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ON THE MULTIPLICITY OF SOLUTIONS OF TWO NONLOCAL VARIATIONAL PROBLEMS*

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Abstract. We study two nonlocal variational problems in this paper. One models microphase separation of diblock copolymers and the other models solid-solid phase transformations that lead to fine structures. We study a parameter range where the problems can be approximated by their asymptotic limits. We find all the local minimum solutions of the limiting problems. Because these local minima are isolated, and hence stable under perturbation, near them there exist local minimum solutions of the original problems.

Key words. nonlocal variational problems, Γ-convergence, isolated local minima, *BV* functions

AMS subject classifications. 49K15, 49K20, 34D15

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1. Introduction.

Two nonlocal variational problems are studied in this paper. The first one is

$$\mathcal{I}_\epsilon(u) = \begin{cases} \displaystyle\int_\Omega \left\{ \frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u) + \frac{1}{2}|(-\gamma^2\Delta)^{-1/2}(u-m)|^2 \right\} dx, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in \mathcal{A}_m \cap W^{1,2}(\Omega), \\ \infty \qquad\qquad\qquad\qquad\qquad\quad \text{if } u \in \mathcal{A}_m \backslash W^{1,2}(\Omega). \end{cases} \tag{1.1}$$

Ω is a bounded and smooth domain in R^d. ϵ and γ are both positive numbers. ϵ is small and γ is fixed.

W is a balanced double-well function with two global minima at -1 and 1, i.e., $W(t) \geq 0$ and $W(t) = 0$ if and only if $t = -1$ or 1. We also assume that W is continuous and there exist $k \geq 2$, $K_1 > 0$, $K_2 > 0$, and $\bar{t} > 1$ such that for all t, $|t| > \bar{t}$,

$$K_1|t|^k \leq W(t) \leq K_2|t|^k. \tag{1.2}$$

$-\gamma^2\Delta$ is the Laplacian operator, multiplied by the constant $-\gamma^2$, with the Neumann boundary condition. The outward normal vector field of $\partial\Omega$ is denoted by n. It is known that the operator

$$-\gamma^2\Delta : \left\{ u \in W^{2,2}(\Omega) : \int_\Omega u = 0,\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0 \right\} \to \left\{ u \in L^2(\Omega) : \int_\Omega u = 0 \right\}$$

is an isomorphism. The inverse of $-\gamma^2\Delta$ is self-adjoint and positive. We denote its positive square root by $(-\gamma^2\Delta)^{-1/2}$. This is a nonlocal operator.

The admissible set of $\mathcal{I}_\epsilon$ is

$$\mathcal{A}_m = \left\{ u \in L^2(\Omega) : \frac{1}{|\Omega|}\int_\Omega u\, dx = m \right\} \tag{1.3}$$

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with the restriction $m \in (-1,1)$. For a measurable set Ω in R^d, $|\Omega|$ denotes its Lebesgue measure. In $\mathcal{A}_m$ we use the metric so that the distance between u and v is $\|u-v\|_2$, the $L^2(\Omega)$-norm of $u-v$. The choice of $L^2(\Omega)$ (as opposed to the choice of $L^1(\Omega)$ in the literature of Γ-convergence) is natural because of the nonlocal part, $(-\gamma^2\Delta)^{-1/2}$, of the functional.

In the study of diblock copolymers, a model was introduced in Ohta and Kawasaki [11] and Bahiana and Oono [1]. It asserts that the free energy of a diblock copolymer takes the form

$$\mathcal{F}_{\epsilon,\sigma}(u) = \int_\Omega \left\{ \frac{\epsilon^2}{2}|\nabla u|^2 + W(u) + \frac{\sigma}{2}|(-\Delta)^{-1/2}(u-m)|^2 \right\} dx,$$
$$\frac{1}{|\Omega|}\int_\Omega u\,dx = m.$$

In a diblock copolymer, a linear-chain molecule consists of two subchains grafted covalently to each other. The subchains are made of two different monomer units, represented by $u=-1$ and $u=1$, respectively. The different subchains tend to segregate below some critical temperature, but as they are chemically bonded, only local microphase separation occurs. The connectivity of the two monomer units leads to the long range interaction term $\frac{\sigma}{2}|(-\Delta)^{-1/2}(u-m)|^2$ in the free energy. The parameter σ is proportional to the inverse of the square root of the total chain length of the copolymer. $\frac{\epsilon^2}{2}|\nabla u|^2$ represents the interfacial energy density at bonding points. The parameter ϵ is proportional to the thickness of interfaces between the two monomers. The double-well potential W prefers segregated monomers to a mixture. m stands for the mass ratio of the two monomer units. When this free energy is being minimized, the first term prefers large blocks of monomers, therefore reducing the combined size of interfaces between the two monomers. The third term, on the other hand, likes rapid oscillation between the two monomers. These two tendencies are competing. The process of reaching a stable configuration is known as microseparation. The results of our paper show that in a parameter range, namely $0<\epsilon\approx\sigma\ll 1$, the monomer components in the copolymer develop blocks of a finite scale. For more references on the mathematical aspects of diblock copolymers we refer the reader to Nishiura and Ohnishi [10], where a different parameter range, $0<\epsilon\ll\sigma\ll 1$, is studied. The functional studied in Müller [9] can also be written in the form of this model with $m=0$. There the parameter range is $0<\epsilon\ll\sigma\approx 1$.

The second problem is

$$(1.4)\quad \mathcal{J}_\epsilon(u) = \begin{cases} \displaystyle\int_\Omega \left\{ \frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u) - \frac{1}{2}u^2 + \frac{1}{2}|(-\gamma^2\Delta+1)^{-1/2}u|^2 \right\} dx, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u \in \mathcal{A}_m \cap W^{1,2}(\Omega), \\ \infty \qquad\qquad\qquad\qquad\qquad\quad \text{if } u \in \mathcal{A}_m \backslash W^{1,2}(\Omega) \end{cases}$$

in the same admissible set $\mathcal{A}_m$, (1.3). W satisfies the same conditions as in $\mathcal{I}_\epsilon$. The operator

$$-\gamma^2\Delta + 1 : \left\{ u \in W^{2,2}(\Omega) : \frac{\partial u}{\partial n}|_{\partial\Omega} = 0 \right\} \to L^2(\Omega)$$

is an isomorphism. Again we denote the positive square root of the inverse of $-\gamma^2\Delta+1$ by $(-\gamma^2\Delta+1)^{-1/2}$.

In studying solid-solid phase transformations, Ren and Truskinovsky proposed in [12] a model of 1-dimensional elastic bars that develop a mixture of two phase variants. Let u be the strain field of a deformed elastic bar. The stored energy is

$$\mathcal{F}_\epsilon(u) = \int_0^1 \left\{ \frac{\epsilon^2}{2}|\nabla u|^2 + W(u) - \frac{\epsilon}{2}u^2 + \frac{\epsilon}{2}|(-\gamma^2\Delta + 1)^{-1/2}u|^2 \right\} dx,$$

$$\int_0^1 u\, dx = m,$$

where m is the total displacement of the bar. $W(u)$ is the local part of the energy density. It is assumed to be nonconvex. It prefers the two phase variants, $u = -1$ and $u = 1$. $\frac{\epsilon^2}{2}|\nabla u|^2$ is the short range self-interaction of the strain field and $-\frac{\epsilon}{2}u^2 + \frac{\epsilon}{2}[(-\gamma^2\Delta + 1)^{-1/2}u]^2$ is the long range self-interaction. Similar to the copolymer model, these two competing factors lead to a mixture of the two phase variants. The result of this paper again shows the characteristic scale of each phase in the mixture.

When we study $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$, we will show that as ϵ tends to 0, $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$ both converge to their limiting problems, defined in (2.1) and (4.1). The convergence falls in the general theory of Γ-limits.

Then we will study the 1-dimensional cases, $\Omega = (0,1)$, of $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$. We find all local minima of the limiting problems. It turns out that these local minima of the limiting problems are isolated, and near them there are local minima of $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$ if ϵ is sufficiently small.

For each positive integer ν, we set

$$x_1 = \frac{1-m}{2\nu},\ x_2 = x_1 + \frac{1+m}{\nu},\ x_3 = x_2 + \frac{1-m}{\nu}, \dots,\ x_\nu = x_{\nu-1} + \frac{1+(-1)^\nu m}{\nu}.$$

We also set $x_0 = 0$ and $x_{\nu+1} = 1$. We define a step function $U_{\nu,1} \in \mathcal{A}_m$ so that

$$U_{\nu,1}(x) = (-1)^i \text{ if } x \in (x_{i-1}, x_i),\ i = 1, 2, \dots, \nu+1. \tag{1.5}$$

In a similar way for each positive integer ν, we set

$$z_1 = \frac{1+m}{2\nu},\ z_2 = z_1 + \frac{1-m}{\nu},\ z_3 = z_2 + \frac{1+m}{\nu}, \dots,\ z_\nu = z_{\nu-1} + \frac{1-(-1)^\nu m}{\nu}.$$

We also set $z_0 = 0$ and $z_{\nu+1} = 1$. We define $U_{\nu,2} \in \mathcal{A}_m$ so that

$$U_{\nu,2}(x) = (-1)^{(i-1)} \text{ if } x \in (z_{i-1}, z_i),\ i = 1, 2, \dots, \nu+1. \tag{1.6}$$

We denote an open ball in $\mathcal{A}_m$ centered at u of radius δ by $B_\delta(u)$, i.e., $B_\delta(u) = \{v \in \mathcal{A}_m : \|v-u\|_2 < \delta\}$.

Our main result of the paper is the following theorem.

THEOREM 1.1. *Let $\Omega = (0,1)$. For each positive integer N we can find $\delta > 0$ such that*

1. *$\{B_\delta(U_{\nu,1}), B_\delta(U_{\nu,2}) : \nu = 1, 2, \dots, N\}$ is a family of $2N$ mutually disjoint open balls in $\mathcal{A}_m$;*
2. *there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$, every ν, $\nu = 1, 2, \dots, N$, there exist a local minimum $u_{\epsilon,\nu,1}$ of $\mathcal{I}_\epsilon$ (or $\mathcal{J}_\epsilon$) in $B_\delta(U_{\nu,1})$ and a local minimum $u_{\epsilon,\nu,2}$ in $B_\delta(U_{\nu,2})$ satisfying $\lim_{\epsilon\to 0}\|u_{\epsilon,\nu,1} - U_{\nu,1}\|_2 = 0$ and $\lim_{\epsilon\to 0}\|u_{\epsilon,\nu,2} - U_{\nu,2}\|_2 = 0$.*

Our second result describes the global minima of $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$.

THEOREM 1.2. *Let $\Omega = (0,1)$ and u_ϵ be a global minimum of $\mathcal{I}_\epsilon$ (or $\mathcal{J}_\epsilon$). There exists a countable set $\mathcal{C} \subset R$ with the following properties.*

(1) *If* $\frac{(1-m^2)^2}{24\gamma^2 c_0} \notin \mathcal{C}$, *there exists a positive integer* ν_* *so that for every* $\delta > 0$ *there is* ϵ_0 *so that if* $\epsilon < \epsilon_0$, u_ϵ *is in the open ball* $B_\delta(U_{\nu_*,1})$ *or* $B_\delta(U_{\nu_*,2})$.

(2) *If* $\frac{(1-m^2)^2}{24\gamma^2 c_0} \in \mathcal{C}$, *there exists a positive integer* ν_* *so that for every* $\delta > 0$ *there is* ϵ_0 *so that if* $\epsilon < \epsilon_0$, u_ϵ *is in one of the following four open balls:* $B_\delta(U_{\nu_*,1})$, $B_\delta(U_{\nu_*,2})$, $B_\delta(U_{\nu_*+1,1})$, *or* $B_\delta(U_{\nu_*+1,2})$.

c_0 *is given in* (2.2), *and* $\mathcal{C}$ *and* ν_* *are defined near the end of section* 3.

If we take N in Theorem 1.1 to be greater than ν_* in Theorem 1.2 and δ in Theorem 1.2 to be the same as the δ in Theorem 1.1, then every global minimum u_ϵ for small ϵ is a local minimum shown to exist in Theorem 1.1.

In the proof of Theorem 1.2 the reader will see that if the functional is $\mathcal{I}_\epsilon$, $|\nu_* - \max\{1, (\frac{(1-m^2)^2}{12\gamma^2 c_0})^{1/3}\}| < 1$. The estimate of ν_* is weaker in the case of $\mathcal{J}_\epsilon$. There ν_* is close to $(\frac{(1-m^2)^2}{12\gamma^2 c_0})^{1/3}$ only if c_0 is small.

If $W \in C^1(R)$, the local minima $u_{\epsilon,\nu,1}$, $u_{\epsilon,\nu,2}$, and the global minimum u_ϵ of $\mathcal{I}_\epsilon$, together with a v and a λ, solve the Euler equation of $\mathcal{I}_\epsilon$:

$$\begin{aligned}
&-\epsilon\Delta u + \frac{1}{\epsilon}W'(u) + v = \lambda,\ x \in \Omega,\\
&-\gamma^2\Delta v = u - m,\ x \in \Omega,\\
&\frac{\partial u}{\partial n}|_{\partial\Omega} = 0,\ \ \frac{\partial v}{\partial n}|_{\partial\Omega} = 0,\\
&\frac{1}{|\Omega|}\int_\Omega u\,dx = m,\ \frac{1}{|\Omega|}\int_\Omega v\,dx = 0.
\end{aligned}$$

For $\mathcal{J}_\epsilon$ the Euler equation is

$$\begin{aligned}
&-\epsilon\Delta u + \frac{1}{\epsilon}W'(u) - u + v = \lambda,\ x \in \Omega,\\
&-\gamma^2\Delta v + v = u,\ x \in \Omega,\\
&\frac{\partial u}{\partial n}|_{\partial\Omega} = 0,\ \frac{\partial v}{\partial n}|_{\partial\Omega} = 0,\\
&\frac{1}{|\Omega|}\int_\Omega u\,dx = m.
\end{aligned}$$

We point out that there is a large literature on the local variational problem, $\mathcal{I}_\epsilon$ without $\frac{1}{2}|(-\gamma^2\Delta)^{-1/2}(u-m)|^2$ term, and its Γ-limit. We refer to Modica [7] for the Γ-limit and Kohn and Sternberg [6] for local minimum solutions. We also refer the reader to Dal Maso [3] for the general Γ-convergence theory.

The presence of the nonlocal terms, $\frac{1}{2}|(-\gamma^2\Delta)^{-1/2}(u-m)|^2$ in $\mathcal{I}_\epsilon$ and $\frac{1}{2}|(-\gamma^2\Delta+1)^{-1/2}u|^2$ in $\mathcal{J}_\epsilon$, gives us local minima with arbitrarily many transitional layers. This contrasts sharply with the local problem, where, according to a result of Carr, Gurtin, and Slemrod [2], every local minimum must be monotone.

We will only present the complete proof of Theorems 1.1 and 1.2 for $\mathcal{I}_\epsilon$. In section 2 we identify the limiting problem of $\mathcal{I}_\epsilon$ and show that the existence of isolated local minima of the limiting problem implies the existence of local minima of $\mathcal{I}_\epsilon$. Then in section 3 we prove that the limiting problem admits many isolated local minima, hence proving Theorem 1.1. Theorem 1.2 is also proved in that section. The study of J_ϵ is quite similar to that of $\mathcal{I}_\epsilon$. We list the modifications one needs in order to obtain the theorems for $\mathcal{J}_\epsilon$ in section 4.

2. The Γ-limit of $\mathcal{I}_\epsilon$. Associated with $\mathcal{I}_\epsilon$ is the variational problem

$$\mathcal{I}_0(u) = \begin{cases} \dfrac{c_0}{2}\|Du\|(\Omega) + \displaystyle\int_\Omega \frac{1}{2}|(-\gamma^2\Delta)^{-1/2}(u-m)|^2\,dx, & \\ & \text{if } u \in \mathcal{A}_m \cap BV(\Omega,\{-1,1\}), \\ \infty & \text{if } u \in \mathcal{A}_m \backslash BV(\Omega,\{-1,1\}) \end{cases} \tag{2.1}$$

for $u \in \mathcal{A}_m$. Here

$$c_0 = \sqrt{2}\int_{-1}^{1}(W(s))^{1/2}ds. \tag{2.2}$$

$BV(\Omega,\{-1,1\}) = \{u \in BV(\Omega) : u(x) = -1$ or 1 for almost everywhere (a.e.) $x \in \Omega\}$. $BV(\Omega)$ is the space of functions of bounded variation. We refer the reader to [4, Chap. 5, pp. 166–226] for its properties. $\|Du\|$ is the absolute value of the distributional derivative Du of u, regarded as a finite nonnegative measure on Ω. $\|Du\|(\Omega)$ is the size of Ω under this measure.

PROPOSITION 2.1.

(1) *For every family* $\{u_\epsilon\} \subset \mathcal{A}_m$ *with* $\lim_{\epsilon\to 0} u_\epsilon = u$,

$$\liminf_{\epsilon\to 0} \mathcal{I}_\epsilon(u_\epsilon) \geq \mathcal{I}_0(u).$$

(2) *For every* $u \in \mathcal{A}_m \cap BV(\Omega,\{-1,1\})$, *there exists a family* $\{u_\epsilon\} \subset \mathcal{A}_m$ *such that* $\lim_{\epsilon\to 0} u_\epsilon = u$, *and*

$$\limsup_{\epsilon\to 0} \mathcal{I}_\epsilon(u_\epsilon) \leq \mathcal{I}_0(u).$$

Proof. We define three functionals on $\mathcal{A}_m$:

$$\mathcal{H}_\epsilon(u) = \begin{cases} \displaystyle\int_\Omega \left\{\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u)\right\} dx & \text{if } u \in \mathcal{A}_m \cap W^{1,2}(\Omega), \\ \infty & \text{if } u \in \mathcal{A}_m \backslash W^{1,2}(\Omega), \end{cases} \tag{2.3}$$

$$\mathcal{H}_0(u) = \begin{cases} \dfrac{c_0}{2}\|Du\|(\Omega) & \text{if } u \in \mathcal{A}_m \cap BV(\Omega,\{-1,1\}), \\ \infty & \text{if } u \in \mathcal{A}_m \backslash BV(\Omega,\{-1,1\}), \end{cases} \tag{2.4}$$

and

$$\mathcal{K}(u) = \int_\Omega \frac{1}{2}[(-\gamma^2\Delta)^{-1/2}(u-m)]^2\,dx, \quad u \in \mathcal{A}_m. \tag{2.5}$$

Then $\mathcal{I}_\epsilon = \mathcal{H}_\epsilon + \mathcal{K}$ and $\mathcal{I}_0 = \mathcal{H}_0 + \mathcal{K}$. After making some minor modifications (change $L^1(\Omega)$ to $L^2(\Omega)$) in the proof of Propositions 1 and 2 in Modica [7], we find (1) for every family $\{u_\epsilon\} \subset \mathcal{A}_m$ with $\lim_{\epsilon\to 0} u_\epsilon = u$,

$$\liminf_{\epsilon\to 0} \mathcal{H}_\epsilon(u_\epsilon) \geq \mathcal{H}_0(u);$$

(2) for every $u \in \mathcal{A}_m \cap BV(\Omega,\{-1,1\})$, there exists a family $\{u_\epsilon\} \subset \mathcal{A}_m$ such that $\lim_{\epsilon\to 0} u_\epsilon = u$, and

$$\limsup_{\epsilon\to 0} \mathcal{H}_\epsilon(u_\epsilon) \leq \mathcal{H}_0(u).$$

Then we note that $\mathcal{K} : \mathcal{A}_m \to R$ is a continuous functional. Hence the two statements about $\mathcal{H}_\epsilon$ and $\mathcal{H}_0$ are carried over to $\mathcal{I}_\epsilon$ and $\mathcal{I}_0$. □

The notion of Γ-convergence is indeed defined by the two properties of Proposition 2.1. So $\mathcal{I}_\epsilon$ Γ-converges to $\mathcal{I}_0$.

Throughout the rest of this paper we assume $\Omega = (0, 1)$.

PROPOSITION 2.2. *Let ϵ_n be a sequence of positive numbers converging to* 0*, and let $\{u_n\}$ be a sequence in $\mathcal{A}_m$. If $\mathcal{I}_{\epsilon_n}(u_n)$ is bounded above in n, then $\{u_n\}$ is relatively compact in $\mathcal{A}_m$ and its cluster points belong to $BV((0,1), \{-1,1\})$.*

Proof. We set

$$\phi(t) = \int_{-1}^{t} W^{1/2}(s) ds. \tag{2.6}$$

Then (1.2) implies

$$|\phi(t)| \le C + C|t|^{\frac{k}{2}+1}.$$

Set $v_n = \phi(u_n)$. As shown in Modica and Mortola [8], v_n is bounded in $W^{1,1}(0,1)$. For by (1.2) and $k \ge 2$, we find

$$|v_n| \le C + C|u_n|^{\frac{k}{2}+1} \le C + CW(u_n).$$

Therefore $\{v_n\}$ is bounded in $L^1(0,1)$. On the other hand,

$$\int_0^1 |v_n'|\, dx = \int_0^1 W^{1/2}(u_n)|u_n'|\, dx$$

$$\le \frac{\sqrt{2}}{2} \left(\int_0^1 \left[\frac{\epsilon_n}{2}|u_n'|^2 + \frac{1}{\epsilon_n} W(u_n) \right] dx \right) \le \frac{\sqrt{2}}{2} \mathcal{I}_{\epsilon_n}(u_n).$$

Therefore $\{v_n\}$ is bounded in $W^{1,1}(0,1)$. The Sobolev imbedding theorem asserts that $\{v_n\}$ is relatively compact in $L^p(0,1)$ for all $1 \le p < \infty$.

Now consider $u_n = \phi^{-1}(v_n)$. (1.2) and (2.6) imply that ϕ^{-1} is continuous and increasing, and there exists $\bar{t} > 0$ such that ϕ^{-1} is Lipschitz continuous for $|t| \ge \bar{t}$. We can find C such that for all t

$$|\phi^{-1}(t)| \le C + C|t|, \ |\phi^{-1}(t)|^p \le C + C|t|^p. \tag{2.7}$$

To prove that $\{u_n\}$ is relatively compact we show that every subsequence of $\{u_n\}$ has an L^2-convergent further subsequence. Let us recall Vitali's convergence theorem [5, p. 203].

Vitali's convergence theorem. *Let $\{f_n\}$ be a sequence in $L^p(\Omega, \mu)$, $1 \le p < \infty$, and f be an μ-measurable function such that $f_n \to f$ μ-a.e.. Then $f \in L^p(\Omega, \mu)$ and $\|f_n - f\|_p \to 0$ if and only if*

1. *for each $\varepsilon > 0$, there exists a μ-measurable set $A_\varepsilon \subset \Omega$ such that $\mu(A_\varepsilon) < \infty$ and $\int_{\Omega \setminus A_\varepsilon} |f_n|^p\, d\mu < \varepsilon$ for all n; and*
2. *for each $\varepsilon > 0$, there is $\delta > 0$ such that for every μ measurable set E, $\mu(E) < \delta$ implies $\int_E |f_n|^p\, d\mu < \varepsilon$ for all n.*

Part 1 of Vitali's convergence theorem is not needed here because $(0,1)$ itself has finite Lebesgue measure. Let $\{u_{n_l}\}$ be a subsequence of $\{u_n\}$. Then there are a subsequence of $\{v_{n_l} = \phi(u_{n_l})\}$, denoted by $\{v_{n_{l_m}}\}$, and $v \in L^p(0,1)$ such that $v_{n_{l_m}} \to v$ in $L^p(0,1)$ and $v_{n_{l_m}} \to v$ a.e. Then $u_{n_{l_m}} \to \phi^{-1}(v)$ a.e. Applying Vitali's convergence theorem to $v_{n_{l_m}}$, we find that for every $\varepsilon > 0$ there is $\delta > 0$ such that for every measurable set E, $|E| < \delta$ implies $\int_E |v_{n_{l_m}}|^p \, dx < \varepsilon$ for all n. Then (2.7) implies

$$\int_E |u_{n_{l_m}}|^p \, dx \le \int_E (C + C|v_{n_{l_m}}|^p) \, dx < C\delta + C\varepsilon.$$

Now Vitali's convergence theorem applied to $\{u_{n_{l_m}}\}$ asserts that $u_{n_{l_m}} \to \phi^{-1}(v)$ in $L^p(0,1)$, particularly in $L^2(0,1)$.

Let u be a cluster point of $\{u_n\}$, i.e., there exists a subsequence $\{u_{n_l}\}$ such that $u_{n_l} \to u$ in $L^2(0,1)$ as $l \to \infty$. Fatou's lemma and the boundedness of $\mathcal{I}_{\epsilon_n}(u_n)$ imply that

$$0 \le \int_0^1 W(u) \le \lim_{l\to\infty} \int_0^1 W(u_{n_l}) \, dx \le \lim_{l\to\infty} \epsilon_{n_l} \mathcal{I}_{\epsilon_{n_l}}(u_{n_l}) = 0.$$

Then for a.e. $x \in (0,1)$, $u(x) = -1$ or 1. If we consider $v_{n_l} = \phi(u_{n_l})$, then the boundedness of $\{v_{n_l}\}$ in $W^{1,1}(0,1)$, proved earlier, implies that $\phi(u)$, the L^1-limit of $\{v_{n_l}\}$, is a BV function [4, Thm. 1, p. 172]. $\phi(u)$ only takes two values, $\phi(-1)$ and $\phi(1)$. Then $\phi(u) = \phi(-1) + \frac{\phi(1)-\phi(-1)}{2}(u+1)$. Hence u is also a BV function. $\square$

A useful property following Propositions 2.1 and 2.2 is that isolated local minima of the Γ-limit persist under small perturbation. It was used in Kohn and Sternberg [6], (see also Dal Maso [3]). We include this property and its proof below for completeness.

PROPOSITION 2.3. *Let $\delta > 0$ and $u_0 \in \mathcal{A}_m$ be such that $\mathcal{I}_0(u_0) < \mathcal{I}_0(u)$ for all $u \in B_\delta(u_0)$ with $u \ne u_0$. Then there exists $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ there exists $u_\epsilon \in B_{\delta/2}(u_0)$ with $\mathcal{I}_\epsilon(u_\epsilon) \le \mathcal{I}_\epsilon(u)$ for all $u \in B_{\delta/2}(u_0)$. In addition* $\lim_{\epsilon\to 0} \|u_\epsilon - u_0\|_2 = 0$.

Proof. Let $u_{\epsilon,n}$ be a sequence in $B_{\delta/2}(u_0)$ so that

$$\lim_{n\to\infty} \mathcal{I}_\epsilon(u_{\epsilon,n}) = \inf_{u\in B_{\delta/2}(u_0)} \mathcal{I}_\epsilon(u).$$

The standard argument shows that after passing to a subsequence, again denoted by $u_{\epsilon,n}$, there exists $u_\epsilon \in \overline{B}_{\delta/2}(u_0)$ such that $u_{\epsilon,n} \to u_\epsilon$ in $L^2(0,1)$, $u_{\epsilon,n} \to u_\epsilon$ weakly in $W^{1,2}(0,1)$, and

$$\mathcal{I}_\epsilon(u_\epsilon) = \lim_{n\to\infty} \mathcal{I}_\epsilon(u_{\epsilon,n}) = \inf_{u\in B_{\delta/2}(u_0)} \mathcal{I}_\epsilon(u).$$

Next we claim $u_\epsilon \in B_{\delta/2}(u_0)$ if ϵ is small enough. Otherwise there exists a sequence $\epsilon_l \to 0$, such that $\|u_{\epsilon_l} - u_0\|_2 = \delta/2$ and

$$\mathcal{I}_{\epsilon_l}(u_{\epsilon_l}) = \inf_{u\in B_{\delta/2}(u_0)} \mathcal{I}_{\epsilon_l}(u).$$

Part 2 of Proposition 2.1 asserts that there exists a sequence v_{ϵ_l} in $B_{\delta/2}(u_0)$, if l is large enough, such that

$$\limsup_{l\to\infty} \mathcal{I}_{\epsilon_l}(v_{\epsilon_l}) \le \mathcal{I}_0(u_0).$$

Therefore,

$$\limsup_{l\to\infty} \mathcal{I}_{\epsilon_l}(u_{\epsilon_l}) \le \limsup_{l\to\infty} \mathcal{I}_{\epsilon_l}(v_{\epsilon_l}) \le \mathcal{I}_0(u_0).$$

Proposition 2.2 then asserts that, after passing to a subsequence, again denoted by u_{ϵ_l}, there exists $\overline{u}_0$ such that $u_{\epsilon_l} \to \overline{u}_0$ in $L^2(0,1)$, and $\|\overline{u}_0 - u_0\|_2 = \delta/2$. Part 1 of Proposition 2.1 now implies

$$\mathcal{I}_0(\overline{u}_0) \le \liminf_{l\to\infty} \mathcal{I}_{\epsilon_l}(u_{\epsilon_l}) \le \mathcal{I}_0(u_0).$$

This contradicts the condition that $\mathcal{I}_0(u_0) < \mathcal{I}_0(u)$ for all $u \in B_\delta(u_0)$ with $u \ne u_0$. Therefore u_ϵ is in the open ball $B_{\delta/2}(u_0)$, i.e., u_ϵ is a local minimum.

To show $u_\epsilon \to u_0$ in $L^2(0,1)$ as $\epsilon \to 0$, we assume that there exists a sequence $\epsilon_l \to 0$ such that $\|u_{\epsilon_l} - u_0\|_2 = \delta_0 < \delta/2$. Then arguing like above, we have $\widetilde{u}_0$ such that, after passing to a subsequence, again denoted by u_{ϵ_l}, $u_{\epsilon_l} \to \widetilde{u}_0$ and $\|\widetilde{u}_0 - u_0\|_2 = \delta_0$. Then by part 1 of Proposition 2.1,

$$\mathcal{I}_0(\widetilde{u}_0) \le \liminf_{l\to\infty} \mathcal{I}_{\epsilon_l}(u_{\epsilon_l}) \le \mathcal{I}_0(u_0),$$

which is again a contradiction. ☐

3. The local minima of $\mathcal{I}_0$.

A function u in $BV((0,1),\{-1,1\})$, up to a set of Lebesgue measure 0, is a step function. $u(x)$ switches between -1 and 1 at finitely many points $x_1, x_2, \dots, x_\nu$, with $0 < x_1 < x_2 < \cdots < x_\nu < 0$. A formal description is as follows.

For $u \in BV((0,1),\{-1,1\})$ we define set $E_u = \{x \in (0,1) : u(x) = -1\}$. The perimeter of E_u in $(0,1)$ is $\|D\chi_{E_u}\|(0,1)$, where χ_{E_u} is the characteristic function of E_u. The measure $\|D\chi_{E_u}\|$ is often written as $\|\partial E_u\|$. Clearly $\|\partial E_u\| = \frac{\|Du\|}{2}$.

The reduced boundary of E_u, a subset of $(0,1)$, is denoted by $\partial^* E_u$ (see [4, section 5.7, pp. 194–207] for the definition and properties of reduced boundaries). The structure theorem for sets of finite perimeter [4, Thm. 1 (iii), p. 189] asserts that $\|\partial E_u\| = H^0 |\partial^* E_u$, the 0-dimensional Hausdorff measure restricted on $\partial^* E_u$. The 0-dimensional Hausdorff measure is the counting measure. Therefore $\partial^* E_u$ is a set of finitely many points in $(0,1)$ and $\|\partial E_u\|(0,1) = \frac{\|Du\|(0,1)}{2}$ is the number of the points in $\partial^* E_u$.

$\partial^* E_u$ is simply $\{x_1, x_2, \dots, x_\nu\}$, the set of points where $u(x)$ switches, and

$$\frac{\|Du\|(0,1)}{2} = \nu.$$

If $u \in \mathcal{A}_m \cap BV((0,1),\{-1,1\})$, $\frac{\|Du\|(0,1)}{2}$ has to be nonzero. Otherwise u would be a constant. Then $u = -1$ for a.e. $x \in (0,1)$ or $u = 1$ for a.e. $x \in (0,1)$. In either case $\int_0^1 u \ne m$. So we have the following mutually disjoint decomposition:

$$\begin{aligned} &\mathcal{A}_m \cap BV((0,1),\{-1,1\}) = \cup_1^\infty A_\nu, \text{ where} \\ &A_\nu = \left\{ u \in \mathcal{A}_m \cap BV((0,1),\{-1,1\}) : \frac{\|Du\|(0,1)}{2} = \nu \right\}. \end{aligned} \tag{3.1}$$

Proposition 3.1. *For every $u \in A_\nu$, $u \ne U_{\nu,1}$, and $u \ne U_{\nu,2}$, we have $\mathcal{I}_0(U_{\nu,1}) = \mathcal{I}_0(U_{\nu,2}) < \mathcal{I}_0(u)$.*

Proof. For each $u \in A_\nu$, let us denote $\partial^* E_u$ by $\{x_1, x_2, \ldots, x_\nu\}$, where $0 < x_1 < x_2 \cdots < x_\nu < 1$. Since $\|Du\|(x_i, x_{i+1}) = 0$ for each i and (x_i, x_{i+1}) is connected, $u = -1$ for a.e. $x \in (x_i, x_{i+1})$ or $u = 1$ for a.e. $x \in (x_i, x_{i+1})$. And it follows from the definition of reduced boundaries [4, p. 194] that $u(x)$ must jump from -1 to 1 or 1 to -1 when x moves from (x_{i-1}, x_i) to (x_i, x_{i+1}). We can further decompose A_ν into two disjoint sets:

$$\begin{aligned} &A_{\nu,1} = \{u \in A_\nu : u = -1 \text{ for a.e. } x \in (0, x_1)\}, \\ &A_{\nu,2} = \{u \in A_\nu : u = 1 \text{ for a.e. } x \in (0, x_1)\}. \end{aligned} \tag{3.2}$$

For $u \in A_{\nu,1}$ the constraint $\int_0^1 u = m$ becomes $-2x_1 + 2x_2 - \cdots + 2(-1)^\nu x_\nu - (-1)^\nu = m$, and for $u \in A_{\nu,2}$ the constraint $\int_0^1 u = m$ becomes $2x_1 - 2x_2 - \cdots - 2(-1)^\nu x_\nu + (-1)^\nu = m$.

Now $A_{\nu,1}$ can be identified with the set

$$\begin{aligned} S_{\nu,1} = \Big\{&(x_1, \ldots, x_\nu) \in R^\nu : 0 < x_1 < \cdots x_\nu < 1, \\ &-x_1 + x_2 - \cdots + (-1)^\nu x_\nu = \frac{m + (-1)^\nu}{2}\Big\}, \end{aligned} \tag{3.3}$$

and $A_{\nu,2}$ can be identified with the set

$$\begin{aligned} S_{\nu,2} = \Big\{&(x_1, \ldots, x_\nu) \in R^\nu : 0 < x_1 < \cdots x_\nu < 1, \\ &x_1 - x_2 - \cdots - (-1)^\nu x_\nu = \frac{m - (-1)^\nu}{2}\Big\}. \end{aligned} \tag{3.4}$$

$S_{\nu,1}$ and $S_{\nu,2}$ are two bounded open subsets of two $\nu - 1$ dimensional hyper-planes of R^ν.

We take $u \in A_{\nu,1} \cong S_{\nu,1}$ and $\{x_1, x_2, \ldots, x_\nu\}$, $x_1 < \cdots < x_\nu$, to be $\partial^* E_u$. We compute $\mathcal{K}(u)$. Let v be the solution of

$$-\gamma^2 v'' = u - m, \ v'(0) = v'(1) = 0, \ \int_0^1 v = 0.$$

Denote the Green's function of this equation by $G(x, y)$. Then

$$\begin{aligned} \mathcal{K}(u) &= \frac{1}{2} \int_0^1 [(-\gamma^2 \Delta)^{-1/2} (u - m)]^2 \, dx \\ &= \frac{1}{2} \int_0^1 (u - m) v \, dx \\ &= \frac{1}{2} \left[\int_0^{x_1} (-1 - m) v + \int_{x_1}^{x_2} (1 - m) v + \cdots + \int_{x_\nu}^1 ((-1)^{\nu+1} - m) v \right]. \end{aligned}$$

Treating $\mathcal{K}$ as a function of $(x_1, x_2, \ldots, x_\nu)$ in $S_{\nu,1}$, we calculate

$$\frac{\partial \mathcal{K}}{\partial x_1} = \frac{1}{2} \left[-2v(x_1) + \int_0^1 (u - m) \frac{\partial v}{\partial x_1} \, dx \right].$$

Since

$$\begin{aligned} \frac{\partial v}{\partial x_1}(x) = \frac{\partial}{\partial x_1} \Bigg[&\int_0^{x_1} (-1 - m) G(x, y) dy + \int_{x_1}^{x_2} (1 - m) G(x, y) dy + \cdots \\ &+ \int_{x_\nu}^1 ((-1)^{\nu+1} - m) G(x, y) dy \Bigg] \\ = -2G(x, x_1), \end{aligned}$$

we deduce

$$\begin{aligned}\frac{\partial \mathcal{K}}{\partial x_1} &= \frac{1}{2}\left[-2v(x_1) + \int_0^1 (u-m)(-2)G(x,x_1)\,dx\right]\\ &= -2v(x_1).\end{aligned}$$

The same argument applied to other x_i yields

$$\nabla \mathcal{K} = 2(-v(x_1), v(x_2), \dots, (-1)^\nu v(x_\nu)). \tag{3.5}$$

Since $\int_0^1 u = m$, or $-x_1 + x_2 - \cdots + (-1)^\nu x_\nu = \frac{m+(-1)^\nu}{2}$, the Lagrange multiplier method asserts that if $(x_1, x_2, \dots, x_\nu)$ is a critical point of $\mathcal{K}$ in $S_{\nu,1}$, there exists λ such that

$$\nabla \mathcal{K} = \lambda(-1, 1, -1, \dots, (-1)^\nu),$$

which, together with (3.5), implies

$$v(x_1) = v(x_2) = \cdots = v(x_\nu). \tag{3.6}$$

On (x_1, x_2), v solves the linear equation $-\gamma^2 v'' = 1 - m$. Then $v(x_1) = v(x_2)$ implies that v is symmetric about $(x_1 + x_2)/2$, and hence $v'(x_1) = -v'(x_2)$. On intervals $(0, x_1)$ and (x_2, x_3), v satisfies the linear equation $-\gamma^2 v'' = -1 - m$. Since v also satisfies the conditions $v(x_1) = v(x_2)$, $v'(x_1) = -v'(x_2)$, $v(x_2) = v(x_3)$, and $v'(0) = 0$, we conclude by solving the equation on $(0, x_1)$ and (x_2, x_3) that v on $(0, x_1)$ is a reflection of v on $(x_2, (x_2 + x_3)/2)$. Hence the length of $(0, x_1)$ is half that of (x_2, x_3). In the next step we compare intervals (x_2, x_3) and (x_4, x_5) and similarly find that they have the same length. By repeating this argument we conclude that the intervals where $u = -1$ all have the same length with the exception of $(0, x_1)$ and $(x_\nu, 1)$ if $u = -1$ there, whose length is half. The same can be said for the intervals where $u = 1$. Taking $-x_1 + x_2 - \cdots + (-1)^\nu x_\nu = \frac{m+(-1)^\nu}{2}$ into consideration, we find

$$x_1 = \frac{1-m}{2\nu}, x_2 = x_1 + \frac{1+m}{\nu}, \dots, x_\nu = x_{\nu-1} + \frac{1+(-1)^\nu m}{\nu}.$$

We have proved that $\mathcal{K}$ has a unique critical point $(x_1, x_2, \dots, x_\nu)$ in $S_{\nu,1}$. We denote the function in $A_{\nu,1}$ whose reduced boundary is $\{x_1, x_2, \dots, x_\nu\}$ by $U_{\nu,1}$ (also defined in (1.5)). We proceed to prove that $U_{\nu,1}$ minimizes $\mathcal{K}$ in $A_{\nu,1}$. We first compute $\mathcal{K}(U_{\nu,1})$. Let v be the solution of

$$-\gamma^2 v'' = U_{\nu,1} - m,\ v'(0) = v'(1) = 0,\ \int_0^1 v = 0.$$

Then

$$\mathcal{K}(U_{\nu,1}) = \frac{1}{2}\int_0^1 (U_{\nu,1} - m)v = \frac{\gamma^2}{2}\int_0^1 |v'|^2.$$

On $(0, x_1)$ $v'(x) = \frac{-1-m}{-\gamma^2}x + v'(0) = \frac{-1-m}{-\gamma^2}x$. Then

$$\frac{\gamma^2}{2}\int_0^{x_1} |v'|^2 = \frac{\gamma^2}{2}\frac{-\gamma^2}{3(-1-m)}(v'(x_1))^3.$$

On (x_1, x_2) $v'(x) = \frac{1-m}{-\gamma^2}(x - x_1) + v'(x_1)$. Then

$$\frac{\gamma^2}{2}\int_{x_1}^{x_2} |v'|^2 = \frac{\gamma^2}{2}\frac{-\gamma^2}{3(1-m)}[(v'(x_2))^3 - (v'(x_1))^3].$$

After finding $\frac{\gamma^2}{2}\int_{x_i}^{x_{i+1}} |v'|^2$ on each (x_i, x_{i+1}) and summing over i, we deduce

$$\mathcal{K}(U_{\nu,1}) = \frac{\gamma^4}{3(1-m^2)}[(v'(x_1))^3 - (v'(x_2))^3 + (v'(x_3))^3 + \cdots + (-1)^{\nu+1}(v'(x_\nu))^3].$$

Since we also know $v'(x_1) = -v'(x_2) = v'(x_3) = \cdots = (-1)^{\nu+1}v'(x_\nu)$ and $v'(x_1) = \frac{-1-m}{-\gamma^2}\frac{1-m}{2\nu} = \frac{1-m^2}{2\gamma^2\nu}$, we find

$$\mathcal{K}(U_{\nu,1}) = \frac{(1-m^2)^2}{24\gamma^2\nu^2}. \tag{3.7}$$

A similar computation in $A_{\nu,2}$ finds $U_{\nu,2}$ (defined by (1.6)) and again

$$\mathcal{K}(U_{\nu,2}) = \frac{(1-m^2)^2}{24\gamma^2\nu^2}. \tag{3.8}$$

We now show that $\mathcal{K}(u) > \mathcal{K}(U_{\nu,1})$ for every $u \in A_{\nu,1} \cong S_{\nu,1}$, $u \neq U_{\nu,1}$. If this is not the case, since there is only one critical point, $U_{\nu,1}$, in $S_{\nu,1}$, there must be a sequence $\{(x_{n,1}, x_{n,2}, \ldots, x_{n,\nu})\}$ converging to a point $(y_1, y_2, \ldots, y_\nu)$ on the boundary of $S_{\nu,1}$ ($S_{\nu,1}$ is considered as a subset of R^ν) such that

$$\lim_{n\to\infty} \mathcal{K}(x_{n,1}, x_{n,2}, \ldots, x_{n,\nu}) \le \mathcal{K}(U_{\nu,1}).$$

For the point $(y_1, y_2, \ldots, y_\nu)$ to be on the boundary of $S_{\nu,1}$, at least two of 0, $y_1, \ldots, y_\nu$, 1 must be identical. Then $(y_1, y_2, \ldots, y_\nu)$ is identified as a point in $S_{\nu',1}$ or $S_{\nu',2}$, corresponding to $A_{\nu',1}$ or $A_{\nu',2}$ for some $\nu' < \nu$. Let us denote this point by $(z_1, z_2, \ldots, z_{\nu'})$ and assume, without the loss of generality, $(z_1, z_2, \ldots, z_{\nu'}) \in S_{\nu',1}$. We ask whether $U_{\nu',1}$ is the strict global minimum of $\mathcal{K}$ in $A_{\nu',1}$. If so,

$$\mathcal{K}(U_{\nu',1}) \le \mathcal{K}(z_1, z_2, \ldots, z_{\nu'}) = \lim_{n\to\infty}\mathcal{K}(x_{n,1}, x_{n,2}, \ldots, x_{n,\nu}) \le \mathcal{K}(U_{\nu,1}),$$

which, since $\nu' < \nu$, is inconsistent with (3.7), where $\mathcal{K}(U_{\nu,1}) = \mathcal{K}(U_{\nu,2})$ decreases in ν. If $U_{\nu',1}$ is not the strict global minimum of $\mathcal{K}$ in $A_{\nu',1}$, we use the same argument on $U_{\nu',1}$ and end up in a $A_{\nu'',1}$ or $A_{\nu'',2}$ with $\nu'' < \nu'$. This process stops at $\nu = 1$, and there, since $A_{1,1}$ has only one element $U_{1,1}$ and $A_{1,2}$ has only one element $U_{1,2}$, we find $\mathcal{K}(U_{1,1}) = \mathcal{K}(U_{1,2}) \le \mathcal{K}(U_{\nu,1})$, which is inconsistent with (3.7) or (3.8). So we have proved that $U_{\nu,1}$ is the strict global minimum of $\mathcal{K}$ in $A_{\nu,1}$. And since for $u \in A_{\nu,1}$, $\mathcal{I}_0(u) = c_0\nu + \mathcal{K}(u)$, Proposition 3.1 is proved. ◻

PROPOSITION 3.2. *Let N be a positive integer. One can find $\delta > 0$ such that*

1. *$\{B_\delta(U_{\nu,1}), B_\delta(U_{\nu,2}) : \nu = 1, 2, \ldots, N\}$ is a family of $2N$ mutually disjoint open balls in $\mathcal{A}_m$;*
2. *for all $u \in B_\delta(U_{\nu,1})$, $\nu = 1, 2, \ldots, N$, with $u \neq U_{\nu,1}$, $\mathcal{I}_0(U_{\nu,1}) < \mathcal{I}_0(u)$, and for all $u \in B_\delta(U_{\nu,2})$ with $u \neq U_{\nu,2}$, $\mathcal{I}_0(U_{\nu,2}) < \mathcal{I}_0(u)$.*

Proof. Let N be a positive integer and $\nu \in \{1, 2, \ldots, N\}$. We consider $U_{\nu,1}$. The study of $U_{\nu,2}$ is the same.

Take δ to be a positive number to be specified later. Let $u \in B_\delta(U_{\nu,1})$, $u \neq U_{\nu,1}$. If $u \in A_\nu$, then Proposition 3.1 implies Proposition 3.2. So we assume $u \in \mathcal{A}_m \backslash A_\nu$. Then if $u \in (\mathcal{A}_m \backslash A_\nu) \backslash BV((0,1),\{-1,1\})$, $\mathcal{I}_0(U_{\nu,1}) < \mathcal{I}_0(u) = \infty$. So we need only to consider $u \in (\mathcal{A}_m \backslash A_\nu) \cap BV((0,1),\{-1,1\})$.

Because of (3.1), the positive integer $\frac{\|Du\|(0,1)}{2}$ is either $\leq \nu - 1$ or $\geq \nu + 1$. We study these two cases separately.

First we prove that the case $\frac{\|Du\|(0,1)}{2} \leq \nu - 1$ does not happen if δ is small enough. We claim that there is $\delta > 0$ such that for all $u \in B_\delta(U_{\nu,1}) \cap BV((0,1),\{-1,1\})$, $\frac{\|Du_n\|(0,1)}{2} \geq \nu$. Otherwise there exist $\delta_n \to 0$ and $u_n \in B_{\delta_n}(U_{\nu,1}) \cap BV((0,1),\{-1,1\})$ such that $\frac{\|Du\|(0,1)}{2} \leq \nu - 1$. Then $u_n \to U_{\nu,1}$ in $L^2(0,1)$ implies (see [4, Thm. 1, p. 172])

$$2\nu = \|DU_{\nu,1}(0,1)\| \leq \liminf_{n\to\infty} \|Du_n\|(0,1) \leq 2(\nu - 1),$$

a contradiction.

Second we consider the case $\frac{\|Du\|(0,1)}{2} \geq \nu + 1$. Here

$$\begin{aligned}\mathcal{I}_0(u) &\geq c_0(\nu+1) + \int_0^1 \frac{1}{2}[(-\gamma^2\Delta)^{-1/2}(u-m)]^2\,dx \\ &= \mathcal{I}_0(U_{\nu,1}) + c_0 + \mathcal{K}(u) - \mathcal{K}(U_{\nu,1}).\end{aligned}$$

Denote the norm of bounded linear operator $(-\gamma\Delta)^{-1/2}$ from $\{u \in L^2(0,1) : \int_0^1 u = 0\}$ to $L^2(0,1)$ by $\|(-\gamma\Delta)^{-1/2}\|$. Estimate

$$\begin{aligned}&|\mathcal{K}(u) - \mathcal{K}(U_{\nu,1})| \\ &= \frac{1}{2}|\, \|(-\gamma\Delta)^{-1/2}(u-m)\|_2^2 - \|(-\gamma\Delta)^{-1/2}(U_{\nu,1}-m)\|_2^2\,| \\ &= \frac{1}{2}|\, \|(-\gamma\Delta)^{-1/2}(u-m)\|_2 - \|(-\gamma\Delta)^{-1/2}(U_{\nu,1}-m)\|_2\,| \\ &\quad \cdot \{\|(-\gamma\Delta)^{-1/2}(u-m)\|_2 + \|(-\gamma\Delta)^{-1/2}(U_{\nu,1}-m)\|_2\} \\ &\leq \frac{1}{2}\|(-\gamma\Delta)^{-1/2}(u-U_{\nu,1})\|_2 \\ &\quad \cdot\{\|(-\gamma\Delta)^{-1/2}(u-U_{\nu,1})\|_2 + 2\|(-\gamma\Delta)^{-1/2}(U_{\nu,m}-m)\|_2\} \\ &\leq \frac{\|(-\gamma\Delta)^{-1/2}\|^2}{2}\|u-U_{\nu,1}\|_2\{\|u-U_{\nu,1}\|_2 + 2\|U_{\nu,m}\|_2 + 2\|m\|_2\} \\ &\leq \frac{\|(-\gamma\Delta)^{-1/2}\|^2}{2}\delta\{\delta + 2 + 2m\}.\end{aligned}$$

We obtain, choosing δ sufficiently small,

$$\mathcal{I}_0(u) \geq \mathcal{I}_0(U_{\nu,1}) + c_0 - \frac{\|(-\gamma\Delta)^{-1/2}\|^2}{2}\delta\{\delta + 2 + 2m\} > \mathcal{I}_0(U_{\nu,1}).$$

Since there are only finitely many ν, we can choose δ so that it is independent of ν. We can also make $\{B_\delta(U_{\nu,1}), B_\delta(U_{\nu,2}) : \nu = 1,2,\ldots,N\}$ mutually disjoint by having δ small enough. □

PROPOSITION 3.3. *$\{U_{\nu,1}, U_{\nu,2} : \nu = 1,2,3\ldots\}$ is the set of all local minima of $\mathcal{I}_0$ (or $\mathcal{J}_0$).*

Proof. According to Proposition 3.2, each element in $\{U_{\nu,1}, U_{\nu,2} : \nu = 1,2,\ldots\}$ is a local minimum of $\mathcal{I}_0$. On the other hand, if u is a local minimum of $\mathcal{I}_0$, it must be in $A_{\nu,i}$ for some $i = 1$ or 2 and $\nu = 1, 2, \ldots$. And in each $A_{\nu,i} \cong S_{\nu,i}$ there is only one critical point $U_{\nu,i}$ of $\mathcal{I}_0$ considered as a function on $S_{\nu,i}$. Then $u = U_{\nu,i}$. $\square$

Theorem 1.1 follows immediately from Propositions 2.3 and 3.2. To prove Theorem 1.2, we note from (3.7) and (3.8) that

$$\mathcal{I}_0(U_{\nu,1}) = \mathcal{I}_0(U_{\nu,2}) = c_0\nu + \frac{(1-m^2)^2}{24\gamma^2\nu^2}. \tag{3.9}$$

Set

$$g(t) = c_0 t + \frac{(1-m^2)^2}{24\gamma^2 t^2}$$

for $t \geq 1$. Denote the global minimum of this convex function by t_0:

$$t_0 = \max\left\{1, \left(\frac{(1-m^2)^2}{12\gamma^2 c_0}\right)^{1/3}\right\}.$$

Let $[t_0]$ be the greatest integer less than or equal to t_0. Compare $g([t_0])$ and $g([t_0+1])$. Depending on $\frac{(1-m^2)^2}{24\gamma^2 c_0}$, we have $g([t_0]) = g([t_0+1])$ or $g([t_0]) \neq g([t_0+1])$. Let $\mathcal{C} \subset R$ be the set so that when $\frac{(1-m^2)^2}{24\gamma^2 c_0} \in \mathcal{C}$, $g([t_0]) = g([t_0+1])$. In this case we set $\nu_* = [t_0]$. And if $\frac{(1-m^2)^2}{24\gamma^2 c_0} \notin \mathcal{C}$, i.e., $g([t_0]) \neq g([t_0+1])$, we set

$$\nu_* = \begin{cases} [t_0] & \text{if } g([t_0]) < g([t_0]+1), \\ [t_0]+1 & \text{if } g([t_0]) > g([t_0]+1). \end{cases}$$

So if $\frac{(1-m^2)^2}{24\gamma^2 c_0} \notin \mathcal{C}$, for every $\nu \neq \nu_*$,

$$\mathcal{I}_0(U_{\nu_*,1}) = \mathcal{I}_0(U_{\nu_*,2}) < \mathcal{I}_0(U_{\nu,1}) = \mathcal{I}_0(U_{\nu,2}),$$

and if $\frac{(1-m^2)^2}{24\gamma^2 c_0} \in \mathcal{C}$, for every ν, $\nu \neq \nu_*$, and $\nu \neq \nu_* + 1$,

$$\mathcal{I}_0(U_{\nu_*,1}) = \mathcal{I}_0(U_{\nu_*,2}) = \mathcal{I}_0(U_{\nu_*+1,1}) = \mathcal{I}_0(U_{\nu_*+1,2}) < \mathcal{I}_0(U_{\nu,1}) = \mathcal{I}_0(U_{\nu,2}).$$

The case $\frac{(1-m^2)^2}{24\gamma^2 c_0} \notin \mathcal{C}$ is generic. $\mathcal{C}$ is a countable set.

Let u_ϵ be a global minimum of $\mathcal{I}_\epsilon$. The existence of u_ϵ follows from the standard argument as in the proof of Proposition 2.3. Propositions 2.1 and 2.2 imply that every sequence $\{u_{\epsilon_n}\}$ of u_ϵ has a convergent subsequence, and if u_0 is the limit of a subsequence, u_0 must be a global minimum of $\mathcal{I}_0$ in $\mathcal{A}_m \cap BV((0,1), \{-1,1\})$. The decomposition (3.1) and Proposition 3.1 imply that $u_0 = U_{\nu,1}$ or $U_{\nu,2}$ for some ν. And by the definition of ν_*, $\nu = \nu_*$ if $\frac{(1-m^2)^2}{24\gamma^2 c_0} \notin \mathcal{C}$ and $\nu = \nu_*$ or ν_*+1 if $\frac{(1-m^2)^2}{24\gamma^2 c_0} \in \mathcal{C}$. This proves Theorem 1.2.

4. The study of $\mathcal{J}_\epsilon$.

All results in sections 2 and 3 are valid for $\mathcal{J}_\epsilon$. In this section we indicate the modifications needed to prove these results for $\mathcal{J}_\epsilon$.

The condition (1.2) implies that for all small ϵ, $\mathcal{J}_\epsilon$ is bounded from below. The lower bound can be made independent of small ϵ.

The Γ-limit of $\mathcal{J}_\epsilon$ is

$$\mathcal{J}_0(u) = \begin{cases} \dfrac{c_0}{2}\|Du\|(\Omega) - \dfrac{|\Omega|}{2} + \displaystyle\int_\Omega \frac{1}{2}|(-\gamma^2\Delta+1)^{-1/2}u|^2\,dx, \\ \qquad\qquad\qquad\qquad \text{if } u \in \mathcal{A}_m \cap BV(\Omega, \{-1,1\}), \\ \infty \qquad\qquad\qquad\quad \text{if } u \in \mathcal{A}_m \backslash BV(\Omega, \{-1,1\}) \end{cases} \tag{4.1}$$

for $u \in \mathcal{A}_m$. Define

$$\mathcal{L}(u) = \int_\Omega \frac{1}{2}\{-u^2 + [(-\gamma^2\Delta + 1)^{-1/2}u]^2\}\,dx \tag{4.2}$$

for $u \in \mathcal{A}_m$. Thus $\mathcal{J}_\epsilon = \mathcal{H}_\epsilon + \mathcal{L}$ and $\mathcal{J}_0 = \mathcal{H}_0 + \mathcal{L}$.

As in the proof of Proposition 3.1, we study $\mathcal{L}(u)$ for $u \in A_{\nu,1} \cong S_{\nu,1}$. Let $\{x_1, x_2, \dots, x_\nu\}$ be $\partial^* E_u$. Let v be the solution of

$$-\gamma^2 v'' + v = u,\ v'(0) = v'(1) = 0,$$

and let $G(x,y)$ be the Green's function of this equation. Then

$$\begin{aligned}\mathcal{L}(u) &= \frac{1}{2}\int_0^1 (-u^2 + uv)\\ &= \frac{1}{2}\left[-1 + \int_0^{x_1}(-1)v + \int_{x_1}^{x_2} v + \cdots + \int_{x_\nu}^1 (-1)^{\nu+1} v\right].\end{aligned}$$

We then find, treating $\mathcal{L}$ as a function of $x_1, x_2, \dots, x_\nu$,

$$\nabla\mathcal{L} = 2(-v(x_1), v(x_2), \dots, (-1)^\nu v(x_\nu)).$$

Again we see that at a critical point $(x_1, \dots, x_\nu)$, $v(x_1) = v(x_2) = \cdots = v(x_\nu)$. The same symmetry argument as in the proof of Proposition 3.1 shows that

$$x_1 = \frac{1-m}{2\nu}, x_2 = x_1 + \frac{1+m}{\nu}, \dots, x_\nu = x_{\nu-1} + \frac{1+(-1)^\nu m}{\nu}.$$

We again obtain $U_{\nu,1}$.

The calculation of $\mathcal{L}(U_{\nu,1})$ is a bit more complex. Let v be the solution of

$$-\gamma^2 v'' + v = U_{\nu,1},\ v'(0) = v'(1) = 0.$$

Then

$$\mathcal{L}(U_{\nu,1}) = -\frac{1}{2} + \frac{1}{2}\int_0^1 U_{\nu,1} v\,dx.$$

On an subinterval of $(0,1)$, say (a,b), where $U_{\nu,1} = \alpha \in \{-1, 1\}$,

$$\frac{1}{2}\int_a^b U_{\nu,1} v = \frac{\alpha}{2}\int_a^b v = \frac{\alpha}{2}\int_a^b (\gamma^2 v'' + \alpha) = \frac{\alpha\gamma^2}{2}[v'(b) - v'(a)] + \frac{\alpha^2}{2}(b-a).$$

This implies

$$\begin{aligned}\mathcal{L}(U_{\nu,1}) &= -\gamma^2[v'(x_1) - v'(x_2) + v'(x_3) - \cdots + (-1)^{\nu+1}v'(x_\nu)]\\ &= -\gamma^2 \nu v'(x_1).\end{aligned}$$

We now need to calculate $v'(x_1)$. On $(0, x_1)$ $v(x) = -1 + C'\cosh(x/\gamma)$ and on (x_1, x_2) $v(x) = 1 + C''\cosh((x - \frac{x_1+x_2}{2})/\gamma)$ for some appropriate C' and C''. They and their derivatives match at x_1. Therefore,

$$\begin{cases} -1 + C'\cosh\left(\dfrac{x_1}{\gamma}\right) = 1 + C''\cosh\left(\dfrac{x_1 - x_2}{2\gamma}\right),\\[2ex] \dfrac{C'}{\gamma}\sinh\left(\dfrac{x_1}{\gamma}\right) = \dfrac{C''}{\gamma}\sinh\left(\dfrac{x_1 - x_2}{2\gamma}\right).\end{cases}$$

Solving this system we find

$$\begin{cases} C' = \dfrac{2\sinh(\frac{x_2-x_1}{2\gamma})}{\sinh(\frac{1}{\gamma\nu})}, \\ \\ C'' = \dfrac{-2\sinh(\frac{x_1}{\gamma})}{\sinh(\frac{1}{\gamma\nu})}. \end{cases}$$

Recall that $x_1 = \frac{1-m}{2\nu}$, $x_2 - x_1 = \frac{1+m}{\nu}$, and $\frac{x_1+x_2}{2} = \frac{1}{\nu}$. Then

$$v'(x_1) = \frac{2\sinh(\frac{1+m}{2\gamma\nu})\sinh(\frac{1-m}{2\gamma\nu})}{\gamma\sinh(\frac{1}{\gamma\nu})}.$$

Going back to $\mathcal{L}(U_{\nu,1})$, we find

$$\mathcal{L}(U_{\nu,1}) = \mathcal{L}(U_{\nu,2}) = -\frac{2\gamma\nu\sinh(\frac{1+m}{2\gamma\nu})\sinh(\frac{1-m}{2\gamma\nu})}{\sinh(\frac{1}{\gamma\nu})},$$

and hence the key formula

$$\mathcal{J}_0(U_{\nu,1}) = \mathcal{J}_0(U_{\nu,2}) = c_0\nu - \frac{2\gamma\nu\sinh(\frac{1-m}{2\gamma\nu})\sinh(\frac{1+m}{2\gamma\nu})}{\sinh(\frac{1}{\gamma\nu})}, \tag{4.3}$$

analogous to (3.9).

We need to show that (1) $\mathcal{L}(U_{\nu,1}) = \mathcal{L}(U_{\nu,2})$ is decreasing in ν in order to prove Proposition 3.1 for $\mathcal{J}_0$, and (2) $\mathcal{J}_0(U_{\nu,1}) = \mathcal{J}_0(U_{\nu,2})$ is convex in ν in order to prove Theorem 1.2 for $\mathcal{J}_\epsilon$.

For $t > 0$ define

$$g(t) = c_0 t - \frac{2\gamma t\sinh(\frac{1-m}{2\gamma t})\sinh(\frac{1+m}{2\gamma t})}{\sinh(\frac{1}{\gamma t})}.$$

Compute g' and g''.

$$\begin{aligned} g'(t) = c_0 - \frac{2\gamma}{t(\sinh\frac{p_1+p_2}{t})^2}\Bigg\{ & t\sinh\frac{p_1}{t}\sinh\frac{p_2}{t}\sinh\frac{p_1+p_2}{t} \\ & -p_1\cosh\frac{p_1}{t}\sinh\frac{p_2}{t}\sinh\frac{p_1+p_2}{t} \\ & -p_2\sinh\frac{p_1}{t}\cosh\frac{p_2}{t}\sinh\frac{p_1+p_2}{t} \\ & +(p_1+p_2)\sinh\frac{p_1}{t}\sinh\frac{p_2}{t}\cosh\frac{p_1+p_2}{t}\Bigg\}, \\ g''(t) = \frac{4\gamma}{(t\sinh\frac{1}{\gamma t})^3}\Bigg\{ & \cosh\frac{1}{\gamma t}\left[p_1^2\left(\sinh\frac{p_2}{t}\right)^2 + p_2^2\left(\sinh\frac{p_1}{t}\right)^2\right] \\ & - 2p_1p_2\sinh\frac{p_1}{t}\sinh\frac{p_2}{t}\Bigg\}, \end{aligned}$$

where $p_1 = \frac{1-m}{2\gamma}$ and $p_2 = \frac{1+m}{2\gamma}$. Since $\cosh\frac{1}{\gamma t} > 1$,

$$g''(t) > \frac{4\gamma}{(t\sinh\frac{1}{\gamma t})^3}\left\{p_1^2\left(\sinh\frac{p_2}{t}\right)^2 + p_2^2\left(\sinh\frac{p_1}{t}\right)^2 - 2p_1p_2\sinh\frac{p_1}{t}\sinh\frac{p_2}{t}\right\} \geq 0.$$

Like (3.9), $g(t)$ is convex in t. So $\mathcal{J}_0(U_{\nu,1}) = \mathcal{J}_0(U_{\nu,2})$ is convex in ν.

To show that $\mathcal{L}(U_{\nu.1}) = \mathcal{L}(U_{\nu,2})$ is decreasing in ν, we set

$$h(t) = -\frac{2\gamma t \sinh(\frac{1-m}{2\gamma t}) \sinh(\frac{1+m}{2\gamma t})}{\sinh(\frac{1}{\gamma t})}.$$

Then $h' = g' - c_0$ and $h'' = g''$. Near $t = \infty$ we can find Taylor's expansion

$$h'(t) = -\frac{(1-m^2)^2}{12\gamma^2}\frac{1}{t^3} + o\left(\frac{1}{t^3}\right).$$

Therefore $\lim_{t\to\infty} h'(t) = 0$. Then $h'' > 0$ implies that $h'(t) < 0$ for all $t > 0$. So $\mathcal{L}(U_{\nu.1}) = \mathcal{L}(U_{\nu,2})$ is decreasing in ν. We also note

$$\lim_{\nu\to\infty} \mathcal{L}(U_{\nu,1}) = \lim_{\nu\to\infty} \mathcal{L}(U_{\nu,2}) = \lim_{t\to\infty} h(t) = -\frac{1-m^2}{2}.$$

The global minimum t_0 of $g(t)$, $t \geq 1$, in the case of $\mathcal{J}_0$, cannot be obtained explicitly. But its existence and uniqueness are guaranteed by the convexity of g and the fact $\lim_{t\to\infty} g'(t) = c_0 > 0$. Taylor's expansion

$$g'(t) = c_0 - \frac{(1-m^2)^2}{12\gamma^2}\frac{1}{t^3} + o\left(\frac{1}{t^3}\right)$$

implies that if c_0 is small, then t_0, and ν_* in Theorem 1.2, is close to $(\frac{(1-m^2)^2}{12\gamma^2 c_0})^{1/3}$. This number is the same as the one for $\mathcal{I}_0$.

Therefore it can be argued heuristically that as c_0 becomes small, the problems $\mathcal{I}_\epsilon$ and $\mathcal{J}_\epsilon$ start to converge.

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