'(u)} + o(\epsilon^3)\} c_j + O(\epsilon^{2.5})|c| = \lambda(\tau \epsilon c_k + O(\epsilon^2)|c|). \quad (4.37)$$

If we consider the  $c_k$  of the largest absolute value, since  $\epsilon^2 \overline{f'(u)} \sim \epsilon^2$ ,  $\lambda = O(\epsilon)$ . On the left side of (4.37),  $\epsilon^2 \overline{f'(u)}$  is the largest term. Because  $\epsilon^2 \overline{f'(u)}$  is multiplied to  $\sum_j c_j$ , a dichotomy appears at this point, unless  $K = 1$ .

**Case 1.**  $\frac{\sum_k c_k}{|c|} \not\rightarrow 0$ . Note that when  $K = 1$  this is the only case. We rewrite (4.37) as

$$\overline{f'(u)} \sum_j c_j + O(\epsilon^{1/2})|c| = \frac{\lambda \tau}{\epsilon} c_k. \quad (4.38)$$

In the limit we have

$$f'(0) \sum_j c_j^0 = \eta \tau c_k^0, \quad \sum_j c_j^0 \neq 0, \quad (4.39)$$

since  $\lim_{\epsilon \rightarrow 0} \overline{f'(u)} = f'(0)$ . Here  $\eta = \lim_{\epsilon \rightarrow 0} \lambda/\epsilon$  and  $\lim_{\epsilon \rightarrow 0} c_j = c_j^0$ . Solving (4.39) we find

$$\eta = \frac{f'(0)K}{\tau}, \quad c_1^0 = c_2^0 = \dots = c_K^0. \quad (4.40)$$

Thus we obtain the asymptotic expansions for one eigen pair  $\lambda_{(0,0)} = \lambda$  and  $\phi_{(0,0)} = \phi$  of (3.14):

$$\lambda_{(0,0)} = \frac{\epsilon f'(0)K}{\tau} + o(\epsilon), \quad \phi_{(0,0)} \approx \sum_j (h_j - \bar{h}_j). \quad (4.41)$$

Note that this  $\lambda_{(0,0)}$  is positive.

**Case 2.**  $\frac{\sum_k c_k}{|c|} \rightarrow 0$ . This occurs when  $K \geq 2$ . To study this case, we rewrite  $L_0(\phi) = \lambda \phi$  as

$$L_0(\psi) = - \sum_j c_j L_0(h_j - \bar{h}_j) + \lambda \sum_j c_j (h_j - \bar{h}_j) + \lambda \psi.$$

Note

$$\sum_j c_j L_0(h_j - \bar{h}_j) = O(\epsilon^2)|c| + \sum_j c_j (\overline{f'(u)} - f'(u)) \bar{h}_j = O(\epsilon^2)|c| + (\sum_j c_j) O(\epsilon) = o(\epsilon)|c|$$

by the assumption, and (4.26), (4.27), (3.18) and (4.28). Hence

$$L_0(\psi) = o(\epsilon)|c| + |\lambda| O(1)|c| + \lambda \psi. \quad (4.42)$$

**Lemma 4.3**  $\|\psi\|_\infty = o(\epsilon)|c| + |\lambda| O(1)|c|$ .

*Proof.* Suppose that the lemma is false. Replacing  $\psi$  by  $\pm \frac{\psi}{\|\psi\|_\infty}$  in (4.42) we obtain  $L_0(\psi) = o(1) + \lambda\psi$  where at some  $y \in [0, 1]$ ,  $\psi(y) = \|\psi\|_\infty = 1$ . We show that  $y - x_j = O(\epsilon)$  for some  $j$ . Otherwise  $-\epsilon^2\psi''(y) \geq 0$ ,  $\epsilon\gamma G_0[\psi] = O(\epsilon)$ ,  $f'(u)\psi(y) \rightarrow f'(0) > 0$ ,  $-\overline{f'(u)}\psi = O(\epsilon)$ , and  $\lambda\psi(y) = O(\epsilon)$ . Hence the equation  $L_0(\psi) = o(1) + \lambda\psi$  is not satisfied. Then we set  $\Psi(t) = \psi(x_j + \epsilon t)$  which satisfies  $-\Psi'' + f'(u)\Psi = o(1)$  in  $(-x_j/\epsilon, (1-x_j)/\epsilon)$ . As  $\epsilon \rightarrow 0$ ,  $\Psi \rightarrow \Psi_\infty \neq 0$  in  $C_{loc}^2(-\infty, \infty)$  and  $\Psi_\infty$  satisfies  $-\Psi_\infty'' + f'(H)\Psi_\infty = 0$ . Hence  $\Psi_\infty$  is proportional to  $H'$ . On the other hand  $\psi \perp h_j$  implies  $\int_{-\infty}^{\infty} \Psi_\infty H' = 0$ . Thus  $\Psi_\infty = 0$  contradicting the fact that  $\Psi_\infty \neq 0$ .  $\square$

With this lemma we return to (4.33) and recall

$$\int_0^1 (f'(u) - f'(H))h_k\psi = O(\epsilon^2)\|\psi\|_2, \quad \int_0^1 \epsilon\gamma G_0[\psi]h_k = O(\epsilon^2)\|\psi\|_2.$$

Re-derive

$$\int_0^1 (f'(0) - f'(u))\overline{h_k}\psi = \|f'(u) - f'(0)\|_1 O(\epsilon)\|\psi\|_\infty = O(\epsilon^2)\|\psi\|_\infty.$$

Therefore

$$\int_0^1 L_0(\psi)(h_k - \overline{h_k}) = o(\epsilon^3)|c| + |\lambda|O(\epsilon^2)|c| \quad (4.43)$$

and from (4.35)

$$\begin{aligned} \sum_j c_j \int_0^1 L_0(h_j - \overline{h_j})(h_k - \overline{h_k}) &= \epsilon^3 s c_k + \sum_j (\gamma \epsilon^3 G_0(x_j, x_k) + \epsilon^2 \overline{f'(u)}) c_j + o(\epsilon^3)|c| \\ &= O(\epsilon^3)|c| + \epsilon^2 \overline{f'(u)} (\sum_j c_j) = o(\epsilon^2)|c|. \end{aligned}$$

This estimate and (4.43), (4.36) turn equation (4.33) to

$$|\lambda|O(\epsilon^2)|c| + o(\epsilon^2)|c| = \lambda(\epsilon\tau c_k + O(\epsilon^2)|c|).$$

From the  $c_k$  of the largest absolute value we find, with the help of Lemma 4.3,

$$\lambda = o(\epsilon), \quad \|\psi\|_\infty = o(\epsilon)|c|. \quad (4.44)$$

So (4.43) is improved to  $o(\epsilon^3)|c|$  and (4.33) reads

$$\epsilon^3 s c_k + \sum_j (\gamma \epsilon^3 G_0(x_j, x_k) + \epsilon^2 \overline{f'(u)}) c_j + o(\epsilon^3)|c| = \lambda \epsilon \tau c_k. \quad (4.45)$$

We sum over  $k$ .  $\sum_k G_0(x_j^0, x_k^0)$  is independent of  $j$ , an issue further addressed in the next section, so we denote it by  $g$ . Then after dividing by  $\epsilon^2|c|$  we obtain, using (4.44),

$$\frac{\sum_j c_j}{|c|} (\epsilon s + \gamma \epsilon g + \overline{f'(u)} K) + o(\epsilon) = o(1) \frac{\sum_j c_j}{|c|}.$$

Since  $\overline{f'(u)} \sim 1$ ,  $\frac{\sum_j c_j}{|c|} = o(\epsilon)$ . Return to (4.45). Divide by  $\epsilon^3$ . Since  $\overline{f'(u)} \sum_j c_j = o(\epsilon)|c|$ ,

$$s c_k + \gamma \sum_j G_0(x_j, x_k) c_j + o(1)|c| = \frac{\lambda}{\epsilon^2} \tau c_k \quad (4.46)$$

for all  $k$ . In the limit we have

$$sc_k^0 + \gamma \sum_j G_0(x_j^0, x_k^0) c_j^0 = \eta \tau c_k^0, \quad \sum_j c_j^0 = 0. \quad (4.47)$$

Here  $\eta = \lim_{\epsilon \rightarrow 0} \lambda/\epsilon^2$  and  $c_j^0 = \lim_{\epsilon \rightarrow 0} c_j$ . In the next section we will solve (4.47) to find  $K - 1$  pairs of  $\eta$  and  $c^0$ . Once they are determined we obtain the asymptotic expansions of  $K - 1$  eigen pairs  $\lambda_{(0,0)} = \lambda$  and  $\phi_{(0,0)} = \phi$  of (3.14):

$$\lambda_{(0,0)} = \epsilon^2 \eta + o(\epsilon^2), \quad \phi_{(0,0)} \approx \sum_j c_j^0 h_j. \quad (4.48)$$

Here the  $\overline{h_j}$  terms drop out in  $\phi_{(0,0)}$  since  $\sum_j c_j^0 \overline{h_j} = \sum_j (1 + e.s.) c_j^0 = e.s. |c|$  is negligible.

## 5 The spectrum of $[G_0(x_j^0, x_k^0)]$

To understand (4.47) we must find the spectrum of the  $K$  by  $K$  matrix  $G_0(x_j^0, x_k^0)$ . Suppose for every  $k$

$$\sum_j G_0(x_j^0, x_k^0) b_j = \Lambda b_k.$$

Note that from (4.47)  $s + \gamma \Lambda = \tau \eta$ . From the formula

$$G_0(x, y) = \begin{cases} \frac{x^2}{2} + \frac{(1-y)^2}{2} - \frac{1}{6} & \text{if } x < y \\ \frac{(1-x)^2}{2} + \frac{y^2}{2} - \frac{1}{6} & \text{if } x > y \end{cases}$$

we see by straight computation that  $\sum_k G_0(x_j^0, x_k^0)$  is independent of  $j$ . This number is an eigenvalue whose associated eigenvector is  $(1, 1, \dots, 1)^T$  where the superscript  $T$  denotes the transpose of a vector. However this eigen pair is discarded since by (4.47) we require that  $\sum_j b_j = 0$ .

To find other eigen pairs we let  $\zeta = \sum_j G_0(x_j^0, \cdot) b_j$ . Then  $\zeta$  satisfies  $-\zeta'' = \sum_j (\delta(\cdot - x_j^0) - 1) b_j = \sum_j \delta(\cdot - x_j^0)$ ,  $\zeta'(0) = \zeta'(1) = 0$ . Moreover  $\zeta(x_k) = \Lambda b_k$  and  $[-\zeta']_{x_j^0} = b_j$ . Then for every  $k$ ,  $[-\zeta']_{x_j^0} = (1/\Lambda) \zeta(x_k^0)$ . We need to express  $[-\zeta']_{x_j^0}$  in terms of  $\zeta(x_k^0)$ . In other words we find a  $K$  by  $K$  matrix  $T$  so that  $(T\vec{\zeta})_j = [-\zeta']_{x_j^0}$  where  $\vec{\zeta} = (\zeta(x_1^0), \zeta(x_2^0), \dots, \zeta(x_K^0))^T$ . This way the original eigenvalue problem is converted to

$$T\vec{\zeta} = \frac{1}{\Lambda} \vec{\zeta}, \quad \text{with } b = \frac{1}{\Lambda} \vec{\zeta}. \quad (5.49)$$

To find  $T$  note that  $-\zeta$  is affine between the  $x_j^0$ 's. From  $(0, x_1^0)$  we deduce

$$-\zeta'(x_1^0-) = \frac{-\zeta(x_1^0) + \zeta(x_0^0)}{a/K} = 0$$

since  $\zeta'(0) = 0$ . From  $(x_1^0, x_2^0)$  we obtain

$$-\zeta'(x_1^0+) = \frac{-\zeta(x_2^0) + \zeta(x_1^0)}{2b/K}$$

Hence

$$[-\zeta']_{x_1^0} = \frac{K}{2b}\zeta(x_1^0) - \frac{K}{2b}\zeta(x_2^0).$$

On the other intervals we find

$$[-\zeta']_{x_j^0} = \left(\frac{K}{2a} + \frac{K}{2b}\right)\zeta(x_j^0) - \begin{cases} \frac{K}{2b}\zeta(x_{j-1}^0) - \frac{K}{2a}\zeta(x_{j+1}^0) & \text{if } j \text{ is even} \\ \frac{K}{2a}\zeta(x_{j-1}^0) - \frac{K}{2b}\zeta(x_{j+1}^0) & \text{if } j \text{ is odd} \end{cases}.$$

When  $K = 2$  we have

$$T = \begin{bmatrix} K/(2b) & -K/(2b) \\ -K/(2b) & K/(2b) \end{bmatrix}.$$

Therefore after discarding the 0 eigenvalue of the matrix we find  $\Lambda = \frac{b}{K}$ . And for (4.47)

$$\eta = \frac{1}{\tau}\left(-\frac{\gamma ab}{K} + \frac{\gamma b}{K}\right), \quad c^0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (5.50)$$

Note that  $\eta > 0$ . When  $K \geq 3$  we have  $T = (\alpha + \beta)I_{K \times K} - Q$  where  $I_{K \times K}$  is the  $K$  by  $K$  identity matrix,  $\alpha = K/(2a)$ ,  $\beta = K/(2b)$  and

$$Q = \begin{bmatrix} \alpha & \beta & & & & \\ \beta & 0 & \alpha & & & \\ & \alpha & 0 & \beta & & \\ & & \beta & 0 & \alpha & \\ & & & & & \dots \end{bmatrix}.$$

The  $K$  distinct eigenvalues of  $Q$  are found in (B.5), Appendix B. One of them,  $\alpha + \beta$ , is discarded, for its eigenvector is  $(1, 1, \dots, 1)^T$ . If we denote the rest of them by  $q_1, q_2, \dots, q_{K-1}$ , we have  $K - 1$   $\Lambda$ 's:

$$\Lambda = \frac{1}{\alpha + \beta - q_j}, \quad j = 1, 2, \dots, K - 1.$$

Therefore  $K - 1$  pairs of

$$\eta = \frac{1}{\tau}\left(-\frac{\gamma ab}{K} + \frac{\gamma}{\alpha + \beta - q_j}\right), \quad c^0 = \vec{\zeta} \quad (5.51)$$

for (4.47) are found.

When concerned with the positivity of  $\eta$ , we consider the smallest  $\Lambda$  which is associated with the smallest  $q_j$ . According to (B.5), the smallest  $q_j$  is  $-\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta}$  where  $\theta = 2\pi/K$ . Hence the smallest  $\Lambda$  is

$$\Lambda = \frac{1}{\alpha + \beta + \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta}} > \frac{1}{2(\alpha + \beta)} = \frac{ab}{K}.$$

Therefore the smallest  $\eta$  of (4.47) is

$$\eta = \frac{s + \gamma\Lambda}{\tau} > \frac{1}{\tau}\left(-\frac{\gamma ab}{K} + \frac{\gamma ab}{K}\right) = 0.$$

Thus the  $\eta$ 's in both (5.50) and (5.51) are positive.

Finally we show that  $L_0$  has exactly  $K$  simple eigen pairs with the asymptotic expansions (4.41) and (4.48). Let  $F$  be the linear subspace generated by small eigenvalues. It is defined non-ambiguously by  $F = \text{span}\{\phi \in L^2(0,1) : \bar{\phi} = 0, L_0(\phi) = \lambda\phi, |\lambda| < \epsilon^{1/2}\}$ . Since the small eigenvalues of  $L_0$  are of order  $\epsilon^2$  or  $\epsilon$ ,  $F$  addresses all the small eigenvalues when  $\epsilon$  is small enough.

First  $\dim F$ , the dimensional of  $F$ , is at most  $K$ . Suppose that this is not the case. There exist two distinct eigen pairs  $(\lambda, \phi)$  and  $(\lambda', \phi')$  with the same asymptotic behavior. That is

$$\lambda = \epsilon^2\eta + o(\epsilon^2), \quad \lambda' = \epsilon^2\eta + o(\epsilon^2), \quad \text{or } \lambda = \epsilon\eta + o(\epsilon), \quad \lambda' = \epsilon\eta + o(\epsilon),$$

$$\phi = \sum_j c_j(h_j - \bar{h}_i) + \psi, \quad \phi = \sum_j c'_j(h_j - \bar{h}_i) + \psi', \quad \lim_{\epsilon \rightarrow 0} c_j = \lim_{\epsilon \rightarrow 0} c'_j = c_j^0.$$

But the two eigenfunctions must be orthogonal, so

$$\begin{aligned} 0 &= \langle \phi, \phi' \rangle \\ &= \sum_{j,k} c_j c'_k \langle h_j - \bar{h}_i, h_k - \bar{h}_i \rangle + O(|c|)\|\psi\|_2\|h_j\|_2 + O(|c|)\|\psi'\|_2\|h_j\|_2 + \|\psi\|_2\|\psi'\|_2 \\ &= \sum_j c_j^2 \int_0^1 h_j^2 dx + o(\epsilon)|c|^2 = \epsilon|c^0|^2 \int_{-\infty}^{\infty} (H'(t))^2 dt + o(\epsilon)|c^0|^2 \end{aligned}$$

by Remark 4.2. This is obviously impossible when  $\epsilon$  is sufficiently small.

Next  $\dim F$  is at least  $K$ . Suppose otherwise that  $\dim F < K$ . Define a subspace of  $\{\phi \in L^2(0,1) : \bar{\phi} = 0\}$ :  $S = \text{span}\{\sum_j c_j^0(h_j - \bar{h}_j) : \text{all } c^0 \text{ found in (5.51)}\}$ . We use a perturbation argument. The asymmetric distance between the closed subspaces  $S$  and  $F$  is

$$d(S, F) = \sup\{d(\varphi, F) : \varphi \in S, \|\varphi\|_2 = 1\}$$

where  $d(x, F) = \inf\{\|x - y\|_2 : y \in F\}$ . Since  $\dim F < \dim S$ , there exists  $\sum_j c_j^0(h_j - \bar{h}_j) \in S$  such that for every eigenvector in  $F$  which may be written as  $\sum_j c'_j(h_j - \bar{h}_j) + \psi$  with  $\|\psi\|_2 = O(\epsilon)|c'|$  according to (4.32),  $\sum_j \frac{c'_j}{|c'|} \frac{c_j^0}{|c^0|} = o(1)$ . Then a straight computation shows

$$\left\langle \frac{\sum_j c'_j(h_j - \bar{h}_j) + \psi}{\|\sum_j c'_j(h_j - \bar{h}_j) + \psi\|_2}, \frac{\sum_j c_j^0(h_j - \bar{h}_j)}{\|\sum_j c_j^0(h_j - \bar{h}_j)\|_2} \right\rangle = o(1).$$

So if we use  $\varphi = \frac{\sum_j c_j^0(h_j - \bar{h}_j)}{\|\sum_j c_j^0(h_j - \bar{h}_j)\|_2}$ ,  $d(\varphi, F) = 1 - o(1)$  and  $d(S, F) = 1 - o(1)$ . The following lemma due to Helffer and Sjöstrand [5] will give us a contradiction.

**Lemma 5.1** *Let  $L$  be a self-adjoint operator on a Hilbert space  $H$ ,  $R$  a compact interval in  $(-\infty, \infty)$  and  $e_1, e_2, \dots, e_K$  normalized linearly independent elements in the domain of  $L$ . Assume that the following are true.*

1.  $L(e_k) = p_k e_k + r_k$ ,  $\|r_k\| \leq \epsilon'$  and  $p_j \in R$ ,  $k = 1, 2, \dots, K$ .
2. There is  $\omega > 0$  so that  $R$  is  $\omega$ -isolated in the spectrum of  $L$ , i.e.  $(\sigma(L) \setminus R) \cap (R + (-\omega, \omega)) = \emptyset$ .

Then  $d(S, F) \leq \frac{K^{1/2}\epsilon'}{\omega\kappa^{1/2}}$  where  $S = \text{span}\{e_1, \dots, e_K\}$ ,  $F$  = the closed subspace associated to  $\sigma(L) \cap R$ , and  $\kappa$  = the smallest eigenvalue of the matrix  $[\langle e_j, e_k \rangle]$ .

Here we take  $L = L_0$ , each  $e_k \propto \sum_j c_j^0(h_j - \bar{h}_j)$  for each one of the  $K$  vectors  $c^0$ , and  $S, F$  as before.  $\omega$  and  $\kappa$  are positive and bounded away from 0 as  $\epsilon \rightarrow 0$ . Set  $p_k = \eta\epsilon^2$  or  $\eta\epsilon$  depending on  $c^0$  and  $R = [-\epsilon^{1/2}, \epsilon^{1/2}]$ . From (4.29) we find

$$L_0\left(\sum_j c_j^0(h_j - \bar{h}_j)\right) - p_k \sum_j c_j^0(h_j - \bar{h}_j) = O(\epsilon)|c^0|,$$

and on the other hand  $\|\sum_j c_j^0(h_j - \bar{h}_j)\|_2 \sim \epsilon^{1/2}|c^0|$  as discussed in Remark 4.2. Therefore  $\|r_k\|_2 = O(\epsilon^{1/2})$ . Consequently  $d(S, F) = o(1)$ , a contradiction. Part 2 of Theorem 1.1 is proved.

## 6 $m \neq (0, 0)$ eigenvalues

Rewrite (3.15) as

$$L_m(\phi) := -\epsilon^2\phi'' + \epsilon\gamma G_m[\phi] + f'(u)\phi = \mu\phi \quad (6.52)$$

where  $\mu = \lambda_m - \epsilon^2 m^2 \pi^2$ . In this section an eigen pair of (6.52) is denoted by  $(\mu, \phi)$ .

**Lemma 6.1** *If  $\mu \rightarrow \mu^0$  as  $\epsilon \rightarrow 0$ , then  $\mu^0 \geq 0$ .*

The proof of this lemma is almost identical to that of Claim 1 in Section 3 and we skip it, because the extra term  $\epsilon\gamma G_m[\phi]$  is of order  $O(\epsilon)$ . The case  $\mu^0 > 0$  does not concern us, so we assume  $\mu \rightarrow 0$ . Decompose  $\phi = \sum_j c_j h_j + \psi$  where  $\psi \perp h_j$ ,  $j = 1, 2, \dots, K$ . Note

$$L_m(h_j) = (f'(u) - f'(H))h_j + \epsilon\gamma G_m[h_j] + e.s..$$

Because of (3.18) and

$$\epsilon\gamma G_m[h_j](x) = \gamma\epsilon^2 G_m\left[\frac{h_j}{\epsilon}\right](x) = \gamma\epsilon^2 G_m(x, x_j) + O(\epsilon^3),$$

we deduce

$$L_m(h_j) = O(\epsilon^2). \quad (6.53)$$

We write (6.52) as

$$\sum_{j=1}^K c_j L_m(h_j) + L_m(\psi) = \mu \sum_j c_j h_j + \mu\psi. \quad (6.54)$$

**Lemma 6.2**  $\langle L_m(\psi), \psi \rangle \geq C\|\psi\|_2^2$  for some  $C > 0$  independent of  $\epsilon$ .

We skip the proof of this lemma since it is similar to that of Claim 2 in Section 3. Multiply (6.54) by  $\psi$  and integrate. Use (6.53) to deduce

$$|c|O(\epsilon^2)\|\psi\|_2 + \langle L_m(\psi), \psi \rangle = \mu\|\psi\|_2^2.$$

Then Lemma 6.2 implies

$$\|\psi\|_2 = O(\epsilon^2)|c|. \quad (6.55)$$

Next we multiply (6.54) by  $h_k$  and integrate. The left side is

$$\begin{aligned} & \int_0^1 \{L_m(\psi)h_k + \sum_j c_j L_m(h_j)h_k\} = \int_0^1 \{L_m(h_k)\psi + \sum_j c_j L_m(h_j)h_k\} \\ = & \int_0^1 \{(f'(u) - f'(H))h_k\psi + \epsilon\gamma G_m[h_k]\psi + e.s.\psi\} + \sum_j c_j \int_0^1 \{(f'(u) - f'(H))h_j h_k + G_m[h_j]h_k + e.s.h_k\}. \end{aligned} \quad (6.56)$$

All terms in (6.56) are estimated.

$$|\int_0^1 (f'(u) - f'(H))h_k\psi| \leq \|(f'(u) - f'(H))h_k\|_\infty \|\psi\|_2 = O(\epsilon^4)|c|$$

by (3.18) and (6.55).

$$\int_0^1 \epsilon\gamma G_m[h_k]\psi = O(\epsilon) \int_0^1 G_m[\psi]h_k = O(\epsilon)\|G_m[\psi]\|_\infty \|h_k\|_1 = O(\epsilon^2)\|\psi\|_2 = O(\epsilon^4)|c|$$

by (6.55). The rest of (6.56) are estimated as in Section 4:

$$\int_0^1 (f'(u) - f'(H))h_j h_k = \epsilon^3 s\delta_{jk} + O(\epsilon^4), \quad \int_0^1 \epsilon\gamma G_m[h_j]h_k = \gamma\epsilon^3 G_m(x_j, x_k) + o(\epsilon^3).$$

Hence (6.56) becomes

$$\epsilon^3 s c_k + \sum_j c_j \gamma \epsilon^3 G_m(x_j, x_k) + o(\epsilon^3)|c|.$$

The right side of (6.54) multiplied by  $h_k$  and integrated is

$$\int_0^1 \mu \sum_j c_j h_j h_k = \sum_j \mu \epsilon c_j \tau \delta_{jk} + e.s.|c| = \mu \epsilon \tau c_k + e.s.|c|.$$

Equating the last two quantities we find that  $\mu = O(\epsilon^2)$  and for every  $k$

$$s c_k + \gamma \sum_j G_m(x_j, x_k) c_j + o(1)|c| = \frac{\mu \tau}{\epsilon^2} c_k. \quad (6.57)$$

So in the limit

$$s c_k^0 + \gamma \sum_j G_m(x_j^0, x_k^0) c_j^0 = \eta \tau c_k^0. \quad (6.58)$$

Here  $\eta = \lim_{\epsilon \rightarrow 0} \mu/\epsilon^2 =$  and  $c_j^0 = \lim_{\epsilon \rightarrow 0} c_j$ . In the next section we will solve (6.58) to determine  $\eta$  and  $c^0$ . Once they are found we obtain the asymptotic expansions for the eigen pair  $\lambda_m = \mu + \epsilon^2 m^2 \pi^2$  and  $\phi_m = \phi$ :

$$\lambda_m = \epsilon^2(\eta + m^2 \pi^2) + o(\epsilon^2), \quad \phi_m \approx \sum_j c_j^0 h_j. \quad (6.59)$$

## 7 The spectrum of $[G_m(x_j^0, x_k^0)]$

When dealing with

$$\sum_j G_m(x_j^0, x_k^0) b_j = \Lambda b_k,$$

we first consider the simplest case  $K = 1$ . Then  $\Lambda = G_m(x_1^0, x_1^0)$ . On  $(0, x_1^0)$

$$G_m(x, x_1^0) = \frac{G_m(x_1^0, x_1^0)}{\cosh \tilde{m} x_1^0} \cosh \tilde{m} x,$$

where  $\tilde{m} = \pi \sqrt{m_1^2 + m_2^2}$ , and on  $(x_1^0, 1)$

$$G_m(x, x_1^0) = \frac{G_m(x_1^0, x_1^0)}{\cosh \tilde{m}(1 - x_1^0)} \cosh \tilde{m}(1 - x)$$

Then

$$1 = [-G'_m(\cdot, x_1^0)]_{x_1^0} = \left\{ \frac{\tilde{m} \sinh \tilde{m} x_1^0}{\cosh \tilde{m} x_1^0} + \frac{\tilde{m} \sinh \tilde{m}(1 - x_1^0)}{\cosh \tilde{m}(1 - x_1^0)} \right\} G_m(x_1^0, x_1^0).$$

Therefore

$$\Lambda = \frac{1}{\tilde{m}(\tanh \tilde{m} a + \tanh \tilde{m} b)},$$

and in (6.58)

$$\eta = \frac{1}{\tau} \left( -\gamma ab + \frac{\gamma}{\tilde{m}(\tanh \tilde{m} a + \tanh \tilde{m} b)} \right), \quad c^0 = 1. \quad (7.60)$$

To see the sign of  $\lambda_m$ , we recall

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_m}{\epsilon^2} = \eta + m^2 \pi^2 = \frac{1}{\tau} \left( -\gamma ab + \frac{\gamma}{\tilde{m}(\tanh \tilde{m} a + \tanh \tilde{m} b)} \right) + m^2 \pi^2.$$

The right side is positive for all  $m \neq (0, 0)$  if  $\gamma$  is small enough, because  $m^2 \pi^2$  dominates the negative term. However when  $\gamma$  is sufficiently large, we may find some large  $\tilde{m}$  that makes the right side negative. To see this we first take  $\tilde{m}$  large enough so that sum of the two terms in the parentheses is negative. Then we take  $\gamma$  large enough so the entire right side is negative.

When  $K \geq 2$ ,  $\sum_j G_m(x_j^0, x_k^0) b_j = \Lambda b_k$  is a more complex problem. Let  $\zeta$  be the solution of  $-\zeta'' + m^2 \pi^2 \zeta = \sum_j \delta(\cdot - x_j^0) b_j$ ,  $\zeta'(0) = \zeta'(1) = 0$ . Hence  $[-\zeta']_{x_k^0} = b_k$  and  $\zeta(x_k^0) = \Lambda b_k$ . Then for every  $k$   $[-\zeta']_{x_k^0} = \frac{1}{\Lambda} \zeta(x_k^0)$ . As in Section 5 we express  $[-\zeta']_{x_k^0} = (T\vec{\zeta})_k$  in order to convert to the new eigenvalue problem  $T\vec{\zeta} = (1/\Lambda)\vec{\zeta}$ . Away from  $x_j^0$ ,  $\zeta = g_1 \cosh \tilde{m} x + g_2 \sinh \tilde{m} x$ . From here we write, in the matrix notation

$$\begin{bmatrix} \zeta(x_{j-1}^0) \\ \zeta(x_j^0) \end{bmatrix} = \begin{bmatrix} \cosh \tilde{m} x_{j-1}^0 & \sinh \tilde{m} x_{j-1}^0 \\ \cosh \tilde{m} x_j^0 & \sinh \tilde{m} x_j^0 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}.$$

We denote the 2 by 2 matrix by  $A_L$ , for the left of  $x_j^0$ . To the right we have similarly

$$\begin{bmatrix} \zeta(x_j^0) \\ \zeta(x_{j+1}^0) \end{bmatrix} = \begin{bmatrix} \cosh \tilde{m} x_j^0 & \sinh \tilde{m} x_j^0 \\ \cosh \tilde{m} x_{j+1}^0 & \sinh \tilde{m} x_{j+1}^0 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

with the 2 by 2 matrix denoted by  $A_R$ . Hence

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = A_L^{-1} \begin{bmatrix} \zeta(x_{j-1}^0) \\ \zeta(x_j^0) \end{bmatrix} \text{ on } (x_{j-1}^0, x_j^0), \quad \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = A_R^{-1} \begin{bmatrix} \zeta(x_j^0) \\ \zeta(x_{j+1}^0) \end{bmatrix} \text{ on } (x_j^0, x_{j+1}^0).$$

Then

$$-\zeta'(x_j^0-) = -\tilde{m}[\sinh \tilde{m}x_j^0, \cosh \tilde{m}x_j^0]A_L^{-1} \begin{bmatrix} \zeta(x_{j-1}^0) \\ \zeta(x_j^0) \end{bmatrix}$$

$$-\zeta'(x_j^0+) = -\tilde{m}[\sinh \tilde{m}x_j^0, \cosh \tilde{m}x_j^0]A_R^{-1} \begin{bmatrix} \zeta(x_j^0) \\ \zeta(x_{j+1}^0) \end{bmatrix}$$

and

$$[-\zeta']_{x_j^0} = \tilde{m}[\sinh \tilde{m}x_j^0, \cosh \tilde{m}x_j^0] \left\{ A_L^{-1} \begin{bmatrix} \zeta(x_{j-1}^0) \\ \zeta(x_j^0) \end{bmatrix} - A_R^{-1} \begin{bmatrix} \zeta(x_j^0) \\ \zeta(x_{j+1}^0) \end{bmatrix} \right\}.$$

We also compute

$$A_L^{-1} = \frac{1}{\sinh \tilde{m}(x_j^0 - x_{j-1}^0)} \begin{bmatrix} \sinh \tilde{m}x_j^0 & -\sinh \tilde{m}x_{j-1}^0 \\ -\cosh \tilde{m}x_j^0 & \cosh \tilde{m}x_{j-1}^0 \end{bmatrix}$$

$$A_R^{-1} = \frac{1}{\sinh \tilde{m}(x_{j+1}^0 - x_j^0)} \begin{bmatrix} \sinh \tilde{m}x_{j+1}^0 & -\sinh \tilde{m}x_j^0 \\ -\cosh \tilde{m}x_{j+1}^0 & \cosh \tilde{m}x_j^0 \end{bmatrix}.$$

So  $T$  is a triangular matrix. The three entries of the  $j$ -th row where  $j \neq 1, K$ , are

$$-\tilde{m}\text{csch} \tilde{m}(x_j^0 - x_{j-1}^0), \quad \tilde{m} \coth \tilde{m}(x_j^0 - x_{j-1}^0) + \tilde{m} \coth \tilde{m}(x_{j+1}^0 - x_j^0), \quad -\tilde{m}\text{csch} \tilde{m}(x_{j+1}^0 - x_j^0).$$

For the first row,

$$\zeta(x) = \frac{\zeta(x_1^0)}{\cosh \tilde{m}x_1^0} \cosh \tilde{m}x$$

and

$$[-\zeta']_{x_1^0} = \tilde{m}(\tanh \tilde{m}x_1^0 + \coth \tilde{m}(x_2^0 - x_1^0))\zeta(x_1^0) - \tilde{m}\text{csch} \tilde{m}(x_2^0 - x_1^0)\zeta(x_2^0).$$

When  $K = 2$  the matrix  $T$  is

$$\begin{aligned} & \tilde{m} \begin{bmatrix} \tanh \tilde{m}a/2 + \coth \tilde{m}b & -\text{csch} \tilde{m}b \\ -\text{csch} \tilde{m}b & \tanh \tilde{m}b/2 + \coth \tilde{m}a \end{bmatrix} \\ &= \tilde{m}(\coth \tilde{m}a + \coth \tilde{m}b)I_{K \times K} - \tilde{m} \begin{bmatrix} \text{csch} \tilde{m}a & \text{csch} \tilde{m}b \\ \text{csch} \tilde{m}b & \text{csch} \tilde{m}a \end{bmatrix}. \end{aligned}$$

The two  $(1/\Lambda)$ 's are

$$\tilde{m}(\coth \tilde{m}a + \coth \tilde{m}b - \text{csch} \tilde{m}a - \text{csch} \tilde{m}b), \quad \tilde{m}(\coth \tilde{m}a + \coth \tilde{m}b - \text{csch} \tilde{m}a + \text{csch} \tilde{m}b)$$

which again lead to  $\eta$  for (6.58). To see the sign of  $\lambda_m$ , we take the smaller  $\Lambda$  so that the smaller  $\lambda_m$  satisfies

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_m}{\epsilon^2} = \eta + m^2\pi^2 = \frac{1}{\tau} \left( -\frac{\gamma ab}{2} + \frac{\gamma}{\tilde{m}(\coth \tilde{m}a + \coth \tilde{m}b - \text{csch} \tilde{m}a + \text{csch} \tilde{m}b)} \right) + m^2\pi^2.$$

As in the  $K = 1$  case the right side is positive for all  $m \neq (0, 0)$  if  $\gamma$  is small, and is negative for some  $m$  if  $\gamma$  is large.

When  $K \geq 3$  we write  $T = dI_{K \times K} - Q$  with

$$Q = \begin{bmatrix} \alpha & \beta & & & \\ \beta & 0 & \alpha & & \\ & \alpha & 0 & \beta & \\ & & \beta & 0 & \alpha \\ & & & & \dots \end{bmatrix}$$

where

$$\alpha = \tilde{m} \operatorname{csch} \frac{2\tilde{m}a}{K}, \quad \beta = \tilde{m} \operatorname{csch} \frac{2\tilde{m}b}{K}, \quad d = \tilde{m} \coth \frac{2\tilde{m}a}{K} + \tilde{m} \coth \frac{2\tilde{m}b}{K}.$$

Because of diagonal domination the matrix  $dI_{K \times K} - Q$  is positive definite. The  $K$  eigenvalues of  $Q$  are found in (B.5), Appendix B. we again denote them by  $q_j$ ,  $j = 1, 2, \dots, K$ . Then  $\Lambda = \frac{1}{d - q_j}$  and for (6.58)  $K$  eigen pairs

$$\eta = \frac{1}{\tau} \left( -\frac{\gamma ab}{K} + \frac{\gamma}{d - q_j} \right), \quad c^0 = \vec{\zeta} \quad (7.61)$$

are found. To see the sign of  $\lambda_m$ , we focus on the smallest  $\eta$ , which is associated with  $q_j = -\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta}$  where  $\theta = 2\pi/K$ . For this  $q_j$

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{\lambda_m}{\epsilon^2} &= \eta + m^2 \pi^2 = \frac{1}{\tau} \left( -\frac{\gamma ab}{K} + \frac{\gamma}{d - q_j} \right) + m^2 \pi^2 \\ &= \frac{1}{\tau} \left( -\frac{\gamma ab}{K} + \frac{\gamma}{d + \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta}} \right) + m^2 \pi^2. \end{aligned}$$

The dependence of the positivity of the right side on  $\gamma$  is still the same, i.e. the right side is positive for all  $m$  if  $\gamma$  is small, and negative for some  $m$  if  $\gamma$  is large. The dependence of the positivity on  $K$  is also clear. When  $K$  is large,  $m^2 \pi^2$  dominates the negative term, so the whole quantity is positive for all  $m$ .

**Remark 7.1** *If  $\gamma$  and  $K$  are held fixed, then the last line is positive if  $|m|$  is sufficiently large. This is consistent with Part 1 of Theorem 1.1.*

We omit the proof that  $L_m$  has exactly  $K$  simple eigen pairs with small eigenvalues, because it is similar to that for  $L_0$ . This concludes the proof of Part 3, Theorem 1.1.

To visualize Part 3, Theorem 1.1, consider the example  $a = 0.4$  and  $K = 6$ . We study  $m = (8, 0)$  and find the  $c^0$  associated with the smallest  $\lambda_{(8,0)}$  by numerically diagonalizing  $Q$ :

$$c^0 = (0.0424, -0.4774, 0.5199, -0.5199, 0.4774, -0.0424)^T.$$

The eigenfunction of  $L$  associated with this  $\lambda_{(8,0)}$  and  $c^0$  is approximately  $\sum_j c_j^0 h_j(x) \cos(8\pi y_1)$ . When  $\gamma$  is sufficiently large, we have  $\lambda_{(8,0)} < 0$ . For the energetic reason the unstable lamellar solution  $u$  may easily be deformed in the direction of this eigenfunction. In Figure 4 we make a cross section of  $D$ , perpendicular to the  $y_2$  direction. The first plot shows  $u$  on this cross section where the black color indicates  $u \approx 1$  and the white color indicates  $u \approx 0$ . The second plot shows  $u$  deformed by the eigenfunction. Note that under this deformation the straight interfaces in  $u$  become wriggled curves. See [7] for a heuristic argument for this change of shape.

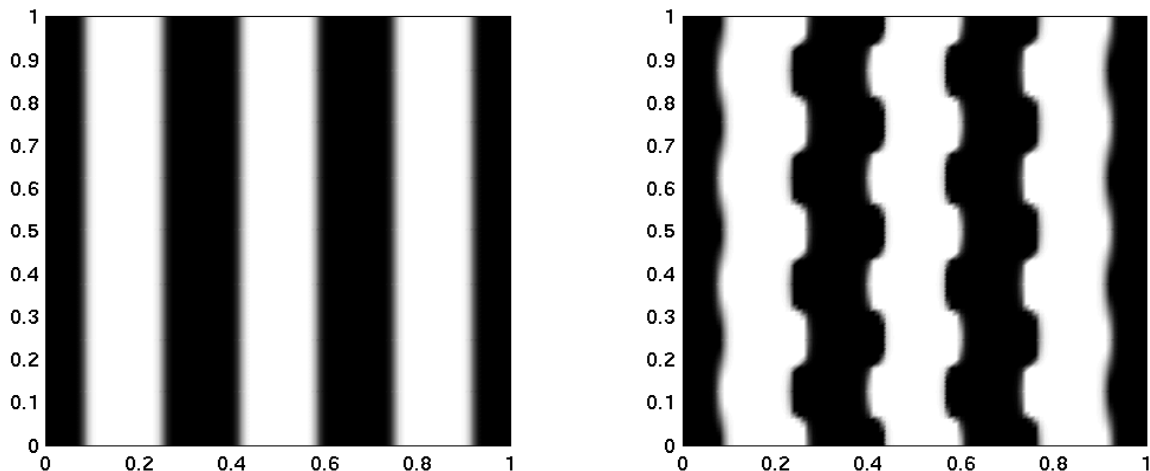


Figure 4: A lamellar solution  $u$  and its deformation in a cross section perpendicular to  $y_2$ .

## 8 The 1-D global minimizer

The integral  $\int_0^1 |(v^0)'|^2 dx$  in the conclusion of Theorem 2.1 may be calculated:

$$\begin{aligned} \int_0^1 |(v^0)'|^2 dx &= K \int_0^{a/K} |(v^0)'|^2 dx + K \int_{a/K}^{1/K} |(v^0)'|^2 dx \\ &= K \int_0^{a/K} (1-a)^2 x^2 dx + K \int_{a/K}^{1/K} a^2 \left(x - \frac{1}{K}\right)^2 dx = \frac{a^3 b^2}{3K^2} + \frac{a^2 b^3}{3K^2} = \frac{a^2 b^2}{3K^2}. \end{aligned}$$

Hence

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-1} I_1(u) = \tau K + \frac{\gamma a^2 b^2}{6K^2}. \quad (8.62)$$

It is shown in [12] that the 1-D global minimizer is a 1-D local minimizer whose number of interfaces  $K_*$  minimizes the right side of (8.62). Note that in some less likely cases two integers,  $K_*$  and  $K_* + 1$  may both minimize the right side of (8.62). Then we may have two global minimizers with  $K_*$  and  $K_* + 1$  interfaces respectively<sup>5</sup>. If we pretend that  $K$  is a positive real number and minimize the right side with respect to  $K$ , then the minimum is achieved at

$$K_* = \left(\frac{a^2 b^2 \gamma}{3\tau}\right)^{1/3}. \quad (8.63)$$

We set  $t = \tilde{m}/K$ . Consider the eigenvalue  $\lambda_m$  that is associated with the smallest  $\eta$  of Section 7.

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_m}{\epsilon^2} \frac{\tau \tilde{m}}{\gamma} = \frac{\tau \tilde{m}(\eta + m^2 \pi^2)}{\gamma}$$

<sup>5</sup>Actually there are 4 global minimizers because of Remark 2.2, if we include solutions of both types.

$$= -abt + \frac{1}{\coth 2at + \coth 2bt + \sqrt{(\operatorname{csch} 2at)^2 + (\operatorname{csch} 2bt)^2 + 2\operatorname{csch} 2at\operatorname{csch} 2bt \cos \theta}} + \frac{\tau K^3 t^3}{\gamma}.$$

For the 1-D global minimizer, we use  $K_*$  in (8.63) for  $K$  to find

$$\lim_{\epsilon \rightarrow 0} \frac{\lambda_m}{\epsilon^2} \frac{\tau \tilde{m}}{\gamma} = \frac{\tau \tilde{m}(\eta + m^2 \pi^2)}{\gamma}$$

$$= -abt + \frac{1}{\coth 2at + \coth 2bt + \sqrt{(\operatorname{csch} 2at)^2 + (\operatorname{csch} 2bt)^2 + 2\operatorname{csch} 2at\operatorname{csch} 2bt \cos \theta}} + \frac{a^2 b^2 t^3}{3}. \quad (8.64)$$

Note that a natural lower bound for the second term in (8.64) is

$$\frac{1}{\coth 2at + \coth 2bt + \operatorname{csch} 2at + \operatorname{csch} 2bt} = \frac{\sinh at \sinh bt}{\sinh t} = abt - \frac{a^2 b^2 t^3}{3} + \frac{(a^2 b^2 + 2a^3 b^3)t^5}{45} + \dots$$

by replacing  $\cos \theta$  by 1. This lower bound is sharp if  $K_*$  is large, i.e.  $\gamma$  is large. The first three terms of the Taylor expansion are given. We observe that the first two terms in the Taylor expansion are *exactly* canceled by the first and the third terms in (8.64). This is certainly no coincidence. The fifth order term is positive. Our numerical tests confirm that the entire (8.64) remains positive. The particular  $K_*$  of (8.63) is barely large enough to overcome the negative third order term in  $\frac{\sinh at \sinh bt}{\sinh t}$ .

To contemplate the physical significance of the shaky stability property of the 1-D global minimizer, we first note that the value (8.63) for  $K_*$  is only an approximate. But the 1-D global minimizer is very close to the borderline of 3-D stability. 1-D local minimizers with larger  $K$  are likely to be stable in 3-D, and 1-D local minimizers with smaller  $K$  are likely to be unstable in 3-D. In the real physical system only the 3-D global minimizer, which is unlikely to be lamellar, is the thermal equilibrium. Other stable solutions of (1.2) are only transient, meta-stable states. In general lamellar phases, including the 1-D global minimizer, are transient. They are vulnerable to perturbations of the form  $\sum_j c_j^0 h_j(x) \cos(m_1 \pi y_1) \cos(m_2 \pi y_2)$  found in Part 3 of Theorem 1.1, which push the straight interfaces in a lamellar state to a wriggled shape, Figure 4. In a forthcoming paper [11] we will actually prove, using bifurcation analysis, that (1.2) admits wriggled solutions for some values of  $\gamma$ .

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## A Proof of Theorem 2.3

**Lemma A.1**  *$u$  has exactly  $K$  transition layers, in the sense that there are exactly  $K$  points,  $x_1, x_2, \dots, x_K$  in  $(0, 1)$ , so that  $u(x_j) = 1/2$ ,  $j = 1, 2, \dots, K$ , and  $\lim_{\epsilon \rightarrow 0} x_j = x_j^0$ .*

The proof of this lemma is similar to that of Proposition 8.2 [15].

**Lemma A.2** *The derivative of  $v = G_0[u - a]$  has exactly  $K - 1$  zeros, denoted by  $y_1, y_2, \dots, y_{K-1}$ , in  $(0, 1)$ , such that  $\lim_{\epsilon \rightarrow 0} y_j = j/K$ .*

*Proof.* The derivative of  $v^0 = G_0[u^0 - a]$  has zeros at  $1/K, 2/K, \dots, (K-1)/K$ . The convergence of  $v'$  to  $(v^0)'$  implies that  $v'$  has exactly  $K-1$  zeros  $y_j$  with the property  $\lim_{\epsilon \rightarrow 0} y_j = j/K$ .  $\square$

We set  $y_0 = 0$  and  $y_K = 1$ . Let  $l_i = y_i - y_{i-1}, i = 1, \dots, K$ . Between two zeros of  $v'$  we integrate the equation  $-v'' = u - a$  and find  $\frac{1}{l_i} \int_{y_{i-1}}^{y_i} u dx = a$ . This allows us to localize the energy of  $u$  on  $(y_{i-1}, y_i)$ . If we set  $l_i \xi + y_{i-1} = x$ ,  $\mathcal{U}_i(\xi) = u(x)$ , and  $\mathcal{V}_i(\xi) = l_i^{-2} v(x) = l_i^{-2} G_0[u - a](x)$ , then  $\int_0^1 \mathcal{U}_i dz = a$ ,  $-\mathcal{V}_i'' = \mathcal{U}_i - a$ ,  $\mathcal{V}_i'(0) = \mathcal{V}_i'(1) = 0$ . More importantly

$$\begin{aligned} I_1(u) &= \sum_{i=1}^K \int_{y_{i-1}}^{y_i} \left\{ \frac{\epsilon^2}{2} |u|^2 + \frac{\gamma \epsilon}{2} |v'|^2 + W(u) \right\} dx \\ &= \sum_{i=1}^K l_i \int_0^1 \left\{ \frac{\epsilon^2}{2l_i^2} |\mathcal{U}_i'|^2 + \frac{l_i^2 \gamma \epsilon}{2} |\mathcal{V}_i'|^2 + W(\mathcal{U}_i) \right\} d\xi = \sum_{i=1}^K l_i J_l(\mathcal{U}_i), \end{aligned} \quad (\text{A.1})$$

if we define a new variational problem:

$$J_l(\mathcal{U}) = \int_0^1 \left\{ \frac{\epsilon^2}{2l^2} |\mathcal{U}'|^2 + \frac{l^3 \gamma \epsilon}{2} \left| \left( -\frac{d^2}{d\xi^2} \right)^{-1/2} (\mathcal{U} - a) \right|^2 + W(\mathcal{U}) \right\} d\xi, \quad \mathcal{U} \in X_a. \quad (\text{A.2})$$

This new  $J_l$  is similar to the original  $I_1$ .  $l$  lies in a compact sub-interval of  $(0, 1)$ , so we take  $l \sim 1$ . We consider a one layer local minimizer  $\mathcal{U}$  that is close to  $\mathcal{U}^0$  which is 0 on  $(0, 1-a)$  and 1 on  $(1-a, 1)$ . The dependence of  $\mathcal{U}$  on  $l$  and  $\epsilon$  is suppressed in the notation. It is proved in Proposition 9.2 of [15] that this local minimizer is unique in a  $L^2$  ball centered at  $\mathcal{U}^0$  of radius  $\delta$ .  $\delta$  is small but independent of  $\epsilon$ . Denote the transition point of  $\mathcal{U}$  by  $\chi$ , i.e.  $\mathcal{U}(\chi) = 1/2$ . This one-layer local minimizer has the following asymptotic expansion.

**Lemma A.3** *Let  $\tilde{\epsilon} = \frac{\epsilon}{l}$  and  $\tilde{\gamma} = l^3 \gamma$ . Then  $\mathcal{U} = H\left(\frac{\cdot - \chi}{\tilde{\epsilon}}\right) + \tilde{\epsilon} \mathcal{Z}$  with  $\|\mathcal{Z} - \mathcal{Z}^0\|_\infty = O(\tilde{\epsilon})$ . Here  $\mathcal{Z}^0 = -\frac{\tilde{\gamma}(\mathcal{V}^0 - \mathcal{V}^0(1-a))}{f'(0)}$ ,  $\mathcal{V}^0 = G_0[\mathcal{U}^0 - a]$ . Note  $\mathcal{Z}^0(1-a) = 0$ .*

*Proof.* See Proposition 8.3 in [15].  $\square$

**Lemma A.4** *Let  $F \in C^2(-\infty, \infty)$  be such that  $F(0) = F(1) = 0$ . Then*

$$\int_0^1 F(\mathcal{U}) d\xi = \tilde{\epsilon} \int_{-\infty}^{\infty} F(H) dt + \tilde{\epsilon} \int_0^{1-a} F'(0) \mathcal{Z}^0 d\xi + \tilde{\epsilon} \int_{1-a}^1 F'(1) \mathcal{Z}^0 d\xi + O(\tilde{\epsilon}^2).$$

*Proof.* See Lemma 8.4 in [15].  $\square$

**Lemma A.5** *Let  $\mathcal{W} = \frac{\partial \mathcal{U}}{\partial l}$ . Then*

$$\mathcal{W}(\xi) = H'\left(\frac{l(\xi - \chi)}{\epsilon}\right) \frac{\xi - \chi}{\epsilon} - \overline{H'\left(\frac{l(\xi - \chi)}{\epsilon}\right) \frac{\xi - \chi}{\epsilon}} + \varphi,$$

*with  $\|\varphi\|_2 = O(1)$ . And  $\varphi = c(h - \bar{h}) + \psi$ , with  $h = H'\left(\frac{l(\xi - \chi)}{\epsilon}\right)$ ,  $h - \bar{h} \perp \psi$ ,  $c = O(\epsilon^{-1/2})$  and  $\|\psi\|_2 = O(\epsilon)$ .*

*Proof.* The brief argument here summarizes the more elaborate proof of the similar Proposition 9.3 in [15]. Differentiate the Euler-Lagrange equation of (A.2) with respect to  $l$  to obtain

$$-\left(\frac{\epsilon}{l}\right)^2 \mathcal{W}'' + \gamma \epsilon l^2 G_0[\mathcal{W}] + f'(\mathcal{U})\mathcal{W} + 4\gamma \epsilon l G_0[u - a] + \frac{2}{l} f(\mathcal{U}) - \frac{2}{l} \overline{f(\mathcal{U})} = \frac{d\overline{f(\mathcal{U})}}{dl}, \quad (\text{A.3})$$

for  $\mathcal{W}$ . Define  $g(\xi) = H'\left(\frac{l(\xi-x)}{\epsilon}\right) \frac{\xi-x}{\epsilon} + e.s.$  so that  $g$  and its derivative vanish at 0 and 1, and  $\varphi = \mathcal{W} - (g - \bar{g})$ .  $g$  satisfies the equation

$$-\left(\frac{\epsilon}{l}\right)^2 g'' + f'(H)g + \frac{2}{l} f(H) = e.s..$$

Subtract this from (A.3) and use the facts  $\|g\|_2 = O(\epsilon^{1/2})$ ,  $\bar{g} = O(\epsilon)$ , and  $(f'(\mathcal{U}) - f'(H))(g - \bar{g}) = O(\epsilon^2)$ , where the last one follows from Lemma A.3, to deduce the equation for  $\varphi$ :

$$-\left(\frac{\epsilon}{l}\right)^2 \varphi'' + \epsilon \gamma l^2 G_0[\varphi] + f'(\mathcal{U})\varphi + O(\epsilon) = \text{Const.} \quad (\text{A.4})$$

where we simply write *Const.* for a constant since its exact value is not needed in this proof. We multiply this equation by  $\varphi$  and integrate:

$$\int_0^1 \left\{ \left(\frac{\epsilon}{l}\right)^2 |\varphi'|^2 + \epsilon \gamma l^2 G_0[\varphi]\varphi + f'(\mathcal{U})\varphi^2 \right\} dz = O(\epsilon) \|\varphi\|_2.$$

By Proposition 9.1 [15] we find  $\|\varphi\|_2 = O(1)$ .

Decompose  $\varphi = c(h - \bar{h}) + \psi$  where  $h = H'\left(\frac{l(\xi-x)}{\epsilon}\right) + e.s.$  and  $h - \bar{h} \perp \psi$ . The exponentially small correction term *e.s.* is added so that  $h$  and  $h'$  vanish at 0 and 1. Then

$$c = \frac{\int_0^1 \varphi(h - \bar{h}) dz}{\|h - \bar{h}\|_2^2} \leq \frac{\|\varphi\|_2}{\|h - \bar{h}\|_2} = O(\epsilon^{-1/2}).$$

The equation satisfied by  $\psi$  is

$$-\left(\frac{\epsilon}{l}\right)^2 \psi'' + f'(\mathcal{U})\psi + O(\epsilon) = \text{Const.}$$

where we have used the fact  $(f(\mathcal{U}) - f(H))h = O(\epsilon^2)$ , again a consequence of Lemma A.3. Argue as in Lemma 4.1 to deduce  $\int_0^1 \left\{ -\left(\frac{\epsilon}{l}\right)^2 \psi'' + f'(\mathcal{U})\psi \right\} \psi d\xi \geq C\|\psi\|_2^2$ , which implies  $\|\psi\|_2 = O(\epsilon)$ .  $\square$

**Lemma A.6** *Let  $E(l) = lJ_l(\mathcal{U})$ . Then  $E(l)$  is strictly convex in  $l$  in any compact subset of  $(0, 1)$ .*

*Proof.* This lemma is similar to Proposition 10.1 in [15]. Differentiating  $E$  with respect to  $l$  yields

$$\frac{\partial E}{\partial l} = \int_0^1 \left\{ -\frac{\epsilon^2}{2l^2} |\mathcal{U}'|^2 + W(\mathcal{U}) + \frac{3\epsilon \gamma l^2}{2} |\mathcal{V}'|^2 \right\} d\xi, \quad (\text{A.5})$$

where  $\mathcal{V} = G_0[\mathcal{U} - a]$ . We have used the fact that  $\mathcal{U}$  is a critical point of  $J_l$ . Differentiate (A.5) with respect to  $l$ :

$$\frac{\partial^2 E}{\partial l^2} = \int_0^1 \left\{ \frac{\epsilon^2}{l^3} |\mathcal{U}'|^2 + 3\epsilon \gamma l |\mathcal{V}'|^2 \right\} d\xi + \int_0^1 \{ 2f(\mathcal{U})\mathcal{W} + 4\epsilon \gamma l^2 \mathcal{V}\mathcal{W} \} d\xi.$$

Call the first integral on the right side  $T_1$  and the second integral  $T_2$ . Multiplying the Euler-Lagrange equation of  $\mathcal{U}$  by  $\mathcal{U} - a$  and integrating by parts, we find the useful integral identity

$$\int_0^1 \left\{ \left(\frac{\epsilon}{l}\right)^2 |\mathcal{U}'|^2 + f(\mathcal{U})(\mathcal{U} - a) + \epsilon \gamma l^2 |\mathcal{V}'|^2 \right\} d\xi = 0.$$

Using this identity and Lemma A.4 we obtain

$$\begin{aligned} T_1 &= \frac{1}{l} \int_0^1 \left\{ -f(\mathcal{U})(\mathcal{U} - a) + 2\epsilon \gamma l^2 |\mathcal{V}'|^2 \right\} d\xi \\ &= \frac{\epsilon}{l^2} \int_{-\infty}^{\infty} -f(H)(H - a) dt + \frac{\epsilon \gamma l a^2 b^2}{3} + \frac{2\epsilon \gamma l a^2 b^2}{3} + O(\epsilon^2) \\ &= \frac{\epsilon}{l^2} \int_{-\infty}^{\infty} -f(H)H dt + \epsilon \gamma l a^2 b^2 + O(\epsilon^2). \end{aligned}$$

Here we have used

$$\int_0^1 |\mathcal{V}'|^2 d\xi = \int_0^1 |\mathcal{V}'_0|^2 d\xi + O(\epsilon) = \frac{a^2 b^2}{3} + O(\epsilon)$$

which follows from (8.17) in [15]. By Lemma A.3 and Lemma A.5

$$\begin{aligned} T_2 &= \int_0^1 (2f(H) + O(\epsilon)) \left( \frac{\xi - \chi}{\epsilon} H' + cH' - c\overline{H}' + \psi \right) d\xi \\ &= \frac{\epsilon}{l^2} \int_{-l\chi/\epsilon}^{l(1-\chi)/\epsilon} 2f(H(t))H'(t)t dt + O(\epsilon^{1.5}) = \frac{\epsilon}{l^2} \int_{-\infty}^{\infty} -2W(H) dt + O(\epsilon^{1.5}). \end{aligned}$$

We have used the estimates

$$\begin{aligned} \int_0^1 \left| \frac{\xi - \chi}{\epsilon} H' \right| d\xi &= \frac{\epsilon}{l^2} \int_{-l\chi/\epsilon}^{l(1-\chi)/\epsilon} |H'(t)t| dt = O(\epsilon), \\ \int_0^1 |f(H)| d\xi &= \frac{\epsilon}{l} \int_{-l\chi/\epsilon}^{l(1-\chi)/\epsilon} |f(H(t))| dt = O(\epsilon) \\ \|2f(H) + O(\epsilon)\|_2 &= O(\epsilon^{1/2}). \\ \int_0^1 f(H)H' d\xi &= e.s.. \end{aligned}$$

Adding  $T_1$  and  $T_2$ , since  $\int_{-\infty}^{\infty} (f(H)H + 2W(H)) dt = 0$  (a consequence of the integral identity  $\int_{-\infty}^{\infty} \{(H')^2 + f(H)H\} dt = 0$  and the first integral of  $H$ ), we arrive at

$$\frac{\partial^2 E}{\partial l^2} = \epsilon l \gamma a^2 b^2 + O(\epsilon^{1.5}), \tag{A.6}$$

proving the lemma.  $\square$

*Proof of Theorem 2.3.* We construct a particular periodic solution  $u_*$  with  $K$  transition layers, and show that  $u = u_*$ . Let  $\mathcal{U}$  be the unique minimum of  $J_l$  in a  $\delta$ -neighborhood of  $\mathcal{U}^0$ , with  $l = 1/K$ , and  $\mathcal{U}^R = \mathcal{U}(1 - \cdot)$  be its reversal. Set  $u_*(x) = \mathcal{U}^R(Kx)$  for  $x \in (0, 1/K)$ . Extend  $u_*$  anti-periodically to  $(0, 1)$ , i.e.  $u_*(x) = \mathcal{U}(Kx - 1)$  for  $x \in (1/K, 2/K)$ ,  $u_*(x) = \mathcal{U}^R(Kx - 2)$  for  $x \in (2/K, 3/K)$ ... Clearly  $u_*$  is periodic with  $K/2$  periods.

For small  $\epsilon$ ,  $u$  and  $u_*$  belong to the same small  $L^2$  neighborhood in which  $u$  is a minimizer. Using the strict convexity of  $E$  in Lemma A.6 and (A.1) we find

$$I(u_*) \geq I(u) = \sum_{i=1}^K l_i J_{l_i}(u(l_i \cdot + y_{i-1})) \geq \sum_{i=1}^K E(l_i) \geq KE\left(\frac{1}{K}\right) = I(u_*).$$

All the inequalities above must be equalities. Therefore  $l_i = 1/K$ ,  $y_i = i/K$  for all  $i$ , and  $l_i J_{l_i}(u(l_i \cdot + y_{i-1})) = E(l_i)$ . Moreover  $u((1/K) \cdot + y_{i-1}) = \mathcal{U}$  when  $i$  is even or  $= \mathcal{U}^R$  when  $i$  is odd by the local uniqueness of  $\mathcal{U}$  and  $\mathcal{U}^R$ , Proposition 9.2 [15]. Thus  $u = u_*$ .  $\square$

## B The matrix $Q$

Consider a matrix  $Q$  like those in Sections 5 and 7 with  $\alpha, \beta > 0$ . In this appendix  $Q$ , whose size is at least 3 by 3, acts on the complex vector space  $\mathbf{C}^K$ . Let  $\vec{q} = (z, tz^2, z^3, tz^4, \dots)^T$  where  $z, t \in \mathbf{C}$  and  $|z| = 1$ . Suppose the eigenvalue problem  $Q\vec{q} = q\vec{q}$  holds for the second through the second last equations, excluding the first and the last. In these  $K - 2$  equations

$$\begin{cases} \alpha tz^{l-1} + \beta tz^{l+1} = qz^l & \text{if } l \text{ is odd} \\ \beta z^{l-1} + \alpha z^{l+1} = qtz^l & \text{if } l \text{ is even} \end{cases} \quad (\text{B.1})$$

They imply

$$t = \pm \frac{\alpha z + \beta \bar{z}}{|\alpha z + \beta \bar{z}|}. \quad (\text{B.2})$$

In particular  $|t| = 1$ . In order to have the first and the last equations satisfied, we let  $\vec{h} = A\vec{q} + B\vec{q}$  and study  $Q\vec{h} = q\vec{h}$ .

If the vector  $\vec{q}$  is extended by  $t$  as the 0th entry and by  $tz^{K+1}$  as the  $(K+1)$ th entry if  $K$  is odd, or by  $z^{K+1}$  as the  $(K+1)$ th entry if  $K$  is even, then the first and the last equations of  $Q\vec{h} = q\vec{h}$  are satisfied if the 0th entry is equal to the first entry and the  $K$ th entry is equal to the  $(K+1)$ th entry. That is

$$\begin{cases} Az + B\bar{z} = At + B\bar{t} \\ Az^K + B\bar{z}^K = Atz^{K+1} + B\bar{t}\bar{z}^{K+1} \end{cases} \quad \text{if } K \text{ is odd;} \quad (\text{B.3})$$

$$\begin{cases} Az + B\bar{z} = At + B\bar{t} \\ Atz^K + B\bar{t}\bar{z}^K = Az^{K+1} + B\bar{z}^{K+1} \end{cases} \quad \text{if } K \text{ is even.} \quad (\text{B.4})$$

They should have nontrivial solutions for  $A$  and  $B$ . In the case of (B.3) this means

$$(z - t)(1 - \bar{t}\bar{z})\bar{z}^K = (\bar{z} - \bar{t})(1 - tz)z^K$$

or, since  $|z| = |t| = 1$ ,  $z^{2K} = 1$ . The case (B.4) gives the same condition. Define  $\theta = \frac{2\pi(j-1)}{K}$ ,  $j = 1, 2, \dots, 2K$ . Then  $z = e^{i\theta/2}$ . From (B.1) we find

$$q = \alpha t\bar{z} + \beta tz = \pm \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta}.$$

Here  $\theta$  ranges from 0 to  $4\pi - (2\pi/K)$ , a too wide range. We restrict  $j$  to  $1, 2, \dots, (K+1)/2$  if  $K$  is odd and  $j = 1, 2, \dots, K/2+1$  if  $K$  is even. Even then we have some unwanted values. When  $z = 1$  and  $t = -1$ , which occur if  $\theta = 0$  and  $q = -(\alpha + \beta)$ , we find  $A + B = 0$  and  $\vec{h} = \vec{0}$  which is not an eigenvector. Also when  $K$  is even,  $z = i$  and  $t = -i$ , which occur if  $\theta = \pi$  and  $q = \beta - \alpha$ , we find  $A - B = 0$  and again  $\vec{h} = \vec{0}$ . In summary the  $K$  distinct eigenvalues of  $Q$  are

$$\begin{aligned} & \alpha + \beta, \pm\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta} \quad (\theta = \frac{2\pi(j-1)}{K}, j = 2, 3, \dots, \frac{K+1}{2}), \quad \text{if } K \text{ is odd;} \\ & \alpha + \beta, \pm\sqrt{\alpha^2 + \beta^2 + 2\alpha\beta \cos \theta} \quad (\theta = \frac{2\pi(j-1)}{K}, j = 2, 3, \dots, \frac{K}{2}), \quad \alpha - \beta, \quad \text{if } K \text{ is even.} \end{aligned} \quad (\text{B.5})$$