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Nucleation in the FitzHugh–Nagumo system: Interface–spike solutions

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Abstract

We find nucleation solutions of N interfaces and K spikes to the one-dimensional FitzHugh–Nagumo system. Each spike sits asymptotically in the middle between two interfaces. We use the Lyapunov–Schmidt reduction method, in which the problem is split into a finite-dimensional problem related to the translation of the K spikes and an infinite-dimensional complement problem. However the complement problem remains near degenerate due to the translation of the N interfaces. To overcome this difficulty we move the interfaces by a small distance and solve the complement problem with the help of a Newton iteration argument.

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1. Introduction

We consider the stationary FitzHugh–Nagumo system

$$\varepsilon^2 \Delta u + f(u) + \delta v = 0, \quad (1.1)$$

$$\Delta v - \gamma v - u = 0, \quad (1.2)$$

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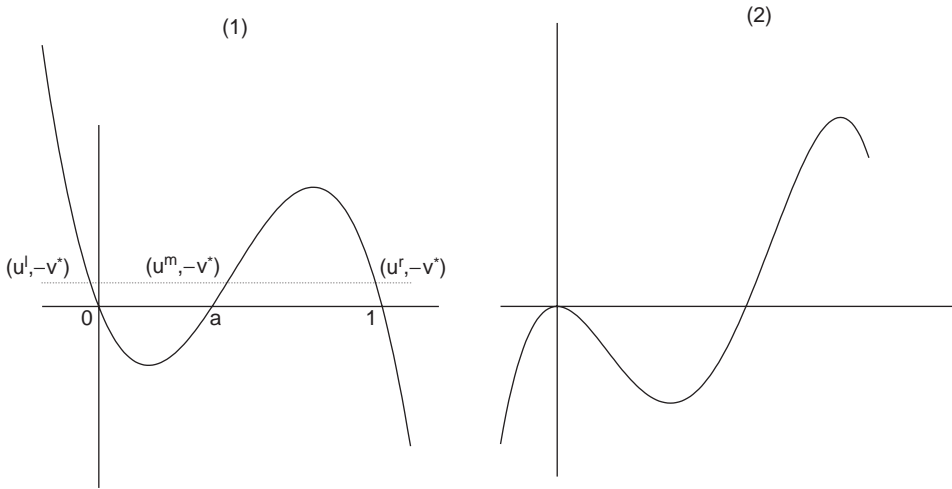


Fig. 1. (1) The graph of f . The height of the dotted line is $-v^*$. (2) The graph of F .

on $(0, 1)$ with the Neumann boundary condition. ε is a small positive parameter, and δ and γ are fixed positive constants. The nonlinear function f is taken to be

$$f(u) = -u(u - a)(u - 1), \quad 0 < a < 1/2. \tag{1.3}$$

Since $a \in (0, 1/2)$, on the graph of f the area of the region below the horizontal axis between 0 and a , is less than the area of the region above the axis between a and 1 , Fig. 1(1). The nonlinearity f is therefore said to be unbalanced. For an unbalanced cubic nonlinearity, the concept of the Maxwell line is often important. It refers to a number v^* so that $f + v^*$ is a balanced nonlinearity. In other words if the level $-v^*$, which is positive here while v^* is negative, is used instead of the horizontal axis, then the two new regions have the same area.

Associated with (1.1)–(1.2) is the functional

$$I(u) = \int_0^1 \left(\frac{\varepsilon^2}{2} |\nabla u|^2 - F(u) + \frac{\delta}{2} |(\gamma - \Delta)^{-1/2} u|^2 \right) dx, \quad u \in W^{1,2}(0, 1). \tag{1.4}$$

Here $F(b) = \int_0^b f(a) da$, Fig. 1(2). The third term in the integrand of (1.4) is a nonlocal expression. For each $h \in L^2(0, 1)$ let z be the solution of $-\Delta z + \gamma z = h$, $z'(0) = z'(1) = 0$. $h \rightarrow z$ defines a linear operator on $L^2(0, 1)$, which we denote by $(\gamma - \Delta)^{-1}$. Then $z = (\gamma - \Delta)^{-1}h$. The operator $(\gamma - \Delta)^{-1}$ is bounded, self-adjoint, and positive, so it has a positive square root, which we denote by $(\gamma - \Delta)^{-1/2}$. If (u, v) is a solution of (1.1)–(1.2), then u is a critical point of (1.4), i.e.

$$-\varepsilon^2 \Delta u - f(u) + \delta(\gamma - \Delta)^{-1}u = 0, \quad u'(0) = u'(1) = 0. \tag{1.5}$$

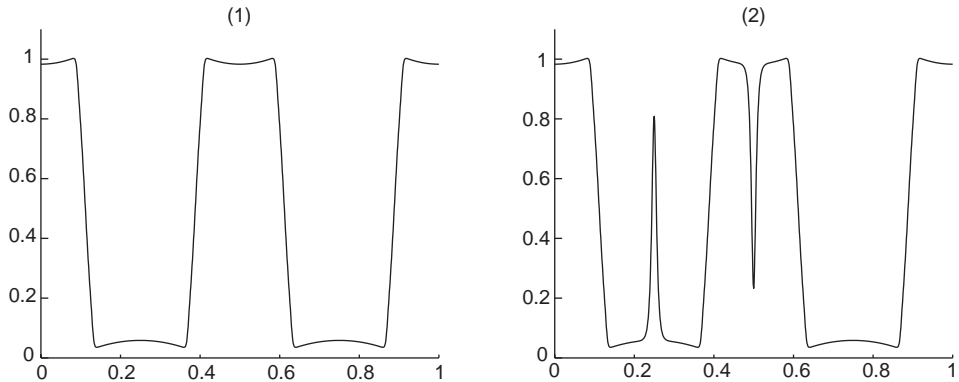


Fig. 2. The graph a local minimizer of 4 interfaces and the graph of a nucleation solution of 4 interfaces and 2 spikes.

Conversely if u is a critical point of (1.4) then, setting $v = (\Delta - \gamma)^{-1}u$, (u, v) is a solution of (1.1)–(1.2).

For each positive integer N , (1.5) admits a solution of N sharp interfaces that is a local minimizer of (1.4), when ε is sufficiently small. Fig. 2(1) shows a local minimizer of 4 internal interfaces. In general these interfaces are periodically positioned and the local minimizer has the internal mirror symmetry so that it may be obtained by extending a piece of the solution with one interface anti-periodically [17].

In this paper we are concerned with a type of saddle points of (1.4). They will be constructed by “adding” spikes on local minimizers of finite interfaces. The width of the spikes is of order ε . Existence of saddle points may be motivated by a *mountain pass* argument between two local minimizers of different numbers of interfaces. To construct saddle points of the particular type we use a rigorous singular perturbation approach. More detailed information on the saddle points will be revealed in the process. Fig. 2(2) shows an example of 4 interfaces and 2 spikes. Note that the interfaces are nearly periodically positioned and the spikes sit almost in the middle of two interfaces.

Such saddle points also help us understand the dynamic counterpart of (1.1)–(1.2). The negative gradient flow of I in the $L^2(0, 1)$ space is the fast inhibitor limit of the dynamic FitzHugh–Nagumo system

$$u_t = \varepsilon^2 \Delta u + f(u) + \delta v, \tag{1.6}$$

$$0 = \Delta v - \gamma v - u, \tag{1.7}$$

with the Neumann boundary condition. The attractor of this system is expected to be made of the solutions of (1.5) and their unstable manifolds. A saddle point of K spikes is unstable whose unstable manifold is at least of dimension K , Theorem 1.2. If in (1.6)–(1.7) the initial value of u is close to the saddle point but with a slightly smaller spike, the spike is likely to disappear in time. On the other hand if the spike is larger, it will probably grow to two interfaces. The latter phenomenon is known as nucleation.

For this reason the saddle points studied in this paper are termed nucleation solutions of (1.5).

The main result is the following existence theorem.

Theorem 1.1. *For every integer $N > 0$ and every integer K , $0 < K < N$, when ε is sufficiently small, there are $\frac{(N-1)!}{(N-1-K)!K!}$ nucleation solutions to (1.5) of N interfaces and K spikes. Each spike sits asymptotically in the middle of two interfaces.*

A nucleation solution of K spikes will be constructed by “placing” K spikes on a local minimizer of N interfaces. Because there are $\frac{(N-1)!}{(N-1-K)!K!}$ ways to place the K spikes between the N interfaces (one can at most put one spike between two adjacent interfaces), we claim that there are $\frac{(N-1)!}{(N-1-K)!K!}$ nucleation solutions for given N and K .

The proof of the theorem is a Lyapunov–Schmidt reduction argument. On a local minimizer of N interfaces, located at x_1, x_2, \dots, x_N , we “add” K spikes at y_1, y_2, \dots, y_K arbitrarily between K prescribed pairs of adjacent interfaces. When $y = (y_1, y_2, \dots, y_K)$ moves, we obtain a manifold of approximate solutions w_y whose dimension is K . In the first step for each $y = (y_1, y_2, \dots, y_K)$ we will solve (1.5) to find g_y in the space “orthogonal” to the manifold. Then in the second step by minimizing I on g_y with respect to y we will find a particular y_* so that $g := g_{y_*}$ also solves (1.5) in the direction of the manifold. This g turns out to be a solution of (1.5).

In the literature when the Lyapunov–Schmidt method was used in this context, the first step, solving an equation in the space “orthogonal” to the manifold, is done by a fixed point argument. Examples include [1–5,10–12,15,19,20,23,24] and the references therein. The minimization argument used in the second step was used in papers like [3,6,7,11,12]. Solutions with only spikes were found in [14,21]. Reinecke and Sweers [22] studied solutions with boundary and internal layers.

Here the situation is complex. Roughly speaking the problem is near degenerate even in the direction perpendicular to the manifold. The first step cannot be done just by a fixed point argument near an approximate solution. Solving the problem in this direction requires some effort. To see this difficulty more clearly, let us consider the critical eigenvalues of the linearized operator of (1.5) at a nucleation solution of N interfaces and K spikes. Here a critical eigenvalue refers to an eigenvalue λ that satisfies $\lambda \rightarrow 0$ as $\varepsilon \rightarrow 0$. We expect that there are $N + K$ critical eigenvalues, corresponding to the translation of the interfaces and the spikes. The presence of these critical eigenvalues means that the nucleation solution is near degenerate. In the Lyapunov–Schmidt reduction method alluded above, the critical eigenvalues related to the translation of the spikes no longer cause trouble in the first step, for they are handled in the second step which is a finite-dimensional problem. However, there are still the critical eigenvalues associated with the translation of the interfaces. They make the first step highly nontrivial.

The key idea in this paper is to “move” the interfaces x_j to x'_j by a proper distance of order ε . From this new function we will launch the Newton iteration. After two iterations we will find an improved function near which we can apply a fixed point argument and solve the problem “orthogonal” to the manifold.

The nucleation solutions are saddle points of (1.4) in the following sense.

Theorem 1.2. *The Morse index of a nucleation solution of K spikes is at least K , i.e. the second variation of I at the nucleation solution has at least K , counting multiplicity, negative eigenvalues.*

The proof of this theorem is relatively simple. Corresponding to each spike we find a unstable direction of perturbation to I . With K spikes, and hence K directions, we construct a subspace on which I'' at the nucleation solution is negative definite. The theorem follows from the variational characterization of the eigenvalues of I'' at the nucleation solution.

We organize the paper as follows. In Section 2, we summarize the properties of the finite interface local minimizers. In Sections 3 and 4, we do the first step of the proof of Theorem 1.1. In Section 3, we construct a K -dimensional manifold of approximate solutions w_y , from which we launch the Newton iteration in Section 4 and solve (1.5) in the direction perpendicular to the manifold to obtain a new manifold of g_y . Then in Section 4 we find g in the new manifold, which solves (1.5), to complete the proof. Theorem 1.2 is also proved in this section. Several technical lemmas are included in the appendix.

2. The finite interface local minimizer u

To make the proofs to the two theorems more readable, we assume, without the loss of generality, that

$$\delta = \gamma = 1. \quad (2.1)$$

We always suppress the dependence on ε in notations. For instance we write I in (1.4) instead of I_ε . However when a quantity is independent of ε , we always emphasize. In the case that a quantity independent of ε arises as a limit as $\varepsilon \rightarrow 0$ of an ε -dependent quantity, we denote the limit with a subscript 0 or a superscript 0. The L^∞ norm is widely used so we simply write $\|\cdot\|$ for it. Other norms are written with subscripts such as $\|\cdot\|_2$ and $\|\cdot\|_{2,2}$ for the L^2 and $W^{2,2}$ norms, respectively. The inner product in $L^2(0, 1)$ is denoted by $\langle \cdot, \cdot \rangle$. The second derivative operator is often denoted by Δ even though we deal with the one-dimensional case. The first derivative operator is occasionally denoted by ∇ .

Let v^* be the particular number so that $f + v^*$ is balanced. Denote its three zeros by u^l , u^m , and u^r , where $u^l < u^m < u^r$, Fig. 1(1). Let $\alpha = u^r - u^l$.

In this section, we summarize some properties of the finite interface local minimizers of I . We will later “add” spikes on them and build nucleation solutions. To save space we take a formal style to describe the local minimizers in this section. More detailed and rigorous statements are found in Fife [9], Mimura et al. [16], Ito [13], and Nishiura and Fujii [18].

Throughout the rest of the paper, we denote an N -interface local minimizer by u , and let $v = (\Delta - 1)^{-1}u$. By convention we assume that the first interface of u goes downward, like in Fig. 2(1). The width of the interfaces is of order ε . The interfaces are defined by points $x_j, j = 1, 2, \dots, N$, where $u(x_j) = u^m$. u has the internal mirror symmetry so that for $x \in (0, 1/N)$,

$$u(x) = u\left(\frac{2}{N} - x\right) = u\left(\frac{2}{N} + x\right) = u\left(\frac{4}{N} - x\right) = u\left(\frac{4}{N} + x\right) = \dots \tag{2.2}$$

Asymptotically $x_j \rightarrow x_j^0$ as $\varepsilon \rightarrow 0$. Meanwhile away from the interfaces $u \rightarrow u_0$, where u_0 is the discontinuous solution of

$$f(u_0) + (\Delta - 1)^{-1}u_0 = 0. \tag{2.3}$$

At x_j^0, u_0 jumps between u^l and u^r , Fig. 3(1). If we take $v_0 = (\Delta - 1)^{-1}u_0$ (Fig. 3(2)), then by (2.3) v_0 satisfies

$$\Delta v_0 - v_0 + f^{-1}(v_0) = 0, \quad v_0'(0) = v_0'(1) = 0. \tag{2.4}$$

Here to define f^{-1} we set \hat{f} to be the restriction of f on $(-\infty, u^l) \cup (u^r, \infty)$ so it is decreasing, and then let f^{-1} be the inverse of \hat{f} . f^{-1} has a jump discontinuity at $-v^*$. Away from the interfaces the difference between u and u_0 is of order ε . In the inner region near $x_j, u(x_j + \varepsilon s)$ approaches locally to H which is a solution of

$$H'' + f(H) + v^* = 0, \quad H(0) = u^m, \quad H(-\infty) = u^l, \quad H(\infty) = u^r, \tag{2.5}$$

if j is even, i.e. when the interface at x_j goes upward. If j is odd, $u(x_j + \varepsilon s)$ approaches locally to $H(-s)$. The ε -order inner expansion is denoted by Q , so that $u(x_j + \varepsilon s) = H(\pm s) + \varepsilon Q(s) + \dots$ Q satisfies

$$Q'' + f'(H)Q + v_0'(x_j^0)s = 0, \quad Q(0) = 0. \tag{2.6}$$

Note that $v_0'(x_j^0)$ changes sign between odd j and even j . Hence Q differs by a sign between odd j and even j .

To better understand u , let us briefly describe the critical eigenvalues of the linearized operator $L(u) := \varepsilon^2 \Delta + f'(u) + (\Delta - 1)^{-1}$. Suppose that λ is an eigenvalue that tends to 0 as ε tends to 0 and φ is an corresponding eigenfunction. λ has the expansion $\lambda = \varepsilon \lambda_1 + O(\varepsilon^2)$, φ has the outer expansion $\varphi = \varphi_0 + \varepsilon \varphi_1 + O(\varepsilon^2)$, and the inner expansion around $x_j, \varphi(x_j + \varepsilon s) = \Phi_0 + \varepsilon \Phi_1 + O(\varepsilon^2)$.

In the leading order the outer expansion φ_0 of φ satisfies

$$f'(u_0)\varphi_0 + (\Delta - 1)^{-1}\varphi_0 = 0 \tag{2.7}$$

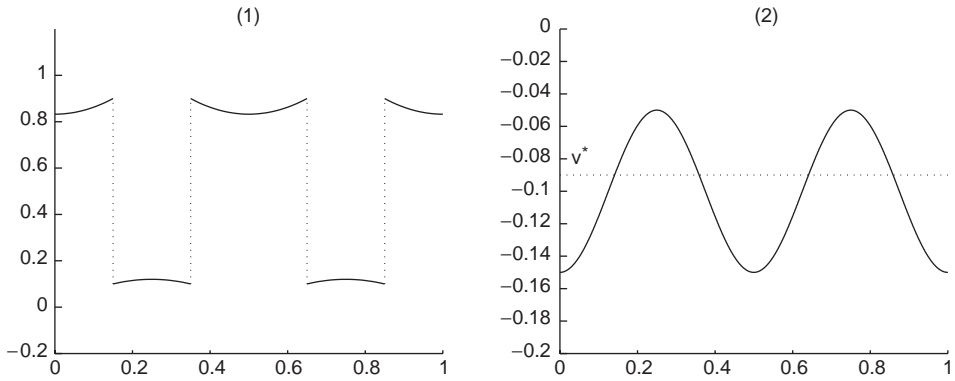


Fig. 3. (1) The graph a of u_0 where $N = 4$. The dotted lines indicate the discontinuity at x_j^0 . (2) The graph of v_0 . The height of the dotted line is v^* .

which implies that $\varphi_0 = 0$. In the inner region near an interface x_j , Φ_0 , the leading order term Φ_0 satisfies

$$\Phi_0'' + f'(H)\Phi_0 = 0, \tag{2.8}$$

so $\Phi_0 = c_j H'$ for some $c_j = O(1)$. In the ε -order φ_1 satisfies

$$f'(u_0)\varphi_1 + (\Delta - 1)^{-1}\varphi_1 - \alpha \sum_{j=1}^N c_j G(x, x_j^0) = 0. \tag{2.9}$$

Here G is the Green function of $1 - \Delta$, namely $G(x, z)$ satisfies

$$-G_{xx}(x, z) + G(x, z) = \delta(x - z), \quad G_x(0, z) = G_x(1, z) = 0. \tag{2.10}$$

If we define \tilde{p}_j to be the solution of

$$f'(u_0)\tilde{p}_j + (\Delta - 1)^{-1}\tilde{p}_j - \alpha G(x, x_j^0) = 0, \tag{2.11}$$

then $\varphi_1 = \sum_{j=1}^N c_j \tilde{p}_j$. The inner term in the ε -order Φ_1 satisfies

$$\begin{aligned} \Phi_1'' + f'(H)\Phi_1 + c_j f''(H)QH' - \alpha \sum_{k=1}^N c_k G(x_j^0, x_k^0) + (\Delta - 1)^{-1}\varphi_1(x_j^0) \\ = \lambda_1 c_j H'. \end{aligned} \tag{2.12}$$

If we multiply (2.12) by H' and integrate over R , then

$$\begin{aligned}
 c_j \int_R f'(H) Q(H')^2 ds - \sum_{k=1}^N c_k (\alpha^2 G(x_j^0, x_k^0) - \alpha(\Delta - 1)^{-1} \tilde{p}_k(x_j^0)) \\
 = \lambda_1 c_j \int_R (H')^2 ds.
 \end{aligned}
 \tag{2.13}$$

Eq. (2.13) is an N -dimensional eigenvalue problem for λ_1 .

Lemma 2.1. *The matrix elements in (2.13)*

$$\delta_{jk} \int_R f''(H) Q(H')^2 ds - \alpha^2 G(x_k^0, x_j^0) + \alpha(\Delta - 1)^{-1} \tilde{p}_j(x_k^0)$$

form a negative definite matrix. Here the constant $\int_R f''(H) Q(H')^2 ds$ is independent of j and is equal to $(-1)^{j+1} \alpha v'_0(x_j^0)$.

Proof. The negativity of the matrix is proved in [17]. There Nishiura considered the more general Fitzhugh–Nagumo system, where the left-hand side of (1.7) is τv_t instead of 0. We take $\tau = 0$ when quoting the result there. To see $\int_R f''(H) Q(H')^2 ds = (-1)^{j+1} \alpha v'_0(x_j^0)$ we differentiate (2.6) with respect to s , multiply by H' and integrate. Then

$$\int_R ((H''' + f'(H)H')Q + f''(H)Q(H')^2 + v'_0(x_j^0)H') dx = 0,$$

which implies $\int_R f''(H) Q(H')^2 ds = (-1)^{j+1} \alpha v'_0(x_j^0)$. Because of the internal mirror symmetry of u and v , $(-1)^{j+1} \alpha v'_0(x_j^0)$ is independent of j . \square

When λ_1 is found from (2.13) we also obtain c_j , the eigenvector of (2.13). Then we can write down a uniform approximation for φ :

$$\varphi \approx \sum_{j=1}^N c_j (H'(\frac{x - x_j}{\varepsilon}) + \varepsilon \tilde{p}_j(x)).
 \tag{2.14}$$

It includes the 0-order inner expansion and ε -order outer expansion. The ε -order inner expansion Φ_1 is not needed in this paper.

We end this section with the remark that u is not a global minimizer. The global minimizer must have an unbounded number of interfaces as $\varepsilon \rightarrow 0$. This phenomenon was studied for the Dirichlet boundary problem in [8].

3. Approximate solutions w_y

We look for a nucleation solution by “placing” spikes between the interfaces of u . First we move each interface of u at x_j by εt_j . Let η be a C^∞ cut-off function so that

$$\eta(x) = \begin{cases} 1 & \text{if } |x| \leq d/2, \\ 0 & \text{if } |x| \geq d, \end{cases} \tag{3.1}$$

where d is a small fixed number independent of ε . Let $t = (t_1, t_2, \dots, t_N)$ be an N vector and set

$$u_t(x) = u(x - \varepsilon \sum_{j=1}^N t_j \eta(x - x_j)). \tag{3.2}$$

We have defined \bar{u}_t by moving each interface of u at x_j by εt_j . Let $x'_j = x_j - \varepsilon t_j$.

We then add spikes. Let $y = (y_1, y_2, \dots, y_K)$ be a K vector. Each y_l is between two interfaces, i.e. $y_l \in (x_j^0, x_{j+1}^0)$. y_l is arbitrary and independent of ε . We let U_l be the homoclinic solution of

$$U'' + f(u(y_l) + U) - f(u(y_l)) = 0, \quad U(\pm\infty) = 0, \quad U'(0) = 0. \tag{3.3}$$

Set

$$w_{t,y} = u_t + \sum_{l=1}^K U_l\left(\frac{x - y_l}{\varepsilon}\right). \tag{3.4}$$

A subtle point in this paper is the choice of t . It is chosen depending on y . Let S from $\{u : u \in W^{2,2}(0, 1), u'(0) = u'(1) = 0\}$ to $L^2(0, 1)$ be defined by

$$S(u) = \varepsilon^2 \Delta u + f(u) + (\Delta - 1)^{-1} u. \tag{3.5}$$

We will choose t so that $S(w_{t,y})$ is not adversely affected by the translation of the interfaces x'_j (see (3.16)). But first we must estimate $S(w_{t,y})$.

Lemma 3.1. $S(w_{t,y}) = O(\varepsilon)$ and more precisely

$$\begin{aligned} S(w_{t,y}) &= \varepsilon v'(x) \sum_{j=1}^N t_j \eta(x - x_j) + \varepsilon \alpha \sum_{j=1}^N (-1)^j G(x, x_j) t_j - \varepsilon \sum_{l=1}^K \left(\int_{\mathbb{R}} U_l(s) ds \right) G(x, y_l) \\ &\quad + \sum_{l=1}^K O(|x - y_l| U_l\left(\frac{x - y_l}{\varepsilon}\right)) + O(\varepsilon^2). \end{aligned}$$

Proof. We define

$$v_t = v(x - \varepsilon \sum_{j=1}^N t_j \eta(x - x_j)). \tag{3.6}$$

Rewrite $S(w_{t,y}) = T_1 + T_2 + T_3 + T_4$ where

$$T_1 = \varepsilon^2 \Delta u_t + f(u_t) + v_t \tag{3.7}$$

$$T_2 = \sum_{l=1}^K \varepsilon^2 \Delta U_l + f(u_t + \sum_{l=1}^K U_l) - f(u_t) \tag{3.8}$$

$$T_3 = (\Delta - 1)^{-1} u_t - v_t \tag{3.9}$$

$$T_4 = \sum_{l=1}^K (\Delta - 1)^{-1} U_l. \tag{3.10}$$

We now estimate these four terms.

Clearly $T_1 = 0$ if x is not in $\cup_j ((-d + x_j, -\frac{d}{2} + x_j) \cup (\frac{d}{2} + x_j, d + x_j))$, because of Eqs. (1.1)–(1.2) that u and v satisfy. Otherwise $T_1 = O(\varepsilon^2)$ because u is bounded in C^2 norm when x is in $\cup_j ((-d + x_j, -\frac{d}{2} + x_j) \cup (\frac{d}{2} + x_j, d + x_j))$. Overall

$$T_1 = O(\varepsilon^2). \tag{3.11}$$

Since U_l is exponentially small away from y_l ,

$$\begin{aligned} T_2 &= f(u + \sum_{l=1}^K U_l) - f(u) - \sum_{l=1}^K (f(u(y_l) + U_l) - f(u(y_l))) + O(e^{-C/\varepsilon}) \\ &= \sum_{l=1}^K O(|x - y_l| U_l) + O(e^{-C/\varepsilon}). \end{aligned} \tag{3.12}$$

Regarding T_3 we have

$$\begin{aligned} T_3 &= (\Delta - 1)^{-1} (u_t - u) + v - v_t \\ &= (\Delta - 1)^{-1} (u_t - u) + \varepsilon v' \sum_{j=1}^N t_j \eta(x - x_j) + O(\varepsilon^2) \\ &= -\varepsilon \sum_{j=1}^N \left(\int_R (H(s - t_j) - H(s)) ds \right) (-1)^j G(x, x_j) + \varepsilon v' \sum_{j=1}^N t_j \eta(x - x_j) + O(\varepsilon^2). \end{aligned}$$

For the first term we note

$$\int_R (H(s - t_j) - H(s)) ds = - \int_R s(H'(s - t_j) - H'(s)) ds = - \int_R t_j H'(s) ds = -\alpha t_j.$$

Therefore

$$T_3 = \varepsilon v' \sum_{j=1}^N t_j \eta(x - x_j) + \varepsilon \alpha \sum_{j=1}^N (-1)^j G(x, x_j) t_j + O(\varepsilon^2). \tag{3.13}$$

And finally

$$T_4 = -\varepsilon \sum_{l=1}^K \left(\int_R U_l ds \right) G(x, y_l) + O(\varepsilon^2). \tag{3.14}$$

The lemma follows from (3.11)–(3.14). \square

Recall \tilde{p}_j defined by (2.11). To see (2.11) more clearly we let $q_j = f'(u_0)\tilde{p}_j$. Then q_j satisfies

$$\Delta q_j - q_j + \frac{q_j}{f'(u_0)} + \alpha \delta_{x_j^0} = 0, \quad q'_j(0) = q'_j(1) = 0.$$

$\delta_{x_j^0}$ is the delta measure centered at x_j^0 . This equation is uniquely solvable because $f'(u_0) < 0$. Moreover $f'(u_0)$ is C^1 on $[0, 1]$, because of (2.3) that defines u_0 . Hence \tilde{p}_j is C^1 on $[0, 1]$ and smooth on $(0, 1) \setminus \{x_j^0\}$. We introduce a cut-off function κ to smooth \tilde{p}_j at x_j^0 . Let $\kappa \in C^\infty(R)$ be such that

$$\kappa(s) = 0 \text{ if } |s| > \mu, \text{ and } \kappa(s) = 1 \text{ if } |s| < \mu/2,$$

to smooth out \tilde{p}_j at x_j^0 . μ is a positive constant independent of ε . Define

$$p_j(x) = \left(1 - \kappa\left(\frac{x - x_j^0}{\varepsilon}\right)\right) \tilde{p}_j(x) + \kappa\left(\frac{x - x_j^0}{\varepsilon}\right) \tilde{p}_j(x_j^0). \tag{3.15}$$

Now motivated by (2.14) we choose t_j so that

$$S(w_{t,y}) \perp H'\left(\frac{x - x'_j}{\varepsilon}\right) + \varepsilon p_j, \quad j = 1, 2, \dots, N. \tag{3.16}$$

Here $x'_j = x_j - \varepsilon t_j$. We denote $H'(\frac{x-x'_j}{\varepsilon})$ by H'_j . When t satisfies (3.16) we denote the corresponding $w_{t,y}$ by w_y .

To see that (3.16) is solvable, we note that by Lemma 3.1

$$\begin{aligned} & \int_0^1 S(w_{t,y})(H'_j + \varepsilon p_j) dx \\ &= \int_0^1 S(w_{t,y})H'_j + \varepsilon \int_0^1 S(w_{t,y})p_j \\ &= o(\varepsilon^2) + \varepsilon^2 \alpha v'_0(x_j^0)t_j + \varepsilon^2 \alpha^2 \sum_{k=1}^N (-1)^k G(x_j^0, x_k^0)t_k - \varepsilon^2 \alpha \sum_{l=1}^K \left(\int_R U_l ds \right) G(x_j^0, y_l) \\ & \quad + \varepsilon^2 \alpha \sum_{k=1}^N (-1)^k t_k \int_0^1 G(x, x_k) p_j dx - \varepsilon^2 \sum_{l=1}^K \left(\int_R U_l ds \right) \int_0^1 G(x, y_l) p_j(x) dx. \end{aligned}$$

Here we have used the estimate

$$\sum_{l=1}^K O(\varepsilon \int_0^1 |x - y_l| |U_l p_j| dx) = O(\varepsilon^3).$$

For $\langle S(w_{t,y}), H'_j + \varepsilon p_j \rangle$ to be zero, we consider the leading order ε^2 terms. After dividing by ε^2 and sending $\varepsilon \rightarrow 0$, we deduce a linear system

$$\begin{aligned} & \alpha v'_0(x_j^0)t_j^0 + \sum_{k=1}^N [\alpha^2 (-1)^k G(x_j^0, x_k^0) - \alpha (-1)^k (\Delta - 1)^{-1} \tilde{p}_j(x_k^0)] t_k^0 \\ &= \sum_{l=1}^K \left(\int_R \tilde{U}_l ds \right) (\alpha G(x_j^0, y_l) - (\Delta - 1)^{-1} \tilde{p}_j(y_l)), \quad j = 1, 2, \dots, N. \end{aligned}$$

Here \tilde{U} is a slightly altered version of U_l . \tilde{U} solves

$$U'' + f(u_0(y_l) + U) - f(u_0(y_l)) = 0, \quad U(\pm\infty) = 0, \quad U'(0) = 0.$$

To solve for t_j^0 we note that the matrix elements are the same as the ones in Lemma 2.1, after we divide each column here by $(-1)^{k+1}$. The matrix in Lemma 2.1 is shown to be negative definite there. Hence the system for t_j^0 here is non-singular and uniquely solvable. After perturbation we find that (3.16) is solvable.

We would like to solve $S(g_y) = 0$ up to U'_l , $l = 1, 2, \dots, K$ in the following sense. Find g_y near w_y with $g_y - w_y \perp U'_l(\frac{x-y_l}{\varepsilon})$, $l = 1, 2, \dots, K$, and a K vector

$c = (c_1, c_2, \dots, c_K)$ so that

$$S(g_y) = \sum_{l=1}^K c_l U'_l \left(\frac{x - y_l}{\varepsilon} \right). \tag{3.17}$$

Let us introduce the projection operator π_y from $\{h \in L^2(0, 1) : h \perp U'_l, l = 1, 2, \dots, K\}$ to itself by

$$\pi_y h = h - \sum_{l=1}^K \left\langle h, \frac{U'_l}{\|U'_l\|_2} \right\rangle U'_l. \tag{3.18}$$

Then (3.17) may be written as

$$\pi_y(S(g_y)) = 0. \tag{3.19}$$

To solve (3.19) we have to complete three steps: two Newton iterations and a contraction mapping argument. Let $L(w_y)$ be the linearized operator of S at w_y , i.e.

$$L(w_y)\phi = \varepsilon^2 \Delta \phi + f'(w_y)\phi + (\Delta - 1)^{-1}\phi. \tag{3.20}$$

First we find some $\phi_{1,y} \perp U'_l \left(\frac{x - y_l}{\varepsilon} \right), l = 1, 2, \dots, K$, so that

$$\pi_y(L(w_y)\phi_{1,y} + S(w_y)) = 0. \tag{3.21}$$

Then we find a $\phi_{2,y} \perp U'_l \left(\frac{x - y_l}{\varepsilon} \right)$ so that

$$\pi_y(L(w_y + \phi_{1,y})\phi_{2,y} + S(w_y + \phi_{1,y})) = 0. \tag{3.22}$$

Finally we use the contraction mapping argument to find $\psi_y \perp U'_l \left(\frac{x - y_l}{\varepsilon} \right)$ so that

$$\pi_y(S(w_y + \phi_{1,y} + \phi_{2,y} + \psi_y)) = 0. \tag{3.23}$$

Then $g_y = w_y + \phi_{1,y} + \phi_{2,y} + \psi_y$.

4. Reduction to g_y

Lemma 4.1. *Eq. (3.21) is uniquely solvable and $\phi_{1,y} = O(\varepsilon)$.*

Proof. We first prove the estimate $\phi_{1,y} = O(\varepsilon)$ assuming (3.21) is solvable. Suppose that the estimate is false. Let $\tilde{\phi}_1 = \phi_{1,y}/\|\phi_{1,y}\|$ where $1/\|\phi_{1,y}\| = o(1/\varepsilon)$. $\tilde{\phi}_1$ satisfies

$$L\tilde{\phi}_1 - \sum_{l=1}^K \langle L\tilde{\phi}_1, U_l' \rangle \frac{U_l'}{\|U_l'\|_2^2} = -\frac{S(w_y)}{\|\phi_{1,y}\|} + \frac{1}{\|\phi_{1,y}\|} \sum_{l=1}^K \langle S(w_y), U_l' \rangle \frac{U_l'}{\|U_l'\|_2^2} \tag{4.1}$$

and $\|\tilde{\phi}_1\| = 1$. For simplicity in this proof we write L for $L(w_y)$. We decompose

$$\tilde{\phi}_1 = \sum_{j=1}^N d_j(H_j' + \varepsilon p_j) + \tilde{\phi}_1^\perp, \quad H_j' + \varepsilon p_j \perp \tilde{\phi}_1^\perp.$$

If we multiply the last equation by $H_k' + \varepsilon p_k$ and integrate over $(0, 1)$, then the left-hand side becomes $O(\varepsilon)$ and the right-hand side becomes

$$\sum_{j=1}^N d_j \int_0^1 (H_j' + \varepsilon p_j)(H_k' + \varepsilon p_k) dx = \sum_{j=1}^N d_j (\varepsilon \delta_{jk} \int_R (H')^2 ds + O(\varepsilon^2)).$$

$\delta_{jk} = 1$ if $j = k$ and 0 otherwise. d_j satisfies the system

$$\sum_{j=1}^N d_j (\varepsilon \delta_{jk} \int_R (H')^2 ds + O(\varepsilon^2)) = O(\varepsilon)$$

from which we conclude that $d_j = O(1)$ and consequently $\tilde{\phi}_1^\perp = O(1)$. We will show that actually $d_j = o(1)$ and $\tilde{\phi}_1^\perp = o(1)$. They contradict $\|\tilde{\phi}_1\| = 1$, and hence follows the lemma.

We first show that $\tilde{\phi}_1^\perp = o(1)$. For this we only need a weaker version

$$L\tilde{\phi}_1 - \sum_{l=1}^K \langle L\tilde{\phi}_1, U_l' \rangle \frac{U_l'}{\|U_l'\|_2^2} = o(1) \tag{4.2}$$

of (4.1), in which

$$\langle L\tilde{\phi}_1, U_l' \rangle = \langle \tilde{\phi}_1, LU_l' \rangle = \langle \tilde{\phi}_1, (f'(w_y) - f'(u(y_l) + U_l))U_l' \rangle = O(\varepsilon^2).$$

One then further simplifies (4.2) to

$$L\tilde{\phi}_1 = o(1). \tag{4.3}$$

On the other hand $L(H'_j + \varepsilon p_j)$ is small. More precisely we note first that

$$\begin{aligned} L(H'_j) &= (f'(w_y) - f'(H_j))H'_j - \varepsilon \alpha G(x, x'_j) + O(\varepsilon^2) \\ &= \varepsilon f''(H_j)Q_j H'_j - \varepsilon \alpha G(x, x'_j) + O(\varepsilon^2). \end{aligned} \tag{4.4}$$

The last equation is valid when j is even. When j is odd, the first term changes to $-f''(H_j)Q_j H'_j$. However Q_j also differs by a sign because of (2.6) and the fact that $v'_0(x_j^0)$ alternates sign while keeping the same absolute value. Next note by (3.15)

$$\begin{aligned} Lp_j &= \varepsilon^2 \Delta p_j + f'(w_y)p_j + (\Delta - 1)^{-1} p_j \\ &= \varepsilon^2 [\varepsilon^{-2} \kappa''(\frac{x - x_j^0}{\varepsilon})(\tilde{p}_j(x_j^0) - \bar{p}_j) - 2\varepsilon^{-1} \kappa'(\frac{x - x_j^0}{\varepsilon})(\tilde{p}_j)_x \\ &\quad + (1 - \kappa(\frac{x - x_j^0}{\varepsilon}))(\tilde{p}_j)_{xx}] + f'(w_y)p_j + (\Delta - 1)^{-1} p_j \\ &= f'(w_y)p_j + (\Delta - 1)^{-1} p_j + O(\varepsilon), \end{aligned}$$

since $(\tilde{p}_j)_{xx}$ is bounded on $(0, 1) \setminus \{x_j^0\}$. By (2.11) we obtain

$$Lp_j = (f'(w_y) - f'(u_0))p_j + \alpha G(x, x_j^0) + O(\varepsilon). \tag{4.5}$$

Hence

$$L(H'_j + \varepsilon p_j) = \varepsilon f''(H_j)Q_j H'_j + \varepsilon(f'(w_y) - f'(u_0))p_j + O(\varepsilon^2). \tag{4.6}$$

For the moment we only need a weaker version $L(H'_j + \varepsilon p_j) = O(\varepsilon)$ of (4.6).

Eq. (4.3) becomes

$$L\tilde{\phi}_1^\perp := \varepsilon^2 \Delta \tilde{\phi}_1^\perp + f'(w_y)\tilde{\phi}_1^\perp + (\Delta - 1)^{-1} \tilde{\phi}_1^\perp = o(1). \tag{4.7}$$

Denote $(\Delta - 1)^{-1} \tilde{\phi}_1^\perp$ by φ . We multiply (4.7) by $\tilde{\phi}_1^\perp$ and integrate. Then

$$\int_0^1 (-\varepsilon^2 |\nabla \tilde{\phi}_1^\perp|^2 + f'(w_y)|\tilde{\phi}_1^\perp|^2 - |\nabla \varphi|^2 - |\varphi|^2) dx = o(1).$$

Note that $f'(w_y)$ is negative except in neighborhoods of the interfaces and the spikes, whose width is of order ε . So we can write the last equation as

$$\int_0^1 (\varepsilon^2 |\nabla \tilde{\phi}_1^\perp|^2 + c(x)|\tilde{\phi}_1^\perp|^2 + |\nabla \varphi|^2 + |\varphi|^2) dx = o(1),$$

where $c(x) > 0$. Hence $\|\tilde{\phi}_1^\perp\|_{1,2} = o(1)$. This implies that $\varphi := (\Delta - 1)^{-1}\tilde{\phi}_1^\perp = o(1)$. And (4.7) is simplified to

$$\varepsilon^2 \Delta \tilde{\phi}_1^\perp + f'(w_y)\tilde{\phi}_1^\perp = o(1). \tag{4.8}$$

To show $\tilde{\phi}_1^\perp = o(1)$, we again argue by contradiction. Without the loss of generality we assume $\|\tilde{\phi}_1^\perp\| = \max \tilde{\phi}_1^\perp = \tilde{\phi}_1^\perp(x_*)$ which is bounded below away from 0. We claim that x_* must lie in an neighborhood, of size ε , of a x'_j or a y_l . Otherwise $\varepsilon^2 \Delta \tilde{\phi}_1^\perp(x_*) \leq 0$ and $f'(w_y)\tilde{\phi}_1^\perp(x_*) < b < 0$, which are inconsistent with (4.8). Suppose x_* is in an ε -neighborhood of x'_j . Then $\tilde{\phi}_1^\perp(x'_j + \varepsilon s)$ approaches in $C^2_{loc}(R)$, as $\varepsilon \rightarrow 0$, to a function Φ which satisfies $\Phi'' + f'(H)\Phi = 0$ on R . Therefore $\Phi = cH'$ with $c \neq 0$. And $\langle H'_j + \varepsilon p_j, \tilde{\phi}_1^\perp \rangle = \varepsilon c \int_R (H')^2 + o(\varepsilon)$. But this contradicts the fact that $H'_j + \varepsilon p_j \perp \tilde{\phi}_1^\perp$. Suppose that x_* is in an ε -neighborhood of y_l . Then $\tilde{\phi}_1^\perp(y_l + \varepsilon s)$ approaches in $C^2_{loc}(R)$, as $\varepsilon \rightarrow 0$, to a function Φ which satisfies $\Phi'' + f'(u_0(y_l) + U)\Phi = 0$ on R . The function U here is the positive solution of

$$U'' + f(u_0(y_l) + U) - f(u_0(y_l)) = 0, \quad U(\pm\infty) = 0, \quad U'(0) = 0.$$

Therefore $\tilde{\phi}_1^\perp = cU'$ with $c \neq 0$. Then $\langle U'_l, \tilde{\phi}_1^\perp \rangle = \varepsilon c \int_R (U')^2 + o(\varepsilon)$. This contradicts the fact that $\tilde{\phi}_1^\perp \perp U'_l$.

Next we show that $d_j = o(1)$. We multiply (4.1) by $H'_k + \varepsilon p_k$ and integrate. The right-hand side becomes

$$-\frac{1}{\|\phi_{1,y}\|} \int_0^1 S(w)(H'_k + \varepsilon p_k) dx + \frac{1}{\|\phi\|} \langle S(w), \frac{U'_l}{\|U'_l\|_2} \rangle \int_0^1 U'_l(H'_k + \varepsilon p_k) dx.$$

The first term is 0 because of (3.16). The second term is

$$\sum_{l=1}^K o\left(\frac{1}{\varepsilon}\right) O(\varepsilon) O(\varepsilon) \frac{1}{\varepsilon} \int_0^1 \varepsilon U'_l p_k dx = o(\varepsilon^2).$$

So the right-hand side now is $o(\varepsilon^2)$. The left-hand side becomes

$$\begin{aligned} &\langle L\tilde{\phi}_1, H'_k + \varepsilon p_k \rangle - \sum_{l=1}^K \langle L\tilde{\phi}_1, \frac{U'_l}{\|U'_l\|_2} \rangle \langle U'_l, H'_k + \varepsilon p_k \rangle \\ &= \langle \tilde{\phi}_1, L(H'_k + \varepsilon p_k) \rangle - \sum_{l=1}^K \langle \tilde{\phi}_1, LU'_l \rangle O\left(\frac{1}{\varepsilon}\right) o(\varepsilon^2) \end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^N d_j \langle H'_j + \varepsilon p_j, L(H_k + \varepsilon p_k) \rangle + \langle \tilde{\phi}_1^\perp, L(H'_k + \varepsilon p_k) \rangle + o(\varepsilon^3) \\ &= \sum_{j=1}^N d_j \langle L(H'_j + \varepsilon p_j), H_k + \varepsilon p_k \rangle + o(\varepsilon^2). \end{aligned}$$

To reach the last line we have used $\langle \tilde{\phi}_1, L(H'_k + \varepsilon p_k) \rangle = o(\varepsilon^2)$, a consequence of $\tilde{\phi}_1 = o(1)$ and (4.6). We obtain a system of equations for d_j :

$$\sum_{j=1}^N d_j \langle L(H'_j + \varepsilon p_j), H_k + \varepsilon p_k \rangle = o(\varepsilon^2).$$

The matrix elements $\langle L(H'_j + \varepsilon p_j), H_k + \varepsilon p_k \rangle$ are computed in Lemma A.1. They are of order ε^2 , and the matrix is negative definite, Lemma 2.1. By solving this system, we deduce $d_j = o(1)$. This completes the proof that $\phi_{1,y} = O(\varepsilon)$.

The existence and uniqueness of $\phi_{1,y}$ are proved by appealing to the Fredholm Alternative. To solve the linear equation

$$\pi_y \circ L\phi = \zeta, \quad \zeta \in L^2(0, 1), \zeta \perp U'_l, \quad l = 1, 2, \dots, K,$$

we apply the operator $\varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1}$ to the equation and rewrite it as

$$\varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1} \circ \pi_y \circ L\phi = \varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1} \zeta.$$

The last two equations are equivalent if we can show that

$$\begin{aligned} &\varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1} : \{ \zeta \in L^2(0, 1) : \zeta \perp U'_l, \quad l = 1, 2, \dots, K \}, \\ &\rightarrow \{ \varphi \in W^{2,2}(0, 1) : \varphi'(0) = \varphi'(1) = 0, \quad \varphi \perp U'_l, \quad l = 1, 2, \dots, K \} \end{aligned}$$

is one-to-one and onto. To see that it is one-to one, we consider $\varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1} \zeta = 0$, which implies that there exist $c_l, l = 1, 2, \dots, K$, such that $(\Delta - 1)^{-1} \zeta = \sum_{l=1}^K c_l U'_l$.

Therefore $\zeta = \sum_{l=1}^K c_l ((\Delta - 1)U'_l)$. Multiply the last equation by $U'_{l'}$ and integrate to find

$$0 = \sum_{l=1}^K c_l \langle (\Delta - 1)U'_l, U'_{l'} \rangle, \text{ which implies that } c_l = 0 \text{ since}$$

$$\langle (\Delta - 1)U'_l, U'_{l'} \rangle = \begin{cases} -\int_0^1 (\|\nabla U'_l\|_2^2 + (U'_l)^2) dx \neq 0 & \text{if } l = l', \\ 0 & \text{if } l \neq l'. \end{cases}$$

Thus $\varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1}\zeta = 0$ becomes $(\Delta - 1)^{-1}\zeta = 0$. Then $\zeta = 0$. To see that $\varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1}$ is onto, we consider the equation $\varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1}\zeta = \varphi$ for a given φ . We look for $c_l, l = 1, 2, \dots, K$, so that

$$\varepsilon^{-2}(\Delta - 1)^{-1}\zeta = \varphi - \sum_{l=1}^K c_l U'_l, \quad \text{i.e.} \quad \zeta = \varepsilon^2(\Delta - 1)\varphi - \sum_{l=1}^K c_l \varepsilon^2((\Delta - 1)U'_l). \tag{4.9}$$

We multiply the last equation by U'_l and integrate. Then

$$0 = \langle (\Delta - 1)\varphi, U'_l \rangle - \sum_{l=1}^K c_l \langle (\Delta - 1)U'_l, U'_l \rangle$$

from which we find c_l . And then ζ follows from (4.9).

The operator $\varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1} \circ \pi_y \circ L$ is defined from the space

$$\{\phi \in W^{2,2}(0, 1) : \phi'(0) = \phi'(1) = 0, \phi \perp U'_l, l = 1, 2, \dots, K\}$$

to itself. Moreover it is a sum of the identity operator and a compact operator, for

$$\begin{aligned} &\varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1} \circ \pi_y \circ L\phi \\ &= \varepsilon^{-2}\pi_y \circ (\Delta - 1)^{-1} \left(L\phi - \sum_{l=1}^K \langle L\phi, \frac{U'_l}{\|U'_l\|_2} \rangle U'_l \right) \\ &= \pi_y(\phi + \varepsilon^{-2}(\Delta - 1)^{-1}(\varepsilon^2\phi + f'(w_y)\phi) + (\Delta - 1)^{-1}\phi) \\ &\quad - \varepsilon^{-2} \sum_{l=1}^K \langle L\phi, \frac{U'_l}{\|U'_l\|_2} \rangle (\Delta - 1)^{-1}U'_l \\ &= \phi - \sum_{l=1}^K \langle \phi, \frac{U'_l}{\|U'_l\|_2} \rangle U'_l + \pi_y(\varepsilon^{-2}(\Delta - 1)^{-1}(\varepsilon^2\phi + f'(w_y)\phi) + (\Delta - 1)^{-1}\phi) \\ &\quad - \varepsilon^{-2} \sum_{l=1}^K \langle L\phi, \frac{U'_l}{\|U'_l\|_2} \rangle (\Delta - 1)^{-1}U'_l. \end{aligned}$$

So the existence and uniqueness of $\phi_{1,y}$ follow if we can show that the homogeneous equation

$$\pi_y(L\phi) = 0 \tag{4.10}$$

only has the trivial solution. Assume this is not true. We have a nontrivial solution ϕ with $\|\phi\| = 1$. we decompose $\phi = \sum_{j=1}^N e_j(H'_j + \varepsilon p_j) + \phi^\perp$, in which $e_j = O(1)$ and $\phi^\perp = O(1)$. Argue as in the first half of this proof to show that ϕ^\perp satisfies $L\phi^\perp = o(1)$ and hence $\phi^\perp = o(1)$. Then as before we find that e_j satisfy the system $\sum_{j=1}^N e_j \langle L(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle = o(\varepsilon^2)$, from which we conclude that $e_j = o(1)$. A contradiction to $\|\phi\| = 1$. \square

Lemma 4.2. *Eq. (3.22) is uniquely solvable and $\phi_{2,y} = \sum_{j=1}^N c_j H'_j + O(\varepsilon^2)$, where $c_j = O(\varepsilon)$.*

Proof. We will only show the estimate for $\phi_{2,y}$. The existence of $\phi_{2,y}$ follows from the same argument as in Lemma 4.1. Let us denote $L(w_y + \phi_{1,y})$ by \tilde{L} . By (3.21) Eq. (3.22) for $\phi_{2,y}$ may be written as

$$\pi_y(\tilde{L}\phi_{2,y} + \frac{1}{2}f''(w)\phi_{1,y}^2 + O(\|\phi_{1,y}\|^3)) = 0.$$

In this proof it suffices to write

$$\pi_y(\tilde{L}\phi_{2,y} + O(\varepsilon^2)) = 0. \tag{4.11}$$

Repeating the proof of Lemma 4.1 with minor modifications, we find

$$\phi_{2,y} = O(\varepsilon). \tag{4.12}$$

In this process, we need the fact that the matrix elements $\langle \tilde{L}(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle$ are the same as those of L in the leading order, which is provided by Lemma A.2. Eq. (4.12) simplifies (4.11) to

$$\tilde{L}\phi_{2,y} = O(\varepsilon^2). \tag{4.13}$$

We decompose $\phi_{2,y}$ into

$$\phi_{2,y} = \sum_{j=1}^N c_j(H'_j + \varepsilon p_j) + \phi_2^\perp. \tag{4.14}$$

Multiplying (4.14) by $H'_k + \varepsilon p_k$ and integrating yield $c_j = O(\varepsilon)$ and hence $\phi_2^\perp = O(\varepsilon)$. It remains to show that $\phi_2^\perp = O(\varepsilon^2)$.

Using the fact that $\tilde{L}(H'_j + \varepsilon p_j) = O(\varepsilon)$, we find from (4.13) and $c_j = O(\varepsilon)$ that

$$\tilde{L}(\phi_2^\perp) = O(\varepsilon^2). \tag{4.15}$$

We again argue by contradiction. Suppose that $\phi_2^\perp = O(\varepsilon^2)$ is false. Then let $\hat{\phi} = \phi_2^\perp / \|\phi_2^\perp\|$ so

$$\tilde{L}(\hat{\phi}) = o(1). \tag{4.16}$$

If we multiply (4.16) by $\hat{\phi}$ and integrate, then

$$\int_0^1 (-\varepsilon^2 |\nabla \hat{\phi}|^2 + f'(w_y + \phi_{1,y}) |\hat{\phi}|^2 - |\nabla \varphi|^2 - \varphi^2) dx = o(1),$$

where $\varphi = (\Delta - 1)^{-1} \hat{\phi}$. Note that $f'(w_y + \phi_{1,y})$ is negative except near the interfaces x'_j and the spikes y_l . So we find $\|\varphi\|_{1,2} = o(1)$. Hence $\varphi := (\Delta - 1)^{-1} \hat{\phi} = o(1)$. Thus (4.15) becomes

$$\varepsilon^2 \Delta \hat{\phi} + f'(w_y + \phi_{1,y}) \hat{\phi} = o(1). \tag{4.17}$$

This equation and the facts that $\hat{\phi} \perp H'_j + \varepsilon p_j$ and $\hat{\phi} \perp U'_l$ imply that $\hat{\phi} = o(1)$. A contradiction to $\|\hat{\phi}\| = 1$. \square

To solve (3.23) we rewrite it as

$$\pi_y(S(w_y + \phi_{1,y} + \phi_{2,y}) + L(w_y + \phi_{1,y} + \phi_{2,y})\psi + M_y\psi) = 0.$$

Here we have defined

$$\begin{aligned} M_y\psi &= f(w_y + \phi_{1,y} + \phi_{2,y} + \psi) - f(w_y + \phi_{1,y} + \phi_{2,y}) \\ &\quad - f'(w_y + \phi_{1,y} + \phi_{2,y})\psi. \end{aligned} \tag{4.18}$$

The operator $L(w_y + \phi_{1,y} + \phi_{2,y})$ has the following properties.

Lemma 4.3. $\pi_y \circ L(w_y + \phi_{1,y} + \phi_{2,y})$ is invertible on $\{\phi \in W^{2,2}(0, 1) : \phi'(0) = \phi'(1) = 0, \phi \perp U'_l, l = 1, 2, \dots, K\}$. For every $\phi \perp U'_l, \|\phi\| \leq \frac{C}{\varepsilon} \|\pi_y \circ L(w_y + \phi_{1,y} + \phi_{2,y})\phi\|$.

Proof. The proof is similar to that of Lemma 4.1 so we only sketch a few steps. We denote $L(w_y + \phi_{1,y} + \phi_{2,y})$ by \tilde{L} . Suppose that the lemma is false. There is ϕ ,

$\|\phi\| = 1$, so that $\pi_y(\tilde{L}\phi) = o(\varepsilon)$. We decompose ϕ to $\phi = \sum_{j=1}^N a_j(H'_j + \varepsilon p_j) + \phi^\perp$.

Then ϕ^\perp satisfies $\tilde{L}\phi^\perp = O(\varepsilon)$ which implies that $\phi^\perp = O(\varepsilon)$. Then one finds that a_j satisfies

$$\sum_{j=1}^N a_j \langle \tilde{L}(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle = o(\varepsilon^2).$$

The matrix elements $\langle \tilde{L}(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle$ are the same as those of L , to the leading order, Lemma A.3. This implies $a_j = o(1)$. A contradiction to $\|\phi\| = 1$. \square

Lemma 4.4. *Let ϕ be the solution of*

$$\pi_y(L(w_y + \phi_{1,y} + \phi_{2,y})\phi + f''(w_y + \phi_{1,y})\phi_{2,y}^2) = 0.$$

Then $\phi = o(\varepsilon)$.

Proof. Since $f''(w_y + \phi_{1,y})\phi_{2,y}^2 = O(\varepsilon^2)$, we find as in Lemma 4.2 that

$$\phi = \sum_{j=1}^N b_j(H'_j + \varepsilon p_j) + \phi^\perp, \tag{4.19}$$

where $b_j = O(\varepsilon)$ and $\phi^\perp = O(\varepsilon^2)$. It suffices to show that $b_j = o(\varepsilon)$.

We multiply $H'_k + \varepsilon p_k$ to the equation

$$\begin{aligned} & \sum_{j=1}^N b_j \tilde{L}(H'_j + \varepsilon p_j) + \tilde{L}\phi^\perp - \sum_{l=1}^K \langle \tilde{L}\phi, \frac{U'_l}{\|U'_l\|_2^2} \rangle U'_l \\ & = -f''(w_y + \phi_{1,y})\phi_{2,y}^2 + \sum_{l=1}^K \langle f''(w_y + \phi_{1,y})\phi_{2,y}^2, \frac{U'_l}{\|U'_l\|_2^2} \rangle U'_l \end{aligned} \tag{4.20}$$

and integrate. Then

$$\begin{aligned} & \sum_{j=1}^N b_j \langle \tilde{L}(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle + \langle \phi^\perp, \tilde{L}(H'_k + \varepsilon p_k) \rangle \\ & - \sum_{l=1}^K \langle \phi, \frac{\tilde{L}U'_l}{\|U'_l\|_2^2} \rangle \langle U'_l, H'_k + \varepsilon p_k \rangle \end{aligned}$$

$$\begin{aligned}
 &= -\langle f''(w_y + \phi_{1,y})\phi_{2,y}^2, H'_k + \varepsilon p_k \rangle + \sum_{l=1}^K \langle f''(w_y + \phi_{1,y})\phi_{2,y}^2, \\
 &\quad \times \frac{U'_l}{\|U'_l\|_2} \rangle \langle U'_l, H'_k + \varepsilon p_k \rangle.
 \end{aligned}$$

The second term on the left-hand side is $O(\varepsilon^4)$ and the third term on the left-hand side is $o(\varepsilon^4)$. The second term on the right-hand side is $o(\varepsilon^4)$. The last equation is now written as

$$\sum_{j=1}^N b_j \langle \tilde{L}(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle = -\langle f''(w_y + \phi_{1,y})\phi_{2,y}^2, H'_k + \varepsilon p_k \rangle + O(\varepsilon^4).$$

However Lemma 4.2 implies that the first term on the right-hand side is

$$\varepsilon c_k^2 \int_R f''(H)(H')^3 + o(\varepsilon^3), \quad c_k = O(\varepsilon).$$

But $\int_R f''(H)(H')^3 = 0$. For if we differentiate $H'' + f(H) = 0$ twice, we find $H'''' + f'(H)H'' + f''(H)(H')^2 = 0$. Multiplying by H' and integrating yield the result. So we find that b_j satisfy the linear system

$$\sum_{j=1}^N b_j \langle \tilde{L}(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle = o(\varepsilon^3).$$

Then we deduce $b_j = o(\varepsilon)$. \square

We write (3.23) in the fixed point form

$$\psi = T_y \psi, \tag{4.21}$$

where

$$T_y \psi = -(\pi_y \circ L(w_y + \phi_{1,y} + \phi_{2,y}))^{-1} \pi_y (S(w_y + \phi_{1,y} + \phi_{2,y}) + M_y \psi) \tag{4.22}$$

and M_y is a nonlinear operator defined by (4.18).

Lemma 4.5. *When c_0 is small enough, T_y is a contraction map on $D(T_y) = \{\psi \in C[0, 1] : \psi \perp U'_l, l = 1, 2, \dots, K, \|\psi\|_\infty \leq c_0 \varepsilon\}$. The unique fixed point $\psi_{\varepsilon,y}$ solves (3.23).*

Proof. We must first show that T_y maps $D(T_y)$ to itself. We know from Lemma 4.3 that

$$\|(\pi_y \circ \tilde{L})^{-1}\omega\| \leq \frac{C}{\varepsilon} \|\omega\|, \quad \omega \perp U'_l. \tag{4.23}$$

Also

$$\begin{aligned} \|\pi_y \circ M_y \psi\| &\leq \|M_y \psi\| + \sum_{l=1}^K \|\langle M_y \psi, U'_l \rangle \frac{U'_l}{\|U'_l\|_2}\| \\ &\leq C\|\psi\|^2 + C \sum_{l=1}^K \|M_y \psi\| \leq C\|\psi\|^2. \end{aligned} \tag{4.24}$$

We have used the fact

$$\|\langle M_y \psi, U'_l \rangle \frac{U'_l}{\|U'_l\|_2}\| \leq \frac{\|M_y \psi\| \cdot \|U'_l\|_1 \cdot \|U'_l\|}{\|U'_l\|_2^2} \leq C\|M_y \psi\|.$$

To estimate $\pi_y \circ S(w_y + \phi_{2,y} + \phi_{2,y})$, we write

$$S(w_y + \phi_{1,y} + \phi_{2,y}) = S(w_y + \phi_{1,y}) + L(w_y + \phi_{1,y})\phi_{2,y} + M_1\phi_{2,y} := M_1\phi_{2,y},$$

where

$$\begin{aligned} M_1\phi_{2,y} &= f(w_y + \phi_{1,y} + \phi_{2,y}) - f(w_y + \phi_{1,y}) - f'(w_y + \phi_{1,y})\phi_{2,y} \\ &= \frac{1}{2}f''(w_y + \phi_{1,y})\phi_{2,y}^2 + O(\|\phi_{2,y}\|^3) \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} \pi_y \circ S(w_y + \phi_{1,y} + \phi_{2,y}) &= \pi_y \circ M_1\phi_{2,y} \\ &= \pi_y(\frac{1}{2}f''(w_y + \phi_{1,y})\phi_{2,y}^2) + O(\|\phi_{2,y}\|^3). \end{aligned} \tag{4.26}$$

Combining (4.23), (4.24), (4.26) and Lemma 4.4 we find

$$T_y \psi = o(\varepsilon) + \frac{C}{\varepsilon}(O(\varepsilon^3) + C(c_0\varepsilon)^2) = o(\varepsilon) + C(c_0)^2\varepsilon. \tag{4.27}$$

Hence T_y maps $D(T_y)$ to itself as long as we choose c_0 sufficiently small.

Next we show that T_y is a contraction. Let ψ_1 and ψ_2 be in $D(T_y)$. Then

$$\|T\psi_1 - T\psi_2\| \leq \frac{C}{\varepsilon} \|\pi_y \circ M_y \psi_1 - \pi_y \circ M_y \psi_2\| \leq \frac{C}{\varepsilon} \|\psi_1 - \psi_2\|^2 \leq Cc_0 \|\psi_1 - \psi_2\|.$$

Hence T_y is a contraction if we take c_0 small. \square

In conclusion we have found a function $g_y := w_y + \phi_{1,y} + \phi_{2,y} + \psi_y$ so that $\pi_y(S(g_y)) = 0$.

5. The reduced problem

In this section we will show that there exists a K vector y_* so that (3.23) becomes

$$S(g_*) = 0. \tag{5.1}$$

To find such a y_* we consider I at g_y and view it as a function of y . We will show that this function is minimized at some y_* at which (5.1) is satisfied.

Lemma 5.1. *With respect to y , $I(g_y)$ is minimized at some y_* . Asymptotically each $y_{*,l}$ in $y_* = (y_{*,1}, y_{*,2}, \dots, y_{*,K})$ lies in the middle of two interfaces, i.e. for each l there exists j such that $y_{*,l} = \frac{x_j^0 + x_{j+1}^0}{2} + o(1)$.*

Proof. We will show that $I(g_y)$ depends on y in the ε -order. Higher orders are negligible. Let $\phi_y = g_y - w_y = \phi_{1,y} + \phi_{2,y} + \psi_y$. We first note the expansion

$$I(g_y) = I(w_y) - \int_0^1 S(w_y)\phi_y dx - \frac{1}{2} \int_0^1 (L(w_y)(\phi_y))\phi_y dx + O(\|\phi_y\|^3). \tag{5.2}$$

Since $S(w_y) = O(\varepsilon)$ and $\phi_y = O(\varepsilon)$, we find that

$$\int_0^1 S(w_y)\phi_y dx = O(\varepsilon^2). \tag{5.3}$$

Also because ϕ_y satisfies the equation $\pi_y \circ S(w_y + \phi_y) = 0$, which may be written as

$$\pi_y \circ (S(w_y) + L(w_y)\phi_y + O(\|\phi_y\|^2)) = 0,$$

we deduce, since $\phi_y \perp U'_l$, that

$$\int_0^1 (L(w_y)(\phi_y))\phi_y dx = - \int_0^1 S(w_y)\phi_y dx + O(\|\phi_y\|^3) = O(\varepsilon^2). \tag{5.4}$$

Hence we obtain from (5.2)–(5.4) that

$$I(g_y) = I(w_y) + O(\varepsilon^2). \tag{5.5}$$

By the remark earlier we will only consider $I(w_y)$.

For this we recall that $w_y = u_t + \sum_{l=1}^K U_l$ where t is determined from y by (3.16). However, the exact dependence of t on y is not needed in this proof. Then

$$\begin{aligned} I(w_y) &= I(u_t + \sum_{l=1}^K U_l) \\ &= I(u_t) + \varepsilon \int_0^1 u'_t \sum_{l=1}^K U'_l dx + \int_0^1 \left(\frac{1}{2} \left| \sum_{l=1}^K U'_l \right|^2 - (F(u_t + \sum_{l=1}^K U_l) - F(u_t)) \right) dx \\ &\quad + \int_0^1 (1 - \Delta)^{-1/2} u_t (1 - \Delta)^{-1/2} \left(\sum_{l=1}^K U_l \right) dx + \frac{1}{2} \int_0^1 |(1 - \Delta)^{-1/2} \sum_{l=1}^K U_l|^2 dx \\ &= I(u_t) + \int_0^1 \left[\frac{1}{2} \left| \sum_{l=1}^K U'_l \right|^2 - (F(u_t + \sum_{l=1}^K U_l) - F(u_t)) \right. \\ &\quad \left. - f(u_t) \sum_{l=1}^K U_l \right] dx + O(\varepsilon^2), \end{aligned}$$

where we have used that fact that u solves (1.5) so

$$\begin{aligned} &\varepsilon \int_0^1 u'_t \sum_{l=1}^K U'_l dx - \int_0^1 f(u_{\varepsilon,y}) \sum_{l=1}^K U_l dx + \int_0^1 (1 - \Delta)^{-1/2} u_t (1 - \Delta)^{-1/2} \\ &\quad \times \left(\sum_{l=1}^K U_l \right) dx = O(\varepsilon^2) \end{aligned}$$

and the fact

$$\int_0^1 |(1 - \Delta)^{-1/2} \sum_{l=1}^K U_l|^2 dx = O(\varepsilon^2).$$

Moreover we deduce

$$\begin{aligned}
 I(w_y) = I(u_t) + \varepsilon \sum_{l=1}^K \int_R \left[\frac{1}{2} |U'_l|^2 - (F(u(y_l) + U_l) - F(u(y_l)) \right. \\
 \left. - f(u(y_l))U_l) \right] ds + o(\varepsilon).
 \end{aligned}
 \tag{5.6}$$

We compare $I(u_t)$ with $I(u)$. Let $\varphi = u_t - u$. Then

$$\begin{aligned}
 I(u_t) = I(u) + \frac{\varepsilon^2}{2} \int_0^1 |\varphi'|^2 dx + \varepsilon^2 \int_0^1 u' \varphi' dx - \int_0^1 (F(u_t) - F(u)) dx \\
 + \int_0^1 (1 - \Delta)^{-1/2} u (1 - \Delta)^{-1/2} \varphi + \frac{1}{2} \int_0^1 |(1 - \Delta)^{-1/2} \varphi|^2 dx.
 \end{aligned}$$

Using the fact that u is a solution of (1.5) and

$$\int_0^1 |(1 - \Delta)^{-1/2} \varphi|^2 dx = O(\varepsilon^2),$$

we find

$$I(u_t) = I(u) + \frac{\varepsilon^2}{2} \int_0^1 |\varphi'|^2 dx - \int_0^1 (F(u_t) - F(u) - f(u)\varphi) dx + O(\varepsilon^2).
 \tag{5.7}$$

Note that φ satisfies

$$\begin{aligned}
 \varphi'' = u''(x - \varepsilon \sum_{j=1}^N t_j \eta(x - x_j))(1 - \varepsilon \sum_{j=1}^N t_j \eta'(x - x_j))^2 \\
 + u'(x - \varepsilon \sum_{j=1}^N t_j \eta(x - x_j))(\varepsilon \sum_{j=1}^N t_j \eta''(x - x_j)).
 \end{aligned}$$

Hence

$$\varepsilon^2 \varphi'' + f(u_t) - f(u) + (v_t - v) = O(\varepsilon^2).
 \tag{5.8}$$

Recall that v_t is defined in (3.6). Therefore

$$\begin{aligned}
 \varepsilon^2 \int_0^1 |\varphi'|^2 dx = \int_0^1 (f(u_t) - f(u))\varphi + \int_0^1 (v_t - v)\varphi + O(\varepsilon^2) \\
 = \int_0^1 (f(u_t) - f(u))\varphi dx + O(\varepsilon^2).
 \end{aligned}$$

Eq. (5.7) now becomes

$$I(u_t) = I(u) + \frac{1}{2} \int_0^1 (f(u_t) - f(u))(u_t - u) dx - \int_0^1 (F(u_t) - F(u) - f(u)(u_t - u)) dx.$$

The two integrals on the right-hand side satisfy

$$\begin{aligned} & \frac{1}{2} \int_0^1 (f(u_t) - f(u))(u_t - u) dx - \int_0^1 (F(u_t) - F(u) - f(u)(u_t - u)) dx \\ &= \varepsilon \sum_{j=1}^N \left[\frac{1}{2} \int_R (f(H(s - t_j)) - f(H(s)))(H(s - t) - H(s)) ds \right. \\ & \quad \left. - \int_R (F(H(s - t_j)) - F(H(s)) - f(H(s))(H(s - t_j) - H(s))) ds \right] + o(\varepsilon). \end{aligned}$$

It is shown in Lemma A.4 that each term in the sum after ε is zero. Therefore,

$$I(u_t) = I(u) + o(\varepsilon).$$

Combining this with (5.5) and (5.6) we find

$$I(g_y) = I(u) + \varepsilon \sum_{l=1}^K \int_R \left[\frac{1}{2} |U_l'|^2 - (F(u(y_l) + U_l) - F(u(y_l)) - f(u(y_l))U_l) \right] ds + o(\varepsilon). \tag{5.9}$$

The first term on the left-hand side of (5.9) is independent of y . We only need to show that the second term is minimized by some $y = y_*$.

There are two cases of y_l . In the first case y_l is between an upward interface and a downward interface. In the second case y_l is between a downward interface and an upward interface. Without the loss of generality we consider the second case. $u(y_l)$ is between u^l and the smaller of the two critical points of f . Lemma A.5 shows that

$$\int_R \left[\frac{1}{2} |U_l'|^2 - (F(u(y_l) + U_l) - F(u(y_l)) - f(u(y_l))U_l) \right] ds$$

is minimized when $u(y_l)$ is maximized. It is known [17] that between the two interfaces x_j^0 and x_{j+1}^0 , u_0 , the outer limit of u , has a maximum at exactly the

middle point $\frac{x_j^0+x_{j+1}^0}{2}$. Therefore, we conclude that $I(g_y)$ is minimized at some $y_* = (y_{*,1}, y_{*,2}, \dots, y_{*,K})$ where $y_{*,l} = \frac{x_j^0+x_{j+1}^0}{2} + o(1)$. \square

When $y = y_*$, we denote $g_y, w_y, \phi_{1,y}, \phi_{2,y}$ and ψ_y by g, w, ϕ_1, ϕ_2 and ψ , respectively.

Lemma 5.2. *g satisfies (1.5).*

Proof. Since y_* is an interior minimum of $I(g_y)$, regarded as a function of y , at $y = y_*$ we have, for each l ,

$$0 = \frac{\partial I(g_y)}{\partial y_l} = \int_0^1 (-\varepsilon^2 \Delta g_y - f(g_y) + (1 - \Delta)^{-1} g_y) \frac{\partial g_y}{\partial y_l} dx = \sum_{m=1}^K c_m \int_0^1 U'_m \frac{\partial g_y}{\partial y_l} dx.$$

Here, we have assumed that at y_* , $-S(g_y) = \sum_{m=1}^K c_m U'_m$, because $\pi_y(S(g_y)) = 0$. The last equation asserts that the coefficients c_m satisfy a linear homogeneous system whose ml matrix entry is $\int_0^1 U'_m \frac{\partial g_y}{\partial y_l} dx$.

Recall that $g_y = w_y + \phi_{1,y} + \phi_{2,y} + \psi_y$ and $U'_m \perp \phi_{1,y} + \phi_{2,y} + \psi_y$. We differentiate $0 = \int_0^1 U'_m (\phi_{1,y} + \phi_{2,y} + \psi_y) dx$ with respect to y_l to obtain

$$\int_0^1 U'_m \frac{\partial(\phi_{1,y} + \phi_{2,y} + \psi_y)}{\partial y_l} = - \int_0^1 \frac{\partial U'_m}{\partial y_l} (\phi_{1,y} + \phi_{2,y} + \psi_y) dx.$$

Therefore, since $\phi_{1,y} + \phi_{2,y} + \psi_y = O(\varepsilon)$,

$$\int_0^1 U'_m \frac{\partial g_y}{\partial y_l} dx = \int_0^1 (U'_m \frac{\partial w_y}{\partial y_l} - \frac{\partial U'_m}{\partial y_l} (\phi_{1,y} + \phi_{2,y} + \psi_y)) dx = \delta_{ml} \int_R (U')^2 ds + O(\varepsilon).$$

Therefore, the coefficient matrix is non-singular and $c_m = 0$, i.e. $S(g) = 0$. \square

We have thus completed the proof of Theorem 1.1.

Proof of Theorem 1.2. Let g be the K spike nucleation solution constructed above. In defining the Morse index of I at g , one views $I'(g)$ as a functional on $W^{1,2}(0, 1)$:

$$I'(g)(\phi) = \int_0^1 (\nabla g \nabla \phi - f(g)\phi + g(1 - \Delta)^{-1} \phi) dx.$$

Then $I''(g)$ is a quadratic form on $W^{1,2}(0, 1)$:

$$I''(g)(\phi, \tilde{\phi}) = \int_0^1 (\nabla\phi\nabla\tilde{\phi} - f'(g)\phi\tilde{\phi} + \phi(1 - \Delta)^{-1}\tilde{\phi}) dx.$$

The eigenvalues of $I''(g)$ may be characterized variationally by

$$\lambda_l = \max_{B_l} \min_{\phi \in B_l} \{I''(g)(\phi, \phi) : \|\phi\|_2 = 1\},$$

where B_l ranges over all l -dimensional subspaces of $W^{1,2}(0, 1)$. We will show that there is a linear subspace of dimension K on which $I''(g)$ is negative definite. From this we conclude that there are at least K negative eigenvalues, i.e. the Morse index of $I''(g)$ is at least K .

To define this subspace, consider the eigenvalue problem

$$\Omega'' + f'(u(y_l) + U_l)\Omega = \Lambda\Omega$$

on the real line. The principal eigenvalue is positive which we denote by Λ_l . Its corresponding eigenfunction is denoted by Ω_l with $\|\Omega_l\| = 1$. Let the subspace be made of functions of the form $\sum_{l=1}^K c_l \Omega_l(\frac{x-y_l}{\epsilon})$. Taking $c_l = O(1)$ we find

$$\begin{aligned} I''(g) & \left(\sum_{l=1}^K c_l \Omega_l\left(\frac{x-y_l}{\epsilon}\right), \sum_{l=1}^K c_l \Omega_l\left(\frac{x-y_l}{\epsilon}\right) \right) \\ & = \int_0^1 (\epsilon^2 |\nabla \sum_{l=1}^K c_l \Omega_l\left(\frac{x-y_l}{\epsilon}\right)|^2 - f'(g) (\sum_{l=1}^K c_l \Omega_l\left(\frac{x-y_l}{\epsilon}\right))^2 \\ & \quad + (\sum_{l=1}^K c_l \Omega_l\left(\frac{x-y_l}{\epsilon}\right))(1 - \Delta)^{-1} \sum_{l=1}^K c_l \Omega_l\left(\frac{x-y_l}{\epsilon}\right)) dx \\ & = \int_0^1 (\epsilon^2 |\nabla \sum_{l=1}^K c_l \Omega_l\left(\frac{x-y_l}{\epsilon}\right)|^2 - f'(g) (\sum_{l=1}^K c_l \Omega_l\left(\frac{x-y_l}{\epsilon}\right))^2) dx + O(\epsilon^2) \\ & = -\epsilon \sum_{l=1}^K \Lambda_l c_l^2 \left(\int_R \Omega_l^2 ds \right) + O(\epsilon^2) < 0. \end{aligned}$$

Theorem 1.2 then follows. \square

Appendix

We again denote $L(w)$, $L(w_y + \phi_{1,y})$, and $L(w_y + \phi_{1,y} + \phi_{2,y})$ respectively by L , \tilde{L} , and $\tilde{\tilde{L}}$. We calculate the matrix elements $\langle L(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle$, $\langle \tilde{L}(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle$ and $\langle \tilde{\tilde{L}}(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle$.

Lemma A.1.

$$\begin{aligned} \langle L(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle &= \varepsilon^2(\delta_{jk} \int_R f''(H) Q(H')^2 ds - \alpha^2 G(x_k, x_j) \\ &\quad + \alpha(\Delta - 1)^{-1} p_j(x_k)) + O(\varepsilon^3). \end{aligned}$$

Proof. For $\langle L(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle$ we note from (4.4)

$$\langle LH'_j, H'_k \rangle = \varepsilon^2 \delta_{jk} \int_R f''(H) Q(H')^2 ds - \varepsilon^2 \alpha^2 G(x_j, x_k) + O(\varepsilon^3). \tag{A.1}$$

Note that $\int_R f''(H) Q(H')^2 ds$ is independent of j because of the remark after (4.4). Next we note that

$$L(\varepsilon p_j) = \varepsilon^3 \Delta p_j + \varepsilon f'(w_y) p_j + \varepsilon(\Delta - 1)^{-1} p_j$$

for which

$$\begin{aligned} \int_0^1 (\varepsilon^3 \Delta p_j + \varepsilon f'(w_y) p_j) H'_k &= \varepsilon \int_0^1 (\varepsilon^2 \Delta H'_k + f'(w_y) H'_k) p_j \\ &= \varepsilon \int_0^1 (f'(w_y) - f'(H_k)) H'_k p_j = O(\varepsilon^3). \end{aligned}$$

So we deduce

$$\langle L\varepsilon p_j, H'_k \rangle = \varepsilon^2 \alpha(\Delta - 1)^{-1} p_j(x_k) + O(\varepsilon^3). \tag{A.2}$$

For $\langle L\varepsilon p_j, \varepsilon p_k \rangle$ we note by (4.5)

$$\langle L\varepsilon p_j, \varepsilon p_k \rangle = -\varepsilon^2 \alpha(\Delta - 1)^{-1} p_k + O(\varepsilon^3). \tag{A.3}$$

From (A.1)–(A.3) we conclude that

$$\begin{aligned} \langle L(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle &= \varepsilon^2 \left(\int_R f''(H) Q(H')^2 ds - \alpha G(x_k, x_j) \right. \\ &\quad \left. + \alpha(\Delta - 1)^{-1} p_j(x_k) \right) + O(\varepsilon^3). \quad \square \tag{A.4} \end{aligned}$$

Lemma A.2.

$$\begin{aligned} \langle \tilde{L}(H'_j + \varepsilon p_j), H'_k + \varepsilon p_k \rangle &= \varepsilon^2 (\delta_{jk} \int_R f''(H) Q(H')^2 ds - \alpha^2 G(x_k, x_j) \\ &\quad + \alpha(\Delta - 1)^{-1} p_j(x_k)) + o(\varepsilon^2). \end{aligned}$$

Proof. The proof is similar to that of Lemma A.1. The main difference is that

$$\begin{aligned} \tilde{L}H'_j &= \varepsilon^2 \Delta H'_j + f'(w_y + \phi_{1,y})H'_j + (\Delta - 1)^{-1} H'_j \\ &= (f'(w_y + \phi_{1,y}) - f'(H_j))H'_j - \varepsilon \alpha G(x, x_j) + O(\varepsilon^2) \\ &= \varepsilon f''(H) Q_j H'_j + f''(H) \phi_{1,y} H'_j - \varepsilon \alpha G(x, x_j) + O(\varepsilon^2). \end{aligned}$$

So in the calculations of $\langle \tilde{L}H'_j, H'_k \rangle$ we have an extra term $\int_0^1 f''(H) \phi_{1,y} H'_j H'_k dx$. Of course it is negligible if $j \neq k$. We will show that when $j = k$, this quantity is $o(\varepsilon^2)$. Then the argument in the proof of the last lemma will yield this one.

At the first glance $\int_0^1 f''(H) \phi_{1,y} (H'_j)^2 dx = O(\varepsilon^2)$ since $\phi_{1,y} = O(\varepsilon)$ and H'_j decays exponentially away from x'_j . To improve this estimate recall Eq. (3.21)

$$\pi_y(L\phi_{1,y} + S(w_y)) = 0.$$

We multiply it by H''_j and integrate. Then

$$\int_0^1 (L\phi_{1,y})H''_j dx + \int_0^1 S(w_y)H''_j dx = O(e^{-C/\varepsilon}). \tag{A.5}$$

From Lemma 3.1 we conclude

$$\int_0^1 S(w_y)H''_j = o(\varepsilon^2). \tag{A.6}$$

And moreover because of the equation $H'''' + f'(H)H'' + f''(H)(H')^2 = 0$ satisfied by H'' ,

$$\begin{aligned} \int_0^1 (L\phi_{1,y})H''_j dx &= \int_0^1 (\varepsilon^2 \Delta H''_j + f'(w_y)H''_j + (\Delta - 1)^{-1} H''_j) \phi_{1,y} dx \\ &= \int_0^1 (-f'(H_j)H''_j - f''(H_j)(H')^2 + f'(w_y)H''_j) \phi_{1,y} dx + o(\varepsilon^2) \end{aligned}$$

$$\begin{aligned}
 &= \int_0^1 (f'(w_y) - f'(H_j))H_j''\phi_{1,y} dx \\
 &\quad - \int_0^1 f''(H_j)\phi_{1,y}(H_j')^2 dx + o(\varepsilon^2) \\
 &= - \int_0^1 f''(H_j)\phi_{1,y}(H_j')^2 dx + o(\varepsilon^2).
 \end{aligned}$$

Combining this to (A.5)–(A.6) we find $\int_0^1 f''(H)\phi_{1,y}(H_j')^2 dx = o(\varepsilon^2)$. \square

Lemma A.3.

$$\begin{aligned}
 \langle \tilde{L}(H_j' + \varepsilon p_j), H_k' + \varepsilon p_k \rangle &= \varepsilon^2(\delta_{jk} \int_R f''(H)Q(H')^2 ds - \alpha^2 G(x_k, x_j) \\
 &\quad + \alpha(\Delta - 1)^{-1} p_j(x_k)) + o(\varepsilon^2).
 \end{aligned}$$

Proof. Like in the proof of the last lemma here we need to show that $\int_0^1 f''(H)\phi_{2,y}(H_j')^2 dx = o(\varepsilon^2)$. We write Eq. (3.22) for $\phi_{2,y}$ as

$$\pi_y(\tilde{L}\phi_{2,y} + O(\|\phi_{1,y}\|^2)) = 0.$$

After multiplying this by H_j'' and integrating, we find

$$\int_0^1 (\tilde{L}\phi_{2,y})H_j'' dx + O(\varepsilon^3) = 0. \tag{A.7}$$

Using the equation for H'' again, we deduce

$$\begin{aligned}
 \int_0^1 (\tilde{L}\phi_{2,y})H_j'' dx &= \int_0^1 (\tilde{L}H_j'')\phi_{2,y} \\
 &= \int_0^1 \varepsilon^2(\Delta H_j'' + f'(w_y + \phi_{1,y})H_j'')\phi_{2,y} dx + o(\varepsilon^2) \\
 &= \int_0^1 (f'(w_y + \phi_{1,y}) - f'(H_j))H_j''\phi_{2,y} dx \\
 &\quad - \int_0^1 f''(H_j)(H_j')^2\phi_{2,y} dx + o(\varepsilon^2) \\
 &= - \int_0^1 f''(H_j)\phi_{2,y}(H_j')^2 dx + o(\varepsilon^2).
 \end{aligned}$$

When this is combined with (A.7), we find $\int_0^1 f''(H)\phi_{2,y}(H_j')^2 dx = o(\varepsilon^2)$. \square

Lemma A.4. *Let $H_t = H(\cdot - t)$. Then*

$$\frac{1}{2} \int_R (f(H_t) - f(H))(H_t - H) - \int_R (F(H_t) - F(H) - f(H)(H_t - H)) = 0.$$

Proof. We let $f^*(a) = f(a) + v^*$ be the balanced cubic nonlinearity and $F^*(a) = F(a) + v^*a + c$ be the corresponding equal depth, double well potential with maximum value 0. The left-hand side in the lemma becomes

$$\begin{aligned} & \frac{1}{2} \int_R (f^*(H_t) - f^*(H)(H_t - H)) - \int_R (F^*(H_t) - F^*(H) - f^*(H)(H_t - H)) \\ &= \frac{1}{2} \int_R (f^*(H_t) - f^*(H)(H_t - H)) + \int_R f^*(H)(H_t - H) \\ &= \frac{1}{2} \int_R (f^*(H_t) + f^*(H)(H_t - H)) = \frac{1}{2} \int_R (f^*(H)H_t - f^*(H_t)H) \\ &= \frac{1}{2} \int_R (f^*(H)H_t - f^*(H)H_{-t}) = \frac{1}{2} \int_R f^*(H)(H_t - H_{-t}). \end{aligned}$$

The last quantity is zero because $f^*(H)$ is odd and $H_t - H_{-t}$ is even. To see that $H_t - H_{-t}$ is even, we note that H is asymmetric with respect to the line u^m , i.e.

$$\frac{H(-s) + H(s)}{2} = u^m.$$

This gives

$$\begin{aligned} H(-s - t) - H(-s + t) &= 2u^m - H(s + t) - (2u^m - H(s - t)) \\ &= H(s - t) - H(s + t). \quad \square \end{aligned}$$

Lemma A.5. *Let z be greater than u^l and less than the smaller of the two critical points of f . Let U_z be the homoclinic solution of*

$$U_z'' + f(U_z + z) - f(z) = 0, \quad U_z(\pm\infty) = 0, \quad U_z'(0) = 0.$$

Define

$$E(z) = \int_R \left(\frac{1}{2} |U_z'|^2 - F_z(U_z) \right) ds,$$

where $F_z(b) = F(b + z) - F(z) - f(z)b$. Then $E(z)$ is decreasing in z .

Similarly if z is greater than the larger of the two critical points of f and less than u^r . Then $E(z)$ is increasing in z .

Proof. We only consider the first part of the lemma. The second part may be proved similarly. The solution U_z has an first integral $\frac{1}{2}|U'_z|^2 + F_z(U_z) = 0$. So we can rewrite $E(z)$ as

$$\begin{aligned} E(z) &= 2 \int_{-\infty}^0 \left(\frac{1}{2}|U'_z|^2 - F_z(U_z) \right) ds = 2 \int_{-\infty}^0 2 \frac{1}{\sqrt{2}} \sqrt{-F_z(U_z)} U'_z ds \\ &= 2\sqrt{2} \int_0^{M_z} \sqrt{-F_z(b)} db \end{aligned}$$

where M_z is the middle zero of F_z and $U_z(0) = M_z$. The lemma will be proved after we show $\frac{\partial(-F_z)}{\partial z} < 0$ and $\frac{dM_z}{dz} < 0$.

Straight calculations show that

$$-F_z(b) = \frac{b^2}{4} (b^2 + (4z - \frac{4(a+1)}{3})b + 6z^2 - 4(a+1)z + 2a). \tag{A.8}$$

Hence

$$\frac{\partial(-F_z)}{\partial z} = \frac{b^2}{4} (4b + 12z - 4(a+1)).$$

One also finds

$$f(b+z) - f(z) = b(-b^2 + (a+1-3z)b + 1 - a - z + 2az - z^2).$$

b is always less than the largest zero of $f(\cdot+z) - f(z)$. Hence $b < a+1-3z$ which is the sum of the two positive zeros. This implies $\frac{\partial(-F_z)}{\partial z} < 0$.

Since M_z is the middle zero of $-F_z$, we deduce from (A.8) that

$$M_z^2 + (4z - \frac{4(a+1)}{3})M_z + 6z^2 - 4(a+1)z + 2a = 0.$$

Implicit differentiation yields

$$\frac{dM_z}{dz} = - \frac{12z - 4(a+1) - 4M_z}{4z - \frac{4}{3}(a+1) + 2M_z}. \tag{A.9}$$

The top of the last fraction is negative since $12z - 4(a+1) = -4(a+1-3z) < 0$ for $a+1-3z$, as the sum of the two positive zeros of $f(\cdot+z) - f(z)$, is positive. Denote

the greatest zero of $-F_z$ by M'_z . Then from (A.8)

$$M_z + M'_z = \frac{4(a+1)}{3} - 4z.$$

The bottom of the fraction on the right-hand side of (A.9) is $M_z - M'_z < 0$. Therefore $\frac{dM_z}{dz} < 0$. \square

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