Counting Peaks of Solutions to Some Quasilinear Elliptic Equations with Large Exponents

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Abstract

We consider the asymptotic behavior of certain solutions to a quasilinear problem with large exponent in the nonlinearity. Starting with the investigation of a Sobolev embedding, we get a sharp estimate for the embedding constant. Then we obtain a crucial L^1-estimate for the $N-$Laplacian operators in R^N. Using these estimates we prove that the solutions obtained by the standard variational method will develop a spiky pattern of peaks as the nonlinear exponent gets large, and we also have a upper bound depending on N only of the number of the peaks. Stronger results for some special convex domains and some special solutions are also achieved.

1 Introduction

In this paper we shall study the asymptotic behavior of certain solutions, as $p \to \infty$, of the quasilinear elliptic equation

$$\left\{ \begin{array}{l} \Delta_N u + u^p = 0 \ \text{ in } \ \Omega \\ u|_{\partial\Omega} = 0, \ u > 0 \text{ in } \ \Omega \end{array} \right. \tag{1.1}$$

where $p > 1$, $N \geq 2$, $\Delta_N u = \operatorname{div}(|\nabla u|^{N-2}\nabla u)$ is the N-Laplacian operator and $\Omega \subset R^N$ is a smooth bounded domain. We shall only focus on the solutions of the problem obtained by the following variational method. Let

$$\mathcal{A}_p = \{v \in W_0^{1,N}(\Omega) : \ \|v\|_{p+1} = 1\}$$

be the admissible set and define

$$J_p : \ \mathcal{A}_p \to R \tag{1.2}$$

by

$$J_p(v) = \int_\Omega |\nabla v|^N.$$

Clearly J_p is bounded from below. Standard arguments show that J_p has at least one nonnegative minimizer in $\mathcal{A}_p$. If we denote such a minimizer by u_p', then a suitable multiple of u_p', say u_p, solves (1.1) and

$$c_p(N) := \inf\{[\int_\Omega |\nabla u|^N]^{1/N} : \ u \in \mathcal{A}_p\} = \frac{\|\nabla u_p\|_{L^N(\Omega)}}{\|u_p\|_{L^{p+1}(\Omega)}}. \tag{1.3}$$

A Hopf type boundary lemma, see M. Guedda and L. Veron [7], shows that u_p is positive in Ω. It is also known that the solutions of (1.1) are $C^{1,\alpha}$ functions. We refer to [7], [17] and [16] for the regularity, comparison principle and Hopf boundary lemma for N-Laplacian operators.

Our goal is to understand the asymptotic behavior of the variational solutions u_p obtained above when p, serving as a parameter, gets large. The case where $N = 2$ is studied in our earlier work [12]. In that article, we proved that $\|u_p\|_{L^\infty}$ are bounded both from below and above as p tends to infinity. We also proved that u_p approach zero except at one or two points. u_p hence develop a pattern of peaks in Ω. In this paper we shall show that our method developed there can be successfully extended to higher dimensional cases with Δ replaced by Δ_N. Our first result is

Theorem 1.1 *Let u_p be a variational solution of (1.1) obtained above. Then there exist positive C_1, C_2, independent of p, such that*

$$0 < C_1 < \|u_p\|_{L^\infty} < C_2 < \infty$$

for p large.

To state the second theorem, let

$$v_p = \frac{u_p}{(\int_\Omega u_p^p)^{1/(N-1)}}. \tag{1.4}$$

For a sequence $\{v_{p_n}\}$ of v_p we define the blow-up set $\mathcal{B}$ of $\{v_{p_n}\}$ to be the subset of $\overline{\Omega}$ such that $x \in \mathcal{B}$ if there exist a subsequence, still denoted by v_{p_n}, and a sequence x_n in Ω with

$$v_{p_n}(x_n) \to \infty \text{ and } x_n \to x. \tag{1.5}$$

We also define, with respect to $\{v_{p_n}\}$,

$$\begin{aligned} S &= \mathcal{B} \cap \Omega, \\ S' &= \mathcal{B} \cap \partial\Omega. \end{aligned} \tag{1.6}$$

We use $\#\mathcal{B}$ ($\#S$, $\#S'$), to denote the cardinality of $\mathcal{B}$ (S, S' respectively). It turns out later that $\mathcal{B}$ (S, S') will be the set of global (interior, boundary) peaks of the subsequence v_{p_n} respectively. We also call them global (interior, boundary) peak sets.

Theorem 1.2 *Let $N \geq 2$. Then for any sequence $\{v_{p_n}\}$ of v_p with $p_n \to \infty$, the global peak set $\mathcal{B}$ of v_{p_n} is not empty and there exists a subsequence of v_{p_n} such that the interior peak set S of the subsequence has the property*

$$0 \leq \#S \leq [\frac{1}{d_N}(\frac{N}{N-1})^{N-1}]$$

where

$$d_N = \inf_{X \neq Y \in R^N} \frac{(|X|^{N-2}X - |Y|^{N-2}Y)(X-Y)}{|X-Y|^N}$$

is a positive number depending on N only.

From the above results, we see that the variational solutions develop a spiky pattern as p approaches infinity and the number of peaks is controlled in Theorem 1.2. If we impose more condition on the domain as well as solutions, we can prove that they develop one single peak in the interior of the domain. We would like to mention that single-peak spiky patterns also appear in the works of W.-M. Ni and I. Takagi [9] [10], W.-M. Ni, X. Pan and I. Takagi [8] and X. Pan [11] where some biological pattern formation problems are considered.

Our paper is organized as follows. In section 2, we prove a crucial sharp estimate for $c_p(N)$ defined in (1.3). Theorem 1.1 will be proved in section 3. In section 4 We extend an estimate of H. Brezis and F. Merle [1] to the N-Laplacian cases using the level set method. Theorem 1.2 will then be proved in section 5. Stronger conclusions for some special convex domains and some special variational solutions u_p are obtained in section 6; namely, $\#S = 1$ and $S' = \emptyset$.

2 An Estimate for $c_p(N)$

Recall $c_p(N)$ defined in (1.3). we first prove

Lemma 2.1 *For every $t \geq 2$ there is D_t such that*

$$\|u\|_{L^t} \leq D_t t^{(N-1)/N} \|\nabla u\|_{L^N}$$

for all $u \in W_0^{1,N}(\Omega)$ *where* Ω *is a bounded domain in* R^N*, furthermore*

$$\lim_{t\to\infty} D_t = (\alpha_N)^{-(N-1)/N}(\frac{N-1}{Ne})^{(N-1)/N}$$

where $\alpha_N = N\omega_{N-1}^{1/(N-1)}$ *and* ω_{N-1} *is the area of unit* $N-1$ *sphere in* R^N*.*

Proof. Let $u \in W_0^{1,N}(\Omega)$. We know

$$\frac{1}{\Gamma(s+1)}x^s \le e^x$$

for all $x \ge 0$, $s \ge 0$ where Γ is the Γ function. From Moser's sharp form of the Trudinger's Inequality (see [5] page 160 and [6]), we have

$$\int_\Omega \exp[\alpha_N(\frac{u}{\|\nabla u\|_{L^N}})^{N/(N-1)}]dx \le C|\Omega|$$

where α_N is defined in Lemma 2.1, C depends on N only and $|\Omega|$ is the Lebesgue measure of Ω. Therefore

$$\begin{aligned}
&\frac{1}{\Gamma(\frac{N-1}{N}t+1)}\int_\Omega u^t dx \\
&= \frac{1}{\Gamma(\frac{N-1}{N}t+1)}\int_\Omega [\alpha_N(\frac{u}{\|\nabla u\|_{L^N}})^{N/(N-1)}]^{\frac{N-1}{N}t}dx(\alpha_N)^{-\frac{N-1}{N}t}\|\nabla u\|_{L^N}^t \\
&\le \int_\Omega \exp[\alpha_N(\frac{u}{\|\nabla u\|_{L^N}})^{N/(N-1)}]dx(\alpha_N)^{-\frac{N-1}{N}t}\|\nabla u\|_{L^N}^t \\
&\le C|\Omega|(\alpha_N)^{-\frac{N-1}{N}t}\|\nabla u\|_{L^N}^t.
\end{aligned}$$

Hence

$$(\int_\Omega u^t dx)^{1/t} \le (\Gamma(\frac{N-1}{N}t+1))^{1/t}(C|\Omega|)^{1/t}\alpha_N^{-(N-2)/N}\|\nabla u\|_{L^N(\Omega)}.$$

Notice according to Stirling's formula

$$(\Gamma(\frac{N-1}{N}t+1))^{1/t} \sim (\frac{N-1}{Ne})^{(N-1)/N}t^{(N-1)/N}.$$

Choosing D_t to be

$$(\Gamma(\frac{N-1}{N}t+1))^{1/t}(C|\Omega|)^{1/t}\alpha_N^{-(N-1)/N}t^{-(N-1)/N}$$

we get the desired result. □

We then prove a sharp estimate for $c_p(N)$.

Lemma 2.2

$$\lim_{p\to\infty} \frac{c_p(N)}{p^{-(N-1)/N}} = (\frac{N}{N-1}\alpha_N e)^{(N-1)/N}.$$

Proof. Without loss of generality, we assume $0 \in \Omega$. Let $L > 0$ be such that $B_L \subset \Omega$ where B_L is the ball of radius L centered at origin. For $0 < l < L$ consider the so called Moser's function

$$m_l(x) = \frac{1}{\omega_{N-1}^{1/N}} \begin{cases} (\log \frac{L}{l})^{(N-1)/N}, & 0 \le |x| \le l \\ \dfrac{\log \frac{L}{|x|}}{[\log \frac{L}{l}]^{1/N}}, & l \le |x| \le L \\ 0, & |x| \ge L. \end{cases}$$

Then $m_l \in W_0^{1,N}(\Omega)$ and $\|\nabla u\|_{L^N} = 1$. Now

$$\begin{aligned} &(\int_\Omega m_l^{p+1}(x)dx)^{1/(p+1)} \\ &\ge (\int_{B_l} m_l^{p+1}(x)dx)^{1/(p+1)} \\ &= \frac{1}{\omega_{N-1}^{1/N}}(\log \frac{L}{l})^{(N-1)/N}(\frac{1}{N}l^N\omega_{N-1})^{1/(p+1)}. \end{aligned}$$

Choosing $l = L\exp(-\frac{N-1}{N^2}(p+1))$, we have

$$\|m_l\|_{p+1}$$

$$\ge \frac{1}{\omega_{N-1}^{1/N}}\exp(-\frac{N-1}{N})(\frac{N-1}{N^2})^{(N-1)/N}(p+1)^{(N-1)/N}(\frac{1}{N}\omega_{N-1}L^N)^{1/(p+1)}.$$

Therefore

$$c_p(N)$$

$$\le \omega_{N-1}^{1/N}\exp(\frac{N-1}{N})(\frac{N^2}{N-1})^{(N-1)/N}(p+1)^{-(N-1)/N}(\frac{1}{N}\omega_{N-1}L^N)^{-1/(p+1)}.$$

Combining this with Lemma 2.1, we get the conclusion. □

By the construction of the variational solutions u_p in section 1, we have

$$c_p(N) = \frac{\|\nabla u_p\|_{L^N(\Omega)}}{\|u_p\|_{L^{p+1}(\Omega)}}.$$

If we multiply equation (1.1) by u_p and integrate both sides on Ω, we have

$$\int_\Omega |\nabla u_p|^N = \int_\Omega u_p^{p+1}.$$

Hence we derive from Lemma 2.2

Corollary 2.3

$$\lim_{p\to\infty} p^{N-1}\int_\Omega u_p^{p+1} = (\frac{N\alpha_N e}{N-1})^{N-1},$$

$$\lim_{p\to\infty} p^{N-1}\int_\Omega |\nabla u_p|^N = (\frac{N\alpha_N e}{N-1})^{N-1}.$$

Define

$$\begin{aligned}\nu_p &= [\int_\Omega u_p^p]^{1/(N-1)},\\ L_0' &= \overline{\lim_{p\to\infty}}\frac{p\nu_p}{e},\\ L_0 &= L_0' d_N^{-1/(N-1)}\end{aligned} \tag{2.1}$$

where d_N is defined in Theorem 1.2. We have the following rough estimates for L_0 and L_0'.

Corollary 2.4 *For any smooth bounded domain* Ω *in* R^N

$$L_0' \le \frac{N}{N-1}\alpha_N, \quad L_0 \le \frac{N}{N-1}\alpha_N d_N^{-1/(N-1)}.$$

Proof. From Corollary 2.3 we have by Holder's inequality

$$L_0' = \overline{\lim_{p\to\infty}}\frac{p\nu_p}{e} \le \overline{\lim_{p\to\infty}} p[\int_\Omega u_p^{p+1}]^{\frac{p}{p+1}\frac{1}{N-1}}|\Omega|^{\frac{1}{p+1}\frac{1}{N-1}}e^{-1} \le \frac{N\alpha_N}{N-1}$$

□

3 Proof of Theorem 1.1

To get a lower bound for $\|u_p\|_{L^\infty}$, we define

$$\lambda = \inf\{\frac{\|\nabla u\|_{L^N}}{\|u\|_{L^N}} : \ u \in W_0^{1,N}(\Omega), \ u \ne 0\}.$$

From Poincaré's Inequality, we have $0 < \lambda < \infty$. For u_p we have

$$\int_\Omega u_p^{p+1} = \int_\Omega |\nabla u_p|^N \ge \lambda^N \int_\Omega u_p^N,$$

$$\int_\Omega (u_p^{p+1} - \lambda^N u_p^N) \ge 0.$$

Therefore

$$\|u_p\|_{L^\infty}^{p+1-N} \ge \lambda^N.$$

Letting $p >> N-1$, we obtain

$$\|u_p\|_{L^\infty} \geq \lambda^{N/(p+1-N)} \geq C_1 > 0.$$

To get a upper bound for $\|u_p\|_{L^\infty}$, let

$$\begin{aligned} \gamma_p &= \max_{x\in\overline{\Omega}} u_p(x), \\ A &= \{x :\ u_p(x) > \gamma_p/2\}, \\ \Omega_t &= \{x :\ u_p(x) > t\}. \end{aligned} \tag{3.1}$$

Both A and Ω_t depend on p. From Lemma 2.1 and Corollary 2.3, we have

$$\|u_p\|_{L^{\frac{Np}{N-1}}} \leq D_{\frac{Np}{N-1}}(\frac{Np}{N-1})^{(N-1)/N}\|\nabla u\|_{L^N}$$

$$\leq C(\frac{Np}{N-1})^{(N-1)/N}p^{-(N-1)/N} < M$$

where M is a constant independent of p. Then

$$(\frac{\gamma_p}{2})^{\frac{Np}{N-1}}|A| \leq \int_\Omega u_p^{\frac{Np}{N-1}} \leq M^{\frac{Np}{N-1}}. \tag{3.2}$$

On the other hand

$$\int_{\Omega_t} u_p^p = -\int_{\Omega_t} \mathrm{div}(|\nabla u_p|^{N-2}\nabla u_p) = \int_{\partial\Omega_t} |\nabla u_p|^{N-1} ds$$

and

$$-\frac{d}{dt}|\Omega_t| = \int_{\partial\Omega_t} \frac{ds}{|\nabla u_p|}$$

where the second is the co-area formula (see Federer [3]). By the Schwartz inequality and the isoperimetric inequality we have

$$(-\frac{d}{dt}|\Omega_t|)^{N-1}\int_{\Omega_t} u_p^p$$

$$\begin{aligned} &= (\int_{\partial\Omega_t}\frac{ds}{|\nabla u_p|})^{N-1}(\int_{\partial\Omega_t}|\nabla u_p|^{N-1}ds) \\ &\geq (\int_{\partial\Omega_t}\frac{ds}{|\nabla u_p|})^{N-1}(\int_{\partial\Omega_t}|\nabla u_p|)^{N-1}|\partial\Omega_t|^{-(N-2)} \\ &\geq |\partial\Omega_t|^{2(N-1)}|\partial\Omega_t|^{-(N-2)} = |\partial\Omega_t|^N \geq C_N|\Omega_t|^{N-1} \end{aligned}$$

where $|\partial\Omega_t|$ denotes the $(N-1)-$dimensional Hausdorff measure of $\partial\Omega_t$ and C_N is the best constant in the isoperimetric inequality (we refer to [3] for more information about the Hausdorff measures and the isoperimetric inequality). Now we define $r(t)$ for $0 \leq t \leq \gamma_p$ such that

$$|\Omega_t| = \frac{1}{N}\omega_{N-1} r^N(t);$$

then

$$\frac{d}{dt}|\Omega_t| = \omega_{N-1} r^{N-1}(t)\frac{dr}{dt}.$$

Hence we have

$$(-\omega_{N-1} r^{N-1}(t)\frac{dr}{dt})^{N-1}\int_{\Omega_t} u_p^p(x)dx \geq C_N(\frac{1}{N}\omega_{N-1} r^N(t))^{N-1};$$

$$(-\frac{dr}{dt})^{N-1}\int_{\Omega_t} u_p^p(x)dx \geq C_N r^{N-1};$$

$$-\frac{dt}{dr} \leq C'_N \frac{1}{r}(\int_{\Omega_t} u_p^p(x)dx)^{1/(N-1)}$$

$$\leq C'_N \frac{1}{r}\gamma_p^{p/(N-1)}|\Omega_t|^{1/(N-1)} = C'_N \gamma_p^{p/(N-1)} r^{1/(N-1)}.$$

Integrating the inequality from 0 to r_0, we have

$$t(0) - t(r_0) \leq C''_N \gamma_p^{p/(N-1)} r_0^{N/(N-1)}.$$

Choosing r_0 so that $t(r_0) = \frac{\gamma_p}{2}$, we get

$$\gamma_p \leq C'_N \gamma_p^{p/(N-1)} r_0^{N/(N-1)};$$

$$\gamma_p \leq C'_N \gamma_p^{p/(N-1)} |A|^{1/(N-1)}.$$

Combining this with (3.2), we obtain

$$\gamma_p \leq C'_N \gamma_p^{p/(N-1)}(\frac{(2M)^{\frac{Np}{N-1}}}{\gamma_p^{\frac{Np}{N-1}}})^{1/(N-1)};$$

$$\gamma_p \leq C^{\frac{1}{1+Np/(N-1)^2 - p/(N-1)}}(2M)^{\frac{Np}{1+Np/(N-1)^2 - p/(N-1)}} \leq C'$$

for p large enough where the last C' is a constant independent of large p. This proves Theorem 1.1.

We derive a consequence of Theorem 1.1 which will be used later.

Corollary 3.1 *There exist C_1 and C_2 independent of p such that*

$$\frac{C_1}{p^{N-1}} \leq \int_\Omega u_p^p \leq \frac{C_2}{p^{N-1}}$$

for large p.

Proof. The first inequality follows from Theorem 1.1 and the first limit of Corollary 2.3; the second inequality follows from the first limit of Corollary 2.3 by an interpolation. □

4 A Priori Estimates for N-Laplacian Operators

In this section we extend the L^1 estimate of H. Brezis and F. Merle [1] to N-Laplacian operators. Due to the nonlinearity of N-Laplacian operators for $N \geq 3$, we use the level set argument here.

Lemma 4.1 *Let u be a $C^{1,\alpha}$ solution of*

$$\begin{cases} -\Delta_N u = f(x) \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $f \in L^1(\Omega)$, $f \geq 0$. Then for every $\delta \in (0, N\omega_{N-1}^{1/(N-1)}) = (0, \alpha_N)$ we have

$$\int_\Omega \exp[\frac{(\alpha_N - \delta)|u(x)|}{\|f\|_{L^1}^{1/(N-1)}}]dx \leq \frac{\alpha_N}{\delta}|\Omega|$$

where $|\Omega|$ denotes the volume of Ω.

Proof. We prove this by the symmetrization method. Consider the symmetrized problem

$$\begin{cases} -\text{div}(|\nabla U|^{N-2}\nabla U) = F(x) \text{ in } \Omega^* \\ U|_{\partial\Omega^*} = 0 \end{cases}$$

where Ω^* is the ball centered at origin with the same volume as Ω and F is the symmetric decreasing rearrangement of f. We refer to G. Talenti [14] and [15] for properties of the rearrangement. According to [15], we have

$$u^* \leq U$$

where u^* is the symmetric decreasing rearrangement of u. U clearly satisfies the following O.D.E.

$$\begin{cases} (|U'|^{N-2}U')' + \dfrac{N-1}{r}|U'|^{N-2}U' + F(r) = 0 \\ U'(0) = 0,\ U(R) = 0. \end{cases}$$

Therefore

$$-U'(r) = \frac{(\int_0^r s^{N-1}F(s)ds)^{1/(N-1)}}{r} \leq \frac{1}{\omega_{N-1}^{1/(N-1)}}\frac{1}{r}\|F\|_{L^1(\Omega^*)}^{1/(N-1)}.$$

Hence

$$|U(r)| \leq \frac{1}{\omega_{N-1}^{1/(N-1)}}\|F\|_{L^1(\Omega^*)}^{1/(N-1)} \log\frac{R}{r};$$

$$\int_{\Omega^*} \exp[(N-\epsilon)\omega_{N-1}^{1/(N-1)}\frac{U}{\|F\|_{L^1}^{1/(N-1)}}]dx \leq \int_{B(R)} \exp\log(\frac{R}{|x|})^{N-\epsilon}dx$$

$$= \omega_{N-1} \int_0^R (\frac{R}{r})^{N-\epsilon} r^{N-1} dr = \epsilon^{-1} \omega_{N-1} R^N.$$

Letting $\epsilon \omega_{N-1}^{1/(N-1)} = \delta$, we have

$$\int_{\Omega^*} \exp[(\alpha_N - \delta) \frac{U(r)}{\|F\|_{L^1}^{1/(N-1)}}] \leq \frac{\omega_{N-1}^{N/(N-1)}}{\delta} R^N.$$

According to the properties of the symmetric decreasing function, we have

$$\|F\|_{L^1(\Omega^*)} = \|f\|_{L^1(\Omega)},$$

$$\int_\Omega \exp[(\alpha_N - \delta) \frac{u(x)}{\|f\|_{L^1(\Omega)}}] dx = \int_{\Omega^*} \exp[(\alpha_N - \delta) \frac{u^*(x)}{\|f\|_{L^1}^{1/(N-1)}}]$$

$$\leq \int_{\Omega^*} \exp[(\alpha_N - \delta) \frac{U(r)}{\|F\|_{L^1}^{1/(N-1)}}] \leq \frac{\omega_{N-1}^{N/(N-1)}}{\delta} R^N$$

$$= \frac{\alpha_N}{\delta} |\Omega|.$$

□

An interesting consequence is

Corollary 4.2 *Let* u_n *be a sequence of* $C^{1,\alpha}$ *solutions of*

$$\begin{cases} \Delta_N u_n + V_n e^{u_n} = 0 \text{ in } \Omega \\ u_n|_{\partial\Omega} = 0 \end{cases}$$

such that

$$\|V_n\|_{L^q} \leq C_1;$$

$$\int_\Omega |V_n| e^{u_n} \leq \epsilon_0 < \frac{\alpha_N}{q'}$$

for some $1 < q < \infty$ *and* $q' = \frac{q}{q-1}$. *Then*

$$\|u_n\|_{L^\infty(\Omega)} \leq C$$

where C *depends on* $N, \ C_1, \ |\Omega|, \ \epsilon_0$ *only.*

Proof. Fix $\delta > 0$ so that $\alpha_N - \delta > \epsilon_0(q' + \delta)$. By Lemma 4.1 we have

$$\int_\Omega \exp[(q' + \delta)|u_n|] \leq C$$

for some C independent of n. Therefore e^{u_n} is bounded in $L^{q'+\delta}(\Omega)$; hence $V_n e^{u_n}$ is bounded in $L^{1+\epsilon_0}(\Omega)$. Then the standard Moser iteration method implies that u_n is bounded in $L^\infty(\Omega)$. □

Next we give a version of Lemma 4.1 without homogeneous boundary condition.

Lemma 4.3 *Let u and φ be $C^{1,\alpha}(\overline{\Omega})$ solutions of*

$$\Delta_N u + f(x) = 0 \text{ in } \Omega,\ \ f > 0$$

and

$$\begin{cases} \Delta_N \varphi = 0 \text{ in } \Omega \\ \varphi|_{\partial\Omega} = u \end{cases}$$

respectively. Then there exists a constant C depending on Ω only such that

$$\int_\Omega \exp[\frac{(\alpha_N - \delta) d_N^{1/(N-1)}}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}(u - \varphi)] \leq \frac{C}{\delta}$$

where d_N is defined in Theorem 1.2.

Proof. Let u_ϵ and φ_ϵ be solutions of the non-degenerate equations

$$\begin{cases} -\mathrm{div}((\epsilon + |\nabla u_\epsilon|^2)^{(N-2)/2} \nabla u_\epsilon) = f \text{ in } \Omega,\ \ f > 0 \\ u_\epsilon|_{\partial\Omega} = u \end{cases}$$

and

$$\begin{cases} -\mathrm{div}((\epsilon + |\nabla \varphi_\epsilon|^2)^{(N-2)/2} \nabla \varphi_\epsilon) = 0 \text{ in } \Omega \\ \varphi_\epsilon|_{\partial\Omega} = u \end{cases}$$

respectively. (These solutions are smooth and obtained easily by the variational method. Furthermore

$$\lim_{\epsilon \to 0} u_\epsilon = u,$$

$$\lim_{\epsilon \to 0} \varphi_\epsilon = \varphi$$

in $C^{1,\beta}$ for some β. See [16].) Let $\Omega_t = \{x \in \Omega :\ u_\epsilon - \varphi_\epsilon > t\}$.

Claim:

$$\frac{\partial u_\epsilon(x)}{\partial \nu} < \frac{\partial \varphi_\epsilon(x)}{\partial \nu}$$

on $\partial\Omega_t$ for almost all $t \geq 0$.

Let $x_0 \in \partial\Omega_t$. For almost all $t > 0$ we can find a ball $B_\delta(x_1) \subset \Omega_t$ with $\overline{B_\delta(x_1)} \cap \overline{\Omega_t} = x_0$ by Sard's theorem. Let $w = u_\epsilon - \varphi_\epsilon - t$. Then w verifies

$$-\sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij} \frac{\partial w}{\partial x_j}) = f > 0$$

where

$$a_{ij} = (\epsilon + |t_i \nabla u_\epsilon + (1 - t_i) \nabla \varphi_\epsilon|^2)^{(N-4)/2} \{\delta_{ij}(\epsilon + |t_i \nabla u_\epsilon + (1 - t_i) \nabla \varphi_\epsilon|^2)$$

$$+(N-2)(t_i\frac{\partial u_\epsilon}{\partial x_i}+(1-t_i)\frac{\partial \varphi_\epsilon}{\partial x_i})(t_i\frac{\partial u_\epsilon}{\partial x_j}+(1-t_i)\frac{\partial \varphi_\epsilon}{\partial x_j})\}$$

and $t_i \in (0,1)$. Because this equation is non-degenerate, we can apply Hopf's lemma. Therefore

$$\frac{\partial w}{\partial \nu} < 0;$$

hence we proves the claim.

Following the standard level set argument, we have

$$\int_{\Omega_t} f(x) = -\int_{\Omega_t} \mathrm{div}((\epsilon+|\nabla u_\epsilon|^2)^{(N-2)/2}\nabla u_\epsilon)+\int_{\Omega_t} \mathrm{div}((\epsilon+|\nabla \varphi_\epsilon|^2)^{(N-2)/2}\nabla \varphi_\epsilon)$$

$$= \int_{\partial\Omega_t}((\epsilon+|\nabla u_\epsilon|^2)^{(N-2)/2}\nabla u_\epsilon-(\epsilon|\nabla\varphi_\epsilon|^2)^{(N-2)/2}\nabla\varphi_\epsilon)\frac{(\nabla u_\epsilon-\nabla\varphi_\epsilon)}{|\nabla u_\epsilon-\nabla\varphi_\epsilon|}$$

$$\geq d_N^\epsilon\int_{\partial\Omega_t}|\nabla u_\epsilon-\nabla\varphi_\epsilon|^{N-1}$$

where

$$d_N^\epsilon = \inf_{X\neq Y\in R^N}\frac{((\epsilon+|X|^2)^{(N-2)/2}X-(\epsilon+|Y|^2)^{(N-2)/2}Y)(X-Y)}{|X-Y|^N}$$

is a positive number,

$$\lim_{\epsilon\longrightarrow 0} d_N^\epsilon = d_N$$

and d_N is defined in Theorem 1.2. Also by the co-area formula we have

$$-\frac{d}{dt}|\Omega_t| = \int_{\partial\Omega_t}\frac{ds}{|\nabla u_\epsilon-\nabla\varphi_\epsilon|}.$$

Hence by the Schwartz inequality and the isoperimetric inequality,

$$(-\frac{d}{dt}|\Omega_t|)^{N-1}\int_{\Omega_t}f(x) \geq (\int_{\partial\Omega_t}\frac{ds}{|\nabla u_\epsilon-\nabla\varphi_\epsilon|})^{N-1}d_N^\epsilon(\int_{\partial\Omega_t}|\nabla u_\epsilon-\nabla\varphi_\epsilon|^{N-1})$$

$$\geq (\int_{\partial\Omega_t}\frac{ds}{|\nabla u_\epsilon-\nabla\varphi_\epsilon|})^{N-1}d_N^\epsilon(\int_{\partial\Omega_t}|\nabla u_\epsilon-\nabla\varphi_\epsilon|)^{N-1}|\partial\Omega_t|^{-(N-2)}$$

$$\geq d_N^\epsilon|\partial\Omega_t|^{2(N-1)}|\partial\Omega_t|^{-(N-2)} = d_N^\epsilon|\partial\Omega_t|^N$$

$$\geq d_N^\epsilon\omega_{N-1}N^{N-1}|\Omega_t|^{N-1} = d_N^\epsilon\alpha_N^{N-1}|\Omega_t|^{N-1}.$$

Define $r(t)$ so that

$$|\Omega_t| = \frac{1}{N}\omega_{N-1}r^{N-1}(t);$$

then

$$\frac{d|\Omega_t|}{dt} = \frac{1}{N}\omega_{N-1}Nr^{N-1}(t)\frac{dr}{dt} = \omega_{N-1}r^{N-1}(t)\frac{dr}{dt}.$$

Hence we have from above

$$(-\omega_{N-1}r^{N-1}(t)\frac{dr}{dt})^{N-1}\int_{\Omega_t}f(x)dx \geq d_N^{\epsilon}N^{N-1}\omega_{N-1}(\frac{1}{N}\omega_{N-1}r^N(t))^{N-1};$$

$$(-\frac{dr}{dt})^{N-1}\int_{\Omega_t}f(x)dx \geq d_N^{\epsilon}\omega_{N-1}r^{N-1};$$

$$\begin{aligned}(-\frac{dt}{dr})^{N-1} &\leq \frac{1}{d_N^{\epsilon}\omega_{N-1}}\frac{1}{r^{N-1}}\int_{\Omega_t}f(x)dx \\ &\leq \frac{1}{d_N^{\epsilon}\omega_{N-1}}\frac{1}{r^{N-1}}\|f\|_{L^1(\Omega)};\end{aligned}$$

$$-\frac{dt}{dr} \leq \frac{1}{(d_N^{\epsilon})^{1/(N-1)}\omega_{N-1}^{1/(N-1)}}\|f\|_{L^1(\Omega)}^{1/(N-1)}\frac{1}{r}.$$

Integrating the last inequality over (r, R) (Note $|\Omega| = \frac{1}{N}\omega_{N-1}R^N$), we have

$$t(r) \leq \frac{1}{(d_N^{\epsilon})^{1/(N-1)}\omega_{N-1}^{1/(N-1)}}\|f\|_{L^1(\Omega)}^{1/(N-1)}\log\frac{R}{r};$$

$$\exp(\frac{(d_N^{\epsilon})^{1/(N-1)}\omega_{N-1}^{1/(N-1)}(N-\epsilon_0)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}t(r)) \leq (\frac{R}{r})^{N-\epsilon_0};$$

$$\begin{aligned}&\int_0^R \exp(\frac{(d_N^{\epsilon})^{1/(N-1)}\omega_{N-1}^{1/(N-1)}(N-\epsilon_0)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}t(r))r^{N-1}dr \\ &\qquad \leq \int_0^R(\frac{R}{r})^{N-\epsilon_0}r^{N-1}dr = \frac{C}{\epsilon_0}.\end{aligned}$$

However, the left hand side of the last inequality,

$$\begin{aligned}&\int_0^R \exp(\frac{(d_N^{\epsilon})^{1/(N-1)}\omega_{N-1}^{1/(N-1)}(N-\epsilon_0)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}t(r))r^{N-1}dr \\ &= \int_{\infty}^0 \exp(\frac{(d_N^{\epsilon})^{1/(N-1)}\omega_{N-1}^{1/(N-1)}(N-\epsilon_0)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}t)\frac{1}{\omega_{N-1}}d|\Omega_t| \\ &= \frac{1}{\omega_{N-1}}\int_{\Omega}\exp(\frac{(d_N^{\epsilon})^{1/(N-1)}\omega_{N-1}^{1/(N-1)}(N-\epsilon_0)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}(u_\epsilon-\varphi_\epsilon))dx.\end{aligned}$$

Letting $\delta = \omega_{N-1}^{1/(N-1)}\epsilon_0$, we have the desired estimate for u_ϵ and φ_ϵ. Finally letting $\epsilon \to 0$, we get the estimate for u and φ themselves. □

In order to have a local analogy of Corollary 4.2, we state a result in J. Serrin [13] which can be proved following the Moser's iteration scheme.

Proposition 4.4 *Let u be a weak solution of*

$$\Delta_N u + f(x) = 0$$

in $B_{2R} \subset \Omega$ and $f \in L^{N/(N-\epsilon)}(B_{2R})$. Then we have

$$\|u\|_{L^\infty(B_R)} \leq CR^{-1}(\|u\|_{L^N(B_{2R})} + KR)$$

where

$$K = (R^\epsilon \|f\|_{L^{N/(N-\epsilon)}(B_{2R})})^{1/(N-1)}$$

and C depends on N only.

Corollary 4.5 *Let*

$$\Delta_N u_n + V_n e^{u_n} = 0 \text{ in } \Omega$$

and

$$\|u_n\|_{L^N(\Omega)} \leq C_1, \ \|V_n\|_{L^q(B_R)} \leq C_2.$$

where $1 < q < \infty$ and B_R is a ball compactly contained in Ω. Assuming

$$\int_{B_R} V_n e^{u_n} \leq \epsilon_0 < \frac{\alpha_N d_N^{1/(N-1)}}{q'}$$

where $q' = q/(q-1)$, we have

$$\|u_n\|_{L^\infty(B_{R/4})} \leq C$$

for some C depending on N, C_1, C_2, R and ϵ_0 only.

Proof. Consider on B_R

$$\begin{cases} \Delta_N \varphi_n = 0 \text{ in } B_R \\ \varphi_n|_{\partial B_R} = u_n|_{\partial B_R}. \end{cases}$$

By the comparison principle in [7], we have

$$\varphi_n \leq u_n; \ \|\varphi_n\|_{L^N(B_R)} \leq C_1.$$

Using Proposition 4.4, we conclude

$$\|\varphi_n\|_{L^\infty(B_{R/2})} \leq C \tag{4.1}$$

for some constant C depending on N, C_1, C_2 and R only. From Lemma 4.3 we also know

$$\int_{B_R} \exp[\frac{(\alpha_N - \delta)d_N^{1/(N-1)}}{\epsilon_0}(u_n - \varphi_n)]$$

$$\leq \int_{B_R} \exp[\frac{(\alpha_N - \delta)d_N^{1/(N-1)}}{\|V_n e^{u_n}\|_{L^1(B_R)}}(u_n - \varphi_n)] \leq \frac{C}{\delta}.$$

Combining this with (4.1), we obtain

$$\int_{B_{R/2}} \exp[\frac{(\alpha_N-\delta)d_N^{1/(N-1)}}{\epsilon_0}u_n] \leq \frac{C}{\delta}. \tag{4.2}$$

Choosing δ small enough so that

$$(\alpha_N-\delta)d_N^{1/(N-1)} > \epsilon_0(q'+\delta),$$

we get from (4.2)

$$\|\exp u_n\|_{L^{q'+\delta}(B_{R/2})} \leq C.$$

Therefore

$$\|V_n \exp u_n\|_{L^{1+\epsilon_1}(B_{R/2})} \leq C$$

for some $\epsilon_1 > 0$. Using Proposition 4.4 again, we finally conclude

$$\|u_n\|_{L^\infty(B_{R/4})} \leq C.$$

□

We close this section with a positive lower bound for d_N.

Proposition 4.6 *Let*

$$d_N = \inf_{X\neq Y\in R^N} \frac{(|X|^{N-2}X-|Y|^{N-2}Y)(X-Y)}{|X-Y|^N}.$$

Then

$$d_N \geq \frac{2}{N}(\frac{1}{2})^{N-2},$$

in particular $d_2 = 1$.

Proof. Without loss of generality, let $0 \leq |Y| \leq |X|$, $X \neq Y$ and $X \neq 0$. Let

$$t = \frac{|Y|}{|X|}, \quad \cos\theta = \frac{<X,Y>}{|X||Y|}.$$

Then

$$\frac{(|X|^{N-2}X-|Y|^{N-2}Y)(X-Y)}{|X-Y|^N} = \frac{1-(t^{N-1}+t)\cos\theta+t^N}{(1-2t\cos\theta+t^2)^{N/2}}.$$

Let

$$f(t,x) = \frac{1-(t^{N-1}+t)x+t^N}{(1-2tx+t^2)^{N/2}}$$

for $0 \leq t \leq 1$ and $-1 \leq x \leq 1$. Fix t and set

$$\frac{\partial f}{\partial x} = 0.$$

Then

$$1-(t^{N-1}+t)x+t^N=\frac{t^{N-2}+1}{N}(1-2tx+t^2).$$

Therefore at the critical points x of $f(t,\ )$,

$$f(t,x)=\frac{t^{N-2}+1}{N}\frac{1}{(1-2tx+t^2)^{(N-2)/2}}$$

$$=\frac{1}{N}\frac{t^{N-2}+1}{(1-2tx+t^2)^{(N-2)/2}}\geq\frac{1}{N}\frac{t^{N-2}+1}{(t+1)^{N-2}}$$

$$\geq\frac{1}{N}\min_{0\leq t\leq 1}\frac{t^{N-2}+1}{(t+1)^{N-2}}.$$

Let

$$g(t)=\frac{t^{N-2}+1}{(t+1)^{N-2}}.$$

Then

$$g'(t)=\frac{(t+1)^{N-3}}{(t+1)^{2(N-2)}}(N-2)(t^{N-3}-1)\leq 0,$$

and

$$\min_{0\leq t\leq 1}g(t)=g(1)=\frac{2}{2^{N-2}}.$$

Hence

$$d_N\geq\frac{2}{N}(\frac{1}{2})^{N-2}.$$

□

Remark 4.7 *An upper bound for #S in Theorem 1.2 can therefore be*

$$\frac{N}{4}(\frac{2N}{N-1})^{N-1},$$

which equals 2 *when* $N=2$.

5 Proof of Theorem 1.2

Recall (1.4) and (2.1)

$$v_p=\frac{u_p}{(\int_\Omega u_p^p)^{1/(N-1)}},\ \nu_p=[\int_\Omega u_p^p]^{1/(N-1)}.$$

Define

$$f_p=\frac{u_p^p}{\int_\Omega u_p^p}=\nu_p^{p-(N-1)}v_p^p. \tag{5.1}$$

Then we have

$$\Delta_N v_p + f_p = 0. \tag{5.2}$$

We first prove $\mathcal{B} \neq \emptyset$ for any sequence $\{v_n\} = \{v_{p_n}\}$ of v_p with $p_n \to \infty$. Let x_n be such that

$$v_n(x_n) = \max_{x\in\overline{\Omega}} v_n(x) = \frac{\max u_n(x)}{(\int_\Omega u_n^{p_n})^{1/(N-1)}}$$

$$\geq \frac{C_1}{(\int_\Omega u_n^{p_n})^{1/(N-1)}} \to \infty$$

by Theorem 1.1 and Corollary 3.1. Therefore cluster points of $\{x_n\}$ belong to $\mathcal{B}$; hence $\mathcal{B} \neq \emptyset$.

Since $\int_\Omega f_p = 1$ and $f_p > 0$,for any sequence of $\{f_p\}$ we can subtract a subsequence

$$\{f_n\} = \{f_{p_n}\}$$

which converges to a measure μ weakly in $M(\Omega)$ where $M(\Omega)$ is the space of real bounded measures on Ω and μ is a positive measure with $\mu(\Omega) \leq 1$. From now on in the rest of this section we shall work on this subsequence $\{f_n\}$ and the corresponding $\{v_n\} = \{v_{p_n}\}$. For any $\delta > 0$, we call $x_0 \in \Omega$ a δ-regular point if there is a function $\varphi \in C_0(\Omega)$, $0 \leq \varphi \leq 1$ with $\varphi = 1$ in a neighborhood of x_0, such that

$$\int_\Omega \varphi d\mu < (\frac{\alpha_N}{L_0 + 3\delta})^{N-1} \tag{5.3}$$

where L_0 is defined in (2.1). We also define δ-irregular set

$$\Sigma(\delta) = \{y_0 : \ y_0 \ is \ not \ a \ \delta - regular \ point\}.$$

Clearly

$$\mu(y_0) \geq (\frac{\alpha_N}{L_0 + 3\delta})^{N-1} \tag{5.4}$$

for all $y_0 \in \Sigma(\delta)$. We shall frequently say 'regular', 'irregular' not mentioning δ if there is no confusion.

Lemma 5.1 *If x_0 is a regular point, then sequence $\{v_n\}$ is uniformly bounded in $L^\infty(B_{R_0(x_0)})$ for some R_0.*

Proof. Let x_0 be a regular point. From (5.3), we can find $R_1 > 0$ such that

$$\int_{B_{R_1}(x_0)} f_n < (\frac{\alpha_N}{L_0 + 2\delta})^{N-1}. \tag{5.5}$$

Applying Lemma 4.1 to f_n on Ω (Notice: $\|f_n\|_{L^1(\Omega)} = 1$), we have

$$\int_\Omega \exp[(\alpha_N - \epsilon)v_n]dx \leq \frac{C}{\epsilon},$$

especially $\|v_n\|_{L^N(B_{R_1}(x_0))} \le C$ for some C independent of n.

Let φ_n be a solution of

$$\begin{cases} -\Delta_N \varphi_n = 0 \text{ in } B_{R_1}(x_0), \\ \varphi_n|_{\partial B_{R_1}(x_0)} = v_n|_{\partial B_{R_1}(x_0)}. \end{cases}$$

Then by Proposition 4.4, we have (Note $\varphi_n \le v_n$ by the comparison principle)

$$\|\varphi_n\|_{L^\infty(B_{R_1/2}(x_0))} \le C.$$

By lemma 4.3 and (5.5), if we choose 'δ' in Lemma 4.3 small enough,

$$\int_{B_{R_1}(x_0)} \exp[(L_0+\delta)d_N^{1/(N-1)}(v_n - \varphi_n)] \le C;$$

hence

$$\int_{B_{R_1/2}(x_0)} \exp[(L_0+\delta)d_N^{1/(N-1)} v_n]dx \le C. \tag{5.6}$$

Let $t = L_0' + d_N^{1/(N-1)}\delta/2$. Observe

$$\log x \le \frac{x}{e}$$

for $x > 0$. We get

$$\begin{aligned} p_n \log \frac{u_n}{\nu_n^{(N-1)/p_n}} &\le \frac{p_n}{e} \frac{u_n}{\nu_n^{(N-1)/p_n}} \\ \le \frac{L_0' + d_N^{1/(N-1)}\delta/3}{\nu_n} \frac{u_n}{\nu_n^{(N-1)/p_n}} &= \frac{t - d_N^{1/(N-1)}\delta/6}{\nu_n^{(N-1)/p_n}} \frac{u_n}{\nu_n} \\ &\le t\frac{u_n}{\nu_n} = tv_n \end{aligned}$$

for n large enough where $\nu_n = \nu_{p_n}$ is defined in (2.1) and the last inequality is based on

$$\lim_{n\to\infty} \nu_n^{(N-1)/p_n} = 1$$

which follows from Corollary 3.1. Hence

$$f_n \le e^{tv_n}.$$

Notice

$$t = L_0' + d_N^{1/(N-1)}\delta/2 = (L_0 + \delta/2)d_N^{1/(N-1)} < (L_0+\delta)d_N^{1/(N-1)};$$

hence with the aid of (5.6) we see that f_n is bounded in $L^q(B_{R_1/2}(x_0))$ where

$$q = \frac{L_0+\delta}{L_0+\delta/2} > 1.$$

Using Proposition 4.4 again, we conclude that for large n there exists $C > 0$ such that

$$\|v_n\|_{L^\infty(B_{R_1/4}(x_0))} \leq C.$$

This proves Lemma 5.1 if we choose $R_0 = R_1/4$. $\square$

Back to the proof of Theorem 1.2, we claim

$S = \Sigma(\delta)$ for any $\delta > 0$ where S is the interior peak set with respect to $\{v_n\}$ defined in (1.6).

Clearly, $S \subset \Sigma$. In fact, letting $x_0 \notin \Sigma$, then we know that x_0 is a regular point. Hence by Lemma 5.1, $\{v_n\}$ is uniformly bounded in a neighborhood of x_0. Therefore $x_0 \notin S$. Conversely, suppose $x_0 \in \Sigma$. Then we have for every $R > 0$

$$\lim_{n\to\infty} \|v_n\|_{L^\infty(B_R(x_0))} = \infty.$$

Otherwise, there would be some $R_0 > 0$ and a subsequence of v_n, again denoted by v_n, such that

$$\|v_n\|_{L^\infty(B_{R_0}(x_0))} < C$$

for some C independent of n. Then

$$f_n = \nu_n^{p_n-(N-1)} v_n^{p_n} < \nu_n^{p_n-(N-1)} C^{p_n}$$

$$\to 0$$

uniformly on $B_{R_0}(x_0)$ as $n \to \infty$. Then

$$\int_{B_{R_0}(x_0)} f_n = \int_{B_{R_0}(x_0)} \nu_n^{p_n-(N-1)} v_n^{p_n} \leq \epsilon_0 < (\frac{\alpha_N}{L_0 + 3\delta})^{N-1}$$

for large n which implies that x_0 is a regular point. This proves the claim.

Back to the measure μ defined earlier in this section. We have from (5.4)

$$1 \geq \mu(\Omega) \geq (\frac{\alpha_N}{L_0 + 3\delta})^{N-1} \#\Sigma(\delta) = (\frac{\alpha_N}{L_0 + 3\delta})^{N-1} \#S.$$

Hence

$$0 \leq \#S \leq (\frac{L_0 + 3\delta}{\alpha_N})^{N-1}.$$

Letting $\delta \to 0$, we get with the aid of Corollary 2.4

$$0 \leq \#S \leq (\frac{L_0}{\alpha_N})^{N-1} \leq (\frac{N}{N-1})^{N-1} d_N^{-1}.$$

This proves Theorem 1.2.

Remark 5.2 *From the proof of Theorem we see that the measure μ is atomic. Actually*

$$\mu = \sum_{k=1}^{\#S} a_k \delta(x_k)$$

where $S = \{x_1, x_2, ..., x_{\#S}\}$ and

$$a_k \geq (\frac{\alpha_N}{L_0})^{N-1}.$$

The subsequence v_n approaches a function G in $C^{1,\alpha}_{loc}(\Omega\backslash S)$ and G is N-harmonic in $\Omega\backslash S$ but singular on S.

Remark 5.3 *It is also clear from the proof of Theorem 1.2 and Corollary 3.1 that the subsequence*

$$u_n \to 0$$

in $L^\infty_{loc}(\Omega\backslash S)$.

6 Further Results

So far, we haven't touched the boundary peak sets S' yet. Our next result shows that when Ω is strictly convex and u_p are generic in some sense, S' is empty; i.e. $\mathcal{B} = S$.

Recall that u_p are solutions of (1.1) obtained by minimizing

$$J_p(v) = \int_\Omega |\nabla v|^N$$

in the class

$$\mathcal{A}_p = \{v \in W^{1,N}_0(\Omega) : \ \|u\|_{p+1} = 1\}.$$

Let

$$J^\epsilon_p : \mathcal{A}_p \to R \tag{6.1}$$

defined by

$$J^\epsilon_p(v) = \int_\Omega (\epsilon + |\nabla v|^2)^{N/2}.$$

We call u_p a generic solution if there exist a sequence ϵ_n of ϵ with

$$\epsilon_n \to 0$$

and a sequence of positive minimizers $\{u'_{p\epsilon_n}\}$ of $J^{\epsilon_n}_p$ such that $\{u'_{p\epsilon_n}\}$ converges to u'_p weakly in $W^{1,N}(\Omega)$ as $\epsilon_n \to 0$ where $u_p = cu'_p$ for some scalar c. Clearly $\{u'_{p\epsilon_n}\}$ is a minimizing sequence of J_p in $\mathcal{A}_p$. Actually any sequence $\{u'_{p\epsilon_n}\}$ of minimizers of $J^{\epsilon_n}_p$ is a minimizing sequence of J_p, so generic solutions exist for all smooth bounded domains.

Theorem 6.1 *Let Ω be a strict convex domain. Assume u_p are generic solutions for all p. Then S', the boundary peak set of $\{u_p\}$, is empty; i.e. $\mathcal{B} = S$.*

Proof. Let

$$\{u'_{p\epsilon_n}\} \to u'_p$$

weakly in $W^{1,N}(\Omega)$ where $u'_p = cu_p$ for some c and $u'_{p\epsilon_n}$ are positive minimizers of $J_p^{\epsilon_n}$. Then $u'_{p\epsilon_n}$ solve

$$\begin{cases} \operatorname{div}((\epsilon + |\nabla u'_{p\epsilon_n}|^2)^{(N-2)/2} \nabla u'_{p\epsilon_n}) + \lambda_n u'_{p\epsilon_n} = 0 \text{ in } \Omega \\ u'_{p\epsilon_n}|_{\partial\Omega} = 0 \end{cases}$$

for some $\lambda_n > 0$.

Therefore using the moving plane method for non-degenerate equations developed by B. Gidas, W.-M. Ni and L. Nirenberg in [4], we can find a neighborhood ω of $\partial\Omega$ and a cone Γ of fixed size both depending on Ω only such that

$$u'_{p\epsilon_n}(x) \le \frac{1}{|\Gamma|} \int_\Omega u'_{p\epsilon_n}(x)dx \tag{6.2}$$

for all $x \in \omega$. We refer to D. G. DeFigueiredo, P. L. Lions and R. D. Nussbaum [2] for details of this trick.

Since $\{u'_{p\epsilon_n}\} \to u'_p$ weakly in $W^{1,N}(\Omega)$, we have

$$\begin{array}{l} u'_{p\epsilon_n} \to u'_p \text{ strongly in } L^1(\Omega); \\ u'_{p\epsilon_n} \to u'_p \text{ almost everywhere.} \end{array} \tag{6.3}$$

Hence passing limit in (6.2), we get

$$u'_p(x) \le \frac{1}{|\Gamma|} \int_\Omega u'_p(x)dx$$

for almost all $x \in \omega$. Therefore

$$u_p(x) \le \frac{1}{|\Gamma|} \int_\Omega u_p(x)dx$$

and

$$v_p(x) \le \frac{1}{|\Gamma|} \int_\Omega v_p(x)dx$$

for almost all $x \in \omega$. But $\int_\Omega v_p(x)dx \le C$ by Lemma 4.1. Therefore v_p are uniformly bounded in $L^\infty(\omega)$; hence $S' = \emptyset$. □

It is interesting to see when the peak set $\mathcal{B}$ contains one point only.

Theorem 6.2 *Let Ω be a strict convex domain and u_{p_n} be a sequence of generic solutions. If we further assume*

$$\int_{\partial\Omega} \frac{ds}{< x - y, n(x) >^{N-1}} < (2d_N)^N (e\alpha_N)^{N-1} (\frac{N-1}{N^2})^{N-1} (\frac{N-1}{N})^{(N-1)^2}$$

for some $y \in \Omega$, then there exists a subsequence of u_{p_n}, again denoted by u_{p_n}, such that the peak set $\mathcal{B}$ of the subsequence equals the interior peak set S and it contains one point only.

Proof. The assertion $\mathcal{B} = S$ follows from Theorem 6.1. We also know $\#\mathcal{B} \geq 1$ from Theorem 1.2.

Now we state a Pohozaev type identity for (1.1). The proof of this integral identity can be found in ([7], Theorem 1.1). Let $u \in L^\infty(\Omega) \cap W_{1,q}(\Omega)$ solve

$$\begin{cases} -\mathrm{div}(|\nabla u|^{q-2}\nabla u) = g(x,u) \text{ in } \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where g is smooth with its growth bounded by $|u|^{\frac{Nq-N+q}{N-q}}$ if $q < N$ or like a polynomial in u if $q = N$. Let $G(x,u) = \int_0^u g(x,r)dr$. Then

$$\int_\Omega NG(x,u)dx + (1-\frac{N}{q})\int_\Omega ug(x,u)dx + \int_\Omega < x-y, \nabla_x G(x,u) > dx \tag{6.4}$$
$$= (1-\frac{1}{q})\int_{\partial\Omega} < x-y, n(x) > |\frac{\partial u}{\partial n}|^q ds$$

for all $y \in R^N$.

Apply it to (1.1). Let 'y' in the integral identity be 'y' in the statement of Theorem 6.2. Without loss of generality, we can assume $y = 0$. Then

$$\frac{N}{p+1}\int_\Omega u_p^{p+1}dx = (1-\frac{1}{N})\int_\Omega < x, n(x) > |\frac{\partial u_p}{\partial n}|^N ds. \tag{6.5}$$

On the other hand

$$\int_\Omega u_p^p dx = \int_{\partial\Omega} |\frac{\partial u_p}{\partial n}|^{N-1} ds.$$

Hence by the Holder's inequality

$$\int_\Omega u_p^p dx$$

$$\leq (\int_{\partial\Omega} \frac{1}{< x, n(x) >^{N-1}} ds)^{1/N} (\int_{\partial\Omega} < x, n(x) > |\frac{\partial u_p}{\partial n}|^N ds)^{(N-1)/N}$$
$$= (\int_{\partial\Omega} \frac{1}{< x, n(x) >^{N-1}} ds)^{1/N} (\frac{N^2}{(N-1)(p+1)}\int_\Omega u^{p+1} dx)^{(N-1)/N}.$$

Therefore

$$(\frac{L_0}{\alpha_N})^{N-1} = \frac{1}{d_N}(\frac{L_0'}{\alpha_N})^{N-1}$$
$$= \frac{1}{d_N}\overline{\lim_{p\to\infty}}(\frac{p}{e\alpha_N}(\int_\Omega u_p^p dx)^{1/(N-1)})^{N-1}$$

$$
\begin{aligned}
&= \frac{1}{d_N} \overline{\lim_{p\to\infty}} \frac{p^{N-1}\int_\Omega u_p^p dx}{(e\alpha_N)^{N-1}} \\
&\le \overline{\lim_{p\to\infty}} \frac{1}{d_N} \frac{1}{(e\alpha_N)^{N-1}} \Big(\int_{\partial\Omega} \frac{1}{< x, n(x) >^{N-1}} ds\Big)^{1/N} \\
&\quad p^{N-1}\Big(\frac{N^2}{N-1}\Big)^{\frac{N-1}{N}} \Big(\frac{1}{p+1}\frac{1}{p^{N-1}}\Big(\frac{N\alpha_N e}{N-1}\Big)^{N-1}\Big)^{\frac{N-1}{N}} \\
&= \frac{1}{d_N}\frac{1}{(e\alpha_N)^{N-1}}\Big(\frac{N^2}{N-1}\Big)^{\frac{N-1}{N}}\Big(\frac{N\alpha_N e}{N-1}\Big)^{\frac{(N-1)^2}{N}}\Big(\int_{\partial\Omega}\frac{1}{< x, n(x) >^{N-1}} ds\Big)^{1/N} \\
&< 2.
\end{aligned}
$$

Hence from the last inequality in the proof of Theorem 1.2 we have $\#S \le 1$; and in turn $\#S = 1$. □

Remark 6.3 *It turns out that when $N = 2$ the assumptions that Ω is strict convex and that u_p are generic solutions are both superfluous for Theorem 6.1 and Theorem 6.2. In our earlier article [12], we proved the corresponding results of Theorem 6.1 and Theorem 6.2 without these two conditions. In that work we used Kelvin transform to take care of non-convex domains and we applied the moving plane method to u_p directly since the equations (1.1) are non-degenerate when $N = 2$.*

Finally we confine ourselves to the problem when $\Omega = B_R$, the ball of radius R centered at origin. We also consider generic solutions. Applying the moving plane method to each approximate solutions $u_{p\epsilon_n}$, of u_p, we conclude that $u_{p\epsilon_n}$ are all radially symmetric, so are u_p. Therefore u_p solve the following O.D.E.

$$
\begin{cases}
(|u'|^{N-2}u')' + \dfrac{N-1}{r}|u'|^{N-2}u' + u^p = 0 \text{ in } (0, R) \\
u'(0) = 0, \;\; u(R) = 0.
\end{cases}
$$

Applying Theorem 1.2, we know $\mathcal{B} = S = \{0\}$; otherwise there would be infinitely many peaks by the symmetry. A straightforward argument shows

$$
f_p \to \delta
$$

in the sense of distribution where f_p is defined in (5.1) and δ is the Dirac mass at 0. We can actually prove the following. We leave the proof to readers.

Theorem 6.4 *Let u_p be generic variational solutions of (1.1) on B_R, the ball of radius R. Then as $p \to \infty$*

$$
v_p = \frac{u_p}{(\int_{B_R} u_p^p)^{1/(N-1)}} \to \frac{1}{\omega_{N-1}^{1/(N-1)}} \log\Big(\frac{R}{r}\Big)
$$

in $C^{1,\alpha}_{loc}(\overline{B}_R \backslash \{0\})$ for some $\alpha > 0$ and also in the sense of distribution on B_R.

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