

## ON MULTIPLE RADIAL SOLUTIONS OF A SINGULARLY PERTURBED NONLINEAR ELLIPTIC SYSTEM\*

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**Abstract.** We study radial solutions of a singularly perturbed nonlinear elliptic system of the FitzHugh–Nagumo type. In a particular parameter range, we find a large number of layered solutions. First we show the existence of solutions whose layers are well separated from each other and also separated from the origin and the boundary of the domain. Some of these solutions are local minimizers of a related functional while the others are critical points of saddle type. Although the local minimizers may be studied by the  $\Gamma$ -convergence method, the reduction procedure presented in this paper gives a more unified approach that shows the existence of both local minimizers and saddle points. Critical points of both types are all found in the reduced finite dimensional problem. The reduced finite dimensional problem is solved by a topological degree argument. Next we construct solutions with odd numbers of layers that cluster near the boundary, again using the reduction method. In this case the reduced finite dimensional problem is solved by a maximization argument.

**Key words.** radial solutions, singularly perturbed elliptic system, Lyapunov–Schmidt reduction,  $\Gamma$ -convergence

**AMS subject classifications.** 35J50, 35J55

**DOI.** 10.1137/050643507

**1. Introduction.** We consider the singularly perturbed elliptic system

$$(1.1) \quad \begin{cases} -\epsilon^2 \Delta u + f(u) + \epsilon \gamma v = 0 & \text{in } \Omega, \\ -\Delta v + v - u = 0 & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega \end{cases}$$

on a smooth bounded domain  $\Omega$ . The perturbation parameter  $\epsilon$  is positive and small. The outward normal derivatives of  $u$  and  $v$  on the boundary of  $\Omega$  are denoted by  $\partial_\nu u$  and  $\partial_\nu v$ , respectively.

The nonlinear function  $f$  in (1.1) is the cubic polynomial

$$(1.2) \quad f(u) = (u - a) \left( u - \frac{a + b}{2} \right) (u - b).$$

It has three zeros  $a$ ,  $\frac{a+b}{2}$ , and  $b$ , in the increasing order. The function is balanced in the sense that

$$(1.3) \quad \int_a^b f(q) dq = 0.$$

The nonlinearity in the system (1.1) is of the FitzHugh–Nagumo type. It was originally proposed to study nerve impulses [10, 18]. The phenomenon that is modeled

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\*Received by the editors October 25, 2005; accepted for publication (in revised form) October 30, 2006; published electronically April 6, 2007.

<http://www.siam.org/journals/sima/38-6/64350.html>

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is the control of the electrical potential across a cell membrane. This control is done by the change of flow of the ionic channels of the cell membrane. This results in the change in potential which is used to send electrical signals between cells. This is readily observed in muscle and other excitable cells. The two variables in the system are the excitable variable  $u$  and the recovery variable  $v$ . The dynamics of the two variables are described by the reaction-diffusion system

$$(1.4) \quad \begin{cases} u_t = \epsilon^2 \Delta u - f(u) - \epsilon \gamma v, \\ \kappa v_t = \Delta v - v + u, \\ \partial_\nu u = \partial_\nu v = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Steady state solutions of (1.1) often have layered structures. In most parts of the domain  $\Omega$ , a solution  $u$  is close to  $a$  or  $b$ . However, there exist small regions in  $\Omega$  where the value of  $u$  changes abruptly from  $a$  to  $b$ . These regions are called transition layers or interfaces.

The parameter range in this paper differs from the more extensively studied one where  $\epsilon$  does not appear in the  $\epsilon \gamma v$  term in the first equation of (1.1) (see, for example, [4, 3, 5, 7, 6, 9, 14, 11, 12, 20, 19, 21, 24]). We will show that the parameter range considered in this paper typically gives solutions with a finite number of interior layers. In the parameter range without  $\epsilon$  in the  $\epsilon \gamma v$  term, the number of interior layers of a solution typically approaches infinity as  $\epsilon \rightarrow 0$  (see [4, 17, 23, 1] for this type of phenomenon). The reason for this difference is that with  $\epsilon$  in  $\epsilon \gamma v$  there is less impact from the coupling effect with  $v$ , and hence there are fewer layers in a solution.

If we solve the second equation in (1.1) for  $v$  in terms of  $u$  with the boundary condition  $\partial_\nu v = 0$  on  $\partial\Omega$ , i.e.,  $v = (1 - \Delta)^{-1}u$ , and substitute the solution into the first equation, we obtain the equation for  $u$ :

$$(1.5) \quad -\epsilon^2 \Delta u + f(u) + \epsilon \gamma (1 - \Delta)^{-1}u = 0 \text{ in } \Omega, \quad \partial_\nu u = 0 \text{ on } \partial\Omega.$$

This integro-differential equation can be viewed as the Euler–Lagrange equation of the functional

$$(1.6) \quad I_\epsilon(u) = \int_\Omega \left\{ \frac{\epsilon^2}{2} |\nabla u|^2 + W(u) + \frac{\epsilon \gamma}{2} |(1 - \Delta)^{-1/2} u|^2 \right\} dx.$$

Here  $W$  is an antiderivative of  $f$ ; i.e.,

$$(1.7) \quad W(u) = \int_a^u f(q) dq.$$

Note that  $W(u) \geq 0$  for all  $u \in (-\infty, \infty)$  and  $W(u) = 0$  if and only if  $u = a$  or  $u = b$ . Note that  $(1 - \Delta)^{-1/2}$  is a nonlocal linear operator. One first defines  $(1 - \Delta)^{-1}$  so that  $v = (1 - \Delta)^{-1}u$  is the solution of

$$-\Delta v + v = u \text{ in } \Omega, \quad \partial_\nu v = 0 \text{ on } \partial\Omega.$$

Since  $(1 - \Delta)^{-1}$  is a positive operator from  $L^2(\Omega)$  to itself, we define  $(1 - \Delta)^{-1/2}$  to be the positive square root of  $(1 - \Delta)^{-1}$ .

The fast inhibitor limit of (1.4) is the parabolic-elliptic system

$$(1.8) \quad \begin{cases} u_t = \epsilon^2 \Delta u - f(u) - \epsilon \gamma v, \\ 0 = \Delta v - v + u, \\ \partial_\nu u = \partial_\nu v = 0 \quad \text{on } \partial\Omega \end{cases}$$

obtained by setting  $\kappa = 0$  in (1.4). This is the gradient flow of  $I_\epsilon$  in the  $L^2(\Omega)$  space. Finding and classifying the critical points of  $I_\epsilon$  (solutions of (1.1)) help us understand the behavior of (1.8).

In this paper we look for radial solutions of (1.1) on a unit ball in  $R^n$ :

$$(1.9) \quad \Omega = \{x \in R^n : |x| < 1\}.$$

The functional  $I_\epsilon$  is therefore defined in the admissible set of radial  $W^{1,2}$  functions:

$$(1.10) \quad \{u \in W^{1,2}(\Omega) : u = u(|x|)\}.$$

In the first part of the paper, we study layered solutions whose interfaces are well separated and away from the origin and the boundary. We prove two theorems.

**THEOREM 1.1.** *Let  $a < 0 < b$  and  $K$  be a positive integer. There exists  $\gamma_0 > 0$  such that for each  $\gamma > \gamma_0$ , there exists  $\epsilon_0 > 0$  so that when  $\epsilon \in (0, \epsilon_0)$  there are four solutions, each of which has  $K$  interfaces. Two of the four solutions, if denoted by  $u_\epsilon^a$ , satisfy  $\lim_{\epsilon \rightarrow 0} u_\epsilon^a(0) = a$ , and the other two, if denoted by  $u_\epsilon^b$ , satisfy  $\lim_{\epsilon \rightarrow 0} u_\epsilon^b(0) = b$ .*

**THEOREM 1.2.**

1. *If  $a < b < 0$  and  $\gamma > \frac{(n-1)\tau}{(b-a)a}$ , there exists  $\epsilon_0 > 0$  such that for each  $\epsilon < \epsilon_0$ , there is a one-interface solution  $u_\epsilon$  with the property  $\lim_{\epsilon \rightarrow 0} u_\epsilon(0) = b$ .*
2. *If  $0 < a < b$  and  $\gamma > \frac{(n-1)\tau}{(b-a)a}$ , there exists  $\epsilon_0 > 0$  such that for each  $\epsilon < \epsilon_0$ , there is a one-interface solution  $u_\epsilon$  with the property  $\lim_{\epsilon \rightarrow 0} u_\epsilon(0) = a$ .*

The constant  $\tau$  here is a positive number, often called the surface tension. It is given in (2.5). The proof of Theorem 1.1 uses a type of Lyapunov–Schmidt reduction procedure tailored for singular perturbation problems. It consists of two steps. First we reduce  $I_\epsilon$  to a functional  $Q_\epsilon$  that is defined on a finite dimensional set. This set is really the coordinates of interfaces. In this step we construct a family of approximate solutions with  $K$  interfaces whose coordinates serve as parameters. The family is a finite dimensional submanifold of the admissible set of  $I_\epsilon$ . Near each approximate solution we find a function that “solves” (1.5) in a direction that is more or less perpendicular to the submanifold. These functions are again parameterized by their interfaces and they form an improved finite dimensional submanifold. The restriction of  $I_\epsilon$  on this new submanifold is  $Q_\epsilon$ , which is viewed as a function of the interfaces. As a consequence of this construction, we show that a critical point of  $Q_\epsilon$  is a solution of (1.5).

In the second step of the proof, we look for critical points of  $Q_\epsilon$ . We show that  $\epsilon^{-1}Q_\epsilon$  converges in  $C^1_{loc}$  to a function  $J$  as  $\epsilon \rightarrow 0$ . When  $\gamma$  is sufficiently large,  $J$  has a minimum. Near this local minimum  $Q_\epsilon$  also has a minimum for small  $\epsilon$ . The topological degree of  $J$  and hence that of  $Q_\epsilon$  are shown to be 0. We then conclude that when  $\gamma$  is large, there are at least two critical points of  $Q_\epsilon$ .

The proof of Theorem 1.2 is similar. After the same reduction procedure, we show that when  $\gamma$  is large, the reduced problem  $J$  has one maximum in  $\mathcal{A}_1^b$  if  $a < b < 0$  and one maximum in  $\mathcal{A}_1^a$  if  $0 < a < b$ .

Another purpose of this paper is to illustrate the power and limitation of the  $\Gamma$ -convergence theory [8, 16, 15, 13] applied to this problem. Consider the case covered in Theorem 1.1. The limit  $J$  of the reduced problem  $Q_\epsilon$  can be easily identified in the  $\Gamma$ -convergence theory. If one can show that the minimum of  $J$  is isolated, then a local minimizer of  $I_\epsilon$  exists according to the theory. For small values of  $K$  ( $K = 1$  or  $K = 2$ ), we are able to show that the minimum is indeed isolated. For general  $K$  this also appears to be true, but we do not have a proof.

Regarding the second critical point of  $J$ , which is found by the topological degree argument on  $Q_\epsilon$ , one in general cannot derive from the  $\Gamma$ -convergence theory that there exists another critical point of  $I_\epsilon$  corresponding to the second critical point of  $J$ . This is because the  $\Gamma$ -convergence theory addresses only isolated local minima of  $J$  but not other types of critical points of  $J$ .

Similarly, in the case covered by Theorem 1.2 the maximum of  $J$  in  $\mathcal{A}_1^a$  and the maximum in  $\mathcal{A}_1^b$  do not yield solutions of (1.1) by the  $\Gamma$ -convergence theory. This paper shows that the nonminimum critical points all correspond to solutions of (1.5). They are saddle points of (1.6).

In the second part of the paper we further demonstrate the effectiveness of the Lyapunov–Schmidt reduction method. We construct solutions with a number of interfaces that cluster near the boundary  $r = 1$ . Namely, we prove the following result.

**THEOREM 1.3.** *Suppose that  $\gamma \geq 0$  and  $a < b$ . For any nonnegative integer  $k$ , there exists an  $\epsilon_0 > 0$  such that for each  $\epsilon \in (0, \epsilon_0]$ , (1.1) has a solution  $u_\epsilon$ , which has  $2k + 1$  interfaces near the boundary  $r = 1$ . Moreover,  $u_\epsilon \rightarrow b$  uniformly in  $B_{1-\delta}(0)$  for any  $\delta > 0$  small if  $b > 0$ , and  $u_\epsilon \rightarrow a$  uniformly in  $B_{1-\delta}(0)$  for any  $\delta > 0$  small if  $a < b \leq 0$ .*

Here the  $2k + 1$  layers are all close to the boundary  $r = 1$ . The distance between two successive interfaces is of order  $\epsilon \log \frac{1}{\epsilon}$ , and the distances between these interfaces and the boundary  $r = 1$  are also of order  $\epsilon \log \frac{1}{\epsilon}$ . After reduction the problem becomes a finite dimensional maximization problem with respect to the interfaces. The solution constructed from this maximization procedure is again of saddle type.

In Theorem 1.3 the nonlocal term  $\epsilon\gamma(1 - \Delta)^{-1}u$  does not play a central role. The existence of solutions with layers clustering near the boundary is valid with ( $\gamma > 0$ ) or without ( $\gamma = 0$ ) the nonlocal term. See also the result in [2] for the unbalanced case. The existence of interior layer solutions (Theorems 1.1 and 1.2) is different. Our results show that to have solutions of multiple interior layers we must have sufficient nonlocality, i.e.,  $\gamma$  must be large enough.

The organization of our paper serves these two purposes. In section 2 we recall how  $J$  is derived from the  $\Gamma$ -convergence theory. We show that the topological degree of  $J$  is always 0 in the case  $a < 0 < b$ . For large  $\gamma$  we show that  $J$  has a minimum and consequently there is another critical point of  $J$ . The main work starts in section 3, where we reduce the study of  $I_\epsilon$  to that of the finite dimensional problem  $Q_\epsilon$ . Then in section 4 we show that  $\epsilon^{-1}Q_\epsilon$  converges to  $J$  in  $C_{loc}^1$  and prove Theorems 1.1 and 1.2.

In section 5 another reduction is used to prove Theorem 1.3. We again derive a reduced functional of the interfaces. This time the interfaces are close to each other and to the boundary (all the distances are of order  $\epsilon \log \frac{1}{\epsilon}$ ). As the interfaces vary in this range the functional varies by a quantity that is much smaller than  $\epsilon$ . This compares differently from the situation discussed in Theorems 1.1 and 1.2. We show that the reduced problem is maximized at an interior point.

The conditions on  $a$ ,  $b$ , and  $\gamma$  in the three theorems are used when we solve the reduced problems. In the case of Theorem 1.1  $J$  has many critical points, and in the case of Theorem 1.2  $J$  has only one critical point. In the case of Theorem 1.3 we will solve the reduced problem by showing that it has an interior maximum point. To achieve this goal, the assumption on the sign of  $b$  is essential.

We use  $C$  to denote constants independent of  $\epsilon$ . Their values may vary from line to line. The  $L^p(\Omega)$  norm,  $p \in [1, \infty]$ , of a function is denoted by  $\|\cdot\|_p$ .

**2. The  $\Gamma$ -limit.** The limiting problem  $J$  is easily identified in the  $\Gamma$ -convergence theory. Other than the expression of  $J$  and its properties, given in Lemmas 2.3 and 2.4,

the details of the  $\Gamma$ -convergence and its consequences are not needed in this paper. Therefore, we omit the proofs of the statements in this section, with the exception of Lemmas 2.3 and 2.4. The interested reader may reconstruct them with the help of the references we provide.

The  $\Gamma$ -limit  $J$  of  $\epsilon^{-1}I_\epsilon$  is defined on the admissible set

$$(2.1) \quad \{u \in BV(\Omega, \{a, b\}) : u = u(|x|)\},$$

where  $BV(\Omega, \{a, b\})$  is the set of functions of bounded variation which take values only in  $\{a, b\}$ . The set (2.1) consists of such functions that are radial.

A function in (2.1) has either a finite number of interfaces or infinitely many interfaces. If it has finite, say,  $K$ , interfaces, there exist  $r_1, r_2, \dots, r_K$ , with  $0 < r_1 < r_2 < \dots < r_K < 1$ , that divide the interval  $(0, 1)$  into  $(0, r_1), (r_1, r_2), \dots, (r_{K-1}, r_K), (r_K, 1)$ , and

$$(2.2) \quad u(r) = a \text{ on } (0, r_1), = b \text{ on } (r_1, r_2), = a \text{ on } (r_2, r_3), \dots$$

or

$$(2.3) \quad u(r) = b \text{ on } (0, r_1), = a \text{ on } (r_1, r_2), = b \text{ on } (r_2, r_3), \dots$$

In the case of (2.2) we say that  $u \in \mathcal{A}_K^a$  and in the case of (2.3) we say that  $u \in \mathcal{A}_K^b$ . On  $\mathcal{A}_K^a$  and  $\mathcal{A}_K^b$  the  $\Gamma$ -limit  $J$  is given by

$$(2.4) \quad J(u) = \omega_{n-1}\tau \sum_{j=1}^K r_j^{n-1} + \frac{\omega_{n-1}\gamma}{2} \int_0^1 |(1 - \Delta)^{-1/2}u|^2 r^{n-1} dr.$$

Here we denote the area of the  $n - 1$  dimensional unit sphere by  $\omega_{n-1}$ . The constant  $\tau$  in (2.4) is given by

$$(2.5) \quad \tau = \int_a^b \sqrt{2W(q)} dq.$$

A function in (2.1) may also have infinite interfaces. Then the interfaces must accumulate at the origin. Otherwise, if there were a cluster point not at the origin, the total length of the interfaces would be infinite and  $u$  could not be in (2.1). Hence there exists a *decreasing* sequence  $r_1, r_2, \dots$ , such that  $1 > r_1 > r_2 > \dots$  and  $\lim_{j \rightarrow \infty} r_j = 0$ , and either

$$(2.6) \quad u = \begin{cases} a & \text{on } (r_1, 1), \\ b & \text{on } (r_2, r_1), \\ a & \text{on } (r_3, r_2) \\ \dots \end{cases}$$

or

$$(2.7) \quad u = \begin{cases} b & \text{on } (r_1, 1), \\ a & \text{on } (r_2, r_1), \\ b & \text{on } (r_3, r_2) \\ \dots \end{cases}$$

In this case  $J$  is defined by (2.4) with  $\sum_{j=1}^K$  replaced by  $\sum_{j=1}^{\infty}$ . Because  $u$  is assumed to have bounded variation, this infinite sum converges.

Now we have  $J$  which is defined in (2.1). But  $I_\epsilon$  is defined in a different set (1.10). We trivially extend both to

$$(2.8) \quad \{u \in L^2(\Omega) : u = u(|x|)\},$$

the radial  $L^2$ -functions, by setting  $I_\epsilon(u) = \infty$  if  $u$  is in (2.8) but not in (1.10) and similarly  $J(u) = \infty$  if  $u$  is in (2.8) but not in (2.1). In (2.8) distance is measured by the  $L^2$  norm  $\|\cdot\|_2$ .

The  $\Gamma$ -convergence of  $\epsilon^{-1}I_\epsilon$  to  $J$  is characterized by the two properties of the following lemma.

LEMMA 2.1. *As  $\epsilon \rightarrow 0$ ,  $\epsilon^{-1}I_\epsilon$   $\Gamma$ -converges to  $J$  in the following sense:*

1. *For every family of functions  $\phi_\epsilon$  in (2.8) with  $\lim_{\epsilon \rightarrow 0} \|\phi_\epsilon - \phi\|_2 = 0$ ,  $\liminf_{\epsilon \rightarrow 0} \epsilon^{-1}I_\epsilon(\phi_\epsilon) \geq J(\phi)$ .*
2. *For every  $\phi$  in (2.8), there is a family of functions  $\phi_\epsilon$  in (2.8) such that  $\lim_{\epsilon \rightarrow 0} \|\phi_\epsilon - \phi\|_2 = 0$  and  $\limsup_{\epsilon \rightarrow 0} \epsilon^{-1}I_\epsilon(\phi_\epsilon) \leq J(\phi)$ .*

One important consequence of the  $\Gamma$ -convergence of  $\epsilon^{-1}I_\epsilon$  is the following existence result.

LEMMA 2.2. *If  $u_0 \in \mathcal{A}_K^a$  (or  $\mathcal{A}_K^b$ , respectively) is an isolated local minimum of  $J$  in  $\mathcal{A}_K^a$  (or  $\mathcal{A}_K^b$ , respectively), for sufficiently small  $\epsilon$ , there exists a local minimizer  $u_\epsilon$  of  $I_\epsilon$ , and  $\lim_{\epsilon \rightarrow 0} \|u_\epsilon - u_0\|_2 = 0$ .*

These two lemmas may be proved by mimicking the argument in Ren and Wei [22]. Lemma 2.2 suggests that we look for minima of  $J$  in  $\mathcal{A}_K^a$  and  $\mathcal{A}_K^b$ . We do this in the rest of this section. Moreover, we will also find critical points of  $J$  that are not local minima.

Let  $G = G(r, s)$  be the Green's function

$$(2.9) \quad -G_{rr} - \frac{n-1}{r}G_r + G = \delta(r-s).$$

Note that  $G(r, s)$  is not symmetric in  $r$  and  $s$ , but  $r^{n-1}G(r, s)$  is. We define

$$(2.10) \quad v(r) = \int_0^1 G(r, s)u(s) ds$$

to be the solution of

$$(2.11) \quad -v_{rr} - \frac{n-1}{r}v_r + v = u.$$

When it is in  $\mathcal{A}_K^a$  or  $\mathcal{A}_K^b$ ,  $u$  is determined by its jump points  $r_1, r_2, \dots, r_K$ , which we term interfaces. Collectively we set  $\mathbf{r} = (r_1, r_2, \dots, r_K)$ . Because  $u$  depends on  $\mathbf{r}$ , we often write  $u = u(r; \mathbf{r})$  and correspondingly  $v = v(r; \mathbf{r})$ .

The nonlocal part of  $J$  may be rewritten as

$$(2.12) \quad \int_0^1 |(1-\Delta)^{-1/2}u(\cdot; \mathbf{r})|^2 r^{n-1} dr = \int_0^1 v(r; \mathbf{r})u(r; \mathbf{r}) r^{n-1} dr.$$

We view  $J$  as a function of  $\mathbf{r}$ :  $J = J(\mathbf{r})$ . Now we compute the derivative of  $J$ . Note that

$$(2.13) \quad \begin{aligned} \frac{\partial}{\partial r_1} \int_0^1 v(r)u(r) r^{n-1} dr &= \frac{\partial}{\partial r_1} \left[ \int_0^{r_1} v(r)a r^{n-1} dr + \int_{r_1}^{r_2} v(r)b r^{n-1} dr + \dots \right] \\ &= (a-b)v(r_1)r_1^{n-1} + \int_0^1 \frac{\partial v(r; \mathbf{r})}{\partial r_1} u(r; \mathbf{r}) r^{n-1} dr. \end{aligned}$$

Because

$$\begin{aligned}
 \frac{\partial v(r; \mathbf{r})}{\partial r_1} &= \frac{\partial}{\partial r_1} \left[ \int_0^{r_1} G(r, s)a \, ds + \int_{r_1}^{r_2} G(r, s)b \, ds + \dots \right] \\
 (2.14) \qquad &= (a - b)G(r, r_1),
 \end{aligned}$$

we find

$$\begin{aligned}
 \frac{\partial}{\partial r_1} \int_0^1 v(r)u(r) r^{n-1} dr &= (a - b)v(r_1)r_1^{n-1} + (a - b) \int_0^1 G(r, r_1)u(r; \mathbf{r}) r^{n-1} dr \\
 &= (a - b)v(r_1)r_1^{n-1} + (a - b) \int_0^1 G(r_1, r)u(r; \mathbf{r}) r_1^{n-1} dr \\
 (2.15) \qquad &= (a - b)v(r_1)r_1^{n-1} + (a - b)v(r_1)r_1^{n-1} = 2(a - b)v(r_1)r_1^{n-1}.
 \end{aligned}$$

We have used the symmetry of  $r^{n-1}G(r, r_1)$ . For a general  $r_j$  we have

$$(2.16) \qquad \frac{\partial}{\partial r_j} \int_0^1 v(r)u(r) r^{n-1} dr = 2(-1)^j(b - a)v(r_j; \mathbf{r})r_j^{n-1}.$$

Therefore

$$(2.17) \qquad \frac{\partial J(\mathbf{r})}{\partial r_j} = \omega_{n-1}[(n - 1)\tau r_j^{n-2} + \gamma(b - a)(-1)^j v(r_j; \mathbf{r})r_j^{n-1}], \quad \mathbf{r} \in \mathcal{A}_K^a.$$

The gradient of  $J$  in  $\mathcal{A}_K^b$  is a bit different:

$$(2.18) \qquad \frac{\partial J(\mathbf{r})}{\partial r_j} = \omega_{n-1}[(n - 1)\tau r_j^{n-2} + \gamma(a - b)(-1)^j v(r_j; \mathbf{r})r_j^{n-1}], \quad \mathbf{r} \in \mathcal{A}_K^b.$$

The existence of critical points of  $J$  depends on  $a$ ,  $b$ , and  $\gamma$ . We consider the following three cases:

- *Case I.*  $a < b < 0$ .
- *Case II.*  $0 < a < b$ .
- *Case III.*  $a < 0 < b$ .

The first two cases are relatively simple. We have the following result.

LEMMA 2.3.

1. *If  $a < b < 0$  and  $\gamma > \gamma_I$  where*

$$(2.19) \qquad \gamma_I = \frac{(n - 1)\tau}{(a - b)b},$$

*there is a local maximum in  $\mathcal{A}_1^b$ . There are no critical points in other classes.*

2. *If  $0 < a < b$  and  $\gamma > \gamma_{II}$  where*

$$(2.20) \qquad \gamma_{II} = \frac{(n - 1)\tau}{(b - a)a},$$

*there is a local maximum in  $\mathcal{A}_1^a$ . There are no critical points in other classes.*

*Proof.* We consider only case I, for case II may be similarly handled. Note that  $u \leq 0$  implies that  $v < 0$ . In each  $\mathcal{A}_K^a$ ,  $K \geq 1$ ,  $\frac{\partial J}{\partial r_1} > 0$ . There is no critical point in  $\mathcal{A}_K^a$ . In  $\mathcal{A}_K^b$  for  $K \geq 2$ ,  $\frac{\partial J}{\partial r_2} > 0$ . So there is no critical point in  $\mathcal{A}_K^b$ ,  $K \geq 2$ . The only class left is  $\mathcal{A}_1^b$ . In this class

$$\frac{\partial J}{\partial r_1} = \omega_{n-1} r_1^{n-2} [(n-1)\tau - \gamma(a-b)v(r_1; r_1)r_1].$$

The quantity inside the brackets is  $(n-1)\tau$  when  $r_1 = 0$  and  $(n-1)\tau - \gamma(a-b)v(1) = (n-1)\tau - \gamma(a-b)b$  when  $r_1 = 1$ . Hence when  $\gamma > \gamma_I$  there is a local maximum of  $J$  in  $\mathcal{A}_1^b$ , where  $\gamma_I$  is given in (2.19). Here since when  $b = 0$ ,  $\gamma_I = \infty$ , the condition  $\gamma > \gamma_I$  can be satisfied only when  $b < 0$ .  $\square$

Case III is the most interesting. We have the following lemma.

LEMMA 2.4. *Suppose  $a < 0 < b$ .*

1. *When  $\gamma$  is sufficiently large,  $J$  attains a global minimum in  $\mathcal{A}_K^a$  (or  $\mathcal{A}_K^b$ )—not on the boundary of  $\mathcal{A}_K^a$  (or  $\mathcal{A}_K^b$ ).*
2. *Given  $\gamma' > 0$  and any compact subset  $\mathcal{K}'$  of  $\mathcal{A}_K^a$  (or  $\mathcal{A}_K^b$ ) one can find a compact subset  $\mathcal{K}$ , such that  $\mathcal{K}' \subset \mathcal{K}$  and for all  $\gamma \in [0, \gamma']$  the topological degree of  $\text{grad } J$  on  $\mathcal{K}$  about  $\vec{0}$  is zero.*
3. *When  $\gamma$  is large, there exist at least two critical points of  $J$  in each  $\mathcal{A}_K^a$  (or  $\mathcal{A}_K^b$ ).*

*Proof.* To prove part 1, we note that as far as the minimum is concerned the condition that  $\gamma$  is large is equivalent to the condition that  $\tau$  is small. Or  $J$  can be considered as a perturbation of the function

$$(2.21) \quad J_0(\mathbf{r}) = \frac{\omega_{n-1}\gamma}{2} \int_0^1 |((1-\Delta)^{-1/2}u(\cdot; \mathbf{r}))^2| r^{n-1} dr, \quad \mathbf{r} \in \mathcal{A}_K^a \text{ (or } \mathcal{A}_K^b).$$

We recall that  $\mathcal{A}_K^a$  (and, similarly,  $\mathcal{A}_K^b$ ) is identified with

$$\{\mathbf{r} = (r_1, \dots, r_K) : 0 < r_1 < \dots < r_K\}$$

so that the boundary of  $\mathcal{A}_K^a$  is not included in  $\mathcal{A}_K^a$ . We study  $J_0$  on the boundary of  $\mathcal{A}_K^a$  (the case  $\mathcal{A}_K^b$  is left to the reader), which consists of three pieces: (1)  $r_1 = 0$ , (2)  $r_K = 1$ , and (3)  $r_j = r_{j+1}$  for some  $j = 1, 2, \dots, K-1$ .

In  $\mathcal{A}_K^a$ ,

$$(2.22) \quad \frac{\partial J_0}{\partial r_j} = \omega_{n-1}\gamma(b-a)(-1)^j v(r_j; \mathbf{r}) r_j^{n-1}.$$

If the minimum of  $J_0$  is achieved on  $r_1 = 0$ , say, at  $\mathbf{r} = (0, r_2, r_3, \dots)$ , then

$$v(r_2; \mathbf{r}) = v(r_3; \mathbf{r}) = \dots = v(r_K; \mathbf{r}) = 0.$$

However,  $v(r_2; \mathbf{r}) = 0$  and  $u(r; \mathbf{r}) = b > 0$  for  $r \in (0, r_2)$  imply that  $v(r; \mathbf{r}) > 0$  for  $r \in [0, r_2)$  by the maximum principal. Then at this  $\mathbf{r}$

$$(2.23) \quad \frac{\partial J_0(\mathbf{r})}{\partial r_1} = -\omega_{n-1}\gamma(b-a)v(0; \mathbf{r}) < 0.$$

This means that the gradient of  $J_0$  points outward at this  $\mathbf{r}$ . Then  $\mathbf{r}$  cannot be a minimum point.

If the minimum of  $J$  is achieved at  $\mathbf{r} = (r_1, r_2, \dots, r_{K-1}, 1)$  on the boundary piece  $r_N = 1$ , then

$$v(r_1; \mathbf{r}) = v(r_2; \mathbf{r}) = \dots = v(r_{K-1}; \mathbf{r}) = 0.$$

Since for  $r \in (r_{K-1}, 1)$ ,

$$u(r; \mathbf{r}) = \begin{cases} a < 0 & \text{if } K \text{ is odd,} \\ b > 0 & \text{if } K \text{ is even,} \end{cases}$$

$v(r_{K-1}; \mathbf{r}) = 0$  implies that  $v(r)$  is negative if  $K$  is odd and positive if  $K$  is even on  $(r_{K-1}, 1]$ . Then

$$(2.24) \quad \frac{\partial J_0}{\partial r_K} = \gamma(b - a)(-1)^K v(1; \mathbf{r}) > 0.$$

Hence the gradient of  $J$  points outward at this  $\mathbf{r}$  and it cannot be the minimum.

If the minimum of  $J$  is achieved at a boundary point  $\mathbf{r}$  on  $r_j = r_{j+1}$ , we have two possibilities. First we may have  $\mathbf{r} = (r_1, r_2, \dots, r_{j-1}, r_j, r_{j+1}, r_{j+2}, \dots, r_K)$  with  $r_1 < r_2 < \dots < r_{j-1} < r_j = r_{j+1} < r_{j+2} < \dots < r_K$ . This means two interfaces coincide but other interfaces stay separate. Then

$$v(r_1; \mathbf{r}) = \dots = v(r_{j-1}; \mathbf{r}) = v(r_{j+2}; \mathbf{r}) = \dots = v(r_K; \mathbf{r}) = 0.$$

When  $r \in (r_{j-1}, r_{j+2})$ ,

$$u(r; \mathbf{r}) = \begin{cases} a < 0 & \text{if } j \text{ is odd,} \\ b > 0 & \text{if } j \text{ is even.} \end{cases}$$

Then, since  $v(r_{j-1}; \mathbf{r}) = v(r_{j+2}; \mathbf{r}) = 0$  for  $r \in (r_{j-1}, r_{j+2})$ ,  $v(r; \mathbf{r})$  is negative if  $j$  is odd and positive if  $j$  is even by the maximal principal. Note that at the minimum  $\mathbf{r}$  the outward normal direction is  $\nu = (0, 0, \dots, 0, 1, -1, 0, \dots, 0)$ , where 1 is the  $j$ th entry and  $-1$  the  $(j + 1)$ th entry. The directional derivative along  $\nu$  is

$$\begin{aligned} \frac{\partial J}{\partial \nu} &= \gamma(b - a)[(-1)^j v(r_j; \mathbf{r})r_j^{n-1} - (-1)^{j+1} v(r_{j+1}; \mathbf{r})r_{j+1}^{n-1}] \\ &= 2\gamma(b - a)(-1)^j v(r_j; \mathbf{r})r_j^{n-1} > 0. \end{aligned}$$

Hence  $\mathbf{r}$  cannot be the minimum.

In this case there is also the possibility that more than two interfaces collapse at one point, where the minimum is attained—for example, at  $\mathbf{r}$  where  $r_1 < r_2 < \dots < r_{j-2} < r_{j-1} = r_j = r_{j+1} < r_{j+2} < \dots < r_K$ . However, this point can be viewed as a point on the boundary of  $\mathcal{A}_{K-1}^a$ . We can make an induction assumption that in every  $\mathcal{A}_N^a$  or  $\mathcal{A}_N^b$  with  $N \leq K - 1$ , the minimum of  $J_0$  is not achieved on the boundary. Therefore this possibility needs no consideration.

Therefore the minimum of  $J_0$  is achieved on a compact subset of  $\mathcal{A}_K^a$ . Hence for large  $\gamma$ , the minimum of  $J$  is also achieved inside  $\mathcal{A}_K^a$ .

To prove part 2, we treat  $\gamma$  in  $J$  as a parameter for the homotopy argument. We consider the topological degree of  $\text{grad } J$ . We are given a compact subset  $\mathcal{K}'$  of  $\mathcal{A}_K^a$  and  $\gamma$  in  $[0, \gamma']$ .

First we show that  $\text{grad } J$  is not  $\vec{0}$  on the boundary of some compact  $\mathcal{K} \supset \mathcal{K}'$ . When  $\gamma = 0$ ,

$$(2.25) \quad \frac{\partial J}{\partial r_j} = (n-1)\omega_{n-1}\tau r_j^{n-2},$$

which is not 0 anywhere in  $\mathcal{A}_K^a$ . When  $\gamma > 0$ , we consider the three pieces of the boundary of  $\mathcal{A}_K^a$  again.

Although on the piece  $r_1 = 0$  of the boundary  $\partial J/\partial r_1 = 0$  if  $n > 2$ , we move slightly away from  $r_1 = 0$  and consider small and positive  $r_1$ . Then

$$(2.26) \quad \frac{\partial J}{\partial r_1} = (n-1)\omega_{n-1}\tau r_1^{n-2} + \gamma O(r_1^{n-1}) > 0.$$

Hence  $\text{grad } J$  is not  $\vec{0}$  when  $r_1$  is positive and small.

On the second piece  $\mathbf{r}_K = 1$ ,

$$(2.27) \quad \frac{\partial J}{\partial r_{K-1}} = (n-1)\omega_{n-1}\tau r_{K-1}^{n-2} + \omega_{n-1}\gamma(b-a)(-1)^{K-1}v(r_{K-1}; \mathbf{r})r_{K-1}^{n-1},$$

$$(2.28) \quad \frac{\partial J}{\partial r_K} = (n-1)\omega_{n-1}\tau + \omega_{n-1}\gamma(b-a)(-1)^K v(1; \mathbf{r}).$$

For  $r \in (r_{K-1}, 1)$ ,  $u(r; \mathbf{r}) = a < 0$  if  $K$  is odd and  $u(r; \mathbf{r}) = b > 0$  if  $K$  is even. If  $\frac{\partial J}{\partial r_{K-1}} = 0$ , then  $v(r_{K-1}; \mathbf{r})$  is negative if  $K$  is odd and positive if  $K$  is even. Then for  $r \in [r_{K-1}, 1]$ ,  $v(r; \mathbf{r})$  is negative if  $K$  is odd and positive if  $K$  is even. In particular,  $v(1; \mathbf{r})$  is negative if  $K$  is odd and positive if  $K$  is even. Then  $\frac{\partial J}{\partial r_K}$  is always positive.

Hence  $\text{grad } J$  is not  $\vec{0}$  on the second piece of the boundary.

On the third piece of the boundary  $r_j = r_{j+1}$ ,

$$(2.29) \quad \frac{\partial J}{\partial r_j} = (n-1)\omega_{n-1}\tau r_j^{n-2} + \omega_{n-1}\gamma(b-a)(-1)^j v(r_j; \mathbf{r})r_j^{n-1},$$

$$(2.30) \quad \frac{\partial J}{\partial r_{j+1}} = (n-1)\omega_{n-1}\tau r_j^{n-2} + \omega_{n-1}\gamma(b-a)(-1)^{j+1} v(r_j; \mathbf{r})r_j^{n-1}.$$

These two partial derivatives cannot simultaneously be 0. Hence  $\text{grad } J$  is not  $\vec{0}$  on the third piece of the boundary.

Now we can find a compact subset  $\mathcal{K} \supset \mathcal{K}'$  of  $\mathcal{A}_K^a$  so that for all  $\gamma \in [0, \gamma']$ ,  $\text{grad } J$  is not  $\vec{0}$  on the boundary of  $\mathcal{K}$ . Consequently we can define the topological degree of  $\text{grad } J$  in  $\mathcal{K}$  about  $\vec{0}$ :

$$(2.31) \quad \text{Deg}(\text{grad } J, \mathcal{K}, \vec{0}).$$

Note that in part 2 of the lemma,  $\gamma$  is allowed to be 0. This is important, because when  $\gamma = 0$ ,  $\text{grad } J \neq \vec{0}$  in  $\mathcal{A}_K^a$ . Hence  $\text{Deg}(\text{grad } J, \mathcal{K}, \vec{0}) = 0$  when  $\gamma = 0$ . By the invariance of the degree under continuous deformation,  $\text{Deg}(\text{grad } J, \mathcal{K}, \vec{0}) = 0$  for all  $\gamma \in [0, \gamma']$ . This proves part 2 of the lemma.

The third part of the lemma follows from parts 1 and 2. For large  $\gamma$ , there is a minimum, say,  $\mathbf{r}_*$ , in  $\mathcal{A}_K^a$ . This gives one critical point of  $J$ . If this is the only critical point of  $J$  in  $\mathcal{A}_K^a$ , we can find an open ball  $\mathcal{B}_\eta(\mathbf{r}_*)$  of radius  $\eta$  centered at  $\mathbf{r}_*$  whose

closure is a subset of  $\mathcal{A}_K^a$ . Let  $\mathcal{K}$  be a compact subset of  $\mathcal{A}_K^a$  given in part 2 of the lemma, and it is large enough to contain  $\mathcal{B}_\eta(\mathbf{r}_*)$  as a subset. Then

$$(2.32) \quad \text{Deg}(\text{grad } J, \mathcal{K}, \vec{0}) = \text{Deg}(\text{grad } J, \overline{\mathcal{B}_\eta(\mathbf{r}_*)}, \vec{0}) + \text{Deg}(\text{grad } J, \mathcal{K} \setminus \mathcal{B}_\eta(\mathbf{r}_*), \vec{0}).$$

We know that  $\text{Deg}(\text{grad } J, \mathcal{K}, \vec{0}) = 0$  from part 2. Also  $\text{Deg}(\text{grad } J, \mathcal{B}_\eta(\mathbf{r}_*), \vec{0}) = 1$  because the minimum  $\mathbf{r}_*$  is the only critical point in  $\mathcal{B}_\eta(\mathbf{r}_*)$ . Therefore  $\text{Deg}(\text{grad } J, \mathcal{K} \setminus \mathcal{B}_\eta(\mathbf{r}_*), \vec{0}) = -1 \neq 0$ . There is another critical point in  $\mathcal{K} \setminus \mathcal{B}_\eta(\mathbf{r}_*)$ .  $\square$

The reader is probably tempted to combine Lemmas 2.2 and 2.4. Since there is a minimum of  $J$  in  $\mathcal{A}_K^a$  when  $\gamma$  is large, one would like to show that this minimum is isolated and then following Lemma 2.2 conclude that  $I_\epsilon$  has a local minimizer near the minimum of  $J$ . When  $K = 1$  or  $K = 2$ , it is indeed easy to show that the local minimum of  $J$  is isolated. However, for general  $K$ , we do not have a proof.

Moreover, in part 3 of Lemma 2.4 we have also found another critical point of  $J$  for large  $\gamma$ . This critical point is in general not a local minimum of  $J$ . Lemma 2.2 is hence not applicable.

Similarly, the local maxima found in Lemma 2.3 are not of much use in the  $\Gamma$ -convergence theory.

To make use of all the critical points of  $J$  found in Lemmas 2.3 and 2.4, we now abandon the  $\Gamma$ -convergence theory and proceed differently. Our new reduction approach may roughly be regarded as a convergence theory at the  $C^2$  level, while the  $\Gamma$ -convergence theory is at the  $C^0$  level. Using this argument we will be able to prove that in case III there are at least two critical points of  $I_\epsilon$  with  $K$  interfaces when  $\gamma$  is sufficiently large (see Theorem 1.1). Similarly, in cases I and II there is a critical point of  $I_\epsilon$  with one interface if  $\gamma$  is large (see Theorem 1.2).

**3. Lyapunov–Schmidt reduction procedure.** The Lyapunov–Schmidt reduction procedure involves the first and second derivatives of  $I_\epsilon$ . For this reason we vaguely regard it as a reduction theory at the  $C^2$  level.

We construct a manifold  $\mathcal{M}$  of approximate solutions parameterized by  $\mathbf{r} = (r_1, r_2, \dots, r_K)$ . First define

$$(3.1) \quad s(r; \mathbf{r}) = a \text{ in } (0, r_1), b \text{ in } (r_1, r_2), a \text{ in } (r_2, r_3), \dots,$$

which gives a profile away from the interfaces. Clearly  $s(\cdot; \mathbf{r}) \in \mathcal{A}_K^a$ . We also identify the domain of  $\mathbf{r}$  with  $\mathcal{A}_K^a$ . From now to the end of the paper we construct the two solutions in Theorem 1.1 that satisfy  $\lim_{\epsilon \rightarrow 0} u_\epsilon^a(0) = a$  and the solution in part 2 of Theorem 1.2. Similar arguments can give the other solutions, starting with an  $s(\cdot; \mathbf{r}) \in \mathcal{A}_K^b$ . We leave the details to the reader.

The interface profile is the solution  $H(t)$  of the differential equation

$$(3.2) \quad -H_{tt} + f(H) = 0, \quad H(-\infty) = a, \quad H(\infty) = b, \quad H(0) = \frac{a+b}{2}.$$

$H(t)$  approaches  $a$  (or  $b$ , respectively) exponentially fast as  $t$  tends to  $-\infty$  (or  $\infty$ , respectively) in the sense that there exist positive  $C_1, C_2$  so that

$$(3.3) \quad 0 < H(t) - a < C_1 e^{C_2 t} \text{ if } t < 0, \text{ and } 0 < b - H(t) < C_1 e^{-C_2 t} \text{ if } t > 0.$$

Near  $r_j$  we use  $H((r - r_j)/\epsilon)$  if  $j$  is odd, or  $H(-(r - r_j)/\epsilon)$  if  $j$  is even.

The outer approximation  $s(\cdot; \mathbf{r})$  and the inner approximation  $H$  must be connected by a smooth cut-off function  $\chi$  to make

$$(3.4) \quad w(r; \mathbf{r}) = \sum_{j=1}^K \chi(r - r_j) H \left( (-1)^{j+1} \frac{r - r_j}{\epsilon} \right) + \left( 1 - \sum_{j=1}^K \chi(r - r_j) \right) s(r; \mathbf{r}),$$

where  $\chi$  is defined to be

$$(3.5) \quad \chi(r) = \begin{cases} 1 & \text{in } (-\epsilon^\alpha, \epsilon^\alpha), \\ 0 & \text{in } \mathbf{R} \setminus (-2\epsilon^\alpha, 2\epsilon^\alpha). \end{cases}$$

The exponent  $\alpha$  in (3.5) satisfies

$$(3.6) \quad 0 < \alpha < 1.$$

$\chi$  satisfies

$$(3.7) \quad \chi = O(1), \quad \chi' = O(\epsilon^{-\alpha}), \quad \chi'' = O(\epsilon^{-2\alpha}).$$

The manifold  $\mathcal{M}$  is

$$(3.8) \quad \mathcal{M} = \{w(\cdot; \mathbf{r}) : \mathbf{r} \in \mathcal{A}_K^a\},$$

which is parameterized by  $\mathbf{r}$  in  $\mathcal{A}_K^a$ .

We define two function spaces  $\mathcal{X}$  and  $\mathcal{Y}$ ,

$$(3.9) \quad \mathcal{X} = \{u \in W^{2,2}(\Omega) : u = u(|x|), \quad u_r(1) = 0\}; \quad \mathcal{Y} = \{q \in L^2(\Omega) : q = q(|x|)\},$$

and a nonlinear operator  $S_\epsilon : \mathcal{X} \rightarrow \mathcal{Y}$  by

$$(3.10) \quad S_\epsilon(u) = -\epsilon^2 \Delta u + f(u) + \epsilon \gamma (1 - \Delta)^{-1} u.$$

Equation (1.5) is  $S_\epsilon(u) = 0$ .

**LEMMA 3.1.**  *$S_\epsilon(w) = O(\epsilon)$  locally uniformly in  $\mathbf{r}$  and  $\gamma$ . More precisely, for each compact subset  $\mathcal{K}$  of  $\mathcal{A}_K^a$  and  $[\gamma_1, \gamma_2]$ ,  $0 < \gamma_1 < \gamma_2$ , there exist  $C > 0$  and  $\epsilon_0 > 0$  such that for all  $\mathbf{r} \in \mathcal{K}$ ,  $\gamma \in [\gamma_1, \gamma_2]$  and  $\epsilon < \epsilon_0$ ,  $\|S_\epsilon(w(\cdot; \mathbf{r}))\|_\infty \leq C\epsilon$ .*

*Proof.* Given  $\mathcal{K}$  and  $[\gamma_1, \gamma_2]$ , we let  $\mathbf{r} \in \mathcal{K}$  and  $\gamma \in [\gamma_1, \gamma_2]$ . Then

$$(3.11) \quad \begin{aligned} S_\epsilon(w) &= -\epsilon^2 \left( w_{rr} + \frac{n-1}{r} w_r \right) + f(w) + \epsilon \gamma (1 - \Delta)^{-1} w \\ &= (-\epsilon^2 w_{rr} + f(w)) - \epsilon^2 \frac{n-1}{r} w_r + \epsilon \gamma (1 - \Delta)^{-1} w \\ &= O(e^{-C/\epsilon}) + O(\epsilon) + O(\epsilon) = O(\epsilon). \end{aligned}$$

The lemma follows.  $\square$

For each  $j = 1, 2, \dots, K$ , let us define

$$(3.12) \quad h_j(r) = H' \left( \frac{r - r_j}{\epsilon} \right) \kappa \left( \frac{r - r_j}{\sqrt{\epsilon}} \right) = H' \left( \frac{r - r_j}{\epsilon} \right) + O(e^{-C/\sqrt{\epsilon}}),$$

where  $\kappa$  is a smooth, even, cut-off function

$$(3.13) \quad \kappa(s) = \begin{cases} 1 & \text{if } |s| \leq 1, \\ 0 & \text{if } |s| \geq 2. \end{cases}$$

Here  $O(e^{-C/\sqrt{\epsilon}})$  is an exponentially small quantity with respect to  $\epsilon$  because of the exponentially fast decay rate of  $H'$ :  $H'(t) \leq C_1 e^{-C_2|t|}$ . Therefore  $h'_j(0) = h'_j(1) = 0$ ,  $\|h'_j - \epsilon^{-1}H''(\frac{\cdot-r_j}{\epsilon})\|_\infty = O(\epsilon^{-C/\sqrt{\epsilon}})$ , and  $\|h''_j - \epsilon^{-2}H'''(\frac{\cdot-r_j}{\epsilon})\|_\infty = O(\epsilon^{-C/\sqrt{\epsilon}})$ . Note that  $h_j$  depends on  $\mathbf{r}$  so we sometimes write it as  $h_j(r; \mathbf{r})$ .

At each  $w(\cdot; \mathbf{r})$  of the manifold we define the space

$$(3.14) \quad \mathcal{F}_{\mathbf{r}} = \{\phi \in \mathcal{X} : \phi \perp h_j, j = 1, 2, \dots, K\},$$

where  $\perp$  is defined from the inner product

$$(3.15) \quad \langle A, B \rangle = \int_0^1 A(r)B(r)r^{n-1}dr.$$

Then  $w_{\mathbf{r}} + \mathcal{F}_{\mathbf{r}}$  is a subset of  $\mathcal{X}$ , which we call the  $\mathbf{r}$ -fiber of  $\mathcal{M}$  in  $\mathcal{X}$ . Define  $\mathcal{E}_{\mathbf{r}}$  to be the subspace

$$(3.16) \quad \mathcal{E}_{\mathbf{r}} = \{q \in \mathcal{Y} : q \perp h_j, j = 1, 2, \dots, K\}$$

of  $\mathcal{Y}$ . Let the projection from  $\mathcal{Y}$  to  $\mathcal{E}_{\mathbf{r}}$  be  $\pi_{\mathbf{r}} : \mathcal{Y} \rightarrow \mathcal{E}_{\mathbf{r}}$ , defined by

$$(3.17) \quad \pi_{\mathbf{r}}(q) = q - \sum_{j=1}^K \frac{\langle q, h_j \rangle}{\|h_j\|_2^2} h_j.$$

At each  $w(\cdot; \mathbf{r})$  we look for a  $\phi(\cdot; \mathbf{r}) \in \mathcal{F}_{\mathbf{r}}$  so that

$$(3.18) \quad \pi_{\mathbf{r}} \circ S_\epsilon(w(\cdot; \mathbf{r}) + \phi(\cdot; \mathbf{r})) = 0.$$

This means that we solve  $S_\epsilon(u) = 0$  in the fiber direction. For each  $\phi \in \mathcal{F}_{\mathbf{r}}$  we expand

$$(3.19) \quad S_\epsilon(w + \phi) = S_\epsilon(w) + L_{\mathbf{r}}(\phi) + R_{\mathbf{r}}(\phi),$$

where the linearized operator of  $S_\epsilon$  at  $w(\cdot; \mathbf{r})$  is denoted by  $L_{\mathbf{r}} : \mathcal{X} \rightarrow \mathcal{Y}$ , defined by

$$(3.20) \quad L_{\mathbf{r}}\phi := -\epsilon^2 \left( \phi_{rr} + \frac{n-1}{r} \phi_r \right) + f'(w(r; \mathbf{r}))\phi + \epsilon\gamma(1 - \Delta)^{-1}\phi,$$

and the remainder is

$$(3.21) \quad R_{\mathbf{r}}(\phi) = f(w + \phi) - f(w) - f'(w)\phi.$$

Then (3.18) is written as

$$(3.22) \quad \pi_{\mathbf{r}} \circ S_\epsilon(w) + \pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\phi) + \pi_{\mathbf{r}} \circ R_{\mathbf{r}}(\phi) = 0.$$

Regarding the linear operator  $\pi_{\mathbf{r}} \circ L_{\mathbf{r}}$ ,

$$(3.23) \quad \pi_{\mathbf{r}} \circ L_{\mathbf{r}} : \mathcal{F}_{\mathbf{r}} \rightarrow \mathcal{E}_{\mathbf{r}}$$

(note that it is defined on  $\mathcal{F}_{\mathbf{r}}$ —not on  $\mathcal{X}$ ), we have the following lemma.

LEMMA 3.2.

1. There exists  $C_1 > 0$  independent of  $\epsilon$  such that  $\|\phi\|_\infty \leq C_1 \|\pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\phi)\|_\infty$  for all  $\phi \in \mathcal{F}_{\mathbf{r}}$ . In particular,  $\pi_{\mathbf{r}} \circ L_{\mathbf{r}}$  is one-to-one from  $\mathcal{F}_{\mathbf{r}}$  to  $\mathcal{E}_{\mathbf{r}}$ .
2.  $\pi_{\mathbf{r}} \circ L_{\mathbf{r}}$  is onto from  $\mathcal{F}_{\mathbf{r}}$  to  $\mathcal{E}_{\mathbf{r}}$ .

*Proof.* To prove part 1 we argue by contradiction. Suppose the conclusion is false. Then there exists  $\psi_\epsilon \in \mathcal{F}_{\mathbf{r}}$  for each  $\epsilon$  such that  $\|\psi_\epsilon\|_\infty = 1$  and along a subsequence of  $\epsilon \rightarrow 0$ ,

$$(3.24) \quad \|\pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\psi_\epsilon)\|_\infty \rightarrow 0.$$

To simplify notation, we write  $\psi$  instead of  $\psi_\epsilon$ . We rewrite (3.24) as

$$(3.25) \quad -\epsilon^2 \left( \psi_{rr} + \frac{n-1}{r} \psi_r \right) + f'(w)\psi + \epsilon\gamma(1-\Delta)^{-1}\psi - \sum_{j=1}^K \beta_j h_j = o(1)$$

for some  $\beta_j \in \mathbf{R}$ . More specifically,  $\beta_j$  are given by

$$(3.26) \quad \beta_j = \frac{\langle L_{\mathbf{r}}(\psi), h_j \rangle}{\|h_j\|_2^2}.$$

We must estimate the size of  $\beta_j$ . To this end we multiply (3.25) by  $h_k$  and integrate. Then

$$(3.27) \quad \int_0^1 \left[ \left( -\epsilon^2 \left( \psi_{rr} + \frac{n-1}{r} \psi_r \right) + f'(w)\psi + \epsilon\gamma(1-\Delta)^{-1}\psi \right) h_k \right] r^{n-1} dr \\ + \sum_{j=1}^K \beta_j \langle h_j, h_k \rangle = o(\epsilon),$$

Simple calculations simplify the second part on the left side, so

$$(3.28) \quad \int_0^1 \left[ \left( -\epsilon^2 \left( \psi_{rr} + \frac{n-1}{r} \psi_r \right) + f'(w)\psi + \epsilon\gamma(1-\Delta)^{-1}\psi \right) h_k \right] r^{n-1} dr \\ + \sum_{j=1}^K \beta_j (\epsilon\tau r_j^{n-1} \delta_{jk} + O(\epsilon^2)) = o(\epsilon),$$

where  $\delta_{jk} = 1$  if  $j \neq k$  and 0 if  $j = k$ . Also we have used the fact that

$$(3.29) \quad \tau = \int_{\mathbf{R}} (H')^2 dt.$$

This  $\tau$  is the same as the one given in (2.5). These two expressions give the same value because of (3.2), which  $H$  satisfies, and its first integral

$$(3.30) \quad -\frac{1}{2}(H'(t))^2 + W(H(t)) = 0.$$

The first part of the left side of (3.28) is estimated as follows:

$$\begin{aligned}
 (3.31) \quad & \int_0^1 \left[ \left( -\epsilon^2 \left( \psi_{rr} + \frac{n-1}{r} \psi_r \right) + f'(w)\psi + \epsilon\gamma(1-\Delta)^{-1}\psi \right) h_k \right] r^{n-1} dr \\
 &= \int_0^1 \left( -\epsilon^2 \left( h_k'' + \frac{n-1}{r} h_k' \right) \psi + f'(w)h_k\psi \right) r^{n-1} dr + \epsilon\gamma \int_0^1 ((1-\Delta)^{-1}\psi)h_k r^{n-1} dr \\
 &= \int_0^1 -\epsilon^2 \frac{n-1}{r} h_k' \psi r^{n-1} dr + O(\epsilon^2) = O(\epsilon^2).
 \end{aligned}$$

This simplifies (3.28) to

$$(3.32) \quad \sum_{j=1}^{2N} \beta_j (\epsilon\tau\delta_{jk} + O(\epsilon^2)) = o(\epsilon).$$

Hence

$$(3.33) \quad \beta_j = o(1).$$

Let  $y \in [0, 1]$  such that, without loss of generality,  $\psi(y) = \|\psi\|_\infty = 1$ . We claim that  $y - r_j = O(\epsilon)$  for some  $j$ . Otherwise, at  $y$ ,

$$\begin{aligned}
 L_{\mathbf{r}}(\psi)(y) &= -\epsilon^2 \Delta\psi(y) + f'(w(y))\psi(y) + \epsilon\gamma((1-\Delta)^{-1}\psi)(y) \\
 &\geq 0 + f'(w(y)) + \epsilon\gamma((1-\Delta)^{-1}\psi)(y) \\
 (3.34) \quad &= f'(w(y)) + O(\epsilon) = f'(a) + o(1).
 \end{aligned}$$

Combining (3.33) and (3.34), we obtain

$$(3.35) \quad \pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\psi)(y) \geq f'(a) - \sum_{j=1}^K \beta_j h_j(y) + o(1) \geq f'(0) + o(1),$$

which contradicts (3.24).

We have thus proved that  $y - \xi_j = O(\epsilon)$  for some  $j$ , along a subsequence of  $\epsilon \rightarrow 0$ . Define  $\Psi(t) = \psi(r_j + \epsilon t)$ . Then (3.25) and (3.33) imply

$$(3.36) \quad -\Psi'' + f'(w_{\mathbf{r}}(r_j + \epsilon t))\Psi = o(1)$$

uniformly on any compact subset of  $\mathbf{R}$ . From here we may pass to the limit and find  $\Psi_\infty$  so that  $\Psi \rightarrow \Psi_\infty$  in  $C_{loc}^2(\mathbf{R})$ . Moreover,  $\Psi_\infty \neq 0$  since  $\Psi((y - \xi_j)/\epsilon) = 1$ , and

$$(3.37) \quad -\Psi_\infty'' + f'(H)\Psi_\infty = 0.$$

The bounded solutions of this equation are scalar multiples of  $H'$ . Hence  $\Psi_\infty = cH'$  for some  $c \neq 0$ .

On the other hand, since  $\psi \in \mathcal{F}_{\mathbf{r}}$  means that  $\psi \perp h_j$ , we deduce that

$$\begin{aligned}
 (3.38) \quad 0 = \langle \psi, h_j \rangle &= \epsilon \int_{-r_j/\epsilon}^{(1-r_j)/\epsilon} \Psi(t)(H'(t) - O(e^{-C/\sqrt{\epsilon}}))(r_j + \epsilon t)^{n-1} dt \\
 &= \epsilon \left( cr_j^{n-1} \int_{\mathbf{R}} (H'(t))^2 dt + o(1) \right),
 \end{aligned}$$

which is impossible, for  $c \neq 0$ . We have thus proved part 1 of the lemma.

To prove part 2 of the lemma we need to solve

$$(3.39) \quad \pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\phi) = p$$

in  $\mathcal{F}_{\mathbf{r}}$  for any given  $p \in \mathcal{E}_{\mathbf{r}}$ . By applying  $\pi_{\mathbf{r}} \circ (1 - \Delta)^{-1}$  to both sides of (3.39) we consider the equation

$$(3.40) \quad \pi_{\mathbf{r}} \circ (1 - \Delta)^{-1} \circ \pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\phi) = \pi_{\mathbf{r}} \circ (1 - \Delta)^{-1} p.$$

The linear operator  $\pi_{\mathbf{r}} \circ (1 - \Delta)^{-1} \circ \pi_{\mathbf{r}} \circ L_{\mathbf{r}}$  on the left side maps from  $\mathcal{F}_{\mathbf{r}}$  to itself. For this operator  $\mathcal{F}_{\mathbf{r}}$  is viewed as a Banach space whose norm is inherited from the  $W^{2,2}(\Omega)$  norm. The operator has the form

$$(3.41) \quad \epsilon^2(\text{identity operator}) + \text{compact operator}.$$

According to the Fredholm alternative, (3.40) is solvable if

$$(3.42) \quad \pi_{\mathbf{r}} \circ (1 - \Delta)^{-1} \circ \pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\phi) = 0$$

has only the trivial solution. To see this we write (3.42) as

$$(3.43) \quad (1 - \Delta)^{-1} \circ \pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\phi) = \sum_{j=1}^K \alpha_j h_j$$

for some  $\alpha_j \in \mathbf{R}$ . Apply  $1 - \Delta$  to the last equation to find

$$(3.44) \quad \pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\phi) = \sum_{j=1}^K \alpha_j (-\Delta h_j + h_j).$$

We multiply it by  $h_k$  and integrate to deduce

$$(3.45) \quad \begin{aligned} 0 &= \sum_{j=1}^K \alpha_j \int_{\Omega} (\nabla h_j \cdot \nabla h_k + h_j h_k) dx \\ &= \alpha_k \int_{\Omega} (|\nabla h_k|^2 + h_k^2) dx, \quad k = 1, 2, \dots, K, \end{aligned}$$

which implies that  $\alpha_j = 0$ ,  $j = 1, 2, \dots, K$ . Then (3.44) becomes

$$(3.46) \quad \pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\phi) = 0.$$

The first part of the lemma implies that  $\phi = 0$ .

Hence (3.40) is solvable; i.e., for any  $p \in \mathcal{E}_{\mathbf{r}}$  there exist  $\phi \in \mathcal{F}_{\mathbf{r}}$  and  $\beta_j \in \mathbf{R}$  such that

$$(3.47) \quad (1 - \Delta)^{-1} \circ \pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\phi) = (1 - \Delta)^{-1} p + \sum_{j=1}^K \beta_j h_j.$$

Apply  $1 - \Delta$  to the last equation to deduce

$$(3.48) \quad \pi_{\mathbf{r}} \circ L_{\mathbf{r}}(\phi) = p + \sum_{j=1}^K \beta_j (-\Delta h_j + h_j).$$

We again multiply by  $h_k$  and integrate to obtain

$$(3.49) \quad 0 = \sum_{j=1}^K \beta_j \int_{\Omega} \nabla h_j \cdot \nabla h_k \, dx = \beta_k \epsilon \int_{\Omega} (|\nabla h_k|^2 + h_k^2) \, dx, \quad k = 1, 2, \dots, K,$$

which implies that  $\beta_j = 0$  for all  $j = 1, 2, \dots, K$ . Then (3.48) becomes (3.39).  $\square$

We are now ready to solve (3.18).

LEMMA 3.3. *There exists  $\phi(\cdot; \mathbf{r}) \in \mathcal{F}_{\mathbf{r}}$  with  $\|\phi(\cdot; \mathbf{r})\|_{\infty} = O(\epsilon)$  so that  $\pi_{\mathbf{r}} \circ S_{\epsilon}(w(\cdot; \mathbf{r}) + \phi(\cdot; \mathbf{r})) = 0$ .*

*Proof.* We write (3.22) in a fixed point form:

$$(3.50) \quad \phi = (\pi_{\mathbf{r}} \circ L_{\mathbf{r}})^{-1}(-\pi_{\mathbf{r}} \circ S_{\epsilon}(w) - \pi_{\mathbf{r}} \circ R_{\mathbf{r}}(\phi)).$$

We define the operator  $T_{\mathbf{r}}$  from  $\mathcal{D}(T_{\mathbf{r}})$  to itself,

$$(3.51) \quad T_{\mathbf{r}}(\phi) = (\pi_{\mathbf{r}} \circ L_{\mathbf{r}})^{-1}(-\pi_{\mathbf{r}} \circ S_{\epsilon}(w) - \pi_{\mathbf{r}} \circ R_{\mathbf{r}}(\phi)),$$

where the domain  $\mathcal{D}(T_{\mathbf{r}})$  of  $T_{\mathbf{r}}$  is

$$(3.52) \quad \mathcal{D}(T_{\mathbf{r}}) = \{\phi \in L^{\infty}(0, 1) : \phi \perp h_j, \quad j = 1, 2, \dots, K\}.$$

Let  $\mathcal{B}_{\mathbf{r}}$  be a closed ball in  $\mathcal{D}(T_{\mathbf{r}})$  defined by

$$(3.53) \quad \mathcal{B}_{\mathbf{r}} = \{\phi \in \mathcal{D}(T_{\mathbf{r}}) : \|\phi\|_{\infty} \leq C_2 \epsilon\},$$

where  $C_2$  is a constant independent of  $\epsilon$  to be determined soon. For every  $\phi \in \mathcal{B}_{\mathbf{r}}$ , by Lemma 3.1

$$(3.54) \quad \begin{aligned} \|T_{\mathbf{r}}(\phi)\|_{\infty} &\leq C_1 \|\pi_{\mathbf{r}} \circ S_{\epsilon}(w)\|_{\infty} + C_1 \|\pi_{\mathbf{r}} \circ R_{\mathbf{r}}(\phi)\|_{\infty} \\ &\leq C_3 \epsilon + C_5 (1 + O(\|\phi\|_{\infty})) \|\phi\|_{\infty}^2 \\ &\leq C_3 \epsilon + C_6 C_2^2 (1 + C_2 \epsilon) \epsilon^2, \end{aligned}$$

where we have estimated  $R_{\mathbf{r}}(\phi)$  as

$$(3.55) \quad \|R_{\mathbf{r}}(\phi)\|_{\infty} \leq 2 \|f(w_{\mathbf{r}} + \phi) - f(w_{\mathbf{r}}) - f'(w_{\mathbf{r}})\phi\|_{\infty} \leq C_4 (1 + O(\|\phi\|_{\infty})) \|\phi\|_{\infty}^2$$

for some  $C_4$  depending only on  $f$ . In (3.54) the constants  $C_3$  and  $C_6$  are again independent of  $\epsilon$ . If we choose  $C_2$  to be sufficiently large, then when  $\epsilon$  is small enough (3.54) is bounded by  $C_2 \epsilon$ . Therefore by choosing such  $C_2$  we see that  $\mathcal{D}(T_{\mathbf{r}})$  maps  $\mathcal{B}_{\mathbf{r}}$  to itself.

Next we prove that  $T_{\mathbf{r}}$  is a contraction mapping in  $\mathcal{D}(T_{\mathbf{r}})$ . Take  $\phi_1$  and  $\phi_2$  in  $\mathcal{D}(T_{\mathbf{r}})$ . Then

$$(3.56) \quad \begin{aligned} \|T_{\mathbf{r}}(\phi_1) - T_{\mathbf{r}}(\phi_2)\|_{\infty} &\leq C_1 \|\pi_{\mathbf{r}} \circ (R_{\mathbf{r}}(\phi_1) - R_{\mathbf{r}}(\phi_2))\|_{\infty} \leq C_7 \|R_{\mathbf{r}}(\phi_1) - R_{\mathbf{r}}(\phi_2)\|_{\infty} \\ &\leq C_8 \|f(w_{\mathbf{r}} + \phi_1) - f(w_{\mathbf{r}} + \phi_2) - f'(w_{\mathbf{r}})(\phi_1 - \phi_2)\|_{\infty} \\ &\leq C_8 \|f'(w_{\mathbf{r}} + \phi_2 + \theta(\phi_1 - \phi_2))(\phi_1 - \phi_2) - f'(w_{\mathbf{r}})(\phi_1 - \phi_2)\|_{\infty} \\ &\leq C_8 \|f'(w_{\mathbf{r}} + \phi_2 + \theta(\phi_1 - \phi_2)) - f'(w_{\mathbf{r}})\|_{\infty} \|\phi_1 - \phi_2\|_{\infty} \\ &\leq O(\|\phi_1\|_{\infty} + \|\phi_2\|_{\infty}) \|\phi_1 - \phi_2\|_{\infty} \\ &\leq C_9 \epsilon \|\phi_1 - \phi_2\|_{\infty}, \end{aligned}$$

which implies that  $T_{\mathbf{r}}$  is a contraction mapping if  $\epsilon$  is sufficiently small. In these estimates  $\theta = \theta(x) \in (0, 1)$  comes from the mean value theorem.  $\square$

**4.  $C^1$ -convergence of the reduced problem.** We now define

$$(4.1) \quad Q_\epsilon(\mathbf{r}) = I_\epsilon(w(\cdot; \mathbf{r}) + \phi(\cdot; \mathbf{r})),$$

where  $w(\cdot; \mathbf{r})$  is the approximate solution constructed in (3.4) and  $\phi(\cdot; \mathbf{r})$  is given in Lemma 3.3. We may view  $Q_\epsilon$  as a function defined on  $\mathcal{A}_K^a$ .

LEMMA 4.1. *If  $\mathbf{r} \in \mathcal{A}_K^a$  is a critical point of  $Q_\epsilon$ , then  $S_\epsilon(w(\cdot; \mathbf{r}) + \phi(\cdot; \mathbf{r})) = 0$ .*

*Proof.* Let  $\mathbf{r}_*$  be a critical point of  $Q_\epsilon$ . Set  $g(r; \mathbf{r}) = (w(\cdot; \mathbf{r}) + \phi(\cdot; \mathbf{r}))$ . At  $\mathbf{r} = \mathbf{r}_*$  we have, for each  $l$ ,

$$\begin{aligned} 0 &= \frac{\partial Q_\epsilon(\mathbf{r}_*)}{\partial r_l} = \int_0^1 (-\epsilon^2 \Delta g + f(g) + \epsilon \gamma (1 - \Delta)^{-1} g) \frac{\partial g}{\partial r_l} r^{n-1} dr \\ &= \sum_{m=1}^K c_m \int_0^1 h_m \frac{\partial g}{\partial r_l} r^{n-1} dr. \end{aligned}$$

Here we have assumed that at  $\mathbf{r}_*$ ,  $S_\epsilon(g) = \sum_{m=1}^K c_m h_m$ , because  $\pi_{\mathbf{r}}(S_\epsilon(g)) = 0$ . The last equation asserts that the coefficients  $c_m$  satisfy a linear homogeneous system whose  $ml$  matrix entry is  $\int_0^1 h_m \frac{\partial g}{\partial r_l} dx$  at  $\mathbf{r} = \mathbf{r}_*$ .

Recall that  $g = w + \phi$  and  $h_m \perp \phi$  for all  $\mathbf{r}$ . We differentiate  $0 = \int_0^1 h_m \phi r^{n-1} dr$  with respect to  $r_l$  to obtain

$$\int_0^1 h_m \frac{\partial \phi(r; \mathbf{r})}{\partial r_l} r^{n-1} dr = - \int_0^1 \frac{\partial h_m(r; \mathbf{r})}{\partial r_l} \phi(r; \mathbf{r}) r^{n-1} dr.$$

Therefore, since  $\phi = O(\epsilon)$ ,

$$\int_0^1 h_m \frac{\partial g}{\partial r_l} r^{n-1} dr = \int_0^1 \left( h_m \frac{\partial w}{\partial r_l} - \frac{\partial h_m}{\partial r_l} \phi \right) r^{n-1} dr = \delta_{ml} r_l^{n-1} \int_R (H'(t))^2 dt + O(\epsilon).$$

Therefore the coefficient matrix is nonsingular. This implies  $c_m = 0$ , i.e.,  $S_\epsilon(g(\cdot; \mathbf{r}_*)) = 0$ .  $\square$

The reduced problem  $Q_\epsilon$ , scaled by  $\epsilon^{-1}$ , converges to  $J$  given in (2.4) in  $C^1$  locally in  $\mathbf{r}$  and locally in  $\gamma$ .

LEMMA 4.2. *Given a compact subset  $\mathcal{K}$  of  $\mathcal{A}_K^a$  and an interval  $[\gamma_1, \gamma_2]$ , with  $0 < \gamma_1 < \gamma_2 < \infty$ , we have that for every  $\delta > 0$  there exists  $\epsilon_0 > 0$  such that when  $\epsilon < \epsilon_0$ ,  $|\epsilon^{-1} Q_\epsilon(\mathbf{r}) - J(\mathbf{r})| < \delta$  and  $|\text{grad } \epsilon^{-1} Q_\epsilon(\mathbf{r}) - \text{grad } J(\mathbf{r})| < \delta$  for all  $\mathbf{r} \in \mathcal{K}$  and  $\gamma \in [\gamma_1, \gamma_2]$ .*

*Proof.* Given  $\mathcal{K}$  and  $[\gamma_1, \gamma_2]$ , we let  $\mathbf{r} \in \mathcal{K}$  and  $\gamma \in [\gamma_1, \gamma_2]$ . We first show the  $C^0$ -convergence. Expand  $I_\epsilon(w(\cdot; \mathbf{r}) + \phi(\cdot; \mathbf{r}))$  to find

$$(4.2) \quad Q_\epsilon(\mathbf{r}) = I_\epsilon(w) + \int_0^1 S_\epsilon(w) \phi r^{n-1} dr + \frac{1}{2} \int_0^1 \phi L_{\mathbf{r}} \phi dx + O(\epsilon^3).$$

The equation  $\pi_{\mathbf{r}} \circ S_\epsilon(w + \phi) = 0$  implies that  $S_\epsilon(w + \phi) = \sum_{j=1}^K \beta_j h_j$  for some  $\beta_j \in \mathbf{R}$ , which can be written as

$$(4.3) \quad S_\epsilon(w) + L_{\mathbf{r}} \phi + O(\epsilon^2) = \sum_{j=1}^K \beta_j h_j.$$

Multiply (4.3) by  $\phi$  and integrate to find

$$(4.4) \quad \int_0^1 S_\epsilon(w)\phi r^{n-1} dr + \int_0^1 \phi L_{\mathbf{r}}\phi r^{n-1} dr + O(\epsilon^3) = 0$$

since  $\phi \perp h_j$ . Substituting (4.4) into (4.2), we deduce

$$(4.5) \quad Q_\epsilon(\mathbf{r}) = I_\epsilon(w) + \frac{1}{2} \int_0^1 S_\epsilon(w)\phi r^{n-1} dr + O(\epsilon^3).$$

By Lemma 3.1 we obtain

$$(4.6) \quad \int_0^1 S_\epsilon(w)\phi r^{n-1} dr = O(\epsilon^2).$$

Now (4.5) becomes

$$(4.7) \quad Q_\epsilon(\mathbf{r}) = I_\epsilon(w(\cdot; \mathbf{r})) + O(\epsilon^2).$$

So we turn our attention to  $I_\epsilon(w(\cdot; \mathbf{r}))$ . Note that

$$\begin{aligned} I_\epsilon(w(\cdot; \mathbf{r})) &= \omega_{n-1} \int_0^1 \left[ \frac{\epsilon^2}{2} |w_r|^2 + W(w) \right] r^{n-1} dr + \frac{\omega_{n-1}\epsilon\sigma}{2} \int_0^1 |(1-\Delta)^{-1/2}w|^2 r^{n-1} dr \\ &= \omega_{n-1}\epsilon\tau \sum_{j=1}^K r_j^{n-1} + \frac{\omega_{n-1}\epsilon\gamma}{2} \int_0^1 |(1-\Delta)^{-1/2}w|^2 r^{n-1} dr + O(\epsilon^2) \\ (4.8) \quad &= \epsilon J(\mathbf{r}) + O(\epsilon^2). \end{aligned}$$

Before arriving at (4.8) we have used the fact that

$$\int_{\mathbf{R}} \left[ \frac{1}{2} (H')^2 + W(H) \right] dt = \tau,$$

which follows from the first integral (3.30) of  $H$  and the definition of  $\tau$ . Therefore,

$$(4.9) \quad Q_\epsilon(\mathbf{r}) = \epsilon J(\mathbf{r}) + O(\epsilon^2),$$

proving the convergence at the  $C^0$  level.

Next we show the convergence of  $\text{grad}(\epsilon^{-1}Q_\epsilon)$ . We calculate

$$\begin{aligned} \frac{\partial Q_\epsilon(\mathbf{r})}{\partial r_j} &= \frac{\partial}{\partial r_j} I_\epsilon(w(\cdot; \mathbf{r}) + \phi(\cdot; \mathbf{r})) \\ &= \int_\Omega S_\epsilon(w + \phi) \frac{\partial(w(\cdot; \mathbf{r}) + \phi(\cdot; \mathbf{r}))}{\partial r_j} dx \\ (4.10) \quad &= \int_\Omega S_\epsilon(w + \phi) \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} dx + \int_\Omega S_\epsilon(w + \phi) \frac{\partial \phi(\cdot; \mathbf{r})}{\partial r_j} dx. \end{aligned}$$

We estimate the second integral in (4.10) first. Note that since  $\pi_{\mathbf{r}}(S_\epsilon(w + \phi)) = 0$ ,

$$S_\epsilon(w + \phi) = \sum_{l=1}^K \beta_l h_l$$

for some  $\beta_l \in \mathbf{R}$ . Since when  $l \neq m$ ,  $h_l$ , and  $h_m$  are supported in disjoint sets, the  $h_l$ 's are perpendicular to each other. We find

$$\beta_l = \frac{\langle S_\epsilon(w + \phi), h_l \rangle}{\|h_l\|_2^2}.$$

To estimate the numerator on the right side we write

$$S_\epsilon(w + \phi) = S_\epsilon(w) + L_{\mathbf{r}}\phi + R_{\mathbf{r}}(\phi).$$

From Lemma 3.1 we find

$$|\langle S_\epsilon(w), h_l \rangle| \leq \|S_\epsilon(w)\|_\infty \|h_l\|_1 = O(\epsilon)O(\epsilon) = O(\epsilon^2).$$

From Lemma 3.3 we have

$$|\langle L_{\mathbf{r}}\phi, h_l \rangle| = |\langle L_{\mathbf{r}}h_l, \phi \rangle| \leq \|L_{\mathbf{r}}h_l\|_1 \|\phi\|_\infty = O(\epsilon^2)O(\epsilon) = O(\epsilon^3),$$

and

$$|\langle R_{\mathbf{r}}(\phi), h_l \rangle| \leq \|R_{\mathbf{r}}(\phi)\|_\infty \|h_l\|_1 = O(\epsilon^2)O(\epsilon) = O(\epsilon^3).$$

Combing the last three estimates we obtain

$$\langle S_\epsilon(w + \phi), h_l \rangle = O(\epsilon^2).$$

Since  $\|h_l\|_2^2$  is of order  $\epsilon$ , we deduce

$$(4.11) \quad \beta_l = O(\epsilon).$$

The fact  $\phi \perp h_l$  implies, after differentiating  $\langle \phi, h_l \rangle = 0$  with respect to  $r_j$ , that

$$(4.12) \quad \int_0^1 \frac{\partial \phi(r; \mathbf{r})}{\partial r_j} h_j r^{n-1} dr + \int_0^1 \frac{\partial h_l(r; \mathbf{r})}{\partial r_j} \phi r^{n-1} dr = 0.$$

Hence the second integral in (4.10) becomes

$$(4.13) \quad \begin{aligned} \int_{\Omega} S_\epsilon(w + \phi) \frac{\partial \phi(\cdot; \mathbf{r})}{\partial r_j} dx &= \sum_{l=1}^K \omega_{n-1} \beta_l \int_0^1 h_j \frac{\partial \phi}{\partial r_j} r^{n-1} dr \\ &= - \sum_{l=1}^K \omega_{n-1} \beta_l \int_0^1 \phi \frac{\partial h_l}{\partial r_j} r^{n-1} dr. \end{aligned}$$

Our estimate of  $\beta_l$ , (4.11), and Lemma 3.3 imply that the last quantity of (4.13) is of order  $\epsilon^2$ :

$$(4.14) \quad \int_{\Omega} S_\epsilon(w + \phi) \frac{\partial \phi(\cdot; \mathbf{r})}{\partial r_j} dx = O(\epsilon^2).$$

It remains to calculate the first integral in (4.10). We again write  $S_\epsilon(w + \phi) = S_\epsilon(w) + L_{\mathbf{r}}\phi + R_{\mathbf{r}}(\phi)$  so that

$$\int_{\Omega} S_\epsilon(w + \phi) \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} dx = \int_{\Omega} \left[ S_\epsilon(w) \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} + L_{\mathbf{r}}\phi \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} + R_{\mathbf{r}}(\phi) \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} \right] dx.$$

We must separate two cases:  $j$  is odd and  $j$  is even. When  $j$  is odd,  $w(r)$  is  $H(\frac{r-r_j}{\epsilon})$  for  $r$  near  $r_j$ . Moreover,

$$(4.15) \quad \frac{\partial w(r; \mathbf{r})}{\partial r_j} = -\epsilon^{-1} H' \left( \frac{r-r_j}{\epsilon} \right) + O(e^{-C/\epsilon}).$$

In this case we argue as in the estimations leading to (4.11) to conclude that

$$\int_{\Omega} L_{\mathbf{r}} \phi \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} dx = O(\epsilon^2), \quad \int_{\Omega} R_{\mathbf{r}}(\phi) \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} dx = O(\epsilon^2).$$

Therefore,

$$(4.16) \quad \begin{aligned} & \int_{\Omega} S_{\epsilon}(w + \phi) \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} dx = \int_{\Omega} S_{\epsilon}(w) \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} dx + O(\epsilon^2) \\ & = \omega_{n-1} \int_0^1 \left[ -\epsilon^2 \left( w_{rr} + \frac{n-1}{r} w_r \right) + f(w) + \epsilon \gamma (1-\Delta)^{-1} w \right] \frac{\partial w(r; \mathbf{r})}{\partial r_j} r^{n-1} dr \\ & \quad + O(\epsilon^2) \\ & = \omega_{n-1} \int_0^1 \left[ -\epsilon^2 \frac{n-1}{r} w_r + \epsilon \gamma (1-\Delta)^{-1} w \right] \left( -\epsilon^{-1} H' \left( \frac{r-r_j}{\epsilon} \right) \right) r^{n-1} dr + O(\epsilon^2) \\ & = \omega_{n-1} \int_0^1 \left[ -\epsilon \frac{n-1}{r} H' \left( \frac{r-r_j}{\epsilon} \right) + \epsilon \gamma (1-\Delta)^{-1} w \right] \left( -\epsilon^{-1} H' \left( \frac{r-r_j}{\epsilon} \right) \right) r^{n-1} dr \\ & \quad + O(\epsilon^2) \\ & = \omega_{n-1} \epsilon [(n-1)\tau r_j^{n-2} - (b-a)\gamma r_j^{n-1} ((1-\Delta)^{-1} w(\cdot; \mathbf{r}))(r_j)] + O(\epsilon^2). \end{aligned}$$

Similarly when  $j$  is even,  $w(r)$  is  $H(-\frac{r-r_j}{\epsilon})$  for  $r$  near  $r_j$  and

$$(4.17) \quad \frac{\partial w(r; \mathbf{r})}{\partial r_j} = \epsilon^{-1} H' \left( -\frac{r-r_j}{\epsilon} \right) + O(e^{-C/\epsilon}).$$

Then

$$(4.18) \quad \begin{aligned} & \int_{\Omega} S_{\epsilon}(w + \phi) \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} dx \\ & = \omega_{n-1} \epsilon [(n-1)\tau r_j^{n-2} + (b-a)\gamma r_j^{n-1} ((1-\Delta)^{-1} w(\cdot; \mathbf{r}))(r_j)] + O(\epsilon^2) \\ & = \omega_{n-1} \epsilon [(n-1)\tau r_j^{n-2} + (b-a)\gamma r_j^{n-1} ((1-\Delta)^{-1} s(\cdot; \mathbf{r}))(r_j)] + O(\epsilon^2). \end{aligned}$$

Recall that  $s \in \mathcal{A}_K^a$  is the outer part of  $w$  defined in (3.1). In conclusion, by (4.10), (4.14), (4.16), (4.18), and the calculations of  $\partial J / \partial r_j$  in section 2, we have that

$$(4.19) \quad \frac{\partial Q_{\epsilon}(\mathbf{r})}{\partial r_j} = \int_{\Omega} S_{\epsilon}(w + \phi) \frac{\partial w(\cdot; \mathbf{r})}{\partial r_j} dx + O(\epsilon^2) = \epsilon \frac{\partial J(\mathbf{r})}{\partial r_j} + O(\epsilon^2).$$

This proves the lemma.  $\square$

We are now ready to prove our main results.

*Proof of Theorem 1.1.* Lemma 2.4, part 1, asserts that  $J$  is minimized at a point in the interior of  $\mathcal{A}_K^a$  if  $\gamma$  is large. By Lemma 4.2 we conclude that  $Q_{\epsilon}$  has a local

minimum  $\mathbf{r}_\epsilon$  in  $\mathcal{A}_K^a$ . As  $\epsilon \rightarrow 0$ ,  $\mathbf{r}_\epsilon$  converges, possibly along a subsequence, to  $\mathbf{r}_* \in \mathcal{A}_K^a$ , which is a minimizer of  $J$ . Choose a small neighborhood  $\mathcal{K}_1$  of  $\mathbf{r}_*$  so that  $\mathbf{r}_\epsilon \in \mathcal{K}_1$ . If there are several critical points of  $Q_\epsilon$  in  $\mathcal{K}_1$ , we are finished. If there is only one, it is an isolated strict local minimum and hence has index 1. On the other hand, there exists a neighborhood  $\mathcal{K}$  of  $\mathcal{K}_1$  on which the degree of  $\text{grad } J$  is zero by Lemma 2.4, part 2. Hence, by continuity, the same is true for  $\text{grad } Q_\epsilon$  for small  $\epsilon$ . Hence there must be another critical point of  $Q_\epsilon$  in  $\mathcal{K}$ , as required.  $\square$

*Proof of Theorem 1.2.* We combine Lemmas 2.3 and 4.2.  $\square$

**5. Solutions with layers near the boundary.** In this section, we construct solutions with multiple layers near the boundary of  $\Omega$  and prove Theorem 1.3. Let  $m^2 = f'(a) = f'(b) = (b - a)^2/2 > 0$ . First, we construct an approximate solution.

Let  $\xi(t)$  be a smooth function, such that  $0 \leq \xi \leq 1$ ,  $\xi(t) = 0$  for  $t \leq \frac{1}{2}$ , and  $\xi = 1$  for  $t \geq \frac{2}{3}$ . Let  $\mathbf{r} = (r_0, \bar{r}_1, r_1, \dots, \bar{r}_k, r_k) \in D_{\epsilon,k}$ , where  $D_{\epsilon,k}$  is the set containing all  $\mathbf{r}$  satisfying

$$1 - M\epsilon \ln \frac{1}{\epsilon} < r_0 < \bar{r}_1 < r_1 < \dots < \bar{r}_k < r_k < 1 - \alpha\epsilon \ln \frac{1}{\epsilon},$$

and

$$\bar{r}_j - r_{j-1} \geq \alpha\epsilon \ln \frac{1}{\epsilon}, \quad r_j - \bar{r}_j \geq \alpha\epsilon \ln \frac{1}{\epsilon}, \quad j = 1, \dots, k,$$

where  $\alpha > 0$  is a small constant and  $M > 0$  is a large constant.

Define

$$v_{\epsilon,j}(r) = (1 - \xi)b + \xi(r)H\left(\frac{r_j - r}{\epsilon}\right), \quad j = 0, 1, \dots, k,$$

and

$$\bar{v}_{\epsilon,j}(r) = (1 - \xi)a + \xi(r)H\left(\frac{r - \bar{r}_j}{\epsilon}\right), \quad j = 1, \dots, k.$$

It is easy to check that  $v_{\epsilon,j}$  and  $\bar{v}_{\epsilon,j}$  satisfy

$$(5.1) \quad -\epsilon^2 \Delta v = -f(v) + O(\epsilon^2 |v'| + \epsilon^2) = -f(v) + O(\epsilon).$$

Let

$$w_{\epsilon,k}(r) = v_{\epsilon,0} + \sum_{j=1}^k (v_{\epsilon,j} + \bar{v}_{\epsilon,j} - a - b).$$

Then, using (5.1), we obtain

$$(5.2) \quad \begin{aligned} & -\epsilon^2 \Delta w_{\epsilon,k} + m^2 w_{\epsilon,k} \\ &= -\epsilon^2 \Delta v_{\epsilon,0} - \sum_{j=1}^k (\epsilon^2 \Delta v_{\epsilon,j} + \epsilon^2 \Delta \bar{v}_{\epsilon,j}) + m^2 w_{\epsilon,k} \\ &= -f(v_{\epsilon,0}) - \sum_{j=1}^k (f(v_{\epsilon,j}) + f(\bar{v}_{\epsilon,j})) + m^2 w_{\epsilon,k} \\ & \quad + \epsilon O\left(\epsilon |v'_{\epsilon,0}| + \epsilon \sum_{j=1}^k (|v'_{\epsilon,j}| + |\bar{v}'_{\epsilon,j}|) + \epsilon\right). \end{aligned}$$

Since  $w_{\epsilon,k}$  does not satisfy  $w'_{\epsilon,k}(1) = 0$ , we need to make a projection as follows. Let  $Pw_{\epsilon,k}$  be the solution of

$$(5.3) \quad \begin{cases} -\epsilon^2 \Delta Pw_{\epsilon,k} + m^2 Pw_{\epsilon,k} = -\epsilon^2 \Delta w_{\epsilon,k} + m^2 w_{\epsilon,k} & \text{in } \Omega, \\ (Pw_{\epsilon,k})'(1) = 0. \end{cases}$$

We denote  $\varphi_\epsilon = w_{\epsilon,k} - Pw_{\epsilon,k}$ . Then  $\varphi_\epsilon$  satisfies

$$(5.4) \quad \begin{cases} -\epsilon^2 \Delta \varphi_\epsilon + m^2 \varphi_\epsilon = 0 & \text{in } \Omega, \\ \varphi'_\epsilon(1) = w'_{\epsilon,k}(1). \end{cases}$$

By the maximum principle, we see  $\varphi_\epsilon < 0$ .

We have the following estimates for  $\varphi_\epsilon$ .

LEMMA 5.1. *For any small  $\theta > 0$ , there is a constant  $C > 0$ , such that*

$$|\varphi_\epsilon(r)| \leq C e^{-m(1-r_k)/\epsilon} e^{-m(1-\theta)(1-r)/\epsilon}.$$

In particular,

$$|\varphi_\epsilon(r)| \leq C e^{-m(1-\theta)|r-r_k|/\epsilon}.$$

*Proof.* Let  $G_\epsilon(Y, y)$  and  $G(Y, y)$  be the Green function of  $-\epsilon^2 \Delta + m^2 I$  in  $\Omega$  and  $-\Delta + m^2 I$  in  $\Omega_{\epsilon,y} = \{Y : \epsilon Y + y \in \Omega\}$  subject to the Neumann boundary condition, respectively. Then

$$G_\epsilon(Y, y) = \frac{1}{\epsilon^n} G\left(\frac{Y-y}{\epsilon}, 0\right).$$

We have

$$\varphi_\epsilon(y) = \epsilon^2 \int_{\partial\Omega} G_\epsilon(Y, y) w'_{\epsilon,k}(1) dY.$$

This, together with  $|w'_{\epsilon,k}(1)| = \epsilon^{-1} |\sum_{j=1}^k H'(\frac{1-r_j}{\epsilon}) - \sum_{j=0}^k H'(\frac{1-r_j}{\epsilon})| \leq C \epsilon^{-1} e^{-m(1-r_k)/\epsilon}$ , gives

$$(5.5) \quad \begin{aligned} |\varphi_\epsilon(y)| &\leq C \epsilon e^{-m(1-r_k)/\epsilon} \int_{\partial\Omega} |G_\epsilon(Y, y)| dY \\ &= C \epsilon e^{-m(1-r_k)/\epsilon} \frac{1}{\epsilon^n} \int_{\partial\Omega} \left| G\left(\frac{Y-y}{\epsilon}, 0\right) \right| dY \\ &= C e^{-m(1-r_k)/\epsilon} \int_{\partial\Omega_{\epsilon,y}} |G(Y, 0)| dY \leq C e^{-m(1-r_k)/\epsilon} e^{-m(1-\theta)(1-r)/\epsilon}, \end{aligned}$$

since  $G(Y, 0) \sim \frac{1}{|Y|^{N-2}}$  as  $|Y| \rightarrow 0$ , and  $|G(Y, 0)| \leq C e^{-m|Y|}$  as  $|Y| \rightarrow +\infty$ .

Since for  $r \in [0, 1]$  we have

$$|r - r_k| \leq |1 - r_k| + |r - 1| = 1 - r_k + 1 - r,$$

as a result

$$|\varphi_\epsilon(y)| \leq C e^{-m(1-r_k)/\epsilon} e^{-m(1-\theta)(1-r)/\epsilon} \leq C e^{-m(1-\theta)|r-r_k|/\epsilon}.$$

So, we have proved the lemma.  $\square$

Define

$$I^*(u) = \frac{\epsilon^2}{2} \int_{\Omega} |Du|^2 + \int_{\Omega} W(u),$$

where  $W(t) = \int_a^t f(s) ds$ .

Next, we estimate  $I^*(Pw_{\epsilon,k})$ . We have

PROPOSITION 5.2.

$$\begin{aligned} I^*(Pw_{\epsilon,k}) &= \epsilon Ar_0^{n-1} + \epsilon A \sum_{j=1}^k (\bar{r}_j^{n-1} + r_j^{n-1}) \\ &\quad - B\epsilon \sum_{j=1}^k (e^{-m(\bar{r}_j - r_{j-1})/\epsilon} + e^{-m(r_j - \bar{r}_j)/\epsilon}) - B_\epsilon \omega_{n-1} \epsilon e^{-2m(1-r_k)/\epsilon} \\ &\quad + \epsilon O \left( \sum_{j=1}^k (e^{-(1+\sigma)m(\bar{r}_j - r_{j-1})/\epsilon} + e^{-(1+\sigma)m(r_j - \bar{r}_j)/\epsilon}) + e^{-2(1+\sigma)m(1-r_k)/\epsilon} + \epsilon \right), \end{aligned}$$

where  $A > 0$  and  $B > 0$  are some constants independent of  $\epsilon$ ,  $B_\epsilon > 0$  is a constant depending on  $\epsilon$ , satisfying  $b_2 \geq B_\epsilon \geq b_1 > 0$  for some constants  $b_2$  and  $b_1$ , and  $\sigma > 0$  is a constant.

From Proposition 5.2, we see that there are three factors that affect the energy of  $Pw_{\epsilon,k}$ :

(i) The contribution from the layers is

$$\epsilon Ar_0^{n-1} + \epsilon A \sum_{j=1}^k (\bar{r}_j^{n-1} + r_j^{n-1}).$$

(ii) The contribution from the interaction between the layers is

$$-B\epsilon \sum_{j=1}^k (e^{-m(\bar{r}_j - r_{j-1})/\epsilon} + e^{-m(r_j - \bar{r}_j)/\epsilon}).$$

(iii) The contribution from the Neumann boundary condition is

$$B_\epsilon \omega_{n-1} \epsilon e^{-2m(1-r_k)/\epsilon}.$$

So, we conclude that the energy will decrease if the layer moves away from the boundary, or the layers move toward each other, or the layer moves toward the boundary. As a result, if  $I^*(Pw_{\epsilon,k})$  attains its maximum, the layers must be suitably separated and stay suitably close to the boundary.

We will prove Proposition 5.2 by proving three lemmas.

Let

$$\hat{f}(t) = f(t + a).$$

Then  $\hat{F}(t) = \int_0^t \hat{f}(s) ds = W(t + a)$ .

Denote  $\psi_{\epsilon,j} = v_{\epsilon,j} - a$ ,  $j = 0, \dots, k$ ,  $\bar{\psi}_{\epsilon,j} = \bar{v}_{\epsilon,j} - a$ ,  $j = 1, \dots, k$ , and  $\tilde{v}_{\epsilon,j} = \psi_{\epsilon,j} + \bar{\psi}_{\epsilon,j} - (b - a) = v_{\epsilon,j} + \bar{v}_{\epsilon,j} - b - a$ ,  $j = 1, \dots, k$ ,  $\tilde{v}_{\epsilon,0} = \psi_{\epsilon,0}$ . So, with this notation, we see that

$$f(v_{\epsilon,j}) = \hat{f}(\psi_{\epsilon,j}), \quad f(\bar{v}_{\epsilon,j}) = \hat{f}(\bar{\psi}_{\epsilon,j}),$$

and

$$W(w_{\epsilon,k}) = \hat{F}(\bar{w}_{\epsilon,k}),$$

where  $\bar{w}_{\epsilon,k} = w_{\epsilon,k} - a = \sum_{j=0}^k \tilde{v}_j$ .

It is easy to see that

$$P\bar{w}_{\epsilon,k} = Pw_{\epsilon,k} - a.$$

Let

$$\hat{I}^*(u) = \frac{\epsilon^2}{2} \int_{\Omega} |Du|^2 + \int_{\Omega} \hat{F}(u).$$

By (5.2) and (5.3), we have

(5.6)

$$\begin{aligned} I^*(Pw_{\epsilon,k}) &= \hat{I}^*(P\bar{w}_{\epsilon,k}) \\ &= \frac{1}{2} \int_{\Omega} (\epsilon^2 |DP\bar{w}_{\epsilon,k}|^2 + m^2 (P\bar{w}_{\epsilon,k})^2) + \int_{\Omega} \hat{F}(P\bar{w}_{\epsilon,k}) - \frac{1}{2} m^2 \int_{\Omega} (P\bar{w}_{\epsilon,k})^2 \\ &= \frac{1}{2} \int_{\Omega} \left( -\hat{f}(\psi_{\epsilon,0}) - \sum_{j=1}^k (\hat{f}(\psi_{\epsilon,j}) + \hat{f}(\bar{\psi}_{\epsilon,j})) + m^2 \hat{w}_{\epsilon,k} \right) P\bar{w}_{\epsilon,k} \\ &\quad + \epsilon O \left( \int_{\Omega} \left( \epsilon |\psi'_{\epsilon,0}| + \epsilon \sum_{j=1}^k (|\psi'_{\epsilon,j}| + |\bar{\psi}'_{\epsilon,j}|) \right) |Pw_{\epsilon,k}| \right) \\ &\quad + \int_{\Omega} \left( \hat{F}(P\bar{w}_{\epsilon,k}) - \frac{m^2}{2} (P\bar{w}_{\epsilon,k})^2 \right) \\ &= \frac{1}{2} \int_{\Omega} \left( -\hat{f}(\psi_{\epsilon,0}) - \sum_{j=1}^k (\hat{f}(\psi_{\epsilon,j}) + \hat{f}(\bar{\psi}_{\epsilon,j})) \right) \bar{w}_{\epsilon,k} + \int_{\Omega} \hat{F}(\bar{w}_{\epsilon,k}) \\ &\quad + \int_{\Omega} \left( -\hat{f}(\bar{w}_{\epsilon,k}) + \frac{1}{2} \left( \hat{f}(\psi_{\epsilon,0}) + \sum_{j=1}^k (\hat{f}(\psi_{\epsilon,j}) + \hat{f}(\bar{\psi}_{\epsilon,j})) \right) + \frac{1}{2} m^2 \bar{w}_{\epsilon,k} \right) \varphi_{\epsilon} \\ &\quad + \int_{\Omega} \left( \hat{F}(P\bar{w}_{\epsilon,k}) - \hat{F}(\bar{w}_{\epsilon,k}) + \hat{f}(\bar{w}_{\epsilon,k}) \varphi_{\epsilon} - \frac{1}{2} m^2 \varphi_{\epsilon}^2 \right) + O(\epsilon^2) \\ &=: \hat{I}_1 + \hat{I}_2 + \hat{I}_3 + O(\epsilon^2). \end{aligned}$$

LEMMA 5.3. *We have*

$$|\hat{I}_3| \leq C\epsilon e^{-(3-\theta)m(1-r_k)/\epsilon},$$

where  $\theta > 0$  is any small constant.

*Proof.* By definition, we have

$$(5.7) \quad \hat{I}_3 = \frac{1}{2} \int_{\Omega} (\hat{f}'(\bar{w}_{\epsilon,k})\varphi_{\epsilon}^2 - m^2\varphi_{\epsilon}^2) + O\left(\int_{\Omega} |\varphi_{\epsilon}|^3\right).$$

Write

$$(5.8) \quad \begin{aligned} \int_{\Omega} (\hat{f}'(\bar{w}_{\epsilon,k})\varphi_{\epsilon}^2 - m^2\varphi_{\epsilon}^2) &= \int_{r_k < r \leq 1} (\hat{f}'(\bar{w}_{\epsilon,k})\varphi_{\epsilon}^2 - m^2\varphi_{\epsilon}^2) + \int_{r \leq r_k} (\hat{f}'(\bar{w}_{\epsilon,k})\varphi_{\epsilon}^2 - m^2\varphi_{\epsilon}^2) \\ &= \int_{r_k < r \leq 1} (\hat{f}'(\bar{w}_{\epsilon,k})\varphi_{\epsilon}^2 - m^2\varphi_{\epsilon}^2) + O\left(\int_{r \leq r_k} \varphi_{\epsilon}^2\right). \end{aligned}$$

Using Lemma 5.1, we obtain for  $r \leq r_k$

$$|\varphi_{\epsilon}(r)| \leq C e^{-m(1-r_k)/\epsilon} e^{-m(1-\theta)(1-r)/\epsilon} \leq C e^{-(2-\theta)m(1-r_k)/\epsilon}.$$

Thus,

$$(5.9) \quad \begin{aligned} \int_{r \leq r_k} |\varphi_{\epsilon}|^2 &\leq C e^{-(2-\theta)2m(1-r_k)/\epsilon} \int_{r \leq r_k} |\varphi_{\epsilon}|^{\theta} \\ &\leq C e^{-(2-\theta)2m(1-r_k)/\epsilon} \int_{r \leq r_k} e^{-\theta(1-\theta)m|r-r_k|/\epsilon} \\ &\leq C \epsilon e^{-(2-\theta)2m(1-r_k)/\epsilon}. \end{aligned}$$

On the other hand, for  $r \geq r_k$ , we have

$$(5.10) \quad \begin{aligned} \hat{f}'(\bar{w}_{\epsilon,k})\varphi_{\epsilon}^2 - m^2\varphi_{\epsilon}^2 &= \hat{f}'(\bar{w}_{\epsilon,k})\varphi_{\epsilon}^2 - \hat{f}'(0)\varphi_{\epsilon}^2 \\ &= O(|\bar{w}_{\epsilon,k}|\varphi_{\epsilon}^2) = O\left(e^{-m(r-r_k)/\epsilon} e^{-(2-\theta)m(1-r_k)/\epsilon} e^{-m(2-\theta)(1-\theta)(1-r)/\epsilon} |\varphi_{\epsilon}^{\theta}|\right) \\ &= O\left(e^{-m(3-\theta)(1-r_k)/\epsilon} |\varphi_{\epsilon}^{\theta}|\right). \end{aligned}$$

Using (5.10), we obtain

$$(5.11) \quad \begin{aligned} \int_{r_k < r \leq 1} (\hat{f}'(\bar{w}_{\epsilon,k})\varphi_{\epsilon}^2 - m^2\varphi_{\epsilon}^2) \\ \leq C e^{-m(3-\theta)(1-r_k)/\epsilon} \int_{\Omega} |\varphi_{\epsilon}^{\theta}| \leq C \epsilon e^{-m(3-\theta)(1-r_k)/\epsilon}. \end{aligned}$$

Combining (5.9) and (5.10), we are led to

$$(5.12) \quad \int_{\Omega} (\hat{f}'(\bar{w}_{\epsilon,k})\varphi_{\epsilon}^2 - m^2\varphi_{\epsilon}^2) = O\left(\epsilon e^{-m(3-\theta)(1-r_k)/\epsilon}\right).$$

Finally,

$$(5.13) \quad \begin{aligned} \int_{\Omega} |\varphi_{\epsilon}|^3 &= \int_{r_k < r \leq 1} |\varphi_{\epsilon}|^3 + \int_{r \leq r_k} |\varphi_{\epsilon}|^3 \leq C \epsilon e^{-m(3-\theta)(1-r_k)/\epsilon} + C \epsilon e^{-(2-\theta)2m(1-r_k)/\epsilon} \\ &\leq C \epsilon e^{-(3-\theta)m(1-r_k)/\epsilon}. \end{aligned}$$

Combining (5.7), (5.12), and (5.13), we obtain the result.  $\square$

LEMMA 5.4. *We have*

$$\begin{aligned} \hat{I}_1 &= \epsilon A r_0^{n-1} + \epsilon A \sum_{j=1}^k (\bar{r}_j^{n-1} + r_j^{n-1}) - B \epsilon \sum_{j=1}^k (e^{-m(\bar{r}_j - r_{j-1})/\epsilon} + e^{-m(r_j - \bar{r}_j)/\epsilon}) \\ &\quad + \epsilon O \left( \sum_{j=1}^k (e^{-(1+\sigma)m(\bar{r}_j - r_{j-1})/\epsilon} + e^{-(1+\sigma)m(r_j - \bar{r}_j)/\epsilon}) + e^{-(2+\sigma)m(1-r_k)/\epsilon} + \epsilon \right), \end{aligned}$$

where  $A = \omega_{n-1} \int_{-\infty}^{+\infty} (W(H(t)) - \frac{1}{2}f(H(t))H(t)) dt > 0$ ,  $B > 0$  is a constant, and  $\sigma > 0$  is a small constant.

*Proof.* It is easy to check that for any bounded  $t_1$  and  $t_2$ ,

$$\hat{F}(t_1 + t_2) - \hat{F}(t_1) - \hat{F}(t_2) = \hat{f}(t_1)t_2 + (\hat{f}(t_2) - \hat{f}'(0)t_2)t_1 + O(|t_1t_2|^2).$$

Thus,

$$\begin{aligned} \hat{F}(\bar{w}_{\epsilon,k}) &= \sum_{j=0}^k \hat{F}(\tilde{v}_{\epsilon,j}) + \sum_{i < j} \hat{f}(\tilde{v}_{\epsilon,i})\tilde{v}_{\epsilon,j} + \sum_{i=1}^{k-1} \left( \hat{f} \left( \sum_{j=i+1}^k \tilde{v}_{\epsilon,j} \right) - \hat{f}'(0) \sum_{j=i+k}^k \tilde{v}_{\epsilon,j} \right) \tilde{v}_{\epsilon,i} \\ (5.14) \quad &+ O \left( \sum_{i \neq j} |\tilde{v}_{\epsilon,i}\tilde{v}_{\epsilon,j}|^2 \right). \end{aligned}$$

Using (5.14), we can write

$$\begin{aligned} \hat{I}_1 &= \sum_{j=0}^k \int_{\Omega} \left( \hat{F}(\tilde{v}_{\epsilon,j}) - \frac{1}{2} \hat{f}(\tilde{v}_{\epsilon,j})\tilde{v}_{\epsilon,j} \right) + \frac{1}{2} \int_{\Omega} \sum_{j=1}^k \left( \hat{f}(\tilde{v}_{\epsilon,j}) - \hat{f}(\psi_{\epsilon,j}) - \hat{f}(\bar{\psi}_{\epsilon,j}) \right) \bar{w}_{\epsilon,k} \\ (5.15) \quad &+ \sum_{i=1}^{k-1} \int_{\Omega} \left( \hat{f} \left( \sum_{j=i+1}^k \tilde{v}_{\epsilon,j} \right) - \hat{f}'(0) \sum_{j=i+1}^k \tilde{v}_{\epsilon,j} \right) \tilde{v}_{\epsilon,i} + O \left( \sum_{i \neq j} \int_{\Omega} |\tilde{v}_{\epsilon,i}\tilde{v}_{\epsilon,j}|^2 \right). \end{aligned}$$

It is easy to prove that

$$(5.16) \quad \int_{\Omega} |\tilde{v}_{\epsilon,i}\tilde{v}_{\epsilon,j}|^2 = \epsilon O \left( e^{-2m|\bar{r}_j - r_i|/\epsilon} \right), \quad j > i.$$

On the other hand, since for any  $t \in (0, b - a)$ ,

$$\hat{f}(t) - \hat{f}'(0)t < 0,$$

we see

$$(5.17) \quad \int_{\Omega} \left( \hat{f} \left( \sum_{j=i+1}^k \tilde{v}_{\epsilon,j} \right) - \hat{f}'(0) \sum_{j=i+1}^k \tilde{v}_{\epsilon,j} \right) \tilde{v}_{\epsilon,i} = -(B + o(1))\epsilon e^{-m(\bar{r}_{i+1} - r_i)/\epsilon},$$

where  $o(1) \rightarrow 0$  as  $\epsilon \rightarrow 0$ , and

$$B = - \int_{-\infty}^{+\infty} (\hat{f}(H(t) - a) - \hat{f}'(0)(H(t) - a)) e^{-mt} dt > 0.$$

Since  $\hat{f}(t) = -\hat{f}(b - a - t)$ , we obtain

$$\begin{aligned} & \hat{f}(\tilde{v}_{\epsilon,j}) - \hat{f}(\psi_{\epsilon,j}) - \hat{f}(\bar{\psi}_{\epsilon,j}) \\ &= -\hat{f}(b - a - \psi_{\epsilon,j} + b - a - \bar{\psi}_{\epsilon,j}) + \hat{f}(b - a - \psi_{\epsilon,j}) + \hat{f}(b - a - \bar{\psi}_{\epsilon,j}) \\ &= O(|(b - a - \psi_{\epsilon,j})|(b - a - \bar{\psi}_{\epsilon,j})). \end{aligned}$$

Thus, for  $i \neq j$ , we see

$$\int_{\Omega} \left( \hat{f}(\tilde{v}_{\epsilon,j}) - \hat{f}(\psi_{\epsilon,j}) - \hat{f}(\bar{\psi}_{\epsilon,j}) \right) \tilde{v}_i = \epsilon O \left( \sum_{j=1}^k (e^{-2m|\bar{r}_j - r_{j-1}|/\epsilon} + e^{-2m|r_j - \bar{r}_j|/\epsilon}) \right).$$

So, we have

$$\begin{aligned} & \sum_{j=0}^k \int_{\Omega} \left( \hat{F}(\tilde{v}_{\epsilon,j}) - \frac{1}{2} \hat{f}(\tilde{v}_{\epsilon,j}) \tilde{v}_{\epsilon,j} \right) + \frac{1}{2} \int_{\Omega} \sum_{j=1}^k \left( \hat{f}(\tilde{v}_{\epsilon,j}) - \hat{f}(\psi_{\epsilon,j}) - \hat{f}(\bar{\psi}_{\epsilon,j}) \right) \bar{w}_{\epsilon,k} \\ &= \sum_{j=0}^k \int_{\Omega} \left( \hat{F}(\tilde{v}_{\epsilon,j}) - \frac{1}{2} \hat{f}(\tilde{v}_{\epsilon,j}) \tilde{v}_{\epsilon,j} \right) + \frac{1}{2} \int_{\Omega} \sum_{j=1}^k \left( \hat{f}(\tilde{v}_{\epsilon,j}) - \hat{f}(\psi_{\epsilon,j}) - \hat{f}(\bar{\psi}_{\epsilon,j}) \right) \tilde{v}_j \\ & \quad + \epsilon O \left( \sum_{j=1}^k (e^{-2m|\bar{r}_j - r_{j-1}|/\epsilon} + e^{-2m|r_j - \bar{r}_j|/\epsilon}) \right) \\ &= \sum_{j=0}^k \int_{\Omega} \hat{F}(\tilde{v}_{\epsilon,j}) - \frac{1}{2} \int_{\Omega} \sum_{j=1}^k \left( \hat{f}(\psi_{\epsilon,j}) + \hat{f}(\bar{\psi}_{\epsilon,j}) \right) \tilde{v}_j \\ (5.18) \quad & + \epsilon O \left( \sum_{j=1}^k (e^{-2m|\bar{r}_j - r_{j-1}|/\epsilon} + e^{-2m|r_j - \bar{r}_j|/\epsilon}) \right). \end{aligned}$$

But from  $\hat{f}(t) = -\hat{f}(b - a - t)$ , we see

$$\hat{f}(\psi_{\epsilon,j})(\bar{\psi}_{\epsilon,j} - (b - a)) = \hat{f}(b - a - \psi_{\epsilon,j})(b - a - \bar{\psi}_{\epsilon,j}),$$

and

$$\hat{f}(\bar{\psi}_{\epsilon,j})(\psi_{\epsilon,j} - (b - a)) = \hat{f}(b - a - \bar{\psi}_{\epsilon,j})(b - a - \psi_{\epsilon,j}).$$

We also have

$$\begin{aligned}
 \hat{F}(\tilde{v}_{\epsilon,j}) &= \hat{F}(b-a-\tilde{v}_{\epsilon,j}) = \hat{F}(b-a-\psi_{\epsilon,j}+b-a-\bar{\psi}_{\epsilon,j}) \\
 &= \hat{F}(b-a-\psi_{\epsilon,j}) + \hat{F}(b-a-\bar{\psi}_{\epsilon,j}) \\
 &\quad + \frac{1}{2}\hat{f}(b-a-\psi_{\epsilon,j})(b-a-\bar{\psi}_{\epsilon,j}) \\
 &\quad + \frac{1}{2}\left(\hat{f}(b-a-\bar{\psi}_{\epsilon,j}) - \hat{f}'(0)(b-a-\bar{\psi}_{\epsilon,j})\right)(b-a-\psi_{\epsilon,j}) \\
 &\quad + \frac{1}{2}\hat{f}(b-a-\bar{\psi}_{\epsilon,j})(b-a-\psi_{\epsilon,j}) \\
 &\quad + \frac{1}{2}\left(\hat{f}(b-a-\psi_{\epsilon,j}) - \hat{f}'(0)(b-a-\psi_{\epsilon,j})\right)(b-a-\bar{\psi}_{\epsilon,j}) \\
 (5.19) \quad &+ O\left(|b-a-\psi_{\epsilon,j}|^2|b-a-\bar{\psi}_{\epsilon,j}|^2\right).
 \end{aligned}$$

Using (5.19), we obtain

$$\begin{aligned}
 &\int_{\Omega} \hat{F}(\tilde{v}_{\epsilon,j}) - \frac{1}{2} \int_{\Omega} \left(\hat{f}(\psi_{\epsilon,j}) + \hat{f}(\bar{\psi}_{\epsilon,j})\right) \tilde{v}_j \\
 &= \int_{\Omega} \left(\hat{F}(\psi_{\epsilon,j}) - \frac{1}{2}\hat{f}(\psi_{\epsilon,j})\psi_{\epsilon,j}\right) + \int_{\Omega} \left(\hat{F}(\bar{\psi}_{\epsilon,j}) - \frac{1}{2}\hat{f}(\bar{\psi}_{\epsilon,j})\bar{\psi}_{\epsilon,j}\right) \\
 (5.20) \quad &- (B + o(1))\epsilon e^{-m(r_j-\bar{r}_j)/\epsilon} + \epsilon O\left(\sum_{j=1}^k e^{-2m(r_j-\bar{r}_j)/\epsilon}\right).
 \end{aligned}$$

Combining (5.15), (5.16), (5.17), (5.18), and (5.20), we obtain

$$\begin{aligned}
 \hat{I}_1 &= \sum_{j=0}^k \int_{\Omega} \left(\hat{F}(\psi_{\epsilon,j}) - \frac{1}{2}\hat{f}(\psi_{\epsilon,j})\psi_{\epsilon,j}\right) + \sum_{j=1}^k \int_{\Omega} \left(\hat{F}(\bar{\psi}_{\epsilon,j}) - \frac{1}{2}\hat{f}(\bar{\psi}_{\epsilon,j})\bar{\psi}_{\epsilon,j}\right) \\
 &\quad - (B + o(1))\epsilon \sum_{j=1}^k (e^{-m(r_j-\bar{r}_j)/\epsilon} + e^{-m(\bar{r}_j-r_{j-1})/\epsilon}) \\
 (5.21) \quad &+ \epsilon O\left(\sum_{j=1}^k (e^{-2m(r_j-\bar{r}_j)/\epsilon} + e^{-2m(\bar{r}_j-r_{j-1})/\epsilon})\right).
 \end{aligned}$$

Finally, let  $\hat{H}(t) = H(t) - a$ . Then we have

$$\begin{aligned}
 (5.22) \quad &\int_{\Omega} \hat{F}(\psi_{\epsilon,j}) = \omega_{n-1}\epsilon r_j^{n-1} \int_{-(1-r_j)/\epsilon}^{r_j/\epsilon} \hat{F}(\hat{H}(t)) dt + O(\epsilon^2) \\
 &= \omega_{n-1}\epsilon r_j^{n-1} \int_{-\infty}^{+\infty} \hat{F}(\hat{H}(t)) dt - \omega_{n-1}\epsilon r_j^{n-1} \int_{-\infty}^{-(1-r_j)/\epsilon} \hat{F}(\hat{H}(t)) dt + O(\epsilon^2),
 \end{aligned}$$

and

$$(5.23) \quad \int_{\Omega} \hat{f}(\psi_{\epsilon,j})\psi_{\epsilon,j} = \omega_{n-1}\epsilon r_j^{n-1} \int_{-\infty}^{+\infty} \hat{f}(\hat{H}(t))\hat{H}(t) dt - \omega_{n-1}\epsilon r_j^{n-1} \times \int_{-\infty}^{-(1-r_j)/\epsilon} \hat{f}(\hat{H}(t))\hat{H}(t) dt + O(\epsilon^2).$$

But from  $\hat{F}(0) = \hat{f}(0) = 0$ , we see

$$(5.24) \quad \left| \int_{-\infty}^{-(1-r_j)/\epsilon} \hat{F}(\hat{H}(t)) dt - \frac{1}{2} \int_{-\infty}^{-(1-r_j)/\epsilon} \hat{f}(\hat{H}(t))\hat{H}(t) dt \right| \leq \int_{-\infty}^{-(1-r_j)/\epsilon} \hat{H}^3(t) dt = O\left(\int_{-\infty}^{-(1-r_j)/\epsilon} e^{3mt} dt\right) \leq C e^{-3m(1-r_j)/\epsilon}.$$

Combining (5.22), (5.23), and (5.24), we are led to

$$(5.25) \quad \int_{\Omega} \left( \hat{F}(\psi_{\epsilon,j}) - \frac{1}{2} \hat{f}(\psi_{\epsilon,j})\psi_{\epsilon,j} \right) = \omega_{n-1}\epsilon r_j^{n-1} \int_{-\infty}^{+\infty} \hat{F}(\hat{H}(t)) dt - \frac{1}{2} \omega_{n-1}\epsilon r_j^{n-1} \int_{-\infty}^{+\infty} \hat{f}(\hat{H}(t))\hat{H}(t) dt + \epsilon O(\epsilon + e^{-3m(1-r_j)/\epsilon}) = \omega_{n-1}\epsilon r_j^{n-1} A + \epsilon O(\epsilon + e^{-3m(1-r_k)/\epsilon}).$$

Similarly, we can obtain

$$(5.26) \quad \int_{\Omega} \left( \hat{F}(\bar{\psi}_{\epsilon,j}) - \frac{1}{2} \hat{f}(\bar{\psi}_{\epsilon,j})\bar{\psi}_{\epsilon,j} \right) = \omega_{n-1}\epsilon \bar{r}_j^{n-1} A + \epsilon O(\epsilon + e^{-3m(1-r_k)/\epsilon}).$$

Combining (5.21), (5.25), and (5.26), we prove this lemma.  $\square$

LEMMA 5.5. *We have*

$$(5.27) \quad \hat{I}_2 = -\epsilon B_{\epsilon} \omega_{n-1} e^{-2m(1-r_k)/\epsilon} + \epsilon O\left(e^{-2m(1-r_k)/\epsilon} + \epsilon\right) + \epsilon O\left(\sum_{j=1}^k (e^{-2(1+\sigma)(r_j-\bar{r}_j)/\epsilon} + e^{-2(1+\sigma)(\bar{r}_j-r_{j-1})/\epsilon})\right),$$

where  $B_{\epsilon}$  is a constant with  $b_2 \geq B_{\epsilon} \geq b_1 > 0$ , and  $\sigma > 0$  is a small constant.

*Proof.* We have

$$(5.28) \quad \hat{I}_2 = \frac{1}{2} \int_{\Omega} \left( -\hat{f}(\psi_{\epsilon,0}) - \sum_{j=1}^k (\hat{f}(\psi_{\epsilon,j}) + \hat{f}(\bar{\psi}_{\epsilon,j})) + m^2 \bar{w}_{\epsilon,k} \right) \varphi_{\epsilon} - \int_{\Omega} \left( \hat{f}(\bar{w}_{\epsilon,k}) - \hat{f}(\psi_{\epsilon,0}) - \sum_{j=1}^k (\hat{f}(\psi_{\epsilon,j}) + \hat{f}(\bar{\psi}_{\epsilon,j})) \right) \varphi_{\epsilon} =: \hat{I}_{2,1} + \hat{I}_{2,2}.$$

It is easy to see that

$$\begin{aligned}
 |\hat{I}_{2,2}| &\leq C\epsilon \left( \sum_{j=1}^k (e^{-m(r_j-\bar{r}_j)/\epsilon} + e^{-m(\bar{r}_j-r_{j-1})/\epsilon}) \right) \|\varphi_\epsilon\|_\infty^{1-\theta} \\
 &\leq C\epsilon \left( \sum_{j=1}^k (e^{-m(\bar{r}_j-r_{j-1})/\epsilon} + e^{-m(r_j-\bar{r}_j)/\epsilon}) \right) e^{-m(1-\theta)(1-r_k)/\epsilon} \\
 (5.29) &\leq C\epsilon \left( \sum_{j=1}^k (e^{-(1+\sigma)m(\bar{r}_j-r_{j-1})/\epsilon} + e^{-(1+\sigma)m(r_j-\bar{r}_j)/\epsilon}) + e^{-(2+\sigma)m(1-r_k)/\epsilon} \right).
 \end{aligned}$$

On the other hand, using (5.2) and (5.4), we see

(5.30)

$$\begin{aligned}
 \hat{I}_{2,1} &= \frac{1}{2} \int_\Omega \left( -\epsilon^2 \Delta \bar{w}_{\epsilon,k} + m^2 \bar{w}_{\epsilon,k} + \epsilon O \left( \epsilon |\psi'_{\epsilon,0}| + \epsilon \sum_{j=1}^k (|\psi'_{\epsilon,j}| + |\bar{\psi}'_{\epsilon,j}|) + \epsilon \right) \right) \varphi_\epsilon \\
 &= \frac{1}{2} \omega_{n-1} \epsilon^2 (-\bar{w}'_{\epsilon,k}(1) \varphi_\epsilon(1) + \bar{w}_{\epsilon,k}(1) \bar{w}'_{\epsilon,k}(1)) \\
 &\quad + \epsilon O \left( \int_\Omega \left( \epsilon |\psi'_{\epsilon,0}| + \epsilon \sum_{j=1}^k (|\psi'_{\epsilon,j}| + |\bar{\psi}'_{\epsilon,j}|) + \epsilon \right) |\varphi_\epsilon| \right) \\
 &= -B_\epsilon \omega_{n-1} \epsilon e^{-2m(1-r_k)/\epsilon} + \epsilon O \left( \int_\Omega \left( \epsilon |\psi'_{\epsilon,0}| + \epsilon \sum_{j=1}^k (|\psi'_{\epsilon,j}| + |\bar{\psi}'_{\epsilon,j}|) + \epsilon \right) |\varphi_\epsilon| \right),
 \end{aligned}$$

where  $B_\epsilon = \frac{1}{2} e^{2m(1-r_k)/\epsilon} (\epsilon \bar{w}'_{\epsilon,k}(1) \varphi_\epsilon(1) - \epsilon \bar{w}_{\epsilon,k}(1) \bar{w}'_{\epsilon,k}(1))$ . By Lemma 5.1, noting that  $\varphi_\epsilon < 0$ ,  $w_{\epsilon,k}(1) \sim e^{-m(1-r_k)/\epsilon}$ , and  $\epsilon w'_{\epsilon,k}(1) \sim -e^{-m(1-r_k)/\epsilon}$ , we see easily that there are  $b_2 > b_1 > 0$ , independent of  $\epsilon$ , such that  $b_2 \geq B_\epsilon \geq b_1$ .

We have

$$\begin{aligned}
 \epsilon \int_\Omega |\psi'_{\epsilon,j}| |\varphi_\epsilon| &= \epsilon \int_{r \leq r_k} |\psi'_{\epsilon,j}| |\varphi_\epsilon| + \epsilon \int_{r \geq r_k} |\psi'_{\epsilon,j}| |\varphi_\epsilon| \\
 &\leq C\epsilon e^{-(2-\theta)m(1-r_k)/\epsilon} + C e^{-m(1-r_k)/\epsilon} \int_{r \geq r_k} e^{-m(r-r_j)/\epsilon} e^{-(1-\theta)m(1-r)/\epsilon} \\
 &\leq C\epsilon e^{-(2-\theta)m(1-r_k)/\epsilon} + C e^{-(2-\theta)m(1-r_k)/\epsilon} \epsilon \ln \frac{1}{\epsilon} \\
 (5.31) &\leq C(e^{-(2+\sigma)m(1-r_k)/\epsilon} + \epsilon),
 \end{aligned}$$

$$(5.32) \quad \epsilon \int_\Omega |\bar{\psi}'_{\epsilon,j}| |\varphi_\epsilon| \leq C(e^{-(2+\sigma)m(1-r_k)/\epsilon} + \epsilon),$$

and

$$(5.33) \quad \epsilon \int_{\Omega} |\varphi_{\epsilon}| \leq C(e^{-(2+\sigma)m(1-r_k)/\epsilon} + \epsilon).$$

Combining (5.30), (5.31), (5.32), and (5.33), we obtain

$$(5.34) \quad \hat{I}_{2,1} = -B_{\epsilon}\omega_{n-1}\epsilon e^{-2m(1-r_k)/\epsilon} + \epsilon O(e^{-(2+\sigma)m(1-r_k)/\epsilon} + \epsilon).$$

So, the result follows from (5.28), (5.29), and (5.34).  $\square$

*Proof of Proposition 5.2.* The proof follows from Lemmas 5.3, 5.4, and 5.5.  $\square$

Now, we look at the reduction. We have

$$(5.35) \quad \begin{aligned} S_{\epsilon}(Pw_{\epsilon,k}) &= -\epsilon^2 \Delta Pw_{\epsilon,k} + f(Pw_{\epsilon,k}) + \epsilon\gamma(1-\Delta)^{-1}Pw_{\epsilon,k} \\ &= f(w_{\epsilon,k}) - f(v_{\epsilon,0}) - \sum_{j=1}^k (f(v_{\epsilon,j}) + f(\bar{v}_{\epsilon,j})) \end{aligned}$$

$$(5.36) \quad + f(Pw_{\epsilon,k}) - f(w_{\epsilon,k}) + m^2(w_{\epsilon,k} - Pw_{\epsilon,k}) + O(\epsilon)$$

$$(5.37) \quad = O\left(e^{-m(1-r_k)/\epsilon} + \sum_{j=1}^k (e^{-m(\bar{r}_j-r_{j-1})/\epsilon} + e^{-m(r_j-\bar{r}_j)/\epsilon}) + \epsilon\right).$$

Similar to the discussion in section 3, we can prove the following result.

LEMMA 5.6. *There is a  $\phi_{\epsilon} \in \mathcal{F}_{\mathbf{r}}$ , such that  $\pi_{\mathbf{r}} \circ S_{\epsilon}(w_{\epsilon,k} + \phi_{\epsilon}) = 0$ . Moreover,*

$$\|\phi_{\epsilon}\|_{\infty} \leq C\left(e^{-m(1-r_k)/\epsilon} + \sum_{j=1}^k (e^{-m(\bar{r}_j-r_{j-1})/\epsilon} + e^{-m(r_j-\bar{r}_j)/\epsilon}) + \epsilon\right).$$

LEMMA 5.7. *We have*

$$\begin{aligned} I_{\epsilon}(Pw_{\epsilon,k} + \phi_{\epsilon}) &= I_{\epsilon}(Pw_{\epsilon,k}) \\ &+ \epsilon O\left(e^{-(2+\sigma)m(1-r_k)/\epsilon} + \sum_{j=1}^k (e^{-(1+\sigma)m(\bar{r}_j-r_{j-1})/\epsilon} + e^{-(1+\sigma)m(r_j-\bar{r}_j)/\epsilon}) + \epsilon\right), \end{aligned}$$

where  $\sigma > 0$  is a small constant.

*Proof.* First, we estimate  $\|\phi_{\epsilon}\|_2$ .

We have

$$\begin{aligned} \|\phi_{\epsilon}\|_2 &\leq C\|S_{\epsilon}(Pw_{\epsilon,k})\|_2 + C\|R_{\mathbf{r}}(\phi_{\epsilon})\|_2 \\ &\leq C\|S_{\epsilon}(Pw_{\epsilon,k})\|_2 + C\|\phi_{\epsilon}^2\|_2 \\ &\leq C\|S_{\epsilon}(Pw_{\epsilon,k})\|_2 + C\|\phi_{\epsilon}\|_{\infty}\|\phi_{\epsilon}\|_2 \\ &\leq C\|S_{\epsilon}(Pw_{\epsilon,k})\|_2 + o(1)\|\phi_{\epsilon}\|_2, \end{aligned}$$

where  $o(1) = \|\phi_{\epsilon}\|_{\infty} \rightarrow 0$  as  $\epsilon \rightarrow 0$ . Thus,

$$(5.38) \quad \|\phi_{\epsilon}\|_2 \leq C\|S_{\epsilon}(Pw_{\epsilon,k})\|_2.$$

We obtain, from (5.37),

$$\begin{aligned}
 \|S_\epsilon(Pw_{\epsilon,k})\|_2 &\leq \|\hat{f}(\bar{w}_{\epsilon,k}) - \sum_{j=0}^k \hat{f}(\tilde{v}_{\epsilon,j})\|_2 \\
 &+ \left\| \sum_{j=1}^k (\hat{f}(\tilde{v}_{\epsilon,j}) - \hat{f}(\psi_{\epsilon,j}) - \hat{f}(\bar{\psi}_{\epsilon,j})) \right\|_2 \\
 (5.39) \quad &+ \|f(Pw_{\epsilon,k}) - f(w_{\epsilon,k}) + m^2(w_{\epsilon,k} - Pw_{\epsilon,k})\|_2 + C\epsilon.
 \end{aligned}$$

It is easy to prove that

$$\begin{aligned}
 &\left\| \hat{f}(\bar{w}_{\epsilon,k}) - \sum_{j=0}^k \hat{f}(\tilde{v}_{\epsilon,j}) \right\|_2^2 \\
 (5.40) \quad &\leq C\epsilon \sum_{j=1}^k (e^{-(1+\sigma)m(r_j-\bar{r}_j)/\epsilon} + e^{-(1+\sigma)m(\bar{r}_j-r_{j-1})/\epsilon}),
 \end{aligned}$$

and

$$\begin{aligned}
 &\left\| \sum_{j=1}^k (\hat{f}(\tilde{v}_{\epsilon,j}) - \hat{f}(\psi_{\epsilon,j}) - \hat{f}(\bar{\psi}_{\epsilon,j})) \right\|_2^2 \\
 (5.41) \quad &\leq C\epsilon \sum_{j=1}^k (e^{-(1+\sigma)m(r_j-\bar{r}_j)/\epsilon} + e^{-(1+\sigma)m(\bar{r}_j-r_{j-1})/\epsilon}).
 \end{aligned}$$

Moreover, from

$$f(Pw_{\epsilon,k}) - f(w_{\epsilon,k}) + m^2(w_{\epsilon,k} - Pw_{\epsilon,k}) = (m^2 - f'(w_{\epsilon,k}))\varphi_\epsilon + O(\varphi_\epsilon^2),$$

we obtain

$$\begin{aligned}
 &\|f(Pw_{\epsilon,k}) - f(w_{\epsilon,k}) + m^2(w_{\epsilon,k} - Pw_{\epsilon,k})\|_2^2 \\
 &\leq C \int_\Omega \varphi_\epsilon^4 + C \int_\Omega (m^2 - f'(w_{\epsilon,k}))^2 \varphi_\epsilon^2 \\
 (5.42) \quad &\leq C\epsilon e^{-3m(1-r_k)/\epsilon} + C \int_\Omega (m^2 - f'(w_{\epsilon,k}))^2 \varphi_\epsilon^2.
 \end{aligned}$$

But

$$\begin{aligned}
 &\int_\Omega (m^2 - f'(w_{\epsilon,k}))^2 \varphi_\epsilon^2 \\
 &\leq C \int_{r \leq r_k} \varphi_\epsilon^2 + C \int_{r \geq r_k} |w_{\epsilon,k} - a| \varphi_\epsilon^2 \\
 (5.43) \quad &\leq C\epsilon e^{-(2+\sigma)m(1-r_k)/\epsilon}.
 \end{aligned}$$

Combining (5.39)–(5.43), we obtain

$$(5.44) \quad \|S_\epsilon(Pw_{\epsilon,k})\|_2^2 \leq C\epsilon e^{-(2+\sigma)m(1-r_k)/\epsilon} + C\epsilon \sum_{j=1}^k (e^{-(1+\sigma)m(r_j-\bar{r}_j)/\epsilon} + e^{-(1+\sigma)m(\bar{r}_j-r_{j-1})/\epsilon}).$$

Using (5.38), we obtain

$$(5.45) \quad \|\phi_\epsilon\|_2^2 \leq C\epsilon e^{-(2+\sigma)m(1-r_k)/\epsilon} + C\epsilon \sum_{j=1}^k (e^{-(1+\sigma)m(r_j-\bar{r}_j)/\epsilon} + e^{-(1+\sigma)m(\bar{r}_j-r_{j-1})/\epsilon}).$$

Similar to section 4, using (5.44) and (5.45), we have

$$\begin{aligned} & I_\epsilon(Pw_{\epsilon,k} + \phi_\epsilon) \\ &= I_\epsilon(Pw_{\epsilon,k}) + O\left(\|S_\epsilon(Pw_{\epsilon,k})\|_2 \|\phi_\epsilon\|_2 + \|\phi_\epsilon\|_2^2\right) \\ &= I_\epsilon(Pw_{\epsilon,k}) \\ &+ \epsilon O\left(\epsilon + e^{-(2+\sigma)m(1-r_k)/\epsilon} + \sum_{j=1}^k (e^{-(1+\sigma)m(r_j-\bar{r}_j)/\epsilon} + e^{-(1+\sigma)m(\bar{r}_j-r_{j-1})/\epsilon})\right). \end{aligned}$$

So we have proved this lemma.  $\square$

*Proof of Theorem 1.3.* We just need to consider the case  $b > 0$ . For  $a < b \leq 0$ , we let  $u_1 = -u$  and  $v_1 = -v$ . Then  $u_1$  and  $v_1$  will satisfy a similar system with  $f_1(t) = (t + a)(t + \frac{a+b}{2})(t + b)$ , and  $-a > -b \geq 0$ . From now on, we always assume that  $b > 0$ .

Consider

$$(5.46) \quad \max_{\mathbf{r}_\epsilon \in D_{\epsilon,k}} Q_\epsilon(\mathbf{r}),$$

where  $Q_\epsilon(\mathbf{r}) = I_\epsilon(Pw_{\epsilon,k} + \phi_\epsilon)$ .

Let  $\mathbf{r}_\epsilon \in D_{\epsilon,k}$  be a maximum point of (5.46). We will prove that  $\mathbf{r}_\epsilon$  is an interior point of  $D_{\epsilon,k}$ . So it is a critical point of  $I_\epsilon(Pw_{\epsilon,k} + \phi_\epsilon)$ .

Let  $L > 0$  be a large number, such that  $mL > 4$ . Choose  $\mathbf{r}_\epsilon^* \in D_{\epsilon,k}$ , such that  $r_k^* = 1 - L\epsilon \ln \frac{1}{\epsilon}$ ,  $\bar{r}_j^* = r_j^* - L\epsilon \ln \frac{1}{\epsilon}$ , and  $r_{j-1}^* = \bar{r}_j^* - L\epsilon \ln \frac{1}{\epsilon}$ . For this  $\mathbf{r}_\epsilon^*$ , using Proposition 5.2, we have

$$(5.47) \quad I^*(Pw_{\epsilon,k}) = \epsilon(2k + 1)A + \epsilon O\left(\epsilon \ln \frac{1}{\epsilon}\right).$$

On the other hand, we have

$$(5.48) \quad \int_\Omega Pw_{\epsilon,k}(1 - \Delta)^{-1}Pw_{\epsilon,k} = \int_\Omega b(1 - \Delta)^{-1}b + O\left(\epsilon \ln \frac{1}{\epsilon}\right).$$

Combining (5.47) and (5.48), we obtain

$$(5.49) \quad Q_\epsilon(\mathbf{r}_\epsilon^*) = \epsilon(2k + 1)A + \frac{1}{2}\gamma\epsilon \int_\Omega b(1 - \Delta)^{-1}b + \epsilon O\left(\epsilon \ln \frac{1}{\epsilon}\right).$$

We have, from  $Q_\epsilon(\mathbf{r}_\epsilon) \geq Q_\epsilon(\mathbf{r}_\epsilon^*)$ ,

$$(5.50) \quad \begin{aligned} &\epsilon A r_{\epsilon,0}^{n-1} + \epsilon A \sum_{j=1}^k (\bar{r}_{\epsilon,j}^{n-1} + r_{\epsilon,j}^{n-1}) + \frac{1}{2}\gamma\epsilon \int_\Omega Pw_{\epsilon,k}(1 - \Delta)^{-1}Pw_{\epsilon,k} \\ &\quad - (B + o(1))\epsilon \sum_{j=1}^k (e^{-m(\bar{r}_{\epsilon,j} - r_{\epsilon,j-1})/\epsilon} + e^{-m(r_{\epsilon,j} - \bar{r}_{\epsilon,j})/\epsilon}) \\ &\quad - (B_\epsilon + o(1))\omega_{n-1}\epsilon e^{-2m(1-r_{\epsilon,k})/\epsilon} + O(\epsilon^2) \\ &\geq \epsilon(2k + 1)A + \frac{1}{2}\gamma\epsilon \int_\Omega b(1 - \Delta)^{-1}b + \epsilon O\left(\epsilon \ln \frac{1}{\epsilon}\right). \end{aligned}$$

Since  $|b - Pw_{\epsilon,k}| \leq C\epsilon^2$  if  $r \leq 1 - 2M\epsilon \ln \frac{1}{\epsilon}$ , it is easy to check that

$$\int_\Omega |(1 - \Delta)^{-1}(b - Pw_{\epsilon,k})|^2 \leq C \int_\Omega |b - Pw_{\epsilon,k}|^2 \leq C\epsilon \ln \frac{1}{\epsilon}.$$

As a result,

$$(5.51) \quad \begin{aligned} &\int_\Omega b(1 - \Delta)^{-1}b - \int_\Omega Pw_{\epsilon,k}(1 - \Delta)^{-1}Pw_{\epsilon,k} \\ &= -2 \int_\Omega (Pw_{\epsilon,k} - b)(1 - \Delta)^{-1}b + \int_\Omega (Pw_{\epsilon,k} - b)(1 - \Delta)^{-1}(Pw_{\epsilon,k} - b) \\ &= 2 \int_\Omega (b - w_{\epsilon,k})(1 - \Delta)^{-1}b + 2 \int_\Omega \varphi_\epsilon(1 - \Delta)^{-1}b + O\left(\epsilon^2 \ln^2 \frac{1}{\epsilon}\right). \end{aligned}$$

But

$$(5.52) \quad \left| \int_\Omega \varphi_\epsilon(1 - \Delta)^{-1}b \right| \leq C \int_\Omega |\varphi_\epsilon| \leq C\epsilon.$$

Combining (5.51) and (5.52), we obtain

$$(5.53) \quad \begin{aligned} &\int_\Omega b(1 - \Delta)^{-1}b - \int_\Omega Pw_{\epsilon,k}(1 - \Delta)^{-1}Pw_{\epsilon,k} \\ &= 2 \int_\Omega (b - w_{\epsilon,k})(1 - \Delta)^{-1}b + O(\epsilon). \end{aligned}$$

So, it follows from (5.50) and (5.53) that

$$\begin{aligned}
 & \epsilon A(1 - r_{\epsilon,0}^{n-1}) + \epsilon A \sum_{j=1}^k (1 - \bar{r}_{\epsilon,j}^{n-1} + 1 - r_{\epsilon,j}^{n-1}) \\
 & + \frac{1}{2} \gamma \epsilon \int_{\Omega} (b - w_{\epsilon,k})(1 - \Delta)^{-1} b \\
 & + (B + o(1)) \epsilon \sum_{j=1}^k (e^{-m(\bar{r}_{\epsilon,j} - r_{\epsilon,j-1})/\epsilon} + e^{-m(r_{\epsilon,j} - \bar{r}_{\epsilon,j})/\epsilon}) \\
 & + (B_{\epsilon} + o(1)) \omega_{n-1} \epsilon e^{-2m(1-r_{\epsilon,k})/\epsilon} \\
 (5.54) \quad & \leq C \epsilon^2 \ln \frac{1}{\epsilon}.
 \end{aligned}$$

Since all the terms in the left-hand side of (5.54) are positive, we obtain

$$1 - r_{\epsilon,0}^{n-1} \leq C \epsilon \ln \frac{1}{\epsilon},$$

and

$$e^{-m(\bar{r}_{\epsilon,j} - r_{\epsilon,j-1})/\epsilon}, \quad e^{-m(r_{\epsilon,j} - \bar{r}_{\epsilon,j})/\epsilon}, \quad e^{-2m(1-r_{\epsilon,k})/\epsilon} \leq C \epsilon \ln \frac{1}{\epsilon}.$$

So  $|\bar{r}_{\epsilon,j} - r_{\epsilon,j-1}| \geq c' \epsilon \ln \frac{1}{\epsilon}$ ,  $|r_{\epsilon,j} - \bar{r}_{\epsilon,j}| \geq c' \epsilon \ln \frac{1}{\epsilon}$ ,  $1 - r_{\epsilon,0} \leq C \epsilon \ln \frac{1}{\epsilon}$ , and  $1 - r_{\epsilon,k} \geq c' \epsilon \ln \frac{1}{\epsilon}$ . This shows that  $r_{\epsilon}$  is an interior point of  $D_{\epsilon,k}$  if  $M > 0$  is large and  $\alpha > 0$  is small.  $\square$

*Remark 5.8.* In the case  $b > 0$ , if  $\gamma = 0$ , or  $\gamma > 0$  and  $a \leq 0$ , then we can use the above techniques to show that (1.1) has a solution, which is close to  $Pw_{\epsilon,k}^*$ , where

$$w_{\epsilon,k}^* = \sum_{j=1}^{k-1} (v_{\epsilon,j} + \bar{v}_{\epsilon,j} - a - b) + \bar{v}_{\epsilon,k}.$$

In this case, similar to (5.54), we have

$$\begin{aligned}
 & \epsilon A(1 - \bar{r}_{\epsilon,k}^{n-1}) + \epsilon A \sum_{j=1}^{k-1} (1 - \bar{r}_{\epsilon,j}^{n-1} + 1 - r_{\epsilon,j}^{n-1}) \\
 & + \frac{1}{2} \gamma \epsilon \int_{\Omega} (a - w_{\epsilon,k}^*)(1 - \Delta)^{-1} a \\
 & + (B + o(1)) \epsilon \sum_{j=1}^{k-1} (e^{-m(\bar{r}_{\epsilon,j} - r_{\epsilon,j-1})/\epsilon} + e^{-m(r_{\epsilon,j} - \bar{r}_{\epsilon,j})/\epsilon}) \\
 & + (B_{\epsilon} + o(1)) \omega_{n-1} \epsilon e^{-2m(1-\bar{r}_{\epsilon,k})/\epsilon} \\
 (5.55) \quad & \leq C \epsilon^2 \ln \frac{1}{\epsilon}.
 \end{aligned}$$

We know that  $a - w_{\epsilon,k}^*$  is always nonpositive. So, if  $a \leq 0$ , the term  $\int_{\Omega}(a - w_{\epsilon,k}^*)(1 - \Delta)^{-1}a$  is nonnegative. Thus, if  $\gamma = 0$ , or  $\gamma > 0$  and  $a \leq 0$ , the left-hand side of (5.55) is always nonnegative. As a result, we can use (5.55) to deduce that  $\mathbf{r}_{\epsilon}$  is an interior point of  $D_{\epsilon,k}$ .

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