Counting Peaks of Solutions to Some Quasilinear Elliptic Equations with Large Exponents

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Abstract

We consider the asymptotic behavior of certain solutions to a quasilinear problem with large exponent in the nonlinearity. Starting with the investigation of a Sobolev embedding, we get a sharp estimate for the embedding constant. Then we obtain a crucial $L^1$-estimate for the $N$-Laplacian operators in $\mathbb{R}^N$. Using these estimates we prove that the solutions obtained by the standard variational method will develop a spiky pattern of peaks as the nonlinear exponent gets large, and we also have a upper bound depending on $N$ only of the number of the peaks. Stronger results for some special convex domains and some special solutions are also achieved.

1 Introduction

In this paper we shall study the asymptotic behavior of certain solutions, as $p \to \infty$, of the quasilinear elliptic equation

$$\begin{cases} 
\Delta_N u + u^p = 0 \text{ in } \Omega \\
u|_{\partial \Omega} = 0, \ u > 0 \text{ in } \Omega
\end{cases}$$

where $p > 1$, $N \geq 2$, $\Delta_N u = \text{div}(|\nabla u|^{N-2}\nabla u)$ is the $N$-Laplacian operator and $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain. We shall only focus on the solutions of
the problem obtained by the following variational method. Let
\[ A_p = \{ v \in W^{1,N}_0(\Omega) : \| v \|_{p+1} = 1 \} \]
be the admissible set and define
\[ J_p : A_p \to \mathbb{R} \]
by
\[ J_p(v) = \int_\Omega |\nabla v|^N. \]
Clearly \( J_p \) is bounded from below. Standard arguments show that \( J_p \) has at least one nonnegative minimizer in \( A_p \). If we denote such a minimizer by \( u'_p \), then a suitable multiple of \( u'_p \), say \( u_p \), solves (1.1) and
\[ c_p(N) := \inf \{ \int_\Omega |\nabla u|^N : u \in A_p \} = \frac{\| \nabla u_p \|_{L^N(\Omega)}}{\| u_p \|_{L^{p+1}(\Omega)}}. \]
A Hopf type boundary lemma, see M. Guedda and L. Veron [7], shows that \( u_p \) is positive in \( \Omega \). It is also known that the solutions of (1.1) are \( C^{1,\alpha} \) functions. We refer to [7], [17] and [16] for the regularity, comparison principle and Hopf boundary lemma for \( N \)-Laplacian operators.

Our goal is to understand the asymptotic behavior of the variational solutions \( u_p \) obtained above when \( p \), serving as a parameter, gets large. The case where \( N = 2 \) is studied in our earlier work [12]. In that article, we proved that \( \| u_p \|_{L^\infty} \) are bounded both from below and above as \( p \) tends to infinity. We also proved that \( u_p \) approach zero except at one or two points. \( u_p \) hence develop a pattern of peaks in \( \Omega \). In this paper we shall show that our method developed there can be successfully extended to higher dimensional cases with \( \Delta \) replaced by \( \Delta_N \). Our first result is

**Theorem 1.1** Let \( u_p \) be a variational solution of (1.1) obtained above. Then there exist positive \( C_1, C_2 \), independent of \( p \), such that
\[ 0 < C_1 < \| u_p \|_{L^\infty} < C_2 < \infty \]
for \( p \) large.

To state the second theorem, let
\[ v_p = \frac{u_p}{(\int_\Omega u_p^{N-1})^{1/(N-1)}}. \]
For a sequence \( \{ v_{p_n} \} \) of \( v_p \) we define the blow-up set \( B \) of \( \{ v_{p_n} \} \) to be the subset of \( \Omega \) such that \( x \in B \) if there exist a subsequence, still denoted by \( v_{p_n} \), and a sequence \( x_n \) in \( \Omega \) with
\[ v_{p_n}(x_n) \to \infty \text{ and } x_n \to x. \]
We also define, with respect to \( \{v_p\} \),

\[
S = B \cap \Omega, \\
S' = B \cap \partial \Omega.
\]  

(1.6)

We use \( \#B(\#S, \#S') \), to denote the cardinality of \( B \) \((S, S' \) respectively). It turns out later that \( B \) \((S, S' \) will be the set of global (interior, boundary) peaks of the subsequence \( v_p \), respectively. We also call them global (interior, boundary) peak sets.

**Theorem 1.2** Let \( N \geq 2 \). Then for any sequence \( \{v_p\} \) of \( v_p \) with \( p_n \to \infty \), the global peak set \( B \) of \( v_p \) is not empty and there exists a subsequence of \( v_p \) such that the interior peak set \( S \) of the subsequence has the property

\[
0 \leq \#S \leq \left[ \frac{1}{d_N \left( \frac{N}{N-1} \right)^{N-1}} \right]
\]

where

\[
d_N = \inf_{X \neq Y \in \mathbb{R}^N} \left[ \frac{|X|^{N-2}X - |Y|^{N-2}Y}(X - Y) \right]
\]

is a positive number depending on \( N \) only.

From the above results, we see that the variational solutions develop a spiky pattern as \( p \) approaches infinity and the number of peaks is controlled in Theorem 1.2. If we impose more condition on the domain as well as solutions, we can prove that they develop one single peak in the interior of the domain. We would like to mention that single-peak spiky patterns also appear in the works of W.-M. Ni and I. Takagi [9] [10], W.-M. Ni, X. Pan and I. Takagi [8] and X. Pan [11] where some biological pattern formation problems are considered.

Our paper is organized as follows. In section 2, we prove a crucial sharp estimate for \( c_p(N) \) defined in (1.3). Theorem 1.1 will be proved in section 3. In section 4 We extend an estimate of H. Brezis and F. Merle [1] to the N-Laplacian cases using the level set method. Theorem 1.2 will then be proved in section 5. Stronger conclusions for some special convex domains and some special variational solutions \( u_p \) are obtained in section 6; namely, \( \#S = 1 \) and \( S' = \emptyset \).

## 2 An Estimate for \( c_p(N) \)

Recall \( c_p(N) \) defined in (1.3). we first prove

**Lemma 2.1** For every \( t \geq 2 \) there is \( D_t \) such that

\[
\|u\|_{L^t} \leq D_t t^{(N-1)/N} \|\nabla u\|_{L^N}
\]
for all \( u \in W^{1,N}_0(\Omega) \) where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \), furthermore

\[
\lim_{t \to \infty} D_t = (\alpha_N)^{-(N-1)/N} \left( \frac{N-1}{Ne} \right)^{(N-1)/N}
\]

where \( \alpha_N = N\omega_{N-1}^{1/(N-1)} \) and \( \omega_{N-1} \) is the area of unit \( N-1 \) sphere in \( \mathbb{R}^N \).

Proof. Let \( u \in W^{1,N}_0(\Omega) \). We know

\[
\frac{1}{\Gamma(s+1)} x^s \leq e^x
\]

for all \( x \geq 0, \ s \geq 0 \) where \( \Gamma \) is the \( \Gamma \) function. From Moser's sharp form of the Trudinger's Inequality (see [5] page 160 and [6]), we have

\[
\int_{\Omega} \exp [\alpha_N \left( \frac{u}{\|\nabla u\|_{L^N}} \right)^{N/(N-1)}] dx \leq C|\Omega|
\]

where \( \alpha_N \) is defined in Lemma 2.1, \( C \) depends on \( N \) only and \( |\Omega| \) is the Lebesgue measure of \( \Omega \). Therefore

\[
\frac{1}{\Gamma\left(\frac{N-1}{N}t+1\right)} \int_{\Omega} u' dx = \frac{1}{\Gamma\left(\frac{N-1}{N}t+1\right)} \int_{\Omega} \left[ \alpha_N \left( \frac{u}{\|\nabla u\|_{L^N}} \right)^{N/(N-1)} \right]^{N/(N-1)} t \, dx \left( \alpha_N \right)^{-\frac{N-1}{N}t} \|\nabla u\|_{L^N}^t
\]

\[
\leq \int_{\Omega} \exp [\alpha_N \left( \frac{u}{\|\nabla u\|_{L^N}} \right)^{N/(N-1)}] dx \left( \alpha_N \right)^{-\frac{N-1}{N}t} \|\nabla u\|_{L^N}^t
\]

\[
\leq C|\Omega| \left( \alpha_N \right)^{-\frac{N-1}{N}t} \|\nabla u\|_{L^N}^t.
\]

Hence

\[
\left( \int_{\Omega} u' dx \right)^{1/t} \leq \left( (\Gamma\left(\frac{N-1}{N}t+1\right))^{1/t} (C|\Omega|)^{1/t} \alpha_N^{-\frac{N-2}{N}} \|\nabla u\|_{L^N(\Omega)} \right).
\]

Notice according to Stirling’s formula

\[
(\Gamma\left(\frac{N-1}{N}t+1\right))^{1/t} \sim \left( \frac{N-1}{Ne} \right)^{(N-1)/N} t^{(N-1)/N}.
\]

Choosing \( D_t \) to be

\[
(\Gamma\left(\frac{N-1}{N}t+1\right))^{1/t} (C|\Omega|)^{1/t} \alpha_N^{-\frac{N-1}{N}t} \|\nabla u\|_{L^N(\Omega)}
\]

we get the desired result. \( \square \)

We then prove a sharp estimate for \( c_p(N) \).
Lemma 2.2

\[
\lim_{p \to \infty} \frac{c_p(N)}{p^{-\frac{N+1}{N}}(N)} = \left( \frac{N}{N-1} \alpha_N \right)^{(N-1)/N}.
\]

Proof. Without loss of generality, we assume 0 ∈ Ω. Let \( L > 0 \) be such that \( B_L \subset \Omega \) where \( B_L \) is the ball of radius \( L \) centered at origin. For 0 < \( l < L \) consider the so called Moser’s function

\[
m_l(x) = \begin{cases} 
\frac{\log \left( \frac{L}{l} \right)}{N}, & 0 \leq |x| \leq l \\
\frac{\log \left( \frac{L}{l} \right)}{N}, & l \leq |x| \leq L \\
0, & |x| \geq L.
\end{cases}
\]

Then \( m_l \in W^{1,N}_0(\Omega) \) and \( \|\nabla u\|_{L^N} = 1 \). Now

\[
\left( \int_{\Omega} m_l^{p+1}(x) dx \right)^{1/(p+1)} \geq \left( \int_{B_l} m_l^{p+1}(x) dx \right)^{1/(p+1)} = \frac{1}{\omega_{N-1}^{1/N}} \left( \log \frac{L}{l} \right)^{(N-1)/N}(\frac{1}{N} l^{-1} \omega_{N-1}^{1/N})^{1/(p+1)}.
\]

Choosing \( l = L \exp\left( -\frac{N-1}{N} (p + 1) \right) \), we have

\[
\|m_l\|_{p+1} \geq \frac{1}{\omega_{N-1}^{1/N}} \exp\left( -\frac{N-1}{N} \right) \left( \frac{N-1}{N^2} \right)^{(N-1)/N} (N+1)^{-1} \left( \frac{1}{N} \omega_{N-1}^{1/N} L^N \right)^{1/(p+1)}.
\]

Therefore

\[
c_p(N) \leq \omega_{N-1}^{1/N} \exp\left( -\frac{N-1}{N} \right) \left( \frac{N-1}{N^2} \right)^{(N-1)/N} (N+1)^{-1} \left( \frac{1}{N} \omega_{N-1}^{1/N} L^N \right)^{-1/(p+1)}.
\]

Combining this with Lemma 2.1, we get the conclusion. □

By the construction of the variational solutions \( u_p \) in section 1, we have

\[
c_p(N) = \frac{\|\nabla u_p\|_{L^N(\Omega)}}{\|u_p\|_{L^{p+1}(\Omega)}}.
\]

If we multiply equation (1.1) by \( u_p \) and integrate both sides on \( \Omega \), we have

\[
\int_{\Omega} |\nabla u_p|^N = \int_{\Omega} u_p^{p+1}.
\]

Hence we derive from Lemma 2.2
Corollary 2.3

\[
\lim_{p \to \infty} p^{N-1} \int_{\Omega} u_{p}^{p+1} = \left( \frac{N \alpha_N e}{N-1} \right)^{N-1},
\]

\[
\lim_{p \to \infty} p^{N-1} \int_{\Omega} |\nabla u_{p}|^{N} = \left( \frac{N \alpha_N e}{N-1} \right)^{N-1}.
\]

Define

\[
\nu_p = \left[ \int_{\Omega} u_{p}^{p+1} \right]^{1/p},
\]

\[
L'_0 = \lim_{p \to \infty} p\nu_p e,
\]

\[
L_0 = L'_0 d_N^{-1/(N-1)}
\]

where \(d_N\) is defined in Theorem 1.2. We have the following rough estimates for \(L_0\) and \(L'_0\).

Corollary 2.4 For any smooth bounded domain \(\Omega\) in \(\mathbb{R}^N\)

\[
L'_0 \leq \frac{N}{N-1} \alpha_N, \quad L_0 \leq \frac{N}{N-1} \alpha_N d_N^{-1/(N-1)}.
\]

Proof. From Corollary 2.3 we have by Holder’s inequality

\[
L'_0 = \lim_{p \to \infty} \frac{p\nu_p}{e} \leq \lim_{p \to \infty} p \left[ \int_{\Omega} u_{p}^{p+1} \right]^{1/(p+1)} \frac{\|u_{p}\|_{L^\infty}}{\Omega} \leq \frac{N \alpha_N}{N-1}.
\]

\(\square\)

3 Proof of Theorem 1.1

To get a lower bound for \(\|u_p\|_{L^\infty}\), we define

\[
\lambda = \inf \{ \|\nabla u\|_{L^\infty} : u \in W^{1,N}_0(\Omega), \ u \neq 0 \}.
\]

From Poincaré’s Inequality, we have \(0 < \lambda < \infty\). For \(u_p\) we have

\[
\int_{\Omega} u_{p}^{p+1} = \int_{\Omega} |\nabla u_{p}|^{N} \geq \lambda^N \int_{\Omega} u_{p}^{N},
\]

\[
\int_{\Omega} (u_{p}^{p+1} - \lambda^N u_{p}^{N}) \geq 0.
\]

Therefore

\[
\|u_p\|_{L^\infty}^{p+1-N} \geq \lambda^N.
\]
Letting $p >> N - 1$, we obtain
\[ \|u_p\|_{L^\infty} \geq \lambda^{N/(p+1-N)} \geq C_1 > 0. \]

To get a upper bound for $\|u_p\|_{L^\infty}$, let
\[ \gamma_p = \max_{x \in \Omega} u_p(x), \]
\[ A = \{ x : u_p(x) > \gamma_p/2 \}, \]
\[ \Omega_t = \{ x : u_p(x) > t \}. \]  

Both $A$ and $\Omega_t$ depend on $p$. From Lemma 2.1 and Corollary 2.3, we have
\[ \|u_p\|_{L^{\frac{Np}{N-1}}} \leq D \frac{Np}{N-1} \|\nabla u\|_{L^N} \]
\[ \leq C \left( \frac{Np}{N-1} \right)^{(N-1)/N} p^{-(N-1)/N} < M \]
where $M$ is a constant independent of $p$. Then
\[ \left( \frac{\gamma_p}{2} \right)^{\frac{N-1}{N}} |A| \leq \int_{\Omega} u_p^{\frac{Np}{N-1}} \leq M^{\frac{Np}{N-1}}. \]  

On the other hand
\[ \int_{\Omega} u_p^p = -\int_{\Omega} \text{div}(|\nabla u_p|^{N-2} \nabla u_p) = \int_{\partial \Omega} |\nabla u_p|^{N-1} ds \]
and
\[ -\frac{d}{dt}|\Omega_t| = \int_{\partial \Omega_t} \frac{ds}{|\nabla u_p|} \]
where the second is the co-area formula (see Federer [3]). By the Schwartz inequality and the isoperimetric inequality we have
\[ (-\frac{d}{dt}|\Omega_t|)^{N-1} \int_{\Omega_t} u_p^p \]
\[ = \left( \int_{\partial \Omega_t} \frac{ds}{|\nabla u_p|} \right)^{N-1} \left( \int_{\partial \Omega_t} |\nabla u_p|^{N-1} ds \right) \]
\[ \geq \left( \int_{\partial \Omega_t} \frac{ds}{|\nabla u_p|} \right)^{N-1} \left( \int_{\partial \Omega_t} |\nabla u_p|^{N-1} |\partial \Omega_t|^{-N+2} \right) \]
\[ \geq |\partial \Omega_t|^{2(N-1)} |\partial \Omega_t|^{-N+2} = |\partial \Omega_t|^N \geq C_N |\Omega_t|^{N-1} \]
where $|\partial \Omega_t|$ denotes the $(N - 1)$-dimensional Hausdorff measure of $\partial \Omega_t$ and $C_N$ is the best constant in the isoperimetric inequality (we refer to [3] for more
information about the Hausdorff measures and the isoperimetric inequality). Now we define $r(t)$ for $0 \leq t \leq \gamma_p$ such that
\[
|\Omega_t| = \frac{1}{N} \omega_{N-1} r^N(t);
\]
then
\[
\frac{d}{dt}|\Omega_t| = \omega_{N-1} r^{N-1}(t) \frac{dr}{dt}.
\]
Hence we have
\[
(-\omega_{N-1} r^{N-1}(t) \frac{dr}{dt})^{N-1} \int_{\Omega_t} u_p(x)dx \geq C_N \left( \frac{1}{N} \omega_{N-1} r^N(t) \right)^{N-1};
\]
\[
(-\frac{dr}{dt})^{N-1} \int_{\Omega_t} u_p(x)dx \geq C_N r^{N-1};
\]
\[
-\frac{dt}{dr} \leq C_N \frac{1}{r} \left( \int_{\Omega_t} u_p(x)dx \right)^{1/(N-1)}
\]
\[
\leq C_N \frac{1}{r} \gamma_p^{p/(N-1)} |\Omega_t|^{1/(N-1)} = C_N \gamma_p^{p/(N-1)} |\Omega_t|^{1/(N-1)}.
\]
Integrating the inequality from 0 to $r_0$, we have
\[
t(0) - t(r_0) \leq C_N \gamma_p^{p/(N-1)} r_0^{N/(N-1)}.
\]
Choosing $r_0$ so that $t(r_0) = \frac{2}{\gamma_p}$, we get
\[
\gamma_p \leq C_N \gamma_p^{p/(N-1)} r_0^{N/(N-1)};
\]
\[
\gamma_p \leq C_N \gamma_p^{p/(N-1)} |\Omega|^{1/(N-1)}.
\]
Combining this with (3.2), we obtain
\[
\gamma_p \leq C_N \gamma_p^{p/(N-1)} \left( \frac{2M}{\gamma_p^{Np}} \right)^{1/(N-1)};
\]
\[
\gamma_p \leq C' \gamma_p^{p/(N-1)} \left( \frac{2M}{\gamma_p^{Np}} \right)^{1/(N-1)} \leq C'
\]
for $p$ large enough where the last $C'$ is a constant independent of large $p$. This proves Theorem 1.1.

We derive a consequence of Theorem 1.1 which will be used later.

**Corollary 3.1** There exist $C_1$ and $C_2$ independent of $p$ such that
\[
\frac{C_1}{p^{N-1}} \leq \int_{\Omega} u_p(x)dx \leq \frac{C_2}{p^{N-1}}
\]
for large $p$.

**Proof.** The first inequality follows from Theorem 1.1 and the first limit of Corollary 2.3; the second inequality follows from the first limit of Corollary 2.3 by an interpolation. $\square$
4 A Priori Estimates for $N$-Laplacian Operators

In this section we extend the $L^1$ estimate of H. Brezis and F. Merle [1] to $N$-Laplacian operators. Due to the nonlinearity of $N$-Laplacian operators for $N \geq 3$, we use the level set argument here.

**Lemma 4.1** Let $u$ be a $C^{1,\alpha}$ solution of
\[
\begin{align*}
-\Delta_N u &= f(x) \text{ in } \Omega \\
u|_{\partial\Omega} &= 0
\end{align*}
\]
where $f \in L^1(\Omega)$, $f \geq 0$. Then for every $\delta \in (0, N \omega^{1/(N-1)} = (0, \alpha_N)$ we have
\[
\int_{\Omega} \exp\left[\frac{(\alpha_N - \delta) |u(x)|}{\|f\|_{L^1(\Omega)}}\right] dx \leq \frac{\alpha_N}{\delta} |\Omega|
\]
where $|\Omega|$ denotes the volume of $\Omega$.

**Proof.** We prove this by the symmetrization method. Consider the symmetrized problem
\[
\begin{align*}
-\text{div}(|\nabla U|^{N-2} \nabla U) &= F(x) \text{ in } \Omega^* \\
U|_{\partial\Omega^*} &= 0
\end{align*}
\]
where $\Omega^*$ is the ball centered at origin with the same volume as $\Omega$ and $F$ is the symmetric decreasing rearrangement of $f$. We refer to G. Talenti [14] and [15] for properties of the rearrangement. According to [15], we have
\[
u^* \leq U
\]
where $u^*$ is the symmetric decreasing rearrangement of $u$. $U$ clearly satisfies the following O.D.E.
\[
\begin{align*}
\left(\frac{N}{r} U^{'N-2} U^{'r} + \frac{N-1}{r} \right) U^{'N-2} U^{'r} + F(r) &= 0 \\
U'(0) &= 0, \quad U(R) = 0.
\end{align*}
\]
Therefore
\[
-U'(r) = \left(\int_0^r s^{N-1} F(s) ds\right)^{1/(N-1)} \leq \frac{1}{\omega_N^{1/(N-1)}} \frac{1}{r} \frac{\|F\|_{L^1(\Omega^*)}}{r}.
\]
Hence
\[
|U(r)| \leq \frac{1}{\omega_N^{1/(N-1)}} \frac{\|F\|_{L^1(\Omega^*)}}{r} \log \frac{R}{r},
\]
\[
\int_{\Omega^*} \exp\left[(N - \epsilon) \omega_N^{1/(N-1)} \frac{U}{\|F\|_{L^1(\Omega^*)}}\right] dx \leq \int_{B(R)} \exp \log \left(\frac{R}{|x|}\right)^{N-\epsilon} dx
\]
\[ \omega_{N-1} \int_0^R \left( \frac{R}{r} \right)^{N-1} r^{N-1} dr = \epsilon^{-1} \omega_{N-1} R^N. \]

Letting \( \epsilon \omega_{N-1}^{1/(N-1)} = \delta \), we have

\[ \int_{\Omega^*} \exp[(\alpha_N - \delta) \frac{U(r)}{\|F\|_{L^1(\Omega)}^{1/(N-1)}}] \leq \frac{\omega_{N-1}^{N/(N-1)}}{\delta} R^N. \]

According to the properties of the symmetric decreasing function, we have

\[ \|F\|_{L^1(\Omega^*)} = \|f\|_{L^1(\Omega)}, \]

\[ \int_\Omega \exp[(\alpha_N - \delta) \frac{u(x)}{\|f\|_{L^1(\Omega)}}] dx = \int_{\Omega^*} \exp[(\alpha_N - \delta) \frac{u^*(x)}{\|f\|_{L^1(\Omega)}^{1/(N-1)}}] \]

\[ \leq \int_{\Omega^*} \exp[(\alpha_N - \delta) \frac{U(r)}{\|F\|_{L^1(\Omega)}^{1/(N-1)}}] \leq \frac{\omega_{N-1}^{N/(N-1)}}{\delta} R^N \]

\[ = \frac{\alpha_N}{\delta} |\Omega|. \]

\[ \square \]

An interesting consequence is

**Corollary 4.2** Let \( u_n \) be a sequence of \( C^{1,\alpha} \) solutions of

\[ \begin{cases} 
\Delta_N u_n + V_n e^{u_n} = 0 \text{ in } \Omega \\
u_n|\partial\Omega = 0
\end{cases} \]

such that

\[ \|V_n\|_{L^q} \leq C_1; \]

\[ \int_\Omega |V_n| e^{u_n} \leq \epsilon_0 < \frac{\alpha_N}{q'} \]

for some \( 1 < q < \infty \) and \( q' = \frac{q}{q-1} \). Then

\[ \|u_n\|_{L^\infty(\Omega)} \leq C \]

where \( C \) depends on \( N, C_1, |\Omega|, \epsilon_0 \) only.

**Proof.** Fix \( \delta > 0 \) so that \( \alpha_N - \delta > \epsilon_0 (q' + \delta) \). By Lemma 4.1 we have

\[ \int_\Omega \exp[(q' + \delta)|u_n|] \leq C \]

for some \( C \) independent of \( n \). Therefore \( e^{u_n} \) is bounded in \( L^{q'+\delta}(\Omega) \); hence \( V_n e^{u_n} \) is bounded in \( L^{1+\epsilon_0}(\Omega) \). Then the standard Moser iteration method implies that \( u_n \) is bounded in \( L^\infty(\Omega) \). \( \square \)

Next we give a version of Lemma 4.1 without homogeneous boundary condition.
Lemma 4.3 Let $u$ and $\varphi$ be $C^{1,\alpha}(\Omega)$ solutions of
\[ \Delta_N u + f(x) = 0 \text{ in } \Omega, \ f > 0 \]
and
\[ \begin{cases} \Delta_N \varphi = 0 \text{ in } \Omega \\ \varphi|_{\partial \Omega} = u \end{cases} \]
respectively. Then there exists a constant $C$ depending on $\Omega$ only such that
\[ \int_{\Omega} \exp\left[ \frac{(\alpha_N - \delta)d_N^{1/(N-1)}}{\|f\|_{L^1(\Omega)}}(u - \varphi) \right] \leq \frac{C}{\delta} \]
where $d_N$ is defined in Theorem 1.2.

Proof. Let $u_\epsilon$ and $\varphi_\epsilon$ be solutions of the non-degenerate equations
\[ \begin{cases} -\text{div}(\epsilon + |\nabla u_\epsilon|^2)^{(N-2)/2}\nabla u_\epsilon) = f \text{ in } \Omega, \ f > 0 \\ u_\epsilon|_{\partial \Omega} = u \end{cases} \]
and
\[ \begin{cases} -\text{div}(\epsilon + |\nabla \varphi_\epsilon|^2)^{(N-2)/2}\nabla \varphi_\epsilon) = 0 \text{ in } \Omega \\ \varphi_\epsilon|_{\partial \Omega} = u \end{cases} \]
respectively. (These solutions are smooth and obtained easily by the variational method. Furthermore
\[ \lim_{\epsilon \to 0} u_\epsilon = u, \]
\[ \lim_{\epsilon \to 0} \varphi_\epsilon = \varphi \]
in $C^{1,\beta}$ for some $\beta$. See [16].) Let $\Omega_t = \{x \in \Omega: u_\epsilon - \varphi_\epsilon > t\}$.

Claim:
\[ \frac{\partial u_\epsilon(x)}{\partial \nu} < \frac{\partial \varphi_\epsilon(x)}{\partial \nu} \]
on $\partial \Omega_t$ for almost all $t \geq 0$.

Let $x_0 \in \partial \Omega_t$. For almost all $t > 0$ we can find a ball $B_\delta(x_1) \subset \Omega_t$ with
$B_\delta(x_1) \cap \Omega_t = x_0$ by Sard’s theorem. Let $w = u_\epsilon - \varphi_\epsilon - t$. Then $w$ verifies
\[ -\sum_{i,j} \frac{\partial}{\partial x_i}(a_{ij}(\epsilon + |\nabla u_\epsilon|^2)^{(N-4)/2}) \frac{\partial w}{\partial x_j} = f > 0 \]
where
\[ a_{ij} = (\epsilon + |t_i \nabla u_\epsilon + (1 - t_i) \nabla \varphi_\epsilon|^2)^{(N-4)/2} \delta_{ij}(\epsilon + |t_i \nabla u_\epsilon + (1 - t_i) \nabla \varphi_\epsilon|^2) \]
\[ + (N - 2)(t_i \frac{\partial u}{\partial x_i} + (1 - t_i) \frac{\partial \varphi}{\partial x_i}) (t_j \frac{\partial u}{\partial x_j} + (1 - t_j) \frac{\partial \varphi}{\partial x_j}) \]
and \( t_i \in (0, 1) \). Because this equation is non-degenerate, we can apply Hopf’s lemma. Therefore
\[
\frac{\partial w}{\partial \nu} < 0;
\]
hence we prove the claim.

Following the standard level set argument, we have
\[
\int_{\Omega_t} f(x) = -\int_{\Omega_t} \text{div}((\epsilon + |\nabla u|)^{\frac{N-2}{2}} \nabla u) + \int_{\Omega_t} \text{div}((\epsilon + |\nabla \varphi|)^{\frac{N-2}{2}} \nabla \varphi)
\]
\[
= \int_{\partial \Omega_t} ((\epsilon + |\nabla u|^2)^{\frac{N-2}{2}} \nabla u \cdot (\nabla u - \nabla \varphi)) \frac{(\nabla u - \nabla \varphi) \cdot (\nabla u - \nabla \varphi)}{|\nabla u - \nabla \varphi|^2}
\]
\[
\geq d_N^* \int_{\partial \Omega_t} |\nabla u - \nabla \varphi|^{N-1}
\]
where
\[
d_N^* = \inf_{X \neq Y \in \mathbb{R}^N} \frac{(\epsilon + |X|^2)^{\frac{N-2}{2}} X - (\epsilon + |Y|^2)^{\frac{N-2}{2}} Y}{|X - Y|^N}
\]
is a positive number,
\[
\lim_{\epsilon \to 0} d_N^* = d_N
\]
and \( d_N \) is defined in Theorem 1.2. Also by the co-area formula we have
\[
-\frac{d}{dt} |\Omega_t| = \int_{\partial \Omega_t} \frac{ds}{|\nabla u - \nabla \varphi|}.
\]
Hence by the Schwartz inequality and the isoperimetric inequality,
\[
\left( -\frac{d}{dt} |\Omega_t| \right)^{N-1} \int_{\Omega_t} f(x) \geq \left( \int_{\partial \Omega_t} \frac{ds}{|\nabla u - \nabla \varphi|} \right)^{N-1} d_N^* \left( \int_{\partial \Omega_t} |\nabla u - \nabla \varphi|^{N-1} \right)
\]
\[
\geq \left( \int_{\partial \Omega_t} \frac{ds}{|\nabla u - \nabla \varphi|} \right)^{N-1} d_N^* \left( \int_{\partial \Omega_t} |\nabla u - \nabla \varphi|^{N-1} |\partial \Omega_t|^{-\frac{N-2}{N-1}} \right)
\]
\[
\geq d_N^* |\partial \Omega_t|^{2(N-1)} |\partial \Omega_t|^{-\frac{N-2}{N-1}} = d_N^* |\partial \Omega_t|^N
\]
\[
\geq d_N^* \omega_{N-1} N^{N-1} |\Omega_t|^{N-1} = d_N^* \omega_{N-1} |\Omega_t|^{N-1}.
\]
Define \( r(t) \) so that
\[
|\Omega_t| = \frac{1}{N} \omega_{N-1} r^{N-1}(t);
\]
then
\[
\frac{d |\Omega_t|}{dt} = \frac{1}{N} \omega_{N-1} N r^{N-1}(t) \frac{dr}{dt} = \omega_{N-1} r^{N-1}(t) \frac{dr}{dt}.
\]
Hence we have from above
\[
(-\omega_{N-1} r^{N-1}) \left( \frac{dr}{dt} \right)^{N-1} \int_{\Omega_t} f(x) dx \geq d_N^N \omega_{N-1} \left( \frac{1}{N} \omega_{N-1} r^N (t) \right)^{N-1};
\]
\[
(-\frac{dr}{dt})^{N-1} \int_{\Omega_t} f(x) dx \geq d_N^N \omega_{N-1} r^{N-1};
\]
\[
(-\frac{dt}{dr})^{N-1} \leq \frac{1}{d_N^N \omega_{N-1}} \frac{1}{r} \int_{\Omega_t} f(x) dx \leq \frac{1}{d_N^N \omega_{N-1}} \frac{1}{r^{N-1}} \lVert f \rVert_{L^1(\Omega)};
\]
\[
-\frac{dt}{dr} \leq \frac{1}{(d_N^N)^{1/(N-1)} \omega_{N-1}^{1/(N-1)}} \frac{1}{\lVert f \rVert_{L^1(\Omega)}}^{1/(N-1)} \frac{1}{r}.
\]
Integrating the last inequality over \((r, R)\) (Note \(|\Omega| = \frac{1}{N} \omega_{N-1} R^N\), we have
\[
t(r) \leq \frac{1}{(d_N^N)^{1/(N-1)} \omega_{N-1}^{1/(N-1)}} \lVert f \rVert_{L^1(\Omega)} \log \frac{R}{r};
\]
\[
\exp \left( \frac{(d_N^N)^{1/(N-1)} \omega_{N-1}^{1/(N-1)} (N - \epsilon_0)}{\lVert f \rVert_{L^1(\Omega)}} \right) t(r) \leq \left( \frac{R}{r} \right)^{N - \epsilon_0};
\]
\[
\int_0^R \exp \left( \frac{(d_N^N)^{1/(N-1)} \omega_{N-1}^{1/(N-1)} (N - \epsilon_0)}{\lVert f \rVert_{L^1(\Omega)}} \right) t(r) \langle r \rangle^{N-1} dr \leq \int_0^R \left( \frac{R}{r} \right)^{N - \epsilon_0} r^{N-1} dr = C \epsilon_0.
\]
However, the left hand side of the last inequality,
\[
\int_0^R \exp \left( \frac{(d_N^N)^{1/(N-1)} \omega_{N-1}^{1/(N-1)} (N - \epsilon_0)}{\lVert f \rVert_{L^1(\Omega)}} \right) t(r) \langle r \rangle^{N-1} dr
\]
\[
= \int_0^\infty \exp \left( \frac{(d_N^N)^{1/(N-1)} \omega_{N-1}^{1/(N-1)} (N - \epsilon_0)}{\lVert f \rVert_{L^1(\Omega)}} \right) \frac{1}{\omega_{N-1}} d\omega_t
\]
\[
= \frac{1}{\omega_{N-1}} \int_\Omega \exp \left( \frac{(d_N^N)^{1/(N-1)} \omega_{N-1}^{1/(N-1)} (N - \epsilon_0)}{\lVert f \rVert_{L^1(\Omega)}} \right) (u_\epsilon - \varphi_\epsilon) dx.
\]
Letting \(\delta = \omega_{N-1}^{1/(N-1)} \epsilon_0\), we have the desired estimate for \(u_\epsilon\) and \(\varphi_\epsilon\). Finally letting \(\epsilon \to 0\), we get the estimate for \(u\) and \(\varphi\) themselves. \(\square\)

In order to have a local analogy of Corollary 4.2, we state a result in J. Serrin [13] which can be proved following the Moser’s iteration scheme.
Proposition 4.4 Let \( u \) be a weak solution of
\[
\Delta_N u + f(x) = 0
\]
in \( B_{2R} \subset \Omega \) and \( f \in L^{N/(N-\epsilon)}(B_{2R}) \). Then we have
\[
\|u\|_{L^\infty(B_R)} \leq CR^{-1}(\|u\|_{L^N(B_{2R})} + KR)
\]
where
\[
K = (R^\epsilon \|f\|_{L^{N/(N-\epsilon)}(B_{2R})})^{1/(N-1)}
\]
and \( C \) depends on \( N \) only.

Corollary 4.5 Let \( \Delta_N u_n + V_ne^{u_n} = 0 \) in \( \Omega \)
and
\[
\|u_n\|_{L^N(\Omega)} \leq C_1, \quad \|V_n\|_{L^q(B_R)} \leq C_2.
\]
where \( 1 < q < \infty \) and \( B_R \) is a ball compactly contained in \( \Omega \). Assuming
\[
\int_{B_R} V_ne^{u_n} \leq \epsilon_0 < \frac{\alpha_N d_N^{1/(N-1)}}{q'}
\]
where \( q' = q/(q-1) \), we have
\[
\|u_n\|_{L^\infty(B_{R/4})} \leq C
\]
for some \( C \) depending on \( N, C_1, C_2, R \) and \( \epsilon_0 \) only.

Proof. Consider on \( B_R \)
\[
\begin{cases}
\Delta_N \varphi_n = 0 \text{ in } B_R \\
\varphi_n|_{\partial B_R} = u_n|_{\partial B_R}.
\end{cases}
\]
By the comparison principle in [7], we have
\[
\varphi_n \leq u_n; \quad \|\varphi_n\|_{L^N(B_R)} \leq C_1.
\]
Using Proposition 4.4, we conclude
\[
\|\varphi_n\|_{L^\infty(B_{R/2})} \leq C \tag{4.1}
\]
for some constant \( C \) depending on \( N, C_1, C_2 \) and \( R \) only. From Lemma 4.3 we also know
\[
\int_{B_R} \exp\left[\frac{(\alpha_N - \delta)d_N^{1/(N-1)}}{\epsilon_0}(u_n - \varphi_n)\right] \leq \int_{B_R} \exp\left[\frac{(\alpha_N - \delta)d_N^{1/(N-1)}}{\|V_n e^{u_n}\|_{L^q(B_R)}}(u_n - \varphi_n)\right] \leq \frac{C}{\delta}.
\]
Combining this with (4.1), we obtain
\[
\int_{B_{R/2}} \exp\left[ (\alpha_N - \delta) d_N^{1/(N-1)} \right] \frac{u_n}{\epsilon_0} \leq \frac{C}{\delta}.
\] (4.2)

Choosing \( \delta \) small enough so that
\[(\alpha_N - \delta) d_N^{1/(N-1)} > \epsilon_0(q' + \delta),\]
we get from (4.2)
\[\| \exp u_n \|_{L^{q'+1}(B_{R/2})} \leq C.\]

Therefore
\[\| V_n \exp u_n \|_{L^{1+\epsilon_1}(B_{R/2})} \leq C\]
for some \( \epsilon_1 > 0 \). Using Proposition 4.4 again, we finally conclude
\[\| u_n \|_{L^\infty(B_{R/4})} \leq C.\]

\[\Box\]

We close this section with a positive lower bound for \( d_N \).

**Proposition 4.6** Let
\[d_N = \inf_{X \neq Y \in \mathbb{R}^N} \frac{(|X|^{N-2}X - |Y|^{N-2}Y)(X - Y)}{|X - Y|^N},\]
Then
\[d_N \geq 2 \left( \frac{1}{N} \right)^{N-2},\]
in particular \( d_2 = 1 \).

**Proof.** Without loss of generality, let \( 0 \leq |Y| \leq |X|, X \neq Y \) and \( X \neq 0 \). Let
\[t = \frac{|Y|}{|X|}, \quad \cos \theta = \frac{<X, Y>}{|X||Y|}.\]

Then
\[\frac{(|X|^{N-2}X - |Y|^{N-2}Y)(X - Y)}{|X - Y|^N} = \frac{1 - (t^{N-1} + t) \cos \theta + t^N}{(1 - 2t \cos \theta + t^2)^{N/2}}.\]

Let
\[f(t, x) = \frac{1 - (t^{N-1} + t)x + t^N}{(1 - 2tx + t^2)^{N/2}}\]
for \( 0 \leq t \leq 1 \) and \(-1 \leq x \leq 1\). Fix \( t \) and set
\[\frac{\partial f}{\partial x} = 0.\]
Then
\[ 1 - (t^{N-1} + t)x + t^N = \frac{t^{N-2} + 1}{N}(1 - 2tx + t^2). \]

Therefore at the critical points \( x \) of \( f(t, \cdot) \),
\[
f(t, x) = \frac{t^{N-2} + 1}{N} \frac{1}{(1 - 2tx + t^2)^{(N-2)/2}} \geq \frac{1}{N} \frac{t^{N-2} + 1}{(t + 1)^{N-2}}.
\]

Let
\[
g(t) = \frac{t^{N-2} + 1}{(t + 1)^{N-2}}.
\]

Then
\[
g'(t) = \frac{(t + 1)^{N-3}}{(t + 1)^{2(N-2)}(N - 2)(t^{N-3} - 1)} \leq 0,
\]
and
\[
\min_{0 \leq t \leq 1} g(t) = g(1) = \frac{2}{2^{N-2}}.
\]

Hence
\[
d_N \geq \frac{2}{N} \left( \frac{1}{2} \right)^{N-2}.
\]

\[ \square \]

**Remark 4.7** An upper bound for \( \#S \) in Theorem 1.2 can therefore be
\[
\frac{N}{4} \left( \frac{2N}{N-1} \right)^{N-1},
\]
which equals 2 when \( N = 2 \).

## 5 Proof of Theorem 1.2

Recall (1.4) and (2.1)
\[
v_p = \frac{u_p}{(\int_{\Omega} u_p^p)^{1/(N-1)}}, \quad \nu_p = \left[ \int_{\Omega} u_p^{p/(N-1)} \right].
\]

Define
\[
f_p = \frac{u_p}{\int_{\Omega} u_p^p} = \nu_p^{p/(N-1)} v_p.
\]

(5.1)
Then we have
\[ \Delta_N v_p + f_p = 0. \] (5.2)

We first prove $\mathcal{B} \neq \emptyset$ for any sequence $\{v_n\} = \{v_{p_n}\}$ of $v_p$ with $p_n \to \infty$. Let $x_n$ be such that
\[ v_n(x_n) = \max_{x \in \Omega} v_n(x) = \frac{\max_{x \in \Omega} u_n(x)}{(\int_{\Omega} u_n^{p_n})^{1/(N-1)}} \geq \frac{C_1}{(\int_{\Omega} u_n^{p_n})^{1/(N-1)}} \to \infty \]
by Theorem 1.1 and Corollary 3.1. Therefore cluster points of $\{x_n\}$ belong to $\mathcal{B}$; hence $\mathcal{B} \neq \emptyset$.

Since $\int_{\Omega} f = 1$ and $f > 0$, for any sequence of $\{f_p\}$ we can subtract a subsequence $\{f_n\} = \{f_{p_n}\}$ which converges to a measure $\mu$ weakly in $M(\Omega)$ where $M(\Omega)$ is the space of real bounded measures on $\Omega$ and $\mu$ is a positive measure with $\mu(\Omega) \leq 1$. From now on in the rest of this section we shall work on this subsequence $\{f_n\}$ and the corresponding $\{v_n\} = \{v_{p_n}\}$. For any $\delta > 0$, we call $x_0 \in \Omega$ a $\delta$-regular point if there is a function $\varphi \in C_0(\Omega)$, $0 \leq \varphi \leq 1$ with $\varphi = 1$ in a neighborhood of $x_0$, such that
\[ \int_{\Omega} \varphi d\mu < \left( \frac{\alpha N}{L_0 + 3\delta} \right)^{N-1} \]
where $L_0$ is defined in (2.1). We also define $\delta$-irregular set
\[ \Sigma(\delta) = \{y_0 : y_0 \text{ is not a } \delta \text{-regular point}\}. \]

Clearly
\[ \mu(y_0) \geq \left( \frac{\alpha N}{L_0 + 3\delta} \right)^{N-1} \]
for all $y_0 \in \Sigma(\delta)$. We shall frequently say ‘regular’, ‘irregular’ not mentioning $\delta$ if there is no confusion.

**Lemma 5.1** If $x_0$ is a regular point, then sequence $\{v_n\}$ is uniformly bounded in $L^\infty(B_{R_0}(x_0))$ for some $R_0$.

**Proof.** Let $x_0$ be a regular point. From (5.3), we can find $R_1 > 0$ such that
\[ \int_{B_{R_1}(x_0)} f_n < \left( \frac{\alpha N}{L_0 + 3\delta} \right)^{N-1}. \] (5.5)

Applying Lemma 4.1 to $f_n$ on $\Omega$ (Notice: $\|f_n\|_{L^1(\Omega)} = 1$), we have
\[ \int_{\Omega} \exp[(\alpha_N - \epsilon) v_n] dx \leq \frac{C}{\epsilon}, \]
especially $\|v_n\|_{L^\infty(B_{R_1}(x_0))} \leq C$ for some $C$ independent of $n$.

Let $\varphi_n$ be a solution of

\[
\begin{aligned}
-\Delta_N \varphi_n &= 0 \text{ in } B_{R_1}(x_0), \\
\varphi_n|_{\partial B_{R_1}(x_0)} &= v_n|_{\partial B_{R_1}(x_0)}.
\end{aligned}
\]

Then by Proposition 4.4, we have (Note $\varphi_n \leq v_n$ by the comparison principle)

$\|\varphi_n\|_{L^\infty(B_{R_1/2}(x_0))} \leq C$.

By lemma 4.3 and (5.5), if we choose $\delta'$ in Lemma 4.3 small enough,

\[
\int_{B_{R_1}(x_0)} \exp[(L_0 + \delta)d_N^{1/(N-1)}(v_n - \varphi_n)] \leq C;
\]

hence

\[
\int_{B_{R_1/2}(x_0)} \exp[(L_0 + \delta)d_N^{1/(N-1)}v_n]dx \leq C. \tag{5.6}
\]

Let $t = L_0' + d_N^{1/(N-1)}\delta/2$. Observe

\[
\log x \leq \frac{x}{e}
\]

for $x > 0$. We get

\[
\begin{aligned}
p_n \log \frac{u_n}{\nu_n^{(N-1)/p_n}} &\leq p_n \frac{u_n}{e \nu_n^{(N-1)/p_n}} \\
&\leq \frac{L_0' + d_N^{1/(N-1)}\delta/3}{\nu_n} \frac{u_n}{\nu_n^{(N-1)/p_n}} = \frac{t - d_N^{1/(N-1)}\delta/6}{\nu_n^{(N-1)/p_n}} \frac{u_n}{\nu_n} \\
&\leq \frac{t}{\nu_n} = tv_n
\end{aligned}
\]

for $n$ large enough where $\nu_n = \nu_{p_n}$ is defined in (2.1) and the last inequality is based on

\[
\lim_{n \to \infty} \nu_n^{(N-1)/p_n} = 1
\]

which follows from Corollary 3.1. Hence

\[
f_n \leq e^{tv_n}.
\]

Notice

\[
t = L_0' + d_N^{1/(N-1)}\delta/2 = (L_0 + \delta/2)d_N^{1/(N-1)} < (L_0 + \delta)d_N^{1/(N-1)};
\]

hence with the aid of (5.6) we see that $f_n$ is bounded in $L^q(B_{R_1/2}(x_0))$ where

\[
q = \frac{L_0 + \delta}{L_0 + \delta/2} > 1.
\]

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Using Proposition 4.4 again, we conclude that for large \( n \) there exists \( C > 0 \) such that
\[
\| v_n \|_{L^\infty(B_{R_1/4}(x_0))} \leq C.
\]
This proves Lemma 5.1 if we choose \( R_0 = R_1/4 \). □

Back to the proof of Theorem 1.2, we claim

\[ S = \Sigma(\delta) \]

for any \( \delta > 0 \) where \( S \) is the interior peak set with respect to \( \{v_n\} \) defined in (1.6).

Clearly, \( S \subset \Sigma \). In fact, letting \( x_0 \not\in \Sigma \), then we know that \( x_0 \) is a regular point. Hence by Lemma 5.1, \( \{v_n\} \) is uniformly bounded in a neighborhood of \( x_0 \). Therefore \( x_0 \not\in S \). Conversely, suppose \( x_0 \in \Sigma \). Then we have for every \( R > 0 \)
\[
\lim_{n \to \infty} \| v_n \|_{L^\infty(B_R(x_0))} = \infty.
\]
Otherwise, there would be some \( R_0 > 0 \) and a subsequence of \( v_n \), again denoted by \( v_n \), such that
\[
\| v_n \|_{L^\infty(B_{R_0}(x_0))} < C
\]
for some \( C \) independent of \( n \). Then
\[
f_n = \nu_p^{(N-1)} v_n^{p_n} < \nu_p^{(N-1)} C^{p_n}
\]
\[
\to 0
\]
uniformly on \( B_{R_0}(x_0) \) as \( n \to \infty \). Then
\[
\int_{B_{R_0}(x_0)} f_n \, dx = \int_{B_{R_0}(x_0)} \nu_p^{(N-1)} v_n^{p_n} \leq \epsilon_0 < \left( \frac{\alpha N}{L_0 + 3\delta} \right)^{N-1}
\]
for large \( n \) which implies that \( x_0 \) is a regular point. This proves the claim.

Back to the measure \( \mu \) defined earlier in this section. We have from (5.4)
\[
1 \geq \mu(\Omega) \geq \left( \frac{\alpha N}{L_0 + 3\delta} \right)^{N-1} \# \Sigma(\delta) = \left( \frac{\alpha N}{L_0 + 3\delta} \right)^{N-1} \# S.
\]
Hence
\[
0 \leq \# S \leq \left( \frac{L_0 + 3\delta}{\alpha N} \right)^{N-1}.
\]
Letting \( \delta \to 0 \), we get with the aid of Corollary 2.4
\[
0 \leq \# S \leq \left( \frac{L_0}{\alpha N} \right)^{N-1} \leq \left( \frac{N}{N - 1} \right)^{N-1} d_{N-1}^{-1}.
\]
This proves Theorem 1.2.
Remark 5.2 From the proof of Theorem we see that the measure $\mu$ is atomic. Actually

$$\mu = \sum_{k=1}^{\#S} a_k \delta(x_k)$$

where $S = \{x_1, x_2, ..., x_{\#S}\}$ and

$$a_k \geq \left(\frac{\alpha N}{L_0}\right)^{N-1}.$$  

The subsequence $v_n$ approaches a function $G$ in $C^{1,\alpha}_{loc}(\Omega \setminus S)$ and $G$ is $N$-harmonic in $\Omega \setminus S$ but singular on $S$.

Remark 5.3 It is also clear from the proof of Theorem 1.2 and Corollary 3.1 that the subsequence $u_n \to 0$ in $L^\infty_{loc}(\Omega \setminus S)$.

6 Further Results

So far, we haven’t touched the boundary peak sets $S'$ yet. Our next result shows that when $\Omega$ is strictly convex and $u_p$ are generic in some sense, $S'$ is empty; i.e. $B = S$.

Recall that $u_p$ are solutions of (1.1) obtained by minimizing

$$J_p(v) = \int_{\Omega} |\nabla v|^N$$

in the class

$$\mathcal{A}_p = \{v \in W^{1,N}_0(\Omega) : \|u\|_{p+1} = 1\}.$$

Let

$$J_p^\epsilon : \mathcal{A}_p \to R$$

defined by

$$J_p^\epsilon(v) = \int_{\Omega} (\epsilon + |\nabla v|^2)^{N/2}.$$  

We call $u_p$ a generic solution if there exist a sequence $\epsilon_n$ of $\epsilon$ with

$$\epsilon_n \to 0$$

and a sequence of positive minimizers $\{u'_{p,\epsilon_n}\}$ of $J_p^\epsilon$ such that $\{u'_{p,\epsilon_n}\}$ converges to $u'_p$ weakly in $W^{1,N}(\Omega)$ as $\epsilon_n \to 0$ where $u_p = cu'_p$ for some scalar $c$. Clearly $\{u'_{p,\epsilon_n}\}$ is a minimizing sequence of $J_p$ in $\mathcal{A}_p$. Actually any sequence $\{u'_{p,\epsilon_n}\}$ of minimizers of $J_p^\epsilon$ is a minimizing sequence of $J_p$, so generic solutions exist for all smooth bounded domains.
Theorem 6.1 Let $\Omega$ be a strict convex domain. Assume $u_p$ are generic solutions for all $p$. Then $S'$, the boundary peak set of $\{u_p\}$, is empty; i.e. $B = S$.

Proof. Let 

$$\{u'_{p, n}\} \to u'_p$$

weakly in $W^{1,N}(\Omega)$ where $u'_p = cu_p$ for some $c$ and $u'_{p, n}$ are positive minimizers of $J_{p, n}$. Then $u'_{p, n}$ solve

$$\begin{cases}
\text{div}(\epsilon + |\nabla u'_{p, n}|^2)^{(N-2)/2} \nabla u'_{p, n} + \lambda_n u'_{p, n} = 0 & \text{in } \Omega \\
u'_{p, n}|_{\partial \Omega} = 0
\end{cases}$$

for some $\lambda_n > 0$.

Therefore using the moving plane method for non-degenerate equations developed by B. Gidas, W.-M. Ni and L. Nirenberg in [4], we can find a neighborhood $\omega$ of $\partial \Omega$ and a cone $\Gamma$ of fixed size both depending on $\Omega$ only such that

$$u'_{p, n}(x) \leq \frac{1}{|\Gamma|} \int_{\Omega} u'_{p, n}(x) dx$$

for all $x \in \omega$. We refer to D. G. DeFigueiredo, P. L. Lions and R. D. Nussbaum [2] for details of this trick.

Since $\{u'_{p, n}\} \to u'_p$ weakly in $W^{1,N}(\Omega)$, we have

$$u'_{p, n} \to u'_p \text{ strongly in } L^1(\Omega); \\
u'_{p, n} \to u'_p \text{ almost everywhere.}$$

(6.3)

Hence passing limit in (6.2), we get

$$u'_p(x) \leq \frac{1}{|\Gamma|} \int_{\Omega} u'_p(x) dx$$

for almost all $x \in \omega$. Therefore

$$u_p(x) \leq \frac{1}{|\Gamma|} \int_{\Omega} u_p(x) dx$$

and

$$v_p(x) \leq \frac{1}{|\Gamma|} \int_{\Omega} v_p(x) dx$$

for almost all $x \in \omega$. But $\int_{\partial \Omega} v_p(x) ds \leq C$ by Lemma 4.1. Therefore $v_p$ are uniformly bounded in $L^\infty(\omega)$; hence $S' = \emptyset$. $\square$

It is interesting to see when the peak set $B$ contains one point only.

Theorem 6.2 Let $\Omega$ be a strict convex domain and $u_{p, n}$ be a sequence of generic solutions. If we further assume

$$\int_{\partial \Omega} \frac{ds}{|x-y|^{N-1}} < (2d_N)^N (\epsilon\alpha_N)^{N-1} \left( \frac{N-1}{N^2} \right)^{N-1} \left( \frac{N-1}{N} \right)^{(N-1)^2}$$

then
for some \( y \in \Omega \), then there exists a subsequence of \( u_{p_n} \), again denoted by \( u_{p_n} \), such that the peak set \( B \) of the subsequence equals the interior peak set \( S \) and it contains one point only.

**Proof.** The assertion \( B = S \) follows from Theorem 6.1. We also know \( \# B \geq 1 \) from Theorem 1.2.

Now we state a Pohozaev type identity for (1.1). The proof of this integral identity can be found in ([7], Theorem 1.1). Let \( u \in L^\infty(\Omega) \cap W^{1,q}(\Omega) \) solve

\[
\begin{align*}
-\text{div}(|\nabla u|^{q-2}\nabla u) &= g(x, u) \quad \text{in } \Omega \\
u|_{\partial \Omega} &= 0
\end{align*}
\]

where \( g \) is smooth with its growth bounded by \( |u|^{\frac{Nq}{N} - \frac{N}{q}} \) if \( q < N \) or like a polynomial in \( u \) if \( q = N \). Let \( G(x, u) = \int_0^u g(x, r)dr \). Then

\[
\int_\Omega NG(x, u)dx + (1 - \frac{N}{q}) \int_\Omega u g(x, u)dx + \int_\Omega < x - y, \nabla_x G(x, u) > dx \quad (6.4)
\]

for all \( y \in R^N \).

Apply it to (1.1). Let ‘\( y \)’ in the integral identity be ‘\( y \)’ in the statement of Theorem 6.2. Without loss of generality, we can assume \( y = 0 \). Then

\[
\frac{N}{p+1} \int_\Omega u_{p+1}^p dx = (1 - \frac{1}{N}) \int_\Omega < x, n(x) > |\frac{\partial u_p}{\partial n}|^N ds. \quad (6.5)
\]

On the other hand

\[
\int_\Omega u_p^p dx = \int_{\partial \Omega} |\frac{\partial u_p}{\partial n}|^{N-1} ds.
\]

Hence by the Holder’s inequality

\[
\int_\Omega u_p^p dx \leq \left( \int_{\partial \Omega} < x, n(x) >^{N-1} ds \right)^{1/N} \left( \int_\Omega < x, n(x) > |\frac{\partial u_p}{\partial n}|^N ds \right)^{(N-1)/N}
\]

\[
= \left( \frac{1}{N-1} \int_{\partial \Omega} < x, n(x) >^{N-1} ds \right)^{1/N} \left( \frac{N^2}{(N-1)(p+1)} \int_\Omega u_{p+1}^p dx \right)^{(N-1)/N}.
\]

Therefore

\[
\left( \frac{L_0}{\alpha N} \right)^{N-1} = \frac{1}{dN} \left( \frac{L_0}{\alpha N} \right)^{N-1}
\]

\[
= \frac{1}{dN \lim_{p \to \infty}} \left( \int_\Omega u_p^p dx \right)^{1/(N-1)}\big|_{N-1}
\]
\[
\lim_{p \to \infty} 1 = \lim_{p \to \infty} \frac{N^2}{N-1} \frac{N\alpha}{N-1} (\int_{\partial \Omega} < x, n(x) > ds)^{N-1} / N
\]

Hence from the last inequality in the proof of Theorem 1.2 we have \#\#S \leq 1; and in turn \#\#S = 1. □

**Remark 6.3** It turns out that when \( N = 2 \) the assumptions that \( \Omega \) is strict convex and that \( u_p \) are generic solutions are both superfluous for Theorem 6.1 and Theorem 6.2. In our earlier article [12], we proved the corresponding results of Theorem 6.1 and Theorem 6.2 without these two conditions. In that work we used Kelvin transform to take care of non-convex domains and we applied the moving plane method to \( u_p \) directly since the equations (1.1) are non-degenerate when \( N = 2 \).

Finally we confine ourselves to the problem when \( \Omega = B_R \), the ball of radius \( R \) centered at origin. We also consider generic solutions. Applying the moving plane method to each approximate solutions \( u_{p\epsilon,n} \), of \( u_p \), we conclude that \( u_{p\epsilon,n} \) are all radially symmetric, so are \( u_p \). Therefore \( u_p \) solve the following O.D.E.

\[
\begin{cases}
    (|u'|^{N-2}u')' + \frac{N-1}{r} |u'|^{N-2}u' + u^p = 0 \text{ in } (0, R) \\
    u'(0) = 0, \quad u(R) = 0.
\end{cases}
\]

Applying Theorem 1.2, we know \( \mathcal{B} = S = \{0\} \); otherwise there would be infinitely many peaks by the symmetry. A straightforward argument shows

\[ f_p \to \delta \]

in the sense of distribution where \( f_p \) is defined in (5.1) and \( \delta \) is the Dirac mass at 0. We can actually prove the following. We leave the proof to readers.

**Theorem 6.4** Let \( u_p \) be generic variational solutions of (1.1) on \( B_R \), the ball of radius \( R \). Then as \( p \to \infty \)

\[ v_p = \frac{u_p}{(\mathcal{F}_R u_p)^{1/(N-1)}} \to \frac{1}{\omega^{1/(N-1)}_{N-1}} \log \left( \frac{R}{r} \right) \]

in \( C^{1,\alpha}_{loc}(\overline{B_R \setminus \{0\}}) \) for some \( \alpha > 0 \) and also in the sense of distribution on \( B_R \).

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References


